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Evolution Problems Associated with Primal Lower Nice Functions

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This paper studies both local and global existence as well as uniqueness of absolutely continuous solutions of some differential inclusions involving subdifferentials of nonconvex functions. The framework is the one of primal lower nice functions in Hilbert spaces. The asymptotic behavior of the related trajectories is also addressed.

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Introduction

In a general Hilbert space H, given a smooth function f from an open subset $\mathcal{O}$ of H into $\mathbb{R}$, nonlinear ordinary differential equations (ODEs for short) of the form $-\dot{u}(t) = \nabla f(u(t))$, $t \geq T_0$, with initial value $u(T_0)=u_0 \in \mathcal{O}$, also known as the "*steepest descent problem*", have been widely studied because of their crucial interest in mechanics, physics, and optimization, especially when f is supposed to be convex. Dropping the smoothness and involving extended real valued proper lower semicontinuous (lsc) convex functions, several authors investigated the existence and further properties of locally absolutely continuous solutions of the differential inclusion

$$\dot{u}(t) + \partial f(u(t)) \ni 0 \quad \text{a.e.,} \quad \text{with} \quad u(0) = u_0 \quad \text{and} \quad f(u_0) < +\infty, \tag{1}$$

where ∂ denotes the subdifferential of convex analysis. Their general strategy (see the books of J.-P. Aubin and A. Cellina [1] and H. Brézis [6], for example, and also references therein) in proving existence results for the problem (1), was to solve related ODEs given by

$$\dot{u}_\lambda(t) + \nabla e_\lambda f(u_\lambda(t)) = 0, \tag{2}$$

where $(e_\lambda f)_{\lambda>0}$ denote the Moreau envelopes of the function f (the notation f_λ is also used by some authors), which are known to be convex and $C^{1,1}$, i.e., differentiable with

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Lipschitz continuous derivatives on H. Then, passing to the limit relying on specific closure properties of the graph of ∂f provided a solution.

Later on, in order to solve partial differential equations of parabolic type but submitted to nonconvex constraints, M. Degiovanni, A. Marino and M. Tosques introduced in their seminal paper [12] (see also [11]) , in the context of Hilbert spaces, the concept of "ϕ-monotone" subdifferential and the class of "*ϕ-convex*" functions. This collection of functions is strictly larger than the convex one, and enables the authors to prove the existence of absolutely continuous *local* solutions of differential inclusions similar to (1), where the subdifferential operator is that of Fréchet. In this new frame that can be far from smoothness and convexity, the local existence results are obtained thanks to regularization techniques via ODEs as above, involving parameterized functions that can be seen as "*local* Moreau envelopes". However, contrary to the convex case, the domain $\mathcal{O}_\lambda$ of the gradient of the regularized function in [12] depends on λ and hence the local existence results required rather subtle developments before the limit process in λ. For more details see [12], Theorems 2.4, 3.1, and 3.2.

Recently, a first relevant simplification of the preceding pioneering work has been proposed by S. Guillaume [14, 17]. It still holds in Hilbert spaces and regards a subclass of "$\phi-$convex" functions, namely the "*qualified strongly convex composite*" functions. These functions are the composition of a proper lsc *convex* function g defined on some Banach space X with a smooth mapping F from H into X having a locally Lipschitz derivative, and this under some qualification condition of Robinson type, see [14, 9, 23]:

$$(R) \qquad \mathbb{R}_+[\operatorname{dom} g - F(u_0) - DF(u_0)H] = X.$$

The approach is less complicated since the approximation $(e_\lambda g) \circ F$ used in place of $e_\lambda f$ in (2) may benefit from the convexity of g and since, in particular, it has a gradient with domain independent of λ because of the $C^{1,1}$ property of $e_\lambda g$ on the whole space X.

By the way, in [14, 17, 15, 16], some assumptions made on the function $f = g \circ F$, as (R) for eg., involve both mappings g and F appearing in (one of) its convex composite representation(s), so the existence and stability results therein depend heavily on the decomposition of f.

In 1998, Combari et al. [9] proved that "*qualified strongly convex composite*" functions on Banach spaces are actually members of the class of *primal lower nice* functions, extending in this way the earlier result of R. A. Poliquin [22, 23] established in finite dimensional spaces. The notion of primal lower nice (pln for short) function was introduced in the finite dimensional setting by R. A. Poliquin in his original and strong paper [22]. As shown in [22], this important class of lsc extended real valued functions benefits from remarkable features such as the coincidence of their proximal and Clarke subdifferentials. In the same paper, a subdifferential characterization of the pln property was also given (alias Corollary 3.4), and analogously to the convex case, it was proved the original fact that pln functions are completely determined by their subgradients in the sense that: if two lsc functions are pln at some point $\bar{x}$ of their domain and have the same proximal subgradients on a neighborhood of $\bar{x}$, then those functions differ from an additive constant near $\bar{x}$; (this being viewed as an integration result). The theory of first and second order epidifferentiability of pln functions was also undertaken in [23].

A few years later, A. B. Levy, R. A. Poliquin and L. Thibault studied in [18] pln functions in general Hilbert spaces and retrieved the subdifferential characterization of these functions. Observe that such a characterization in terms of operators provides a particularly useful way to test the pln property of functions. A similar characterization is not available for qualified convex composite functions. The authors of [18] also retrieved the coincidence of proximal and Clarke subdifferentials of pln functions. Earlier, L. Thibault and D. Zagrodny [27] extended Poliquin's integration theorem for pln functions to the setting of infinite dimensional Hilbert spaces, while some linked and more general inclusions of subdifferentials have been recently studied by F. Bernard et al. in [4]. Following the route opened by the paper [24] of R. A. Poliquin and R. T. Rockafellar concerning "prox-regular" functions on finite dimensional spaces and by some results on pln functions in [27], quite recent works by F. Bernard and L. Thibault [3, 2] demonstrated some local regularity properties of the Moreau envelopes and the related proximal mappings of pln functions. All these facts have strongly motivated the current study.

So our intention, in this paper, is to shed a new light on plausible uses of pln functions which are known to be a subclass of ϕ-convex functions. Without using any results from convex analysis or monotone operators theory, we aim at studying specific properties of (1) in the context of pln functions f. We will show that, in this setting, existence results arise in a natural and fluent way, in particular because of Proposition 2.8 establishing that the domain of the gradients of the local Moreau envelopes of f does not depend on λ. Several ideas from [6] and [12] will be used in our development.

In the first section, we recall some needed notions and establish some useful "*closure properties*" of the graph of the (proximal) subdifferential of pln functions. The second section is devoted to the study of the local existence and uniqueness of solutions of the evolution problem (1) but for f pln. The third section then addresses the global existence issue along with the asymptotic behavior of the trajectories.

1. Definitions and preliminary results

Let us fix some notations and recall a few fundamental definitions.

Throughout all the paper, H stands for a *real Hilbert space*, $\langle\cdot,\cdot\rangle$ is its inner product and $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$ is the associated norm.

We will denote by $B(x,\varepsilon)$ (resp. $B[x,\varepsilon]$) the *open* (resp. *closed*) ball of H centered at x with radius $\varepsilon > 0$ and by $\mathbb{B} := B[0,1]$ its closed unit ball.

Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a function and let $x \in \operatorname{dom} f$, i.e., $f(x) < +\infty$.

The *proximal* subdifferential of f at x is the set $\partial_P f(x)$ of all elements $v \in H$ for which *there exist* $\varepsilon > 0$ and $r > 0$ such that

$$\langle v, y - x\rangle \le f(y) - f(x) + r\|y - x\|^2 \quad \text{for all } y \in B(x,\varepsilon).$$

The *Fréchet* subdifferential $\partial_F f(x)$ of f at x is defined by $v \in \partial_F f(x)$ provided that for each $\varepsilon > 0$, there exists some $\eta > 0$ such that for all $y \in B(x,\eta)$,

$$\langle v, y - x\rangle \le f(y) - f(x) + \varepsilon\|y - x\|.$$

When $f(x) = +\infty$, by convention, $\partial_P f(x) = \emptyset = \partial_F f(x)$.

Recall also that the *limiting* subdifferential (see [8] and [19]), also known as Mordukhovich subdifferential, of f at x is given by

$$\partial_L f(x) := \{w - \lim\ v_n\ :\ v_n \in \partial_F f(x_n),\ x_n \to_f x\},$$

where $x_n \to_f x$ means $\|x_n - x\| \to 0$ with $f(x_n) \to f(x)$. Clearly, one always has $\partial_P f(x) \subset \partial_F f(x) \subset \partial_L f(x)$.

The pln functions, introduced by Poliquin in [22] can be defined as follows.

Definition 1.1. (see [22, 18]) Let $f : H \to \mathbb{R}\cup\{+\infty\}$ be a proper function (i.e., $f \not\equiv +\infty$) and consider $u_0 \in \operatorname{dom} f$. The function f is said to be primal lower nice (pln for short) at u_0, if there exist positive constant real numbers s_0, c_0, Q_0 such that for all $x \in B[u_0, s_0]$, for all $q \geq Q_0$ and all $v \in \partial_P f(x)$ with $\|v\| \leq c_0 q$, one has

$$f(y) \geq f(x) + \langle v, y - x\rangle - \frac{q}{2}\|y - x\|^2 \tag{3}$$

for each $y \in B[u_0, s_0]$.

Remark 1.2. It is straightforward to observe that each extended real valued convex function is pln at each point of its domain as well as functions that are convex up to a square. Another example of pln functions is given by qualified convex composite functions. For more details, one is invited to see [22, 24] for the finite dimensional setting, and [27, 18, 9, 14] for a study in arbitrary Hilbert spaces.

By the way, it is worth mentioning that the pln behavior of f can be characterized by some "local linear hypomonotonicity of $\partial_P f$" that we recall in Proposition 1.3 below. This characterization is due to Poliquin [22] in the finite dimensional setting and to Levy-Poliquin-Thibault [18, Corollary 2.3] in the Hilbert context.

Proposition 1.3 (from [18, Proposition 2.2 and Corollary 2.3]). *Let $f : H \to \mathbb{R}\cup\{+\infty\}$ be a lsc function that is finite at u_0. The following are equivalent:*

(a) *f is pln at u_0.*

(b) *There exist positive constants s_0, c_0, Q_0, such that*

$$\langle v_1 - v_2, x_1 - x_2\rangle \geq -q\|x_1 - x_2\|^2 \tag{4}$$

for any $v_i \in \partial_P f(x_i)$ with $\|v_i\| \leq c_0 q$ whenever $q \geq Q_0$ and $x_i \in B[u_0, s_0]$, $i=1,2$.

Clearly, if f is pln at u_0 with constants s_0, c_0, Q_0, summing up two suitable inequalities (3) one directly obtains that

$$\langle v_1 - v_2, x_1 - x_2\rangle \geq -q\|x_1 - x_2\|^2$$

whenever $v_i \in \partial_P f(x_i)$ with $\|v_i\| \leq c_0 q$ whenever $q \geq Q_0$ and $x_i \in B[u_0, s_0]$, $i=1,2$.
The implication $(b) \Rightarrow (a)$ is less immediate. For more explanations we refer the reader to [18, Proposition 2.2 and Corollary 2.3].

Let us underline that such a useful subdifferential test of the pln property is available neither for the qualified convex composite property nor for ϕ-convexity (see the introduction) with general functions ϕ.

Remark 1.4. Furthermore, note according to Definition 1.1 that, if f is pln at u_0 with constants s_0, c_0, Q_0, then for any positive real numbers ν_0, η_0 such that $\nu_0 + \eta_0 = s_0$, the function f remains pln at each point of $B[u_0, \eta_0] \cap \operatorname{dom} f$, with constants ν_0, c_0, Q_0.

Another result due to Levy-Poliquin-Thibault is of great importance. It ensures that, if a function $f : H \to \mathbb{R} \cup \{+\infty\}$ is pln at $u_0 \in \operatorname{dom} f$ then for all x in a neighborhood of u_0, the proximal subdifferential of f at x agrees with the Clarke subdifferential of f at x, i.e., $\partial_P f(x) = \partial_C f(x)$. See [8] for the definition of the Clarke subdifferential.

Remark 1.5. In particular, whenever f is pln at some point $u_0 \in \operatorname{dom} f$, we have for all x in some neighborhood of u_0

$$\partial_P f(x) = \partial_F f(x) = \partial_L f(x) = \partial_C f(x).$$

In this case, we simply denote by $\partial f(x)$ the common subdifferential, and by $\partial^0 f(x)$ its element of minimum norm for $x \in \operatorname{dom} \partial f$, i.e., $\|\partial^0 f(x)\| = \min\{\|v\| : v \in \partial f(x)\}$. Further, it entails that the definition of the pln property is independent of the involved subdifferential operator.

In addition, the graph of the (proximal) subdifferential of a pln function enjoys some useful "closure" properties.

Proposition 1.6. *Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc function. Assume that f is pln at $x \in \operatorname{dom} f$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence converging strongly to x in H and let $(v_n)_{n\in\mathbb{N}}$ be a sequence converging weakly to some v in H with $v_n \in \partial_F f(x_n)$ for large $n \in \mathbb{N}$. Then*

$$v \in \partial_P f(x) \quad \text{and} \quad \lim_{n\to+\infty} f(x_n) = f(x).$$

Proof. By Remark 1.5, we may suppose that $\partial_P f(x_n) = \partial_F f(x_n)$ for all n. Take positive real numbers s, c and Q such that

$$f(y) \geq f(x') + \langle z, y - x' \rangle - \frac{q}{2}\|y - x'\|^2 \tag{5}$$

whenever $y, x' \in B[x, s], q \geq Q$ and $z \in \partial_P f(x')$ with $\|z\| \leq cq$. Clearly for n large enough, one has $x_n \in B[x, s]$ and $K := \sup_{n\in\mathbb{N}} \|v_n\| < +\infty$ since $(v_n)_{n\in\mathbb{N}}$ is weakly convergent. Hence for any large integer n and any $y \in B[x, s]$, via (5) one has

$$f(y) \geq f(x_n) + \langle v_n, y - x_n \rangle - \frac{1}{2}\max\{Q, Kc^{-1}\}\|y - x_n\|^2. \tag{6}$$

Passing to the inferior limit when $n \to \infty$ in the previous inequality, and making use of the lower semicontinuity of f one gets

$$f(y) \geq f(x) + \langle v, y - x \rangle - \frac{1}{2}\max\{Q, Kc^{-1}\}\|y - x\|^2 \quad \text{for all } y \in B[x, s].$$

This means that $v \in \partial_P f(x)$ as claimed. Further, in the special case when $y = x$ in (6), one obtains

$$f(x) \geq f(x_n) + \langle v_n, x - x_n \rangle - \frac{1}{2}\max\{Q, Kc^{-1}\}\|x - x_n\|^2$$

for each large n. This directly leads to $f(x) \geq \limsup_{n\to+\infty} f(x_n)$ which combined with the lower semicontinuity of f at x entails

$$f(x) \geq \limsup_{n\to+\infty} f(x_n) \geq \liminf_{n\to+\infty} f(x_n) \geq f(x).$$

□

We then derive the following.

Corollary 1.7. *Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc function that is pln at $x \in \operatorname{dom} f$. If there exists a sequence $(x_n)_n$ in $\operatorname{dom} \partial_F f$ with $x_n \overset{\|\cdot\|}{\to} x$ and $\sup_{n\in\mathbb{N}} \|\partial_F^0 f(x_n)\| < +\infty$, then $x \in \operatorname{dom} \partial f$ and $\lim_{n\to+\infty} f(x_n) = f(x)$.*

Proof. Fix any subsequence (x_n') of (x_n). Take a subsequence (x_n'') of (x_n') such that $(\partial_F^0 f(x_n''))$ converges weakly to some v. By Proposition 1.6, $\partial f(x) \neq \emptyset$ and one has $f(x_n'') \to f(x)$. So all the sequence $(f(x_n))$ converges to $f(x)$. □

For any $x \in \operatorname{dom} \partial_F f$, the element $\partial_F^0 f(x)$ of minimum norm is connected with the function $d(x) := d(0, \partial_F f(x))$ via the equality $d(x) = \|\partial_F^0 f(x)\|$. The lsc property of the natural extension of that function established in the lemma below will be of importance in our analysis.

Lemma 1.8. *Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc function. Assume that f is pln at $y \in \operatorname{dom} f$. Then, the function $d(\cdot) : H \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$d(x) := \begin{cases} +\infty & \text{if } x \notin \operatorname{dom} \partial_F f \\ d(0, \partial_F f(x)) & \text{otherwise,} \end{cases}$$

is lsc at y with respect to the strong topology of H.

Proof. Clearly, when $x \in \operatorname{dom} \partial_F f$, the set $\partial_F f(x)$ is nonempty, convex, and closed in the Hilbert space H, and $d(x) = \|\partial_F^0 f(x)\|$, where $\partial_F^0 f(x)$ is the element of minimum norm of $\partial_F f(x)$. Consider any sequence $(y_n)_{n\geq 1}$ that converges in $(H, \|\cdot\|)$ to y. We will show that

$$d(y) \leq \liminf_{n\to+\infty} d(y_n). \tag{7}$$

If $\liminf_{n\to+\infty} d(y_n) = +\infty$, then (7) is obvious. So, let us suppose that $D := \liminf_{n\to+\infty} d(y_n) \in \mathbb{R}$. Consider some subsequence $(y_{n_k})_{k\geq 1}$ such that $D = \lim_{k\to+\infty} d(y_{n_k})$. The sequence $(d(y_{n_k}))_k$ is then bounded in $\mathbb{R}$, with $d(y_{n_k}) = \|\partial_F^0 f(y_{n_k})\|$ for each integer k. So, up to a subsequence that we do not relabel, $(\partial_F^0 f(y_{n_k}))_k$ converges weakly to some vector v in H. Applying Proposition 1.6, we conclude that $v \in \partial_F f(y)$. Next, the weak lower semicontinuity of the norm of H ensures that

$$\|\partial_F^0 f(y)\| \leq \|v\| \leq \liminf_{k\to+\infty} \|\partial_F^0 f(y_{n_k})\| = \lim_{k\to+\infty} d(y_{n_k}) = D,$$

that is, $\|\partial_F^0 f(y)\| = d(y) \leq D = \liminf_{n\to+\infty} d(y_n)$. Thus (7) holds true for any sequence converging strongly to y. The proof is complete. □

The following other closure property will be frequently used in the study of our evolution problems.

Lemma 1.9. *Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc function which is pln at $u_0 \in \operatorname{dom} f$ with positive constants s_0, c_0, Q_0. Let T_0 and T be real numbers with $T_0 < T$, and let $\eta_0 \in]0, s_0[$. Consider $v(\cdot) \in L^2([T_0, T]; H)$ and let $u(\cdot)$ be a mapping from $[T_0, T]$ into H. Let $(u_n(\cdot))_n$ be a sequence of mappings from $[T_0, T]$ into H and $(v_n(\cdot))_n$ be a sequence in $L^2([T_0; T]; H)$. Assume:*

(i) $\{u_n(t) : n \in \mathbb{N}\} \subset B[u_0, \eta_0] \cap \operatorname{dom} f$ *for almost every* $t \in [T_0, T]$,
(ii) $(u_n)_n$ *converges almost everywhere to some mapping* u *with* $u(t) \in \operatorname{dom} f$ *for almost every* $t \in [T_0, T]$,
(iii) $v_n \to v$, *with respect to the weak topology of* $L^2([T_0, T]; H)$, *and*
(iv) *for each* $n \geq 1$, $v_n(t) \in \partial f(u_n(t))$ *for almost every* $t \in [T_0, T]$.

Then, for almost all $t \in [T_0, T]$, $v(t) \in \partial f(u(t))$.

Proof. Denote by N a Lebesgue-null subset of $[T_0, T]$ such that (i), (ii), and (iv) hold for all $t \in [T_0, T]\backslash N$ and all $n \in \mathbb{N}$. By virtue of (iii), the sequence $(v_n(\cdot))_{n\geq 1}$ is bounded in $L^2([T_0, T]; H)$. Obviously, it amounts to saying that $(\|v_n(\cdot)\|_H)_{n\geq 1}$ is bounded in $L^2([T_0, T])$. Hence, one finds some element $g \in L^2([T_0, T])$ and some subsequence of $(v_n(\cdot), \|v_n(\cdot)\|_H)$ (that we do not relabel) which converges weakly in $L^2([T_0, T]; H \times \mathbb{R})$ to $(v(\cdot), g(\cdot))$.

Then, the Mazur's lemma provides a sequence $(h_n(\cdot))_n$ of $L^2([T_0, T]; H\times\mathbb{R})$ converging strongly to $(v(\cdot), g(\cdot))$ in $L^2([T_0, T]; H\times\mathbb{R})$ and satisfying

$$h_n(\cdot) \in co\{(v_k(\cdot), \|v_k(\cdot)\|_H) : k \geq n\} \quad \text{for each integer } n \geq 1.$$

This means that, given any $n \in \mathbb{N}$, there exist a finite set $K_n \subset \{k\in\mathbb{N} : k\geq n\}$ and real constants $\alpha_{n,k} \geq 0$ for each $k \in K_n$, such that $\sum_{k\in K_n} \alpha_{n,k} = 1$ and

$$h_n(\cdot) = \sum_{k\in K_n} \alpha_{n,k}(v_k(\cdot), \|v_k(\cdot)\|_H) \quad \text{in } L^2([T_0, T]; H \times \mathbb{R}).$$

Next, one finds an increasing map $\sigma : \mathbb{N} \to \mathbb{N}$ and a Lebesgue-null set $N_\sigma \subset [T_0, T]$ such that for each $t \in [T_0, T]\backslash N_\sigma$ the sequence $(h_{\sigma(n)}(t))_{n\geq 1}$ converges to $(v(t), g(t))$ with respect to the strong topology of $H \times \mathbb{R}$. Putting $\mathcal{N} := N_\sigma \cup N$, for each $t \in [T_0, T]\backslash\mathcal{N}$, one has

$$\sup_{n\in\mathbb{N}} \sum_{k\in K_{\sigma(n)}} \alpha_{\sigma(n),k}\|v_k(t)\|_H =: S_t \in \mathbb{R}. \tag{8}$$

Now, fix any $\varepsilon > 0$ and arbitrary $t \in [T_0, T]\backslash\mathcal{N}$. By virtue of (ii) and the lower semicontinuity of f at $u(t)$, one can choose some integer $n_{\varepsilon,t} \geq 1$ such that, for each integer $n \geq n_{\varepsilon,t}$,

$$f(u_n(t)) \geq f(u(t)) - \varepsilon \quad \text{and} \quad \|u_n(t) - u(t)\| \leq \varepsilon. \tag{9}$$

Then, fix arbitrary $n \geq n_{\varepsilon,t}$ and $x \in B[u_0, s_0]$. In view of (i) and (iv), for each $k \in K_{\sigma(n)}$, the pln property of f at u_0 ensures that

$$f(x) \geq f(u_k(t)) + \langle v_k(t), x - u_k(t)\rangle - \frac{1}{2}(Q_0 + c_0^{-1}\|v_k(t)\|)\|x - u_k(t)\|^2.$$

Hence, making use of (9), we obtain that

$$\begin{aligned} f(x) \ &\geq \ f(u(t)) - \varepsilon + \langle v_k(t), x - u(t)\rangle - \varepsilon\|v_k(t)\| \\ &\qquad - \frac{1}{2}(Q_0 + c_0^{-1}\|v_k(t)\|)(\|x - u(t)\|^2 + 2\varepsilon\|x - u(t)\| + \varepsilon^2). \end{aligned}$$

Next, multiplying the last inequality by $\alpha_{\sigma(n),k}$ for each $k \in K_{\sigma(n)}$ and summing up on $k \in K_{\sigma(n)}$ yield

$$\begin{aligned} f(x) \ &\geq \ f(u(t)) - \varepsilon + \langle \sum_{k\in K_{\sigma(n)}} \alpha_{\sigma(n),k} v_k(t), x - u(t)\rangle - \varepsilon(\sum_{k\in K_{\sigma(n)}} \alpha_{\sigma(n),k}\|v_k(t)\|) \\ &\qquad - \frac{1}{2}(Q_0 + c_0^{-1}(\sum_{k\in K_{\sigma(n)}} \alpha_{\sigma(n),k}\|v_k(t)\|))(\|x - u(t)\|^2 + 2\varepsilon\|x - u(t)\| + \varepsilon^2). \end{aligned}$$

So in the light of (8), it follows that

$$\begin{aligned} f(x) \ &\geq \ f(u(t)) + \langle \sum_{k\in K_{\sigma(n)}} \alpha_{\sigma(n),k} v_k(t), x - u(t)\rangle - \frac{1}{2}(Q_0 + c_0^{-1}S_t)\|x - u(t)\|^2 \\ &\qquad - \varepsilon - \varepsilon S_t - \frac{1}{2}(Q_0 + c_0^{-1}S_t)(2\varepsilon\|x - u(t)\| + \varepsilon^2), \end{aligned}$$

that holds for all $n \geq n_{\varepsilon,t}$. Letting $n \to +\infty$, we clearly see that

$$\begin{aligned} f(x) \ &\geq \ f(u(t)) + \langle v(t), x - u(t)\rangle - \frac{1}{2}(Q_0 + c_0^{-1}S_t)\|x - u(t)\|^2 \\ &\qquad - \varepsilon(1 + S_t + \frac{1}{2}(Q_0 + c_0^{-1}S_t))(2\|x - u(t)\| + \varepsilon), \end{aligned}$$

for all $x \in B[u_0, s_0]$ and any $\varepsilon > 0$. Finally, making $\varepsilon \downarrow 0$ guarantees that

$$f(x) \geq f(u(t)) + \langle v(t), x - u(t)\rangle - \frac{1}{2}(Q_0 + c_0^{-1}S_t)\|x - u(t)\|^2 \tag{10}$$

for each $x \in B[u_0, s_0]$. To conclude, just recall that $u(t) \in B[u_0, \eta_0]$ by (i) and (ii), so that $v(t) \in \partial_P f(u(t))$ whenever $t \in [T_0, T]\backslash\mathcal{N}$. The proof is complete. □

2. Autonomous evolution problems

We will first study uniqueness and local existence of solutions, in a sense to be made clear, of the evolution problem

$$(P)\begin{cases} -\dot{u}(t) \in \partial_P f(u(t)) & \text{a.e. in } I \\ u(T_0) = u_0 \end{cases}$$

where $T_0 \in [0, +\infty[$ is given as the initial time, I is some line segment in $\mathbb{R}$ starting from T_0, and $u_0 \in \operatorname{dom} f$ is some point where f is pln.

As already mentioned, in Hilbert spaces, the differential inclusions involving a subdifferential operator have been studied in several frameworks.

H. Brézis [6] gave an overview of the case when the function f is proper lsc and convex, using results on maximal monotone operators and the associated semigroups of contractions, (see also [1]).

Beyond the convex setting, Degiovanni-Marino-Tosques [12] introduced the concepts of ϕ-convexity of functions and ϕ-monotonicity of subdifferentials to study the corresponding evolution problems, using variational tools.

And, involving the (Clarke) subdifferential of a qualified strongly convex composite function, S. Guillaume [14, 17] also obtained various uniqueness and existence results.

In this section our objective is to show how the local Moreau envelopes lead to a fluent development establishing local existence results for (P) when the function f is pln at u_0.

2.1. *A priori* comparison of local solutions

To begin with, let us establish a lemma pointing out *a priori* comparison between *absolutely continuous solutions* of (P) for pln functions f. The proof of the lemma follows classical arguments.

Let us denote:

(E_1) $f : H \to \mathbb{R} \cup \{+\infty\}$ is a proper lsc function that is pln at some point $u_0 \in \operatorname{dom} f$ with constants s_0, c_0, Q_0.

Lemma 2.1. *Assume* (E_1). *Let real numbers* $T_0 \geq 0$ *and* $T > T_0$. *Let* $\tau \in]T_0, T]$ *and* $u(\cdot)$, $v(\cdot)$ *from* $[T_0, \tau]$ *into* $B(u_0, s_0)$ *be two absolutely continuous solutions of* (P) *over* $[T_0, \tau]$. *Then*

(a) *for any* $s, t \in [T_0, \tau]$ *with* $s \leq t$ *one has*

$$\|v(t)-u(t)\| \leq \|v(s)-u(s)\| \exp[Q_0(t-s) + c_0^{-1} \int_s^t (\|\dot{u}(r)\| + \|\dot{v}(r)\|)dr];$$

(b) *for all* $s, t \in [T_0, \tau]$ *in the domain of* $\dot{u}$ *with* $s \leq t$,

$$\|\dot{u}(t)\| \leq \|\dot{u}(s)\| \exp[Q_0(t-s) + 2c_0^{-1} \int_s^t \|\dot{u}(r)\|\, dr].$$

Proof. Denote by $\mathcal{N}$ a Lebesgue-null subset of $[T_0, \tau]$ out of which the inclusion of (P) holds for u and v. Fix any $t \in [T_0, \tau] \setminus \mathcal{N}$ and observe that

$$\frac{d}{dt}\|v(t) - u(t)\|^2 = 2\langle \dot{v}(t) - \dot{u}(t), v(t) - u(t)\rangle.$$

Since $\{u(t), v(t)\} \subset B(u_0, s_0)$, the linear hypomonotonicity (4) of the subdifferential associated with the pln property of f at u_0 entails

$$\begin{aligned} &\langle -\dot{v}(t) - (-\dot{u}(t)), v(t) - u(t)\rangle \\ \geq\ & -(Q_0 + c_0^{-1}(\| - \dot{v}(t)\| + \| - \dot{u}(t)\|))\|v(t) - u(t)\|^2 \\ \geq\ & -(Q_0 + c_0^{-1}(\|\dot{u}(t)\| + \|\dot{v}(t)\|))\|v(t) - u(t)\|^2. \end{aligned}$$

The conclusion of (a) is then an immediate consequence of Gronwall's lemma.

Now fix s, t in the domain of $\dot{u}$ with $s \le t < \tau$ and fix a sequence $\delta_n \downarrow 0$ with $\delta_n < \tau - t$. To obtain (b) it suffices to apply (a) with $v(\cdot) = u(\cdot + \delta_n)$ over $[T_0, t]$, to divide by δ_n, and to pass to the limit in n, observing that $\int_s^t \|\dot{u}(r + \delta_n)\| \, dr = \int_{s+\delta_n}^{t+\delta_n} \|\dot{u}(r)\| \, dr \to \int_s^t \|\dot{u}(r)\| \, dr$. □

2.2. Uniqueness and local existence of solutions of (P)

The existence theorems to be obtained in this work, involve the Moreau envelopes and proximal mappings of suitable functions. Let us recall the definitions and some related properties.

Definition 2.2 (see [20] and [26]). Let X be a normed vector space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc function. Consider positive real numbers λ and ε. The Moreau envelope of f is the function from X into $\mathbb{R} \cup \{-\infty, +\infty\}$ defined by

$$e_\lambda f(x) := \inf_{y \in X} \{f(y) + \frac{1}{2\lambda}\|x - y\|^2\}, \quad x \in X,$$

and the associated proximal map is the set-valued operator

$$P_\lambda f(x) := \operatorname{argmin}_{y \in X} \{f(y) + \frac{1}{2\lambda}\|x - y\|^2\}, \quad x \in X.$$

The local Moreau envelope of f associated with λ and ε is defined by

$$e_{\lambda,\varepsilon} f(x) := \inf_{\|y\| \le \varepsilon} \{f(y) + \frac{1}{2\lambda}\|x - y\|^2\}, \quad x \in X,$$

and the corresponding local proximal mapping is

$$P_{\lambda,\varepsilon} f(x) := \operatorname{argmin}_{\|y\| \le \varepsilon} \{f(y) + \frac{1}{2\lambda}\|x - y\|^2\}, \quad x \in X.$$

Observe the elementary facts that for any $\lambda > 0$, the function $e_\lambda f$ is (see [21]) the infimum convolution of f and $\frac{1}{2\lambda}\| \cdot \|^2$, i.e., $e_\lambda f = f \Box \frac{1}{2\lambda}\| \cdot \|^2$. Further, given any $\lambda, \varepsilon > 0$ and $x \in X$, one can write

$$e_{\lambda,\varepsilon} f(x) = e_\lambda(f + \psi(\cdot, B[0, \varepsilon]))(x),$$

and

$$P_{\lambda,\varepsilon} f(x) = P_\lambda(f + \psi(\cdot, B[0, \varepsilon]))(x).$$

Here, for any subset S of X, $\psi(\cdot, S)$ denotes the indicator function of S, i.e., $\psi(x, S) = 0$ if $x \in S$ and $\psi(x, S) = +\infty$ otherwise.

The lemma below will play a crucial role in the sequel. It is due to Correa-Jofré-Thibault.

Lemma 2.3 (from [10]). *Let X be a reflexive Banach space. If the infimum convolution*

$$(f \Box g)(a) = \inf_{y \in X} \{f(y) + g(a - y)\}$$

is exact, that is, if the preceding infimum is attained at some $\bar{y}$, then

$$\partial_F (f \Box g)(a) \subset \partial_F f(\bar{y}) \cap \partial_F g(a - \bar{y}).$$

In addition, handling Moreau envelopes, it will be helpful to keep in mind Lemma 4.2 from Thibault-Zagrodny [27] that states:

Lemma 2.4. *Let X be a normed vector space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function with $f(\bar{x}) < +\infty$. Let s be a positive number such that f is bounded from below over $B[\bar{x}, s]$ and let*

$$(T.Z) \qquad e_{\lambda,\bar{x},s} f(x) := \inf_{y \in B[\bar{x},s]} \{f(y) + \frac{\lambda}{2}\|x - y\|^2\} \quad \text{for all } x \in X.$$

Then, there exists $\lambda_0 > 0$ such that, for each $x \in B[\bar{x}, \frac{s}{4}]$ and each $\lambda \geq \lambda_0$, the infimum above is equal to the infimum over all points y in the open ball $B(\bar{x}, \frac{3s}{4})$, and any point attaining the infimum in $(T.Z)$ (whenever such point exists) must belong to $B(\bar{x}, \frac{3s}{4})$.

The following results from Bernard et al. [3, 4] (see also [2], Chap. 3, Prop. 3.3.4, 3.3.5) provide useful regularity properties of Moreau envelopes and proximal mappings of a pln function in the Hilbert setting. We also refer the reader to the important paper by Poliquin and Rockafellar [24] which contains several similar earlier results related to prox-regular functions in the finite dimensional context.

Proposition 2.5 (alias Proposition 3.2 in [4] and Proposition 3.3.4 in [2]). *Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc function that is pln at $0 \in \operatorname{dom} f$ with parameters ε, c, τ such that $\varepsilon < c$ and f be minorized on $B[0, \varepsilon]$. (Such an ε always exists by the lsc property of f).*

Then, there exists $r_0 > 0$ such that for any $r \geq r_0$,

$$P_{\frac{1}{r},\varepsilon} f = (I + \frac{1}{r} T_{t_r})^{-1} \quad \text{on } B(0, \frac{\varepsilon}{4}) \tag{11}$$

where $t_r := r\varepsilon$ and given $t > 0$, T_t is the truncation whose graph is

$$\operatorname{gph} T_t := \{(x, x^*) \in \operatorname{gph} \partial f : \|x\| < \varepsilon \quad \text{and} \quad \|x^*\| \leq t\}.$$

Moreover, both mappings in (11) are nonempty, single-valued and Lipschitz continuous on $B(0, \frac{\varepsilon}{4})$.

The lemma below points out the Lipschitz modulus of the local proximal mappings appearing in (11) above. Its content appears in [26] inside the proof of Proposition 13.37 and it was explicitely stated and proved in [3, Lemma 3.1].

Lemma 2.6. *Let $\bar{r} \in [0, +\infty[$ and $T : H \rightrightarrows H$ such that $(\bar{r}I + T)$ be monotone. Then, for any $r > \bar{r}$, $(I + r^{-1}T)^{-1}$ is monotone, single-valued, and Lipschitz continuous on its domain, with $\frac{r}{r-\bar{r}}$ as Lipschitz modulus.*

Besides, the following proposition from [3, Proposition 5.1] (see also [2]) characterizes the $\mathcal{C}^1$ regularity of the Moreau envelopes.

Proposition 2.7. *Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc function minorized by a quadratic function. Consider $\lambda > 0$ and let U be an open subset of H. The following properties are equivalent:*

(a) $e_\lambda f$ *is* $\mathcal{C}^1$ *on* U*;*
(b) $P_\lambda f$ *is nonempty, single-valued and continuous on* U*.*

When these properties hold, $\nabla e_\lambda f = \lambda^{-1}(I - P_\lambda f)$ *on* U*.*

As already mentioned, those features of Moreau envelopes and proximal mappings will be key tools in establishing local existence of an absolutely continuous solution of (P) and they lead to the statement of the following proposition.

Proposition 2.8. *Let* $f : H \to \mathbb{R} \cup \{+\infty\}$ *be a proper lsc function. Assume that* f *is pln at* $u_0 \in \operatorname{dom} f$ *with constants* s_0*,* c_0*,* Q_0*, such that*

$$\inf\{f(x) : x \in B[u_0, s_0]\} \in \mathbb{R} \quad \text{and} \quad s_0 < c_0.$$

Consider $(x_0, y_0) \in \operatorname{gph} \partial f$ *with* $\|x_0 - u_0\| < \frac{s_0}{16}$*. Define*

$$\bar{f}(\cdot) := f(\cdot) + \psi(\cdot, B[x_0, \frac{s_0}{2}]),\ Q := \max(2\|y_0\| c_0^{-1}; Q_0) \quad \text{and} \quad c := c_0 - \|y_0\| Q^{-1}.$$

Then, there exits some threshold $\bar{\lambda}_0 \in]0, \frac{s_0}{32(\|y_0\|+1)}[$ *(depending on* u_0 *via* x_0*,* y_0*,* s_0*,* Q_0*,* $f(x_0), \inf_{B[u_0,s_0]} f$*) such that, for any* $\lambda \in]0, \bar{\lambda}_0]$*:*

(a) $e_\lambda \bar{f}$ *is* $\mathcal{C}^{1,1}$ *on* $B(u_0, \frac{s_0}{32})$*,*
(b) $P_\lambda \bar{f}$ *is nonempty, single-valued, and Lipschitz continuous on* $B(u_0, \frac{s_0}{32})$ *with Lipschitz modulus* $\bar{k} := (1 - \frac{s_0}{2c})^{-1}$*,*
(c) $P_\lambda \bar{f}(x_0 + \lambda y_0) = x_0$*,*
(d) $\nabla e_\lambda \bar{f} = \lambda^{-1}(I - P_\lambda \bar{f})$ *on* $B(u_0, \frac{s_0}{32})$*,*
(e) $\|\nabla e_\lambda \bar{f}(x_0)\| \le (1 - \frac{s_0}{2c})^{-1} \|y_0\|$*,*
(f) $P_\lambda \bar{f}(B(u_0, \frac{s_0}{32})) \subset B(u_0, \frac{7}{16} s_0)$*.*

Moreover, given any $x \in B(u_0, \frac{s_0}{32})$

(g) $\nabla e_\lambda \bar{f}(x) \in \partial f(P_\lambda \bar{f}(x))$*,*
(h) $\|x - x_0\| \ge [1 - \lambda(Q_0 + c_0^{-1}((1 - \frac{s_0}{2c})^{-1} \|y_0\| + \|\nabla e_\lambda \bar{f}(x)\|))] \|P_\lambda \bar{f}(x) - P_\lambda \bar{f}(x_0)\|$*,*
(i) *given any other* $y \in B(u_0, \frac{s_0}{32})$ *one has* $\langle \nabla e_\lambda \bar{f}(x) - \nabla e_\lambda \bar{f}(y), P_\lambda \bar{f}(x) - P_\lambda \bar{f}(y) \rangle \ge -\bar{k}^{-2} \max\{Q_0, c_0^{-1} \|\nabla e_\lambda \bar{f}(x)\|, c_0^{-1} \|\nabla e_\lambda \bar{f}(y)\|\} \|x - y\|^2$*.*
(j) *The operator* $\nabla e_\lambda \bar{f}(\cdot)$ *satisfies the linear hypomonotonicity* (4) *around* u_0 *with constants* $\bar{s}$*,* $\bar{c}$*,* $\bar{Q}$*, where* $\bar{s}$ *is any number in* $]0, \frac{s_0}{32}[$*,* $\bar{c} := c_0/\bar{k}^2$*, and* $\bar{Q} := Q_0 \bar{k}^2$*.*

Proof. Let us define $g := f + \psi(\cdot, B[u_0, s_0])$ and for all $x \in H$, put

$$F(x) := g(x + x_0) - f(x_0) - \langle y_0, x \rangle.$$

Clearly, g is proper, lsc and bounded from below on H, with $\operatorname{dom} g = \operatorname{dom} f \cap B[u_0, s_0]$. Further it is not difficult to see that F is lsc on H, $F(0) = 0$ and for any $q \ge Q$ and any $(x, x^*) \in \partial_P F$ with $\|x\| < \frac{s_0}{2} =: \varepsilon$ and $\|x^*\| \le cq$, we have

$$x^* + y_0 \in \partial_P f(x + x_0) \quad \text{and} \quad \|x^* + y_0\| \le c_0 q.$$

Then the pln property of f at u_0 ensures that F is pln at 0 with constants ε, c and Q. Since $y_0 \in \partial f(x_0)$, one also has $0 \in \partial F(0)$. Further, note that $\varepsilon < c$, according to the choices of c and Q, and that F is bounded from below on $B[0,\varepsilon]$. As regards the Moreau envelopes, for any $x \in H$ and $\lambda > 0$, one can write

$$\begin{aligned} & e_\lambda(F+\psi(\cdot, B[0,\varepsilon]))(x) = \inf_{u\in H}\{F(u)+\psi(u,B[0,\varepsilon]) + \frac{1}{2\lambda}\|x-u\|^2\} \\ = \;& -f(x_0)+\langle y_0,x_0\rangle + \inf_{z\in H}\{g(z)-\langle y_0,z\rangle+\psi(z,B[x_0,\varepsilon])+\frac{1}{2\lambda}\|x+x_0-z\|^2\}, \end{aligned}$$

and hence

$$\begin{aligned} & e_\lambda(F+\psi(\cdot,B[0,\varepsilon]))(x)+f(x_0) \\ = \;& \inf_{z\in H}\{f(z)+\psi(z,B[u_0,s_0]\cap B[x_0,\varepsilon])-\langle y_0,z\rangle+\frac{1}{2\lambda}\|x+x_0-z\|^2\}+\langle y_0,x_0\rangle. \end{aligned}$$

Since $\|x_0-u_0\| < \frac{s_0}{16}$ and $\varepsilon = \frac{s_0}{2}$, one has $B[u_0,s_0]\cap B[x_0,\varepsilon] = B[x_0,\varepsilon]$.

Thus,

$$\begin{aligned} & e_\lambda(F+\psi(\cdot,B[0,\varepsilon]))(x) \\ = \;& -f(x_0)-\langle y_0,x\rangle-\frac{\lambda}{2}\|y_0\|^2+\inf_{z\in H}\{f(z)+\psi(z,B[x_0,\varepsilon])+\frac{1}{2\lambda}\|x+x_0+\lambda y_0-z\|^2\} \\ = \;& -f(x_0)-\frac{\lambda}{2}\|y_0\|^2-\langle y_0,x\rangle+e_\lambda\bar f(x+x_0+\lambda y_0), \end{aligned}$$

so that, for any $x \in H$ and any $\lambda > 0$,

$$e_\lambda\bar f(x)=e_\lambda(F+\psi(\cdot,B[0,\varepsilon]))(x-(x_0+\lambda y_0))+\langle y_0,x-(x_0+\lambda y_0)\rangle+f(x_0)+\frac{\lambda}{2}\|y_0\|^2. \tag{12}$$

However, due to Proposition 2.5 above, there exists some threshold $\lambda_0>0$ (depending on u_0 via $x_0, y_0, s_0, Q_0, f(x_0), inf_{B[u_0,s_0]}f$) such that for any $\lambda \in]0,\lambda_0]$,

$$P_\lambda(F+\psi(\cdot,B[0,\varepsilon]))(\cdot) = (I+\lambda T_{\frac{\varepsilon}{\lambda}})^{-1}(\cdot) \quad \text{on } B(0,\frac{\varepsilon}{4}) \tag{13}$$

with both mappings in (13) being nonempty, single-valued and Lipschitz continuous on $B(0,\frac{\varepsilon}{4})\ (= B(0,\frac{s_0}{8}))$, where $T_{\frac{\varepsilon}{\lambda}} := \{(x,x^*)\in\partial_P(F+\psi(\cdot,B[0,\varepsilon])) : \|x\|<\varepsilon, \|x^*\|\le\frac{\varepsilon}{\lambda}\}$. It follows from Proposition 2.5 that for each $\lambda\in]0,\lambda_0]$, the function $e_\lambda(F+\psi(\cdot,B[0,\varepsilon]))(\cdot)$ is $\mathcal{C}^{1,1}$ on $B(0,\frac{\varepsilon}{4})$, which combined with (12) entails that $e_\lambda\bar f(\,\cdot\,)$ is $\mathcal{C}^{1,1}$ on $B(x_0+\lambda y_0,\frac{s_0}{8})$.

Define $\Lambda_0 := \min\{\lambda_0, \frac{s_0}{32(\|y_0\|+1)}\}$. Obviously, given any $\lambda \in]0,\Lambda_0[$, one has $B[u_0,\frac{s_0}{32}] \subset B(x_0+\lambda y_0,\frac{s_0}{8})$ and hence

$$e_\lambda\bar f(\cdot) \text{ is } \mathcal{C}^{1,1} \quad \text{on } B(u_0,\frac{s_0}{32}). \tag{14}$$

Since $\bar f$ is lsc and bounded from below, Proposition 2.7 above then ensures that for each $\lambda \in]0,\Lambda_0[$, $P_\lambda\bar f(\cdot)$ is nonempty, single-valued, and continuous on $B(u_0,\frac{s_0}{32})$ with

$$\nabla e_\lambda\bar f = \lambda^{-1}(I-P_\lambda\bar f) \quad \text{on } B(u_0,\frac{s_0}{32}). \tag{15}$$

Furthermore, given any $\lambda \in]0, \Lambda_0[$, direct computation starting from (12) yields for all $u \in H$

$$x_0 + P_\lambda(F + \psi(\cdot, B[0,\varepsilon]))(u) = P_\lambda \bar{f}(u + x_0 + \lambda y_0). \tag{16}$$

As $0 \in \partial F(0)$, we have $0 \in P_\lambda(F+\psi(\cdot, B[0,\varepsilon]))(0)$ by (13) and hence $P_\lambda(F+\psi(\cdot, B[0,\varepsilon]))(0) = 0$ since $P_\lambda(F + \psi(\cdot, B[0,\varepsilon]))(\cdot)$ is single valued on $B[0,\varepsilon]$ as we saw just after (13). Combining this with (16) we obtain

$$P_\lambda \bar{f}(x_0 + \lambda y_0) = x_0 \quad \text{whenever } \lambda \in]0, \Lambda_0[. \tag{17}$$

In addition, recall that the pln property of F at 0 with constants ε, c, Q implies the monotonicity of the operator $qI + T_{cq}$ for each $q \geq Q$, the graph of T_{cq} being defined in Proposition 2.5 above. In particular, the operator $(\frac{\varepsilon}{c}\lambda^{-1})I + T_{\frac{\varepsilon}{\lambda}}$ is monotone whenever $\lambda \in]0, Q^{-1}c^{-1}\varepsilon[$.

Observe now that the assumption $s_0 < c_0$ and the choice of c ensure that $s_0 < 2c$, i.e., $\frac{\varepsilon}{c} < 1$. Consequently, the equality (13) and Lemma 2.6 above, guarantee that for each $\lambda \in]0, \min\{\Lambda_0, Q^{-1}c^{-1}\varepsilon\}[$, the map $P_\lambda(F + \psi(\cdot, B[0,\varepsilon]))(\cdot)$ has $\bar{k} := (1 - \frac{\varepsilon}{c})^{-1}$ as Lipschitz modulus on $B(0, \frac{\varepsilon}{4})$ (modulus clearly independent of λ). Due to (16), it follows that for each $\lambda \in]0, \min\{\Lambda_0, Q^{-1}c^{-1}\varepsilon\}[$, $P_\lambda \bar{f}(\cdot)$ is nonempty single-valued and Lipschitz continuous on $B(x_0 + \lambda y_0, \frac{\varepsilon}{4})$ with modulus $\bar{k}$, where $\varepsilon = \frac{s_0}{2}$.

Since for each $\lambda \in]0, \min\{\Lambda_0, Q^{-1}c^{-1}\varepsilon\}[$, $B[u_0, \frac{s_0}{32}] \subset B(x_0 + \lambda y_0, \frac{s_0}{8})$, we conclude that for such λ, $P_\lambda \bar{f}(\cdot)$ is nonempty, single valued, and Lipschitz continuous with modulus

$$\bar{k} = (1 - \frac{s_0}{2c})^{-1} \quad \text{on } B[u_0, \frac{s_0}{32}]. \tag{18}$$

To get (e), it suffices to observe, thanks to (15), (17), and (18), that

$$\begin{aligned} \|\nabla e_\lambda \bar{f}(x_0)\| &= \|\lambda^{-1}(x_0 - P_\lambda \bar{f}(x_0))\| = \|\lambda^{-1}(P_\lambda \bar{f}(x_0 + \lambda y_0) - P_\lambda \bar{f}(x_0))\| \\ &\leq \lambda^{-1}(1 - \frac{s_0}{2c})^{-1}\|x_0 + \lambda y_0 - x_0\| = (1 - \frac{s_0}{2c})^{-1}\|y_0\|. \end{aligned}$$

We proceed now to establish (f) and (g) for λ within some threshold. As f is bounded from below on $B[x_0, \frac{s_0}{2}]$ with $f(x_0) \in \mathbb{R}$, Lemma 2.4 above provides some real number $\Lambda_0' > 0$ such that

$$\text{for all } \lambda \in]0, \Lambda_0'[, \quad P_\lambda \bar{f}(B[x_0, \frac{s_0}{8}]) \subset B[x_0, \frac{3s_0}{8}].$$

Let us define $\bar{\lambda}_0 := \min\{\, \Lambda_0,\ Q^{-1}c^{-1}\varepsilon,\ \Lambda_0' \,\}$.

Thus, as $B(u_0, \frac{s_0}{32}) \subset B[x_0, \frac{s_0}{8}]$ and $B[x_0, \frac{3s_0}{8}] \subset B(u_0, \frac{7s_0}{16})$, conclusion (f) holds for each $\lambda \in]0, \bar{\lambda}_0[$.

To finish with (g), note first that, for any $x \in B(u_0, \frac{s_0}{32})$ and any $\lambda \in]0, \bar{\lambda}_0[$, via (18) and (f), the element $P_\lambda \bar{f}(x) \in B(u_0, \frac{7s_0}{16}) \subset B(x_0, \frac{s_0}{2})$, so f and $\bar{f}$ are two lsc functions that coincide on an open neighborhood of $P_\lambda \bar{f}(x)$. It follows that

$$\partial \bar{f}(P_\lambda \bar{f}(x)) = \partial f(P_\lambda \bar{f}(x)). \tag{19}$$

On the other hand, by virtue of Lemma 2.3 combined with the single valuedness of $P_\lambda \bar{f}$ on $B(u_0, \frac{s_0}{32})$ (according to what just precedes (15)), we obtain that, for each $\lambda \in]0, \bar{\lambda}_0[$, we have $\nabla e_\lambda \bar{f}(x) \in \partial \bar{f}(P_\lambda \bar{f}(x))$, and via (19), this says that property (g) is true. Given any $\lambda \in]0, \bar{\lambda}_0[$, in view of (14), (15), (17), (18) and the latter analysis, conclusions $(a) - (g)$ hold.

Now, let us show that the estimation (h) is a straight consequence of $(b) - (g)$ above. For any $x \in B(u_0, \frac{s_0}{32})$, making use of (e), (g), and the pln property of f at u_0, we can write

$$\begin{aligned} & \langle \nabla e_\lambda \bar{f}(x) - \nabla e_\lambda \bar{f}(x_0), P_\lambda \bar{f}(x) - P_\lambda \bar{f}(x_0)\rangle \\ \geq \; & -(Q_0 + c_0^{-1}(\|\nabla e_\lambda \bar{f}(x)\| + \|\nabla e_\lambda \bar{f}(x_0)\|))\|P_\lambda \bar{f}(x) - P_\lambda \bar{f}(x_0)\|^2 \\ \geq \; & -(Q_0 + c_0^{-1}(\|\nabla e_\lambda \bar{f}(x)\| + (1 - \frac{s_0}{2c})^{-1}\|y_0\|))\|P_\lambda \bar{f}(x) - P_\lambda \bar{f}(x_0)\|^2, \end{aligned}$$

and hence, due to (d), we deduce that

$$\begin{aligned} & \langle x - x_0, P_\lambda \bar{f}(x) - P_\lambda \bar{f}(x_0)\rangle \\ \geq \; & [1 - \lambda(Q_0 + c_0^{-1}(\|\nabla e_\lambda \bar{f}(x)\| + (1 - \frac{s_0}{2c})^{-1}\|y_0\|))]\|P_\lambda \bar{f}(x) - P_\lambda \bar{f}(x_0)\|^2, \end{aligned}$$

which clearly leads to (h).

To see (i), fix x and y as in its statement. In view of assertions (f) and (g) above, for $\beta_\lambda(x, y) := \max\{Q_0, c_0^{-1}\|\nabla e_\lambda f(x)\|, c_0^{-1}\|\nabla e_\lambda f(y)\|\}$, we have

$$\langle \nabla e_\lambda \bar{f}(x) - \nabla e_\lambda \bar{f}(y), P_\lambda \bar{f}(x) - P_\lambda \bar{f}(y)\rangle \geq -\beta_\lambda(x, y)\|P_\lambda \bar{f}(x) - P_\lambda \bar{f}(y)\|^2$$

according to the pln property of f at u_0 with constants s_0, c_0, Q_0. Then, the $\bar{k}-$Lipschitz continuity of $P_\lambda \bar{f}(\cdot)$ obtained in (b) above yields

$$\langle \nabla e_\lambda \bar{f}(x) - \nabla e_\lambda \bar{f}(y), P_\lambda \bar{f}(x) - P_\lambda \bar{f}(y)\rangle \geq -\bar{k}^2 \beta_\lambda(x, y)\|x - y\|^2,$$

that is, (i).

It remains to establish the statement (j), that is, the linear hypomonotonicity of $\nabla e_\lambda \bar{f}(\cdot)$ around u_0 with constants $\bar{s} \in]0, \frac{s_0}{32}[$, $\bar{c} = c_0 \bar{k}^{-2}$ and $\bar{Q} = Q_0 \bar{k}^2$. Consider an arbitrary $\bar{s} \in]0, \frac{s_0}{32}[$. Then, fix any $q \geq \bar{Q}$ and any x, $y \in B[u_0, \bar{s}]$ with $\|\nabla e_\lambda \bar{f}(x)\| \leq \bar{c}\, q$ and $\|\nabla e_\lambda \bar{f}(y)\| \leq \bar{c}\, q$. One has

$$\begin{aligned} & \langle \nabla e_\lambda \bar{f}(x) - \nabla e_\lambda \bar{f}(y), x - y\rangle \\ = \; & \langle \nabla e_\lambda \bar{f}(x) - \nabla e_\lambda \bar{f}(y), x - P_\lambda \bar{f}(x)\rangle + \langle \nabla e_\lambda \bar{f}(x) - \nabla e_\lambda \bar{f}(y), P_\lambda \bar{f}(x) - P_\lambda \bar{f}(y)\rangle \\ & + \langle \nabla e_\lambda \bar{f}(x) - \nabla e_\lambda \bar{f}(y), P_\lambda \bar{f}(y) - y\rangle. \end{aligned}$$

Making use of (d) and (i) above, it comes

$$\begin{aligned} & \langle \nabla e_\lambda \bar{f}(x) - \nabla e_\lambda \bar{f}(y), x - y\rangle \\ \geq \; & \langle \nabla e_\lambda \bar{f}(x) - \nabla e_\lambda \bar{f}(y), \lambda \nabla e_\lambda \bar{f}(x)\rangle - q\|x - y\|^2 + \langle \nabla e_\lambda \bar{f}(x) - \nabla e_\lambda \bar{f}(y), -\lambda \nabla e_\lambda \bar{f}(y)\rangle \\ = \; & \lambda\|\nabla e_\lambda \bar{f}(x) - \nabla e_\lambda \bar{f}(y)\|^2 - q\|x - y\|^2 \\ \geq \; & -q\|x - y\|^2. \end{aligned}$$

Thus as claimed, we conclude that $\nabla e_\lambda \bar{f}(\cdot)$ is linearly hypomonotone around u_0 with constants $\bar{s} \in]0, \frac{s_0}{32}[$, $\bar{c} = c_0 \bar{k}^{-2}$ and $\bar{Q} = Q_0 \bar{k}^2$. The proof is complete. $\square$

The preceding proposition is at the core of our study. Handling evolution equations with the Moreau envelopes of a pln function, it will allow us to develop various arguments which are much less sophisticated than those in [12, 13], and this because all the features pointed out in Proposition 2.8, are true on a *fixed* neighborhood of the datum u_0 (that is, independent of λ). Note also that Proposition 2.8 has its own interest.

All the preliminary material being introduced, let us state and prove the local existence theorem in the homogeneous case.

Theorem 2.9 (Local existence and smoothing effect). *Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc function. Consider $T_0 \in [0, +\infty[$ and let $u_0 \in \operatorname{dom} f$ be such that f is pln at u_0 with constants s_0, c_0, Q_0. Then, there exist a real number $T \in]T_0, +\infty[$ and a unique absolutely continuous mapping $u : [T_0, T] \to B(u_0, s_0)$ which is a solution of the problem*

$$(P) \begin{cases} \dot{u}(t) + \partial f(u(t)) \ni 0 & \text{for a.e. } t \in [T_0, T] \\ u(T_0) = u_0. \end{cases}$$

Further, this solution satisfies:

1. *$u(t) \in \operatorname{dom} \partial f$, for all $t \in]T_0, T]$, ("smoothing effect") and $-\dot{u}(t) = \partial^0 f(u(t))$ for almost every $t \in]T_0, T]$ ("lazy system");*
2. *$u(\cdot)$ is $\frac{1}{2}$-Hölder continuous $[T_0, T]$ and $\dot{u}(\cdot) \in L^2([T_0, T]; H)$;*
3. *the mappings $u(\cdot)$ and $f \circ u(\cdot)$ are Lipschitz continuous on each closed interval $[\tau, T] \subset]T_0, T]$, the function $f \circ u(\cdot)$ is nonincreasing and absolutely continuous on $[T_0, T]$, and one has*

$$f(u(s)) - f(u(t)) = \int_s^t \|\dot{u}(r)\|^2 dr \quad \text{for all } T_0 \le s \le t \le T. \tag{20}$$

In the particular case when $u_0 \in \operatorname{dom} \partial f$, the solution $u(\cdot)$ above and the function $f \circ u(\cdot)$ are actually Lipschitz continuous on all $[T_0, T]$ and for any $s_0^{'} \in]0, s_0]$ such that $s_0^{'} < c_0$ with $\inf\{f(x) : x \in B[u_0, s_0^{'}]\} \in \mathbb{R}$,

$$\begin{aligned} c &:= c_0 - \|\partial^0 f(u_0)\| [\max(Q_0, 2\|\partial^0 f(u_0)\| c_0^{-1})]^{-1}, \quad \text{and} \\ K &:= f(u_0) - \inf\{f(x) : x \in B[u_0, s_0^{'}]\}, \end{aligned}$$

the mapping $u(\cdot)$ also satisfies:

4. *$\dot{u} \in L^\infty([T_0, T]; H)$ with for almost every $t \in [T_0, T]$*

$$\|\dot{u}(t)\| \le \|\partial^0 f(u_0)\| \exp\left[(1 - \frac{s_0^{'}}{2c})^{-2} (Q_0(t - T_0) + 2c_0^{-1}(t - T_0)^{\frac{1}{2}} K^{\frac{1}{2}})\right]; \tag{21}$$

5. *for all $t \in [T_0, T]$,*

$$\|\partial^0 f(u(t))\| \le \|\partial^0 f(u_0)\| \exp\left[(1 - \frac{s_0^{'}}{2c})^{-2} (Q_0(t - T_0) + 2c_0^{-1}(t - T_0)^{\frac{1}{2}} K^{\frac{1}{2}})\right]. \tag{22}$$

Proof. • Uniqueness

The uniqueness statement of the absolutely continuous solution of (P) taking values in $B(u_0, s_0)$ on $[T_0, T]$ follows directly from the comparison result of Lemma 2.1 applied on

$[T_0, T]$ with $s = T_0$.

• Existence

Let us establish the local existence. To do so, the idea is to study a family of ordinary differential equations involving Moreau envelopes of some restriction of f to an appropriate neighborhood of u_0, and then pass to the limit on the approximate solutions. This regularization process is suggested by the properties of the Moreau envelopes of a pln function listed in Proposition 2.8 and also by closure properties of ∂f brought to light in Lemma 1.9 and Proposition 1.6.

I) General case $u_0 \in \operatorname{dom} f$

First, let us choose a real number $s_0^{'} \in]0, s_0]$ such that

$$\inf\{f(x) : x \in B[u_0, s_0^{'}]\} \in \mathbb{R} \quad \text{and} \quad s_0^{'} < c_0.$$

This is clearly possible invoking the lower semicontinuity of f at u_0. Next, by virtue of the density of $\operatorname{dom} \partial_F f$ in $\operatorname{dom} f$ with respect to the strong topology of H (see for instance [5]), fix $(x_0, y_0) \in \operatorname{gph} \partial f$ such that $\|x_0 - u_0\| < \frac{s_0^{'}}{16}$ and let

$$\bar{f} := f + \psi(\cdot, B[x_0, \frac{s_0^{'}}{2}]),\ Q := \max(Q_0, 2\|y_0\| c_0^{-1}) \quad \text{and} \quad c := c_0 - \|y_0\| Q^{-1}$$

(be defined like in Proposition 2.8). Denote by $\bar{\lambda}_0 \in]0, \frac{s_0^{'}}{32(\|y_0\|+1)}[$, the threshold provided by Proposition 2.8 such that $(a) - (i)$ therein hold whenever $\lambda \in]0, \bar{\lambda}_0[$. To simplify, put $\eta_0 := \frac{s_0^{'}}{32}$. Clearly, f and $\bar{f}$ are pln at each point of $B[u_0, \eta_0]$ with constants $s_0 - \eta_0$, c_0, Q_0 (see Remark 1.4).

$*$ **Approximation scheme**

Now fix any $\lambda \in]0, \bar{\lambda}_0[$ and consider the approximate problem

$$(P_\lambda) \begin{cases} \text{find a scalar } T_\lambda > T_0 \text{ and a mapping} \\ u_\lambda : [T_0, T_\lambda[\to B(u_0, \eta_0) \text{ of class } \mathcal{C}^1 \text{ such that} \\ \dot{u}_\lambda(t) + \nabla e_\lambda \bar{f}(u_\lambda(t)) = 0 \text{ for all } t \in [T_0, T_\lambda[\\ u_\lambda(T_0) = u_0. \end{cases}$$

The $\mathcal{C}^1$ regularity of $e_\lambda \bar{f}$ on $B(u_0, \eta_0)$, along with the Lipschitz property of $\nabla e_\lambda \bar{f}$ on $B(u_0, \eta_0)$ being guaranteed by Proposition 2.8, the classical Cauchy-Lipschitz theorem enables us to find some real number $T_\lambda > T_0$ and a unique $\mathcal{C}^1$ mapping $u_\lambda : [T_0, T_\lambda[\to B(u_0, \eta_0)$ solution of the problem

$$\begin{cases} (E_\lambda) \quad \dot{y}(\cdot) + \nabla e_\lambda \bar{f}(y(\cdot)) = 0 \quad \text{on } [T_0, T_\lambda[\\ \qquad\quad y(T_0) = u_0, \end{cases}$$

and defined on its maximal interval of existence. Notice immediately that, taking the inner product in (E_λ) with $-\dot{u}_\lambda(r)$ for each $r \in [T_0, T_\lambda[$, one gets

$$-\|\dot{u}_\lambda(r)\|^2 = \langle \nabla e_\lambda \bar{f}(u_\lambda(r)), \dot{u}_\lambda(r) \rangle = \frac{d}{dr}[(e_\lambda \bar{f}) \circ u_\lambda](r), \tag{23}$$

which implies that $r \mapsto e_\lambda \bar{f}(u_\lambda(r))$ is nonincreasing on $[T_0, T_\lambda[$. Further, for any $t \in [T_0, T_\lambda[$, by definition of $e_\lambda \bar{f}$ we have $e_\lambda \bar{f}(u_\lambda(t)) \geq \inf_{B[u_0,s_0^{'}]} f$, which combining with (23) yields

$$\int_{T_0}^{t} \|\dot{u}_\lambda(r)\|^2 dr = e_\lambda \bar{f}(u_0) - e_\lambda \bar{f}(u_\lambda(t)) \leq \bar{f}(u_0) - \inf_{B[u_0,s_0^{'}]} f = f(u_0) - \inf_{B[u_0,s_0^{'}]} f. \tag{24}$$

Let us define

$$K := f(u_0) - \inf_{B[u_0,s_0^{'}]} f.$$

We observe that, for each $t \in [T_0, T_\lambda[$, we have

$$\|u_\lambda(t) - u_0\| = \| \int_{T_0}^{t} \dot{u}_\lambda(r) dr \| \leq \int_{T_0}^{t} \|\dot{u}_\lambda(r)\| dr \leq (t - T_0)^{1/2} (\int_{T_0}^{t} \|\dot{u}_\lambda(r)\|^2 dr)^{1/2}$$

and hence by (24)

$$\|u_\lambda(t) - u_0\| \leq (t - T_0)^{1/2} K^{1/2} \tag{25}$$

and in the same way

$$\|u_\lambda(t) - u_\lambda(s)\| \leq (t - s)^{1/2} K^{1/2} \quad \text{for all } T_0 \leq s \leq t < T_\lambda. \tag{26}$$

So, choosing some number $T \in]T_0, +\infty[$, such that

$$K^{1/2}(T - T_0)^{1/2} < \frac{\eta_0}{2}, \text{ i.e., } (T - T_0)^{1/2} [f(u_0) - \inf_{B[u_0,s_0^{'}]} f]^{1/2} < \frac{s_0^{'}}{2^6}, \tag{27}$$

the maximality of T_λ entails that $T < T_\lambda$, and this holds for any $\lambda \in]0, \bar{\lambda}_0[$. Indeed, if it was not the case for some $\lambda \in]0, \bar{\lambda}_0[$, observing by virtue of (26) that $\zeta_\lambda := \lim_{t \uparrow T_\lambda} u_\lambda(t)$ exists and that $\|\zeta_\lambda - u_0\| \leq \frac{\eta_0}{2}$ by (25) and (27), one would be able to extend $u_\lambda(\cdot)$ on a right neighborhood of T_λ with values staying in $B(u_0, \eta_0)$.

Now, we restrict our study on $[T_0, T]$.

$*$ **Convergence of the approximate solutions**

Given any $t \in [T_0, T]$, one can write according to the first equality in (24)

$$e_\lambda \bar{f}(u_\lambda(t)) \leq e_\lambda \bar{f}(u_0) \leq \bar{f}(u_0) = f(u_0), \tag{28}$$

and, by definition of proximal mappings, one can also write

$$\begin{aligned} e_\lambda \bar{f}(u_\lambda(t)) &= \bar{f}(P_\lambda \bar{f}(u_\lambda(t)) + \frac{1}{2\lambda} \|u_\lambda(t) - P_\lambda \bar{f}(u_\lambda(t))\|^2 \\ &= f(P_\lambda \bar{f}(u_\lambda(t)) + \frac{\lambda}{2} \|\nabla e_\lambda \bar{f}(u_\lambda(t))\|^2, \end{aligned} \tag{29}$$

the second equality resulting from the definition of $\bar{f}$, the fact that $u_\lambda([T_0, T])$ is contained in $B(u_0, \eta_0)$, and the assertions (f) and (d) in Proposition 2.8. Hence, it follows from (28) and (29) that for each $t \in [T_0, T]$ one has

$$\|u_\lambda(t) - P_\lambda \bar{f}(u_\lambda(t))\|^2 \leq 2\lambda (f(u_0) - f(P_\lambda \bar{f}(u_\lambda(t)))),$$

where $P_\lambda \bar f(u_\lambda(t)) \in B(u_0, \frac{7s_0'}{16})$ via (f) in Proposition 2.8. Then,

$$\forall t \in [T_0, T],\ \|u_\lambda(t) - P_\lambda \bar f(u_\lambda(t))\|^2 \le 2\lambda K. \tag{30}$$

Keep now in mind that, by (25) and (27)

$$\text{for each } \lambda \in]0, \bar\lambda_0[, \quad u_\lambda([T_0, T]) \subset B(u_0, \frac{\eta_0}{2}) \tag{31}$$

and consider any $t \in [T_0, T]$ and $\{\lambda, \mu\} \subset]0, \bar\lambda_0[$. We have:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_\lambda(t) - u_\mu(t)\|^2 &= \langle \dot u_\lambda(t) - \dot u_\mu(t), u_\lambda(t) - u_\mu(t)\rangle \\
&= \langle -\nabla e_\lambda \bar f(u_\lambda(t)) + \nabla e_\mu \bar f(u_\mu(t)), P_\lambda \bar f(u_\lambda(t)) - P_\mu \bar f(u_\mu(t))\rangle \\
&\quad + \langle -\nabla e_\lambda \bar f(u_\lambda(t)) + \nabla e_\mu \bar f(u_\mu(t)), u_\lambda(t) - P_\lambda \bar f(u_\lambda(t))\rangle \\
&\quad + \langle -\nabla e_\lambda \bar f(u_\lambda(t)) + \nabla e_\mu \bar f(u_\mu(t)), P_\mu \bar f(u_\mu(t)) - u_\mu(t)\rangle.
\end{aligned}$$

By virtue of (30), we can write

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_\lambda(t) - u_\mu(t)\|^2 &\le (\|\nabla e_\lambda \bar f(u_\lambda(t))\| + \|\nabla e_\mu \bar f(u_\mu(t))\|)(\sqrt{\lambda} + \sqrt{\mu})(2K)^{1/2} \\
&\quad - \langle \nabla e_\lambda \bar f(u_\lambda(t)) - \nabla e_\mu \bar f(u_\mu(t)), P_\lambda \bar f(u_\lambda(t)) - P_\mu \bar f(u_\mu(t))\rangle.
\end{aligned}$$

Next, due to the fact that

$$\{P_\lambda \bar f(u_\lambda(t)), P_\mu \bar f(u_\mu(t))\} \subset B(u_0, \frac{7s_0'}{16})$$

along with the inclusions

$$\nabla e_\lambda \bar f(u_\lambda(t)) \in \partial f(P_\lambda \bar f(u_\lambda(t))), \quad \nabla e_\mu \bar f(u_\mu(t)) \in \partial f(P_\mu \bar f(u_\mu(t))), \tag{32}$$

according to (f) and (g) in Proposition 2.8, the pln property of f at u_0 yields

$$\begin{aligned}
& -\langle \nabla e_\lambda \bar f(u_\lambda(t)) - \nabla e_\mu \bar f(u_\mu(t)), P_\lambda \bar f(u_\lambda(t)) - P_\mu \bar f(u_\mu(t))\rangle \\
\le\ & (Q_0 + c_0^{-1}(\|\nabla e_\lambda \bar f(u_\lambda(t))\| + \|\nabla e_\mu \bar f(u_\mu(t)\|))\|P_\lambda \bar f(u_\lambda(t)) - P_\mu \bar f(u_\mu(t))\|^2 \\
\le\ & 2(Q_0 + c_0^{-1}(\|\nabla e_\lambda \bar f(u_\lambda(t))\| + \|\nabla e_\mu \bar f(u_\mu(t))\|))\ [\ (\ \|P_\lambda \bar f(u_\lambda(t)) - u_\lambda(t)\| \\
& \qquad + \|P_\mu \bar f(u_\mu(t)) - u_\mu(t)\|\ )^2 + \|u_\lambda(t) - u_\mu(t)\|^2\ ] \\
\le\ & 2(Q_0 + c_0^{-1}(\|\nabla e_\lambda \bar f(u_\lambda(t))\| + \|\nabla e_\mu \bar f(u_\mu(t))\|))\|u_\lambda(t) - u_\mu(t)\|^2 \\
& \qquad + 4K(\sqrt{\lambda} + \sqrt{\mu})^2(Q_0 + c_0^{-1}(\|\nabla e_\lambda \bar f(u_\lambda(t))\| + \|\nabla e_\mu \bar f(u_\mu(t))\|)),
\end{aligned}$$

where the last inequality follows from (30).

In the end, as $\nabla e_\alpha \bar f(u_\alpha(t)) = -\dot u_\alpha(t)$ for $\alpha \in \{\lambda, \mu\}$, we obtain

$$\begin{aligned}
\frac{d}{dt}\|u_\lambda(t) - u_\mu(t)\|^2 &\le 2(2K)^{1/2}(\sqrt{\lambda} + \sqrt{\mu})(\|\dot u_\lambda(t)\| + \|\dot u_\mu(t)\|) \\
&\quad + 8K(\sqrt{\lambda} + \sqrt{\mu})^2(Q_0 + c_0^{-1}(\|\dot u_\lambda(t)\| + \|\dot u_\mu(t)\|)) \\
&\quad + 4(Q_0 + c_0^{-1}(\|\dot u_\lambda(t)\| + \|\dot u_\mu(t)\|))\|u_\lambda(t) - u_\mu(t)\|^2,
\end{aligned}$$

holding for any $t\in[T_0,T]$. Now recall that $u_\lambda(T_0)=u_0=u_\mu(T_0)$. Then, Gronwall's lemma ensures that for each $t\in[T_0,T]$

$$\begin{aligned}&\|u_\lambda(t)-u_\mu(t)\|^2\\ \le\;& \Big(\int_{T_0}^t a_{\lambda,\mu}(r)dr\Big)\exp\Big(4\Big(Q_0(t-T_0)+c_0^{-1}\Big(\int_{T_0}^t\|\dot u_\lambda(r)\|dr+\int_{T_0}^t\|\dot u_\mu(r)\|dr\Big)\Big)\Big),\end{aligned}$$

with

$$\begin{aligned}a_{\lambda,\mu}(r)\;:=\;&2(\sqrt{\lambda}+\sqrt{\mu})(2K)^{1/2}(\|\dot u_\lambda(r)\|+\|\dot u_\mu(r)\|)\\ &+8K(\sqrt{\lambda}+\sqrt{\mu})^2(Q_0+c_0^{-1}\|\dot u_\lambda(r)\|+c_0^{-1}\|\dot u_\mu(r)\|).\end{aligned}$$

It is easily seen by (24) that, for each $t\in[T_0,T]$,

$$\begin{aligned}&\int_{T_0}^t a_{\lambda,\mu}(r)dr\\ \le\;& 2^{5/2}K(t-T_0)^{1/2}(\sqrt{\lambda}+\sqrt{\mu})[1+\sqrt{2}(\sqrt{\lambda}+\sqrt{\mu})(Q_0(t-T_0)^{1/2}+2c_0^{-1}\sqrt{K})],\end{aligned}$$

and hence, given any $\lambda,\mu\in]0,\bar\lambda_0[$, one has

$$\begin{aligned}&\sup_{t\in[T_0,T]}\|u_\lambda(t)-u_\mu(t)\|\\ \le\;& 2^{\frac54}K^{\frac12}(T-T_0)^{\frac14}(\sqrt{\lambda}+\sqrt{\mu})^{\frac12}[1+\sqrt{2}(\sqrt{\lambda}+\sqrt{\mu})(Q_0(T-T_0)^{\frac12}+2c_0^{-1}\sqrt{K})]^{\frac12}r\\ &*\exp[2(Q_0(T-T_0)+2c_0^{-1}(T-T_0)^{\frac12}\sqrt{K})].\end{aligned}\tag{33}$$

As a consequence, the uniform Cauchy's criterion holds and this entails the uniform convergence on $[T_0,T]$ of the family $\{u_\lambda(\cdot),\lambda\in]0,\bar\lambda_0[\}$ to some continuous mapping $u:[T_0,T]\to H$, i.e., $\|u_\lambda(\cdot)-u(\cdot)\|_\infty \underset{\lambda\downarrow 0}{\longrightarrow} 0$.

∗Immediate properties of the limit mapping

Obviously, $u(T_0)=u_0$ and owing to the inclusion $\{u_\lambda([T_0,T]):\lambda\in]0,\bar\lambda_0[\}\subset B(u_0,\frac{\eta_0}{2})$, one has $u([T_0,T])\subset B[u_0,\frac{\eta_0}{2}]\subset B(u_0,s_0)$. In addition, since by (26)

$$\|u_\lambda(t_2)-u_\lambda(t_1)\|\le(t_2-t_1)^{1/2}K^{1/2}$$

for all $T_0\le t_1\le t_2\le T$ and all $\lambda\in]0,\bar\lambda_0[$, we conclude that $u(\cdot)$ is $\frac12$-Hölder continuous, and hence absolutely continuous on $[T_0,T]$. The space H being reflexive, $u(\cdot)$ is almost everywhere differentiable in $[T_0,T]$, the map $\dot u\in L^1([T_0,T];H)$, and for any $t\in[T_0,T]$ one has $u(t)=u_0+\int_{T_0}^t\dot u(r)dr$.

Note that in view of (30) the convergence

$$\|P_\lambda\bar f(u_\lambda(\cdot))-u(\cdot)\|_\infty \underset{\lambda\downarrow 0}{\longrightarrow} 0 \text{ on } [T_0,T]\tag{34}$$

also holds. As a result, the lower semicontinuity of f combined with (28) and the inequality $f(P_\lambda\bar f(u_\lambda(t)))\le e_\lambda\bar f(u_\lambda(t))$ (due to (29)) guarantees that $f(u(t))\le f(u_0)$ for all $t\in$

$[T_0, T]$. Thus $u([T_0, T]) \subset \operatorname{dom} f \cap B[u_0, \frac{\eta_0}{2}]$ and f is pln at $u(t)$ for each $t \in [T_0, T]$. Further, since by (24)

$$\sup_{\lambda \in]0, \bar{\lambda}_0[} \|\dot{u}_\lambda\|_{L^2([T_0,T];H)} \leq K^{1/2}, \tag{35}$$

there is some sequence $(\lambda_n)_{n\geq 1} \subset]0, \bar{\lambda}_0[$ converging to 0 such that $(\dot{u}_{\lambda_n})_{n\geq 1}$ converges weakly to some $\xi \in L^2([T_0, T]; H)$. It is not difficult to see that $\xi = \dot{u}$ almost everywhere in $[T_0, T]$, so that $\dot{u} \in L^2([T_0, T]; H)$.

Moreover, for all $n \in \mathbb{N}$ and all $t \in [T_0, T]$, via (24) and (29), one has

$$\int_{T_0}^{t} \|\dot{u}_{\lambda_n}(r)\|^2 dr \leq f(u_0) - e_{\lambda_n}\bar{f}(u_{\lambda_n}(t)) \leq f(u_0) - f(P_{\lambda_n}\bar{f}(u_{\lambda_n}(t))). \tag{36}$$

Invoking (34), the weak convergence of $(\dot{u}_{\lambda_n})_{n\geq 1}$ to $\dot{u}$ in $L^2([T_0, T]; H)$, and the lower semicontinuity of f with respect to the strong topology of H along with that of the norm with respect to the weak topology of $L^2([T_0, T]; H)$, taking the superior limit on n in (36) yields

$$\int_{T_0}^{t} \|\dot{u}(r)\|^2 dr \leq f(u_0) - f(u(t)) \quad \text{for all } t \in [T_0, T]. \tag{37}$$

$*$ **Identification of the solution of (P) on** $[T_0, T]$

We proceed now to show that

$$-\dot{u}(t) \in \partial f(u(t)) \quad \text{for a.e. } t \in [T_0, T]. \tag{38}$$

To summarize, we know that

(a') $\{P_{\lambda_n}\bar{f}(u_{\lambda_n}(t)) : t \in [T_0, T], n \in \mathbb{N}\} \subset B(u_0, \frac{7s_0'}{16}) \cap \operatorname{dom} f$,
(b') $u([T_0, T]) \subset B[u_0, \frac{\eta_0}{2}] \cap \operatorname{dom} f$ and $\|P_{\lambda_n}\bar{f}(u_{\lambda_n}(\cdot)) - u(\cdot)\|_\infty \underset{n\to\infty}{\longrightarrow} 0$,
(c') $\dot{u}_{\lambda_n}(\cdot) = -\nabla e_{\lambda_n}\bar{f}(u_{\lambda_n}(\cdot)) \underset{n\to\infty}{\longrightarrow} \dot{u}(\cdot)$ weakly in $L^2([T_0, T]; H)$, and
(d') for each $n \in \mathbb{N}$, $-\dot{u}_{\lambda_n}(t) \in \partial f(P_{\lambda_n}\bar{f}(u_{\lambda_n}(t)))$ for all $t \in [T_0, T]$ via (32).

Consequently, the closure property of Lemma 1.9 guarantees the inclusion (38).

$*$ **Absolute continuity of $f \circ u$ and equality (20)**

Let us continue with the absolute continuity of $f \circ u$. Fix any $t \in [T_0, T]$ where $-\dot{u}(t) \in \partial f(u(t))$, and fix any $\lambda \in]0, \bar{\lambda}_0[$. Knowing that $u(t) \in B(u_0, \eta_0)$, by (a), (b) and (g) of Proposition 2.8 we may write

$$\begin{aligned} & \langle -\dot{u}(t) - \nabla e_\lambda \bar{f}(u(t)), \nabla e_\lambda \bar{f}(u(t))\rangle \\ = \; & \lambda^{-1}\langle -\dot{u}(t) - \nabla e_\lambda \bar{f}(u(t)), u(t) - P_\lambda \bar{f}(u(t))\rangle \\ \geq \; & -(Q_0 + c_0^{-1}(\| - \dot{u}(t)\| + \|\nabla e_\lambda \bar{f}(u(t))\|))\lambda^{-1}\|u(t) - P_\lambda \bar{f}(u(t))\|^2. \end{aligned} \tag{39}$$

But making $\mu \downarrow 0$ in (33) gives for some constant K_1 that

$$\sup_{s\in[T_0,T]} \|u_\lambda(s) - u(s)\| \leq K_1\sqrt{\lambda}$$

and this with (30) yield for some constant K_2 (independent of λ)

$$\sup_{s\in[T_0,T]} \|u(s) - P_\lambda \bar{f}(u(s))\|^2 \le K_2\lambda.$$

Putting this inequality in (39), we obtain

$$\|\nabla e_\lambda \bar{f}(u(t))\|^2 - (c_0^{-1}K_2 + \|\dot{u}(t)\|)\|\nabla e_\lambda \bar{f}(u(t))\| - K_2(Q_0 + c_0^{-1}\|\dot{u}(t)\|) \le 0$$

and hence after easy computation

$$\|\nabla e_\lambda \bar{f}(u(t))\| \le K_3(1 + \|\dot{u}(t)\|), \tag{40}$$

where K_3 denotes some constant (independent of λ and t).

Since $\|\dot{u}(\cdot)\| \in L^2([T_0,T])$, some sequence $(\nabla e_{\lambda'_n} \bar{f}(u(\cdot)))_{n\in\mathbb{N}}$ converges weakly in $L^2([T_0,T]; H)$ to some mapping $w(\cdot)$. Further by virtue of the Lipschitz behavior of $\nabla e_{\lambda'_n} \bar{f}(u(\cdot))$ on $B(u_0,\eta_0)$ (see (a) of Proposition 2.8) the function $\nabla e_{\lambda'_n} \bar{f}(u(\cdot))$ is absolutely continuous on $[T_0,T]$, and for all $s,t \in [T_0,T]$

$$e_{\lambda'_n} \bar{f}(u(t)) - e_{\lambda'_n} \bar{f}(u(s)) = \int_s^t \langle \nabla e_{\lambda'_n} \bar{f}(u(r)), \dot{u}(r)\rangle dr.$$

Taking the limit on n, we see that

$$f(u(t)) - f(u(s)) = \int_s^t \langle w(r), \dot{u}(r)\rangle dr \quad \text{for all } s,t \in [T_0,T].$$

This translates the absolute continuity of $f \circ u$ on $[T_0,T]$.

Fix now any $t \in]T_0,T[$ where both mappings $f\circ u(\cdot)$ and $u(\cdot)$ are derivable, with $-\dot{u}(t) \in \partial_P f(u(t))$ and consider any $\zeta \in \partial_P f(u(t))$. For any real number h with $0 < |h| < \min\{t - T_0, T - t\}$ we have

$$f\circ u(t+h) - f\circ u(t) \ge \langle \zeta, u(t+h) - u(t)\rangle - \frac{1}{2}(Q_0 + c_0^{-1}\|\zeta\|)\|u(t+h) - u(t)\|^2.$$

Taking successively $h > 0$ and $h < 0$, dividing by h, and passing to the limit $h \to 0$ we successively obtain

$$(f\circ u)'(t) \ge \langle \zeta, \dot{u}(t)\rangle \quad \text{and} \quad (f\circ u)'(t) \le \langle \zeta, \dot{u}(t)\rangle,$$

and hence $(f\circ u)'(t) = \langle \zeta, \dot{u}(t)\rangle$. In particular

$$(f\circ u)'(t) = \langle \partial^0 f(u(t)), \dot{u}(t)\rangle = -\|\dot{u}(t)\|^2. \tag{41}$$

Therefore

$$f(u(s)) - f(u(t)) = \int_s^t \|\dot{u}(r)\|^2 dr \quad \text{for all } T_0 \le s \le t \le T$$

which is (20).

The *"lazy system" property* also follows from (41) which shows that $\|-\dot{u}(t)\| \le \|\partial^0 f(u(t))\|$, and hence $\partial^0 f(u(t)) = -\dot{u}(t)$ for a.e. $t \in [T_0, T]$.

$*$ Lipschitz behavior of u and $f \circ u$, and "smoothing effect"

To complete the study in the case $u_0 \in \operatorname{dom} f$, we need to prove the three points mentioned above.

$\boldsymbol{\alpha}$) First, fix any $s, t \in [T_0, T[$ where u is derivable with $s \le t$ and fix any $h_0 \in]0, T - t[$. For each $h \in]0, h_0[$ and for almost all $r \in [s, t]$, using the linear hypomonotonicity (4) of ∂f on $B(u_0, s_0)$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dr}\|u(r+h) - u(r)\|^2 &= \langle \dot{u}(r+h) - \dot{u}(r), u(r+h) - u(r)\rangle \\ &\le (Q_0 + c_0^{-1}(\|\dot{u}(r+h)\| + \|\dot{u}(r)\|))\|u(r+h) - u(r)\|^2, \end{aligned}$$

and by Gronwall's lemma

$$\begin{aligned} &\|u(t+h) - u(t)\|^2 \\ \le\ &\|u(s+h) - u(s)\|^2 * \exp\left(Q_0(t-s) + c_0^{-1}\left(\int_s^t \|\dot{u}(r)\|dr + \int_s^t \|\dot{u}(r+h)\|dr\right)\right). \end{aligned}$$

Then

$$\begin{aligned} &\|h^{-1}[u(t+h) - u(t)]\| \\ \le\ &\|h^{-1}[u(s+h) - u(s)]\| * \exp(Q_0(t-s) + c_0^{-1}\int_s^t \|\dot{u}(r)\|dr + c_0^{-1}\int_{s+h}^{t+h} \|\dot{u}(r)\|dr) \end{aligned}$$

and, letting $h \downarrow 0$, we see that

$$\|\dot{u}(t)\| \le \|\dot{u}(s)\| \exp(Q_0(t-s) + 2c_0^{-1}\int_s^t \|\dot{u}(r)\|dr). \tag{42}$$

Observing by (37) that

$$\int_s^t \|\dot{u}(r)\|dr \le (t-s)^{\frac{1}{2}}\left(\int_s^t \|\dot{u}(r)\|^2 dr\right)^{\frac{1}{2}} \le (t-s)^{\frac{1}{2}} K^{\frac{1}{2}}$$

we obtain

$$\|\dot{u}(t)\| \le \|\dot{u}(s)\| \exp(Q_0(t-s) + 2c_0^{-1}(t-s)^{\frac{1}{2}} K^{\frac{1}{2}}). \tag{43}$$

For any $\tau \in]T_0, T[$, taking some $s \in]T_0, \tau]$ where u is derivable, (43) entails that $u(\cdot)$ is Lipschitz continuous on $[\tau, T]$ as well as the function $f \circ u(\cdot)$ according to (20).

$\boldsymbol{\beta}$) In view of the "lazy system" property above, fix now any $\bar{s} \in [T_0, T[$ with $\dot{u}(\bar{s}) = -\partial^0 f(u(\bar{s}))$. The lower semicontinuity of the function $d(\cdot)$ in Lemma 1.8 combined with (42) says that for any $t \in [\bar{s}, T]$, the set $\partial f(u(t)) \neq \emptyset$ and

$$\|\partial^0 f(u(t))\| \le \|\partial^0 f(u(\bar{s}))\| \exp(Q_0(t-\bar{s}) + 2c_0^{-1}\int_{\bar{s}}^t \|\dot{u}(r)\|dr). \tag{44}$$

Consequently, $u(t) \in \operatorname{dom} \partial f$ for all $t \in]T_0, T]$ ("smoothing effect").

II) Case $u_0 \in \operatorname{dom} \partial f$

Let us now focus on the special situation when $u_0 \in \operatorname{dom} \partial f$. All the preceding analysis holds with $(u_0, \partial^0 f(u_0))$ in place of (x_0, y_0).

$*$ **Lipschitz continuity of $u(\cdot)$ and $f \circ u(\cdot)$ and proof of (21).**

Let us refine the study of the approximation scheme. First, according to (j) of Proposition 2.8, recall that for all $\lambda \in]0, \bar{\lambda}_0[$, the operator $\nabla e_\lambda \bar{f}(\cdot)$ satisfies the linear hypomonotonicity (4) around u_0 with constants $\bar{s} := \frac{\eta_0}{2}$, $\bar{c} := c_0\, \bar{k}^{-2}$ and $\bar{Q} := Q_0\, \bar{k}^2$ where $\bar{k} = (1 - \frac{s_0^{'}}{2c})^{-1}$ is the Lipschitz modulus of $P_\lambda \bar{f}(\cdot)$ given by (b) from the same proposition.

As a consequence, following the same process as in step α) above with $u_\lambda(\cdot)$ in place of $u(\cdot)$ (resp. $e_\lambda \bar{f}$ in place of f), and since u_λ is $\mathcal{C}^1$ on $[T_0, T]$ with $u_\lambda([T_0, T]) \subset B(u_0, \bar{s})$ (see (31)), it is not difficult to see that for any $s, t \in [T_0, T]$ with $s \le t$

$$\begin{aligned} \|\dot{u}_\lambda(t)\| &\le \|\dot{u}_\lambda(s)\| \exp(\bar{Q}(t-s) + 2\bar{c}^{-1}(t-s)^{\frac{1}{2}}(\int_s^t \|\dot{u}_\lambda(r)\|^2 dr)^{\frac{1}{2}}) \\ &\le \|\dot{u}_\lambda(s)\| \exp(\bar{Q}(t-s) + 2\bar{c}^{-1}(t-s)^{\frac{1}{2}} K^{\frac{1}{2}}), \end{aligned} \tag{45}$$

the last inequality being due to (24). Then, as $-\dot{u}_\lambda(\cdot) = \nabla e_\lambda \bar{f}(u_\lambda(\cdot))$, and $\bar{c} = c_0\, \bar{k}^{-2}$, $\bar{Q} = Q_0\, \bar{k}^2$, one actually has

$$\|\nabla e_\lambda \bar{f}(u_\lambda(t))\| \le \|\nabla e_\lambda \bar{f}(u_\lambda(s))\| \exp[(1 - \frac{s_0^{'}}{2c})^{-2}(Q_0(t-s) + 2c_0^{-1}(t-s)^{\frac{1}{2}} K^{\frac{1}{2}})] \tag{46}$$

for all $\lambda \in]0, \bar{\lambda}_0[$ and all $T_0 \le s \le t \le T$. However, by (c) from Proposition 2.8, we know that

$$\begin{aligned} \|\nabla e_\lambda \bar{f}(u_\lambda(T_0))\| &= \|\nabla e_\lambda \bar{f}(u_0)\| = \|\lambda^{-1}(u_0 - P_\lambda \bar{f}(u_0))\| \\ &= \|\lambda^{-1}(P_\lambda \bar{f}(u_0 + \lambda \partial^0 f(u_0)) - P_\lambda \bar{f}(u_0))\|. \end{aligned} \tag{47}$$

Further, via (h) and (e) from Proposition 2.8, we also have

$$\|P_\lambda \bar{f}(u_0 + \lambda \partial^0 f(u_0)) - P_\lambda \bar{f}(u_0)\| \le \frac{\lambda \|\partial^0 f(u_0)\|}{1 - \lambda(Q_0 + c_0^{-1}(1 + (1 - \frac{s_0^{'}}{2c})^{-1})\|\partial^0 f(u_0)\|)}$$

for $\lambda \in]0, \bar{\lambda}_0[$ small enough to ensure the positivity of

$$1 - \lambda(Q_0 + c_0^{-1}(1 + (1 - \frac{s_0^{'}}{2c})^{-1})\|\partial^0 f(u_0)\|).$$

Combining all those facts, we can write

$$\begin{aligned} \|\dot{u}_\lambda(t)\| &= \|\nabla e_\lambda \bar{f}(u_\lambda(t))\| \\ &\le \frac{\|\partial^0 f(u_0)\| \exp[(1 - \frac{s_0^{'}}{2c})^{-2}(Q_0(t-T_0) + 2c_0^{-1}(t-T_0)^{\frac{1}{2}} K^{\frac{1}{2}})]}{1 - \lambda(Q_0 + c_0^{-1}(1 + (1 - \frac{s_0^{'}}{2c})^{-1})\|\partial^0 f(u_0)\|)} \end{aligned} \tag{48}$$

for each $t \in [T_0, T]$ and any $\lambda \in]0, \bar{\lambda}_0[$ small enough. Finally, in the case when $u_0 \in \operatorname{dom} \partial f$, one has

$$\begin{aligned} \|u_\lambda(t) - u_\lambda(s)\| &= \|\int_s^t \dot{u}_\lambda(r)dr\| \leq \int_s^t \|\dot{u}_\lambda(r)\|dr \\ &\leq \frac{\|\partial^0 f(u_0)\| \int_s^t \exp\left[(1 - \frac{s_0'}{2c})^{-2}(Q_0(r - T_0) + 2c_0^{-1}(r - T_0)^{\frac{1}{2}} K^{\frac{1}{2}})\right]dr}{1 - \lambda(Q_0 + c_0^{-1}(1 + (1 - \frac{s_0'}{2c})^{-1})\|\partial^0 f(u_0)\|)} \end{aligned}$$

for all $T_0 \leq s \leq t \leq T$, and letting $\lambda \downarrow 0$ yields

$$\|u(t) - u(s)\| \leq \|\partial^0 f(u_0)\| \int_s^t \exp[(1 - \frac{s_0'}{2c})^{-2}(Q_0(r - T_0) + 2c_0^{-1}(r - T_0)^{\frac{1}{2}} K^{\frac{1}{2}})]dr. \quad (49)$$

Then, for each $t \in [T_0, T]$ where $u(\cdot)$ is derivable, it is not difficult to see that (49) entails

$$\|\dot{u}(t)\| \leq \|\partial^0 f(u_0)\| \exp[(1 - \frac{s_0'}{2c})^{-2}(Q_0(t - T_0) + 2c_0^{-1}(t - T_0)^{\frac{1}{2}} K^{\frac{1}{2}})].$$

Putting $\gamma_0 := \exp[(1 - \frac{s_0'}{2c})^{-2}(Q_0(T - T_0) + 2c_0^{-1}(T - T_0)^{\frac{1}{2}} K^{\frac{1}{2}})]$, one easily deduces that

$$\|u(t) - u(s)\| \leq (t - s)\gamma_0 \|\partial^0 f(u_0)\| \quad \text{for all } t, s \in [T_0, T] \quad \text{with } s \leq t, \quad (50)$$

and hence, the solution $u(\cdot)$ of (P) is Lipschitz continuous on $[T_0, T]$. Further (20) and (50) ensure that for all $T_0 \leq s \leq t \leq T$

$$|f(u(t)) - f(u(s))| = \int_s^t \|\dot{u}(r)\|^2 dr \leq (\gamma_0 \|\partial^0 f(u_0)\|)^2 (t - s),$$

which means that $f \circ u(\cdot)$ is Lipschitz continuous on $[T_0, T]$.

$*$ **Demonstration of (22)**

From now on, we fix any sequence $(\lambda_n)_n$ in $]0, \bar{\lambda}_0]$ with $\lambda_n \downarrow 0$ and

$$1 - \lambda_n(Q_0 + c_0^{-1}(1 + (1 - \frac{s_0'}{2c})^{-1})\|\partial^0 f(u_0)\|) > 0.$$

It clearly follows from (48) that, given any fixed $t \in [T_0, T]$, the sequence $(\dot{u}_{\lambda_n}(t))_{n \geq 1}$ is bounded in H. Then, one finds some subsequence of $(-\dot{u}_{\lambda_n}(t))_n$ that converges weakly in H to some element v_t that satisfies

$$\|v_t\| \leq \|\partial^0 f(u_0)\| \exp[(1 - \frac{s_0'}{2c})^{-2}(Q_0(t - T_0) + 2c_0^{-1}(t - T_0)^{\frac{1}{2}} K^{\frac{1}{2}}]$$

in view of (48) and the weak lower semicontinuity of $\|\cdot\|_H$. Next, invoking the closure feature from Proposition 1.6 allows us to conclude that $v_t \in \partial f(u(t))$ for each $t \in [T_0, T]$, and hence (22) is immediate. □

Remark 2.10. Let us mention that the inequality (22) will enable us to derive some continuity property of the function $t \mapsto \|\partial^0 f(u(t))\|$ on $[T_0, T[$. That is the reason why, instead of using merely the upper bound of $\|\nabla e_\lambda \bar{f}(u_0)\|$ given by (h) in Proposition 2.8 to exploit (46), we preferred the transformation in (47) allowing us to involve (e) in the same proposition. This leads to a longer proof but a finer estimation!

Remark 2.11. As regards, the interval of definition of the local solution of (P) obtained in Theorem 2.9, it will be helpful to keep in mind the following.

Given any initial value $u_0 \in \operatorname{dom} f$ where f is pln with constants $s_0^{'}$, c_0, Q_0 such that

$$s_0^{'} < c_0 \quad \text{and} \quad \inf\{f(x) : x \in B[u_0, s_0^{'}]\} \in \mathbb{R},$$

according to (27) above, we learn that for any positive real number Δ satisfying

$$\Delta^{\frac{1}{2}} \left[\, f(u_0) - \inf_{B[u_0, s_0^{'}]} f \,\right]^{\frac{1}{2}} < \frac{s_0^{'}}{2^6},$$

there exists a unique absolutely continuous mapping $u(\cdot)$ from $[T_0,\, T_0+\Delta]$ into $B[u_0,\, \frac{s_0^{'}}{2^6}] \cap \operatorname{dom} f$ such that

$$\begin{cases} -\dot{u}(t) \in \partial f(u(t)) & \text{for a.e. } t \in [T_0,\, T_0+\Delta] \\ u(T_0) = u_0, \end{cases}$$

and this solution satisfies conclusions *1.-5.* from Theorem 2.9 on $[T_0,\, T_0+\Delta]$.

Let us investigate additional features of the local solution of the homogeneous problem (P) above, before addressing the questions of global existence and related asymptotic behavior. Our analysis allows us to derive the following properties. They are known to hold globally in the convex case, see [6] Theorems 3.1 and 3.2.

Proposition 2.12. *The assumptions of Theorem* 2.9 *are made with* $u_0 \in \operatorname{dom} f$. *Let* $u : [T_0, T] \to B[u_0, \eta_0]$ *denote the absolutely continuous solution of* (P) *on* $[T_0, T]$, *where* $\eta_0 := \frac{s_0'}{2^6}$ *with* $s_0' \le s_0$ *given as in Remark* 2.11. *Then, the following properties hold:*

(a) *for all* $s, t \in [T_0, T]$ *with* $s \le t$

$$\|u(t) - u(s)\| \le (t-s)^{\frac{1}{2}} (f(u(s)) - f(u(t)))^{\frac{1}{2}};$$

(b) *the mapping* $t \mapsto \partial^0 f(u(t))$ *is continuous on the right over* $]T_0, T[$;

(c) *for all* $t \in]T_0, T[$, *the right derivatives* $\dot{u}_+(t)$ *and* $(f \circ u)_+^{'}(t)$ *exist with*

$$\dot{u}_+(t) = -\partial^0 f(u(t)) \quad \textit{and} \quad (f \circ u)_+^{'}(t) = -\|\partial^0 f(u(t))\|^2;$$

(d) *for all* $s, t \in]T_0, T[$ *such that* $s \le t$, *one has*

$$\|\partial^0 f(u(t))\| \le \|\partial^0 f(u(s))\| \exp(Q_0(t-s) + 2c_0^{-1}(t-s)^{\frac{1}{2}}(f(u(s)) - f(u(t)))^{\frac{1}{2}}).$$

If in addition one assumes $u_0 \in \operatorname{dom} \partial f$, *then the assertions* (b), (c), *and* (d) *remain true on* $[T_0, T[$ *in place of* $]T_0, T[$.

Proof. a) Since $u(\cdot)$ is absolutely continuous, the assertion (a) follows from (20) and the Cauchy-Schwarz inequality.

b) To establish the item (b), we first assume $u_0 \in \operatorname{dom} \partial f$. Thanks to Lemma 1.8, we already know that, for any $s \in [T_0, T[$, one has

$$\|\partial^0 f(u(s))\| \le \liminf_{t \downarrow s} \|\partial^0 f(u(t))\|. \tag{51}$$

On the other hand, by virtue of estimation (22) from Theorem 2.9, recall that for any $t \in [T_0, T]$,

$$\|\partial^0 f(u(t))\| \leq \|\partial^0 f(u_0)\| \exp[(1 - \frac{s_0^{'}}{2c})^{-2}(Q_0(t - T_0) + 2c_0^{-1}(t - T_0)^{\frac{1}{2}} K^{\frac{1}{2}})],$$

where $s_0^{'}$, c, K are constant real numbers. Hence,

$$\limsup_{t \downarrow T_0} \|\partial^0 f(u(t))\| \leq \|\partial^0 f(u_0)\| = \|\partial^0 f(u(T_0))\|. \tag{52}$$

As a result, from (51) with $s = T_0$ and (52), we deduce that the function $\|\partial^0 f(u(\cdot))\|$ is continuous on the right at T_0.

Now suppose merely $u_0 \in \operatorname{dom} f$. Given any fixed $s \in]T_0, T[$, it follows from Theorem 2.9 that $u(s) \in \operatorname{dom} \partial f$ and f is pln at $u(s)$. Because of the autonomous behavior of the studied evolution problem, and since the mapping $u(\cdot)$ is continuous at s, applying Theorem 2.9 with initial data s and $u(s)$, we find some number $\varepsilon(s) > 0$ with $s + \varepsilon(s) < T$ such that the mapping $u(\cdot)$ coincides with the solution of the problem

$$\dot{x}(t) + \partial f(x(t)) \ni 0 \quad \text{for a.e. } t \in [s, s + \varepsilon(s)], \quad x(s) = u(s),$$

and satisfy some inequality starting with s as initial time and constants depending on s. So, by the same process we obtain the right continuity of $\|\partial^0 f(u(\cdot))\|$ at s, for each $s \in]T_0, T[$. So the function $t \mapsto \|\partial^0 f(u(t))\|$ is continuous on the right at each point of $[T_0, T[$.

Besides, given any $s \in]T_0, T[$ (resp. $s \in [T_0, T[$ if $u_0 \in \operatorname{dom} \partial f$) and any sequence $(t_n)_{n \geq 1} \subset [s, T[$ with $t_n \to s$, we have just obtained that

$$\|\partial^0 f(u(t_n))\| \underset{n \to \infty}{\longrightarrow} \|\partial^0 f(u(s))\| \quad \text{and then} \quad \sup_{n \geq 1} \|\partial^0 f(u(t_n))\| < +\infty. \tag{53}$$

Therefore, there exist some $\xi \in H$ and a subsequence $(\partial^0 f(u(t_{n_k})))_{k \in \mathbb{N}}$ that converges weakly in H to ξ when $k \to +\infty$. The function f being pln at $u(s)$, thanks to Proposition 1.6, one has $\xi \in \partial f(u(s))$. Hence, making use of (53), we get

$$\|\xi\| \leq \liminf_{k \to +\infty} \|\partial^0 f(u(t_{n_k}))\| = \|\partial^0 f(u(s))\|.$$

Necessarily, $\xi = \partial^0 f(u(s))$. Finally, by uniqueness of the weak cluster point of $(\partial^0 f(u(t_n)))_n$, the whole sequence is weakly convergent to $\partial^0 f(u(s))$ in H. Since we also have the convergence in (53), the sequence $(\partial^0 f(u(t_n)))_n$ actually converges strongly (i.e., in $(H, \|\cdot\|)$) to $\partial^0 f(u(s))$, that justifies the right continuity of $\partial^0 f(u(\cdot))$ on $]T_0, T[$ (resp. $[T_0, T[$ if $u_0 \in \operatorname{dom} \partial f$) and proves (b).

c) Now, we are able to demonstrate the assertion (c). Let us fix $t \in]T_0, T[$ (resp. $[T_0, T[$ if $u_0 \in \operatorname{dom} \partial f$). By absolute continuity of $u(\cdot)$ and $f \circ u(\cdot)$ on $[T_0, T]$, for all $h > 0$ with $t + h \leq T$, we have

$$u(t + h) - u(t) = \int_t^{t+h} \dot{u}(r) dr, \quad \text{and} \quad f(u(t + h)) - f(u(t)) = \int_t^{t+h} (f \circ u)^{'}(r) dr.$$

Then, from conclusions *1.* and *3.* of Theorem 2.9, it comes

$$h^{-1}[u(t+h)-u(t)] = -h^{-1}\int_t^{t+h} \partial^0 f(u(r))dr,$$

and

$$h^{-1}[f(u(t+h))-f(u(t))] = -h^{-1}\int_t^{t+h} \|\partial^0 f(u(r))\|^2 dr.$$

The continuity on the right of the mappings $\partial^0 f(u(\cdot))$ and $\|\partial^0 f(u(\cdot))\|$ ensures the existence of the right derivatives $\dot{u}_+(t)$ and $(f\circ u)'_+(t)$, with

$$\dot{u}_+(t) = -\partial^0 f(u(t)) \quad \text{and} \quad (f\circ u)'_+(t) = -\|\partial^0 f(u(t))\|^2,$$

for each $t \in]T_0, T[$ (resp. $[T_0, T[$ if $u_0 \in \operatorname{dom}\partial f$) as claimed.
d) To end the proof, let us check (d). For any fixed $s \in]T_0, T[$ such that $-\dot{u}(s) = \partial^0 f(u(s))$, like for (44), according to (b) of Lemma 2.1 and the "lazy system" property from *1.* in Theorem 2.9, we can write

$$\|\partial^0 f(u(t))\| \le \|\partial^0 f(u(s))\| \exp(Q_0(t-s) + 2c_0^{-1}(t-s)^{\frac{1}{2}}(f(u(s))-f(u(t)))^{\frac{1}{2}}) \tag{54}$$

for almost every $t \in [s, T]$. The lower semicontinuity of $\|\partial^0 f(u(\cdot))\|$ on $]T_0, T]$ (see Lemma 1.8) ensures that the inequality (54) holds for all $t \in [s, T]$, that is, for all $s \in]T_0, T[$ with $-\dot{u}(s) = \partial^0 f(u(s))$ and all $t \in [s, T]$ one has

$$\|\partial^0 f(u(t))\| \le \|\partial^0 f(u(s))\| \exp(Q_0(t-s) + 2c_0^{-1}(t-s)^{\frac{1}{2}}(f(u(s))-f(u(t)))^{\frac{1}{2}}).$$

Fix s, $t \in]T_0, T[$ (resp. $[T_0, T[$ if $u_0 \in \operatorname{dom}\partial f$) with $s < t$. Observing that the point s is the limit of some sequence $(s_n)_{n\ge 1} \subset]s, t[$ with $-\dot{u}(s_n) = \partial^0 f(u(s_n))$, the right continuity of $\|\partial^0 f(u(\cdot))\|$ and $f\circ u(\cdot)$ at s directly leads to the expected inequality (e). $\square$

Remark 2.13. Note that in the special case when the involved function f is proper lsc and convex, as already mentioned, it is pln at each point of its domain with any positive constants s_0, c_0, Q_0. The time T found in Theorem 2.9 being independent of the parameters c_0, Q_0, letting $Q_0 \downarrow 0$ and $c_0 \uparrow \infty$ in (d), one retrieves the well known nonincreasing behavior of $\|\partial^0 f(u(\cdot))\|$ on $[T_0, T[$, and we will see in the results to come, that the present solution is actually defined on $[T_0, +\infty[$, in this setting.

3. Existence of global solutions

Two propositions will open this section. They set a "bridge" between local and global solutions of our dynamical system.

Given a non compact interval J we will say (as usual) that $u(\cdot)$ is locally absolutely continuous on J when it is absolutely continuous on each compact subinterval $[\alpha, \beta] \subset J$.

Proposition 3.1. *Let $f : H \to \mathbb{R}\cup\{+\infty\}$ be a proper lsc function. Let $T_0 \in [0, +\infty[$ and $u_0 \in \operatorname{dom} f$ be such that f is pln at u_0 with constants s_0, c_0, Q_0. Denote by E any subset of H containing $B[u_0, s_0]$.*

Then, there exist an element $T \in]T_0, +\infty]$ and a mapping $u : [T_0, T[\to E$ which is locally absolutely continuous on $[T_0, T[$, satisfies:

(a) $\dot{u}(t) + \partial_F f(u(t)) \ni 0$ *for a.e.* $t \in [T_0, T[$;
(b) $u(T_0) = u_0$ *and* $u([T_0, T[) \subset \operatorname{dom} f$;

and u *cannot be extended in a right neighborhood of* T.

Proof. Denote by Λ_0 the collection of all pairs (τ, x) such that $\tau \in]T_0, +\infty]$ and $x(\cdot) : [T_0, \tau[\to E$ is locally absolutely continuous on $[T_0, \tau[$ and have features (a) and (b). Clearly, Theorem 2.9 guarantees the nonemptyness of the set Λ_0. Then, endow Λ_0 with the partial ordering "$\preceq$" defined by

$$(T_1, x_1) \preceq (T_2, x_2) \iff \begin{cases} T_1 \leq T_2, \\ x_1(t) = x_2(t) \quad \text{for all } t \in [T_0, T_1[, \end{cases}$$

for $(T_i, x_i) \in \Lambda_0$, $i = 1, 2$. Let $\{(T_\alpha, x_\alpha),\ \alpha \in A\}$ be a totally ordered family in Λ_0. If $\tilde{T} := \sup_{\alpha \in A} T_\alpha$, then the mapping $\tilde{x} : [T_0, \tilde{T}[\to E$ given by

$$\tilde{x}(t) := x_\alpha(t) \quad \text{if } t \in [T_0, T_\alpha[$$

is well defined and is locally absolutely continuous on $[T_0, \tilde{T}[$. It is not difficult to see that $\tilde{x}(\cdot)$ also verifies properties (a) and (e) on $[T_0, \tilde{T}[$.

Indeed, consider a sequence $(\theta_n)_{n \geq 1} \subset]T_0, \tilde{T}[$ that converges increasingly to $\tilde{T}$. One has

$$[T_0, \tilde{T}[= \bigcup_{n \geq 1} [T_0, \theta_n].$$

Given any integer $n \geq 1$, by definition of the supremum $\tilde{T}$, there is some $\alpha_n \in A$ such that $\theta_n < T_{\alpha_n}$. Hence $[T_0, \theta_n] \subset [T_0, T_{\alpha_n}[$ which implies that for each $t \in [T_0, \theta_n]$, $\tilde{x}(t) = x_{\alpha_n}(t)$. In particular, $\tilde{x}([T_0, \theta_n[) \subset \operatorname{dom} f$ and the absolute continuity of x_{α_n} on $[T_0, \theta_n]$ entails that of $\tilde{x}$ on $[T_0, \theta_n]$. Consequently, $\tilde{x}([T_0, \tilde{T}[\subset \operatorname{dom} f$ and the mapping $\tilde{x}$ is locally absolutely continuous on $[T_0, \tilde{T}[$.

Besides, given any $n \in \mathbb{N}$, there exists some Lebesgue-null set $N_n \subset [T_0, \theta_n]$ such that $\dot{\tilde{x}}(t) + \partial_F f(\tilde{x}(t)) \ni 0$ for all $t \in [T_0, \theta_n] \backslash N_n$. Hence, for all $t \in [T_0, \tilde{T}[\backslash (\bigcup_{n \geq 1} N_n)$ one has $\dot{\tilde{x}}(t) + \partial_F f(\tilde{x}(t)) \ni 0$. Thus the pair $(\tilde{T}, \tilde{x})$ belongs to Λ_0. Furthermore, $(T_\alpha, x_\alpha) \preceq (\tilde{T}, \tilde{x})$ for all $\alpha \in A$, which means that $(\tilde{T}, \tilde{x})$ is an upper bound for the family $\{(T_\alpha, x_\alpha),\ \alpha \in A\}$. According to the Zorn's lemma, there exists a maximal element in Λ_0 that we call (T, u). The mapping u:$[T_0, T[\to E$ is locally absolutely continuous on $[T_0, T[$ and has no other continuation satisfying (a) and (b) than itself. $\square$

Now, let us address the related global existence issue.

Theorem 3.2. *Let* T_0 *be a nonnegative real number and let* f:$H \to \mathbb{R} \cup \{+\infty\}$ *be proper and lsc. Assume that* $(G1)$ *and* $(G2)$ *below hold:*

$(G1)$ f *is pln at each point of its domain,*
$(G2)$ *there exits a positive real number* α *such that*

$$f(x) \geq -\alpha(1 + \|x\|) \quad \textit{for all } x \in H.$$

Then, given $u_0 \in dom\, f$, there exists a unique mapping $u : [T_0, +\infty[\to H$ which is locally absolutely continuous on $[T_0, +\infty[$ and which satisfies

$$\begin{cases} \dot{u}(t) + \partial f(u(t)) \ni 0 \quad \textit{for a.e. } t \in [T_0, +\infty[, \\ u(T_0) = u_0, \;\; u([T_0, +\infty[) \subset dom\, f. \end{cases}$$

In addition, the mapping $u(\cdot)$ is such that:

- *for all $s, t \in [T_0, +\infty[$ with $s \leq t$,*

$$f(u(s)) - f(u(t)) = \int_s^t \|\dot{u}(r)\|^2 dr, \tag{55}$$

  hence, $f \circ u$ is nonincreasing on $[T_0, +\infty[$, and $\dot{u} \in L^2_{loc}([T_0, +\infty[; H)$;
- *for all $s, t \in [T_0, +\infty[$ such that $s \leq t$*

$$\|u(s) - u(t)\| \leq (t - s)^{\frac{1}{2}} [f(u(s)) - f(u(t))]^{\frac{1}{2}}; \tag{56}$$

- *for all $t \in]T_0, +\infty[$*

$$u(t) \in dom\, \partial f, \quad \textit{(global "smoothing effect")},$$

  and the right derivatives $\dot{u}_+(t)$ and $(f \circ u)'_+(t)$ exist and satisfy:

$$\dot{u}_+(t) = -\partial^0 f(u(t)) \quad \textit{and} \quad (f \circ u)'_+(t) = -\|\partial^0 f(u(t))\|^2.$$

Proof. Fix any $u_0 \in \operatorname{dom} f$.

$*$ **Uniqueness**

Fix any $T_1 \in [T_0, +\infty[$ and $\theta \in]T_1, +\infty]$, and let u_1 and u_2 be two mappings that are locally absolutely continuous on $[T_1, +\infty[$ and that satisfy

$$\begin{cases} \dot{x}(t) + \partial_F f(x(t)) \ni 0 \quad \text{a.e. } t \in [T_1, \theta[\\ x(T_1) = x_1, \quad \text{where } x_1 \in \operatorname{dom} f, \\ x(t) \in \operatorname{dom} f \quad \text{for all } t \in [T_1, \theta[. \end{cases}$$

By right continuity of u_1 and u_2 at T_1, and by Theorem 2.9, the set

$$\mathcal{T} := \{t \in]T_1, \theta[: u_1(s) = u_2(s) \quad \text{for all } s \in [T_1, t]\}$$

is nonempty. Let us define $\tau := \sup \mathcal{T}$. If $\tau < \theta$, then by continuity of u_i at τ, $i = 1, 2$, one has $u_1(\tau) = u_2(\tau) =: u_\tau$ and by assumption $u_\tau \in \operatorname{dom} f$. Then, f is pln at u_τ, and thanks to Theorem 2.9, it is possible to find some $\varepsilon_\tau > 0$ with $\tau + \varepsilon_\tau < \theta$ such that the inclusion $\dot{x}(t) + \partial_F f(x(t)) \ni 0$ a.e. in $[\tau, \tau + \varepsilon_\tau]$, with the initial condition $x(\tau) = u_\tau$, admits a unique absolutely continuous solution. The latter uniqueness property entails that $u_1(t) = u_2(t)$ for all $t \in [\tau, \tau + \varepsilon_\tau]$. Hence, u_1 and u_2 coincide on $[T_1, \tau + \varepsilon_\tau]$ which is in contradiction with the definition of τ. Necessarily, $\tau = \theta$ and $u_1(\cdot) = u_2(\cdot)$ on $[T_1, \theta[$.
Global uniqueness is obtained for the particular choice $T_1 = T_0$ and $\theta = +\infty$.

∗ **Existence**

Denote by $u : [T_0, T[\to H$, the mapping supplied by Proposition 3.1 with $E = H$ as the maximal locally absolutely continuous solution of the problem $\dot{u}(t) + \partial f(u(t)) \ni 0$ a.e. in $[T_0, T[$, $u(T_0) = u_0$ satisfying (a) and (b) therein.

I) Denote by $\mathcal{I}$ the set of all $t \in]T_0, T[$ such that for all $s \in [T_0, t]$

$$f(u(s)) - f(u(t)) = \int_s^t \|\dot{u}(r)\|^2 dr, \quad u(]T_0, t[) \subset \operatorname{dom} \partial f,$$

and (b) and (c) of Proposition 2.12 hold on $]T_0, t[$. By Theorem 2.9 and Proposition 2.12 the set $\mathcal{I}$ is nonempty. We claim that $\tau := \sup \mathcal{I} = T$. Indeed suppose that $\tau < T$.

Fix $\bar{s}', \bar{c}, \bar{Q}$ as the constants of the pln property of f at $\bar{u} := u(\tau) \in \operatorname{dom} f$ with $\bar{s}' < \bar{c}$ and put $\bar{s} := \bar{s}'/2$. The infimum $\inf_{B[\bar{u}, 2\bar{s}]} f$ being finite because of (G_2), choose $\Delta > 0$ such that

$$\Delta^{1/2}[f(u_0) - \inf_{B[\bar{u}, 2\bar{s}]} f]^{1/2} < \frac{\bar{s}}{2^6}$$

and choose also by continuity of u some $\bar{t} \in]T_0, \tau[$ with

$$\tau < \bar{t} + \Delta < T \quad \text{and} \quad \|u(\bar{t}) - \bar{u}\| < \bar{s}.$$

Clearly f is pln at $u(\bar{t})$ with the constants $\bar{s}, \bar{c}, \bar{Q}$ (see Remark 1.4). Noting that $f(u(\bar{t})) \leq f(u_0)$, Remark 2.11 provides an absolutely continuous mapping $v(\cdot)$ from $[\bar{t}, \bar{t} + \Delta]$ into $\operatorname{dom} f$ such that $v(\bar{t}) = u(\bar{t})$ and $\dot{v}(t) \in -\partial f(v(t))$ for a.e. $t \in [\bar{t}, \bar{t} + \Delta]$. Further according to the same Remark 2.11 $v(t) \in \operatorname{dom} \partial f$ for all $t \in]\bar{t}, \bar{t} + \Delta]$ and for all $\bar{t} \leq s \leq t \leq \bar{t} + \Delta$

$$f(v(s)) - f(v(t)) = \int_s^t \|\dot{v}(r)\|^2 dr$$

and by Proposition 2.12, the assertions (b) and (c) therein hold on the interval $]\bar{t}, \bar{t} + \Delta[$ for the mapping $v(\cdot)$.

The uniqueness part above says that $u(t) = v(t)$ for all $t \in [\bar{t}, \bar{t} + \Delta[$. By the latter equality and by the properties of $v(\cdot)$ above we have that $u(]T_0, \bar{t} + \Delta[) \subset \operatorname{dom} \partial f$, (b) and (c) of Proposition 2.12 hold true on $]T_0, \bar{t} + \Delta[$ for $u(\cdot)$,

$$f(u(s)) - f(u(t)) = \int_s^t \|\dot{u}(r)\|^2 dr \quad \text{for all } T_0 \leq s \leq t < \bar{t} + \Delta,$$

and $\dot{u}(t) \in -\partial f(u(t))$ for a.e. $t \in]T_0, \bar{t} + \Delta[$. In view of the inequality $\tau < \bar{t} + \Delta$, we obtain a contradiction with the very definition of τ. So $\tau = T$, i.e.,

$$u(]T_0, T[) \subset \operatorname{dom} \partial f \quad \text{and} \quad f(u(s)) - f(u(t)) = \int_s^t \|\dot{u}(r)\|^2 dr \tag{57}$$

for all $T_0 \leq s \leq t < T$, and (b) and (c) of Proposition 2.12 hold for the mapping $u(\cdot)$ on the interval $]T_0, T[$.

II) We claim now that $T = +\infty$. By contradiction, suppose that $T < +\infty$. Then, for any $T_0 \le s \le t < T$, by (57) we have

$$\|u(t) - u(s)\|^2 \le (t-s)\int_s^t \|\dot{u}(r)\|^2 dr \le (t-s)[f(u_0) - f(u(t))] \tag{58}$$

and in view of requirement (G_2) we obtain with $s = T_0$

$$\|u(t) - u_0\|^2 \le (T - T_0)(f(u_0) + \alpha\|u(t)\| + \alpha). \tag{59}$$

This inequality clearly implies $\sup_{t\in[T_0,T[}\|u(t)\| < +\infty$ and via (G_2) the supremum $\sup_{t\in[T_0,T[}(-f(u(t)))$ is also finite. Combining this with (58), the Cauchy criterion guarantees that the limit $\lim_{t\uparrow T} u(t) =: \bar{u}$ exists in $(H, \|\cdot\|)$. By (57) the function $f \circ u$ is nonincreasing on $[T_0, T[$ and by the lower semicontinuity of f we obtain $f(\bar{u}) \le f(u_0)$. Thus $\bar{u} \in \operatorname{dom} f$ and f is pln at $\bar{u}$. Fix $\bar{s}$ and Δ as in step I) and $\bar{t} \in]T_0, T[$ such that

$$T < \bar{t} + \Delta \quad \text{and} \quad \|u(\bar{t}) - \bar{u}\| < \bar{s}.$$

By Remark 2.11 we obtain an absolutely continuous mapping v from $[\bar{t}, \bar{t}+\Delta]$ into H such that $v(\bar{t}) = u(\bar{t})$, $v([\bar{t}, \bar{t}+\Delta]) \subset \operatorname{dom} f$ and $\dot{v}(t) \in -\partial f(v(t))$ for a.e. $t \in [\bar{t}, \bar{t}+\Delta]$. By the uniqueness part above we have $u(t) = v(t)$ for all t $\in [\bar{t}, T[$. Putting $w(t) := u(t)$ for $t \in [t_0, T[$ and $w(t) := v(t)$ for $t \in [T, \bar{t}+\Delta[$, the mapping w is easily seen to be locally absolutely continuous on $[T_0, \bar{t}+\Delta[$ and clearly

$$w([T_0, \bar{t}+\Delta[) \subset \operatorname{dom} f \quad \text{and} \quad \dot{w}(t) \in -\partial f(w(t)) \quad \text{for a.e. } t \in [T_0, \bar{t}+\Delta[.$$

Necessarily $T = +\infty$ and $u(\cdot)$ is a locally absolutely continuous solution on $[T_0, +\infty[$ of our problem. So via (57) we obtain (55), $u(t) \in \operatorname{dom}\partial f$ for all $t \in]T_0, +\infty[$, as well as (b) and (c) of Proposition 2.12 for all $t \in]T_0, +\infty[$ by the last sentence of step I). Finally (56) follows from the equality (55) and the Cauchy-Schwarz inequality. □

4. Asymptotic behavior of global trajectories

Here, our aim is to examine the asymptotic behavior of the solutions of differential inclusions involving the subdifferential of a pln function supposed to be lsc and bounded from below on H.

Throughout this subsection, the function $f : H \to \mathbb{R} \cup \{+\infty\}$ is proper and lsc on the Hilbert space H, and we suppose that

$(G3)$ f is *pln at each point* of $\operatorname{dom} f$ and *bounded from below* on H.

Consider $T_0 \in [0, +\infty[$ and $u_0 \in \operatorname{dom} f$. In all this section $u : [T_0, +\infty[\to H$ will denote the unique global solution of the problem

$$(P_\infty) \quad \begin{cases} -\dot{u}(t) \in \partial f(u(t)) & \text{for a.e. } t \in [T_0, +\infty[, \\ u(T_0) = u_0, \ u([T_0, +\infty[) \subset \operatorname{dom} f, \end{cases}$$

given by Theorem 3.2.

The following proposition collects some classical facts.

Proposition 4.1. *Assume that $(G3)$ holds. Then*

(a) $\lim_{t\to+\infty} f(u(t))$ *exists in* $\mathbb{R}$ *and* $\lim_{t\to+\infty} f(u(t)) = \inf_{t\geq T_0} f(u(t))$;

(b)

$$\int_{T_0}^{+\infty} \|\dot{u}(r)\|^2 dr < +\infty;$$

(c) *if* $(u_\infty, v_\infty) \in H \times H$ *is a* $(\|\cdot\| \times w)-$*sequential cluster point of* $\{(u(t), \dot{u}(t)),\ t \geq T_0\}$ *when* $t \to +\infty$ *then*

$$\lim_{t\to+\infty} f(u(t)) = f(u_\infty) \quad \textit{and} \quad -v_\infty \in \partial f(u_\infty).$$

Proof. The assertion (a) follows from the boundedness requirement $(G3)$ on f and from the fact that $f \circ u$ is nonincreasing according to Theorem 3.2.

By Theorem 3.2 again, we know that

$$f(u_0) - f(u(t)) = \int_{T_0}^{t} \|\dot{u}(r)\|^2 dr$$

for all $t \in]T_0, +\infty[$. So (b) follows from (a), letting $t \uparrow +\infty$.

The last assertion (c) is a straightforward consequence of (a) and Proposition 1.6. □

The following lemma is also classical.

Lemma 4.2. *Let* $\lambda(\cdot)$ *stand for the Lebesgue measure on* $\mathbb{R}$ *and let* $v \in L^2([T_0, +\infty[; H)$. *Then*

(a) *for any* $\varepsilon > 0$ *one has*

$$\lambda(\{t \geq T_0 : \|v(t)\| \geq \varepsilon\}) < +\infty \quad \textit{and} \quad \lambda(\{t \geq T_0 : \|v(t)\| < \varepsilon\}) = +\infty;$$

(b) *if* $\xi \in H$ *is such that* $\lambda(\{t \geq T_0 : \|\dot{u}(t) - \xi\| \leq \varepsilon\}) = +\infty$ *for all* $\varepsilon > 0$, *then necessarily* $\xi = 0$.

Proof. The inequality in (a) is obvious and the equality is a direct consequence of it. The assertion (b) is obtained by contradiction with the particular $\varepsilon = \frac{\|\xi\|}{2}$. □

Notation. As usual (see e.g. [7]) define

$$\overline{\omega_{T_0}} := \bigcap_{T\geq T_0} cl_{\|\cdot\|}(u([T, +\infty[))$$

as the (strong) limit set of the trajectory $u(\cdot)$ (starting from u_0 at T_0).

We first observe that this limit set $\overline{\omega_{T_0}}$ is obviously nonempty, connected and compact in $(H, \|\cdot\|)$ whenever the mapping $u(\cdot)$ is strongly relatively compact on $[T_0, +\infty[$, that is, $cl_{\|\cdot\|}(\{u(t) : t \geq T_0\})$ is a compact subset of H with respect to the strong topology. The strong relative compactness is of course satisfied when the sublevel set $\{f \leq f(u_0)\}$ is compact in $(H, \|\cdot\|)$.

The proof of the next theorem makes use of the technical lemma below, inspired by [7].

Lemma 4.3. *Let $(s_n)_{n\in\mathbb{N}}\subset]T_0,+\infty[$ be any sequence satisfying $s_n \underset{n\to\infty}{\longrightarrow} +\infty$. Then, there exists a strictly increasing mapping $\nu(\cdot):\mathbb{N}\to\mathbb{N}$ such that:*

$\forall\delta>0$, $\exists m(\delta)\in\mathbb{N}$, $\forall n\geq m(\delta)$, $\exists t_{n,\delta}\geq T_0$ *such that*

(a) $s_{\nu(n)}-\delta<t_{n,\delta}<s_{\nu(n)}+\delta$,
(b) $\dot{u}(t_{n,\delta})$ *exists,* $-\dot{u}(t_{n,\delta})\in\partial f(u(t_{n,\delta}))$, *and*
(c) $\|\dot{u}(t_{n,\delta})\|<\delta$.

Proof. Fix any sequence $(s_n)_{n\geq 1}\subset]T_0,+\infty[$ such that $\lim_{n\to+\infty}s_n=+\infty$. It is not difficult to see that there exists an increasing mapping $\phi:\mathbb{N}\to\mathbb{N}$ such that the subsequence $(s_{\phi(n)})_{n\in\mathbb{N}}$ is increasing to $+\infty$. In particular, $s_{\phi(n)} \underset{n\to\infty}{\longrightarrow} +\infty$, $(s_{\phi(n)})_n$ is not a Cauchy sequence, and hence

$$\exists\varepsilon>0,\ \forall n\in\mathbb{N},\ \exists p>n,\ \exists q>n\ :\ |s_{\phi(p)}-s_{\phi(q)}|\geq\varepsilon.$$

As a consequence, making use of the increasing behavior of $(s_{\phi(n)})_n$, for each integer $n\geq 1$, one finds some $p>n$ such that $s_{\phi(p)}\geq s_{\phi(n)}+\varepsilon$. This enables us to generate an increasing mapping ψ from $\mathbb{N}$ into $\mathbb{N}$ satisfying

$$s_{\phi\circ\psi(n+1)}\geq s_{\phi\circ\psi(n)}+\varepsilon,\quad \text{for all } n\geq 1.$$

Hence, define $\nu(\cdot):\mathbb{N}\to\mathbb{N}$ by $\nu(n)=\phi\circ\psi(n)$, $n\geq 1$. Further, denoting by $\mathcal{N}$ a Lebesgue null subset of $[T_0,+\infty[$ out of which $-\dot{u}(t)\in\partial f(u(t))$, consider any $\delta>0$ and define

$$A_\delta:=\{t\geq T_0\ :\ t\notin\mathcal{N},\ \|\dot{u}(t)\|<\delta\}.$$

Put $\alpha:=\min\{\varepsilon,\delta\}$ and observe that for each $n\in\mathbb{N}$,

$$s_{\nu(n+1)}-\frac{\alpha}{2}\geq s_{\nu(n)}+\frac{\alpha}{2}. \tag{60}$$

We claim that only finitely many of the intervals $\{]s_{\nu(n)}-\delta,\ s_{\nu(n)}+\delta[\}_{n\geq 1}$ can fail to intersect A_δ. Indeed, assuming that the claim is false, one could find some increasing mapping $k:\mathbb{N}\to\mathbb{N}$ such that, for each $n\in\mathbb{N}$,

$$]s_{\nu(k(n))}-\delta,\ s_{\nu(k(n))}+\delta[\cap A_\delta=\emptyset.$$

But, given any $n\geq 1$, one has $]s_{\nu(k(n))}-\frac{\alpha}{2},\ s_{\nu(k(n))}+\frac{\alpha}{2}[\subset]s_{\nu(k(n))}-\delta,\ s_{\nu(k(n))}+\delta[$ and

$$s_{\nu(k(n+1))}-\frac{\alpha}{2}\geq s_{\nu(k(n)+1)}-\frac{\alpha}{2}\geq s_{\nu(k(n))}+\frac{\alpha}{2}, \tag{61}$$

the last inequality being due to (60). Therefore,

$$\bigcup_{n\geq 1}]s_{\nu(k(n))}-\frac{\alpha}{2},\ s_{\nu(k(n))}+\frac{\alpha}{2}[\subset\ [T_0,+\infty[\setminus A_\delta. \tag{62}$$

By virtue of (61), note also that the intervals of the preceding union do not intersect one another, which ensures that

$$\lambda(\bigcup_{n\geq 1}]s_{\nu(k(n))}-\frac{\alpha}{2},\ s_{\nu(k(n))}+\frac{\alpha}{2}[)=\sum_{n=1}^{+\infty}\alpha=+\infty.$$

This equality and (62) entail that $\lambda([T_0, +\infty[\setminus A_\delta) = +\infty$ leading to a contradiction with (a) of Lemma 4.2 since $\dot{u} \in L^2([T_0, +\infty[; H)$ by (b) of Proposition 4.1. So the claim holds true. In other words, there exists a rank $m(\delta) \in \mathbb{N}$ such that for all $n \geq m(\delta)$, there exists $t_{n,\delta} \in A_\delta$ with $|s_{\nu(n)} - t_{n,\delta}| < \delta$. The conclusion follows from the above definitions of the set $\mathcal{N}$ and A_δ. $\square$

The next propositions shed some light on the asymptotic behavior of the dynamical system associated with the problem (P_∞), under assumption (G3).

Proposition 4.4. *If* $u_\infty \in \overline{\omega_{T_0}}$, *then*

$$\lim_{t\to+\infty} f(u(t)) = f(u_\infty) = \inf_{t\geq T_0} f(u(t)) \quad \textit{and} \quad 0 \in \partial f(u_\infty).$$

Proof. Assume that $u_\infty \in \overline{\omega_{T_0}}$ and fix $(s_n)_{n\geq 1}$ in $[T_0, +\infty[$ such that

$$s_n \underset{n\to\infty}{\longrightarrow} +\infty \quad \text{and} \quad u(s_n) \underset{n\to\infty}{\longrightarrow} u_\infty \ \text{ in } (H, \|\cdot\|).$$

We invoke Lemma 4.3 to find some increasing mapping $\nu(\,\cdot\,) : \mathbb{N}\to\mathbb{N}$ and generate a sequence of integers $(\sigma(k))_{k\geq 1}$ increasing to $+\infty$, along with elements $t_k^{'}{:=}t_{\sigma(k),\,k^{-1}}$ satisfying

$$|t_k^{'} - s_{\nu(\sigma(k))}| < k^{-1}, \ -\dot{u}(t_k^{'}) \in \partial f(u(t_k^{'})) \quad \text{and} \quad \|\dot{u}(t_k^{'})\| < k^{-1} \qquad \text{for each } k \in \mathbb{N}.$$

Then, we deduce that $\|u(s_{\nu(\sigma(k))}) - u(t_k^{'})\| \leq k^{-\frac{1}{2}}(f(u_0) - \inf_H f)^{\frac{1}{2}}$ and it remains to apply (c) and (a) of Proposition 4.1 to obtain, in an easy way, the expected conclusion. $\square$

Proposition 4.5. *Assumption (G3) is made. Suppose also that the following "generalized Palais-Smale condition" is satisfied:*

$$(GPS) \qquad \begin{array}{l} \textit{Given } (x_n)_n \subset H, \\ \sup\limits_n f(x_n) < +\infty \quad \textit{and} \quad d(0, \partial f(x_n)) \to 0 \\ \Rightarrow (x_n)_n \textit{ has a convergent subsequence in } (H, \|\cdot\|). \end{array}$$

Then, the limit set $\overline{\omega_{T_0}}$ *is nonempty, and any element* $u_\infty \in \overline{\omega_{T_0}}$ *is a critical point of* f, *that is,* $0 \in \partial f(u_\infty)$. *Moreover,* $\lim_{t\to+\infty} f(u(t)) = f(u_\infty) = \inf_{t\geq T_0} f(u(t))$.

Proof. In view of Proposition 4.4, only the nonemptyness of $\overline{\omega_{T_0}}$ has to be proved under assumption (GPS). So, observe that Lemma 4.3 enables us to generate some sequence $t_k \to +\infty$ with $-\dot{u}(t_k) \in \partial f(u(t_k))$ and $\|\dot{u}(t_k)\| \underset{k\to\infty}{\longrightarrow} 0$. Thus $d(0, \partial f(u(t_k))) \underset{k\to\infty}{\longrightarrow} 0$ and, as $\sup_k f(u(t_k)) \leq f(u_0)$, the (GPS) condition provides some subsequence of $(u(t_k))_k$ which converges strongly to a vector $u_\infty \in H$ which clearly lies in $\overline{\omega_{T_0}}$. $\square$

Remark 4.6. It is immediate to note that if some element $u_\infty \in \overline{\omega_{T_0}}$ is a strict local minimum of f, then there exists a radius $r \in]0, +\infty[$ such that

$$\overline{\omega_{T_0}} \cap B[u_\infty, r] = \{u_\infty\}.$$

In this case, the next result makes sense.

Proposition 4.7. *If $u(\cdot)$ is strongly relatively compact and if $u_\infty \in \overline{\omega_{T_0}}$ is a strict local minimum of f, then the trajectory $u(\cdot)$ has a limit in $(H, \|\cdot\|)$ when the time t goes to $+\infty$ and*

$$\lim_{t\to+\infty} u(t) = u_\infty.$$

Proof. Due to the above remark, the point u_∞ is isolated in $\overline{\omega_{T_0}}$. Further, owing to the comments just before Lemma 4.3, we know that the latter set is connected. Necessarily $\overline{\omega_{T_0}}=\{u_\infty\}$, which implies that $\lim_{t\to+\infty} u(t)$ exists and is nothing but u_∞. □

References

[1] J.-P. Aubin, A. Cellina: Differential Inclusions, Springer, Berlin (1984).

[2] F. Bernard: Etude des Fonctions Prox-régulières en Dimension Infinie, Thèse de Doctorat, Université Montpellier 2 (2003).

[3] F. Bernard, L. Thibault: Prox-regular functions in Hilbert spaces, J. Math. Anal. Appl. 303 (2005) 1–14.

[4] F. Bernard, L. Thibault, D. Zagrodny: Integration of primal lower nice functions in Hilbert spaces, J. Optimization Theory Appl. 124(3) (2005) 561–579.

[5] J. M. Borwein, D. Preiss: A smooth variational principle with applications to subdifferentiability and to differentiability of convex functions, Trans. Amer. Math. Soc. 303(2) (1987) 517–527.

[6] H. Brézis: Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert, (in French), North-Holland Mathematics Studies 5, Notas de Matemática (50), North-Holland, Amsterdam (1973).

[7] R. E. Bruck: Asymptotic convergence of nonlinear contraction semigroups in Hilbert space, J. Funct. Anal. 18 (1975) 15–26.

[8] F. H. Clarke, Y. S. Ledyaev, R. J. Stern, P. R. Wolenski: Nonsmooth Analysis and Control Theory, Graduate Texts in Mathematics 178, Springer, New York (1998).

[9] C. Combari, A. Elhilali Alaoui, A. B. Levy, R. A. Poliquin, L. Thibault: Convex composite functions in Banach spaces and the primal lower-nice property, Proc. Amer. Math. Soc. 126(12) (1998) 3701–3708.

[10] R. Correa, A. Jofré, L. Thibault: Characterization of lower semicontinuous convex functions, Proc. Amer. Math. Soc. 116(1) (1992) 67–72.

[11] E. De Giorgi, M. Degiovanni, A. Marino, M. Tosques: Evolution equations for a class of nonlinear operators, Atti Accad. Naz. Lincei, VIII. Ser., Rend., Cl. Sci. Fis. Mat. Nat. 75(1-2) (1983) 1–8.

[12] M. Degiovanni, A. Marino, M. Tosques: Evolution equations with lack of convexity, Nonlinear Anal. 9(12) (1985) 1401–1443.

[13] M. Degiovanni, M. Tosques: Evolution equations for (φ, f)-monotone operators, Boll. Un. Mat. Ital., VI. Ser., B 5(2) (1986) 537–568.

[14] S. Guillaume: Problèmes d'Optimisation et d'Évolution en Analyse non Convexe de Type Convexe Composite, Thèse de Doctorat, Université Montpellier 2 (1996).

[15] S. Guillaume: Evolution equations governed by the subdifferential of a convex composite function in finite dimensional spaces, Discrete Contin. Dyn. Syst. 2(1) (1996) 23–52.

[16] S. Guillaume: Méthode de plus grande pente en analyse convexe composite, C. R. Acad. Sci., Paris, Sér. 322(1) (1996) 9–14.

[17] S. Guillaume: Subdifferential evolution inclusion in nonconvex analysis, Positivity 4(4) (2000) 357–395.

[18] A. B. Levy, R. A. Poliquin, L. Thibault: Partial extensions of Attouch's theorem with applications to proto-derivatives of subgradient mappings, Trans. Amer. Math. Soc. 347(4) (1995) 1269–1294.

[19] B. S. Mordukhovich, Y. H. Shao: Nonsmooth sequential analysis in Asplund spaces, Trans. Amer. Math. Soc. 348(4) (1996) 1235–1280.

[20] J. J. Moreau: Proximité et dualité dans un espace Hilbertien, Bull. Soc. Math. France 93 (1965) 273–299.

[21] J. J. Moreau: Fonctionnelles Convexes, Second edition, Facoltà di Ingegneria Università di Roma "Tor Vergata" (2003).

[22] R. A. Poliquin: Integration of subdifferentials of nonconvex functions, Nonlinear Anal. 17(4) (1991) 385–398.

[23] R. A. Poliquin: An extension of Attouch's theorem and its application to second-order epi-differentiation of convexly composite functions, Trans. Amer. Math. Soc. 332(2) (1992) 861–874.

[24] R. A. Poliquin, R. T. Rockafellar: Prox-regular functions in variational analysis, Trans. Amer. Math. Soc. 348(5) (1996) 1805–1838.

[25] R. A. Poliquin, R. T. Rockafellar, L. Thibault: Local differentiability of distance functions, Trans. Amer. Math. Soc. 352(11) (2000) 5231–5249.

[26] R. T. Rockafellar, R. J.-B. Wets: Variational Analysis, Springer, Berlin (1998).

[27] L. Thibault, D. Zagrodny: Integration of subdifferentials of lower semicontinuous functions on Banach spaces, J. Math. Anal. Appl. 189(1) (1995) 33–58.