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Helly's Intersection Theorem on Manifolds of Nonpositive Curvature

Yuri S. Ledyaev*
Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008, USA
ledyaev@wmich.edu

Jay S. Treiman†
Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008, USA
jay.treiman@wmich.edu

Qiji J. Zhu†
Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008, USA
qiji.zhu@wmich.edu

Dedicated to the memory of Simon Fitzpatrick.

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We give a generalization of the classical Helly's theorem on intersection of convex sets in $\mathbb{R}^N$ for the case of manifolds of nonpositive curvature. In particular, we show that if any $N+1$ sets from a family of closed convex sets on N-dimensional Cartan-Hadamard manifold contain a common point, then all sets from this family contain a common point.

1. Introduction

The classical Helly's theorem on intersection of convex sets in finite-dimensional Euclidean space $\mathbb{R}^N$, along with Carathéodory's and Radon's theorems, is one of the important results in combinatorial convex geometry. It claims (see [13, 9] for the precise statement) that, if for a family $\{C_a\}_{a\in A}$ of convex sets any $N+1$ sets have a common point, then all of the sets C_a have a common point. Note that in the case when the index set A is not finite we should assume that sets C_a are closed and at least one of them is bounded.

This paper contains a generalization of the Helly's theorem for the case of convex sets on a Riemannian manifold M of nonpositive curvature. Since the traditional proofs of the Helly's theorem in $\mathbb{R}^N$ rely heavily on the linear structure of Euclidean space they cannot be used in our setting.

Our proof uses a variational argument to establish solvability (see [6, 7]) of some nonsmooth inequality and exploits convexity of distance functions for convex sets on manifolds

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of nonpositive curvature. It is based on the classical Carathéodory's theorem for convex hulls of sets. The first use of this theorem for the proof of Helly's theorem in $\mathbb{R}^N$ was suggested by Rademacher and Shoenberg in [23], later Eggleston established that Helly's and Carathéodory's theorems are equivalent in $\mathbb{R}^N$ [12]. We should also mention a paper [24] by Rockafellar which relates some generalization of Helly's theorem to solvability of system of convex functions on $\mathbb{R}^n$. Excellent surveys of Helly's theorem and related results can be found in [9, 11].

When $M = \mathbb{R}^N$ our proof is rather straightforward and we present it below to illustrate the main idea. To simplify the exposition we assume that all sets are compact and are contained in some bounded set. In the rest of the introduction we discuss how some of the recently developed tools in nonsmooth and variational analysis on smooth manifolds [18, 19, 20] can be used to provide a similar proof of Helly's theorem for general Riemannian manifolds of nonpositive curvature.

Theorem 1.1. *Let $\{C_a\}_{a\in A}$ be a family of compact convex subsets of $\mathbb{R}^N$ with $\cup_{a\in A} C_a$ bounded. Suppose that, any $N+1$ sets C_a have a common point. Then all of the sets C_a have a common point.*

Proof. Define

$$f = \sup_{a\in A} d_{C_a},$$

where $d_{C_a}(x) = \inf\{\|x - c\| : c \in C_a\}$ is the distance function to the set C_a. Then f is convex on $\mathbb{R}^N$ and Lipschitz with Lipschitz constant 1 (since all the distance functions d_{C_a} have the same properties). Without loss of generality we assume that $\cup_{a\in A} C_a \subset \overline{B_{R_0}(0)}$ for some positive R_0, where

$$B_R(x) := \{y \in \mathbb{R}^N : \|y - x\| < R\}$$

denotes the open ball centered at x with radius R.

Then for sufficiently large R

$$\min_{\mathbb{R}^N} f = \min_{\overline{B_R(0)}} f,$$

and the minimum is attained at some point $x \in B_R(0)$.

If $f(x) = 0$ then $x \in \bigcap_{a\in A} C_a$ and theorem is proven. We consider the nontrivial case when $f(x) = 2r > 0$ and choose $\varepsilon \in (0, r/2(1+R))$. Since f attains its minimum at x we have and f is convex

$$0 \in \partial_F f(x) = \partial f(x).$$

Here ∂ and ∂_F signify the convex and Fréchet subdifferentials, respectively.

Using the representation formula for a Fréchet subgradient of the supremum function from [21] and Carathéodory's theorem we conclude that there exist $a_1, \ldots, a_{N+1} \in A$, nonnegative numbers $\lambda_1, \ldots, \lambda_{N+1}$ with $\sum_{n=1}^{N+1} \lambda_n = 1$, $x_n \in B_\varepsilon(x)$ with $d_{C_{a_n}}(x_n) > f(x) - \varepsilon$ and $x_n^* \in \partial_F d_{C_{a_n}}(x_n) = \partial d_{C_{a_n}}(x_n)$ such that the subgradient 0 of f at x is approximately represented as $\sum_{n=1}^{N+1} \lambda_n x_n^*$. More precisely,

$$\|\sum_{n=1}^{N+1} \lambda_n x_n^*\| < \varepsilon. \tag{1}$$

By the assumption of the theorem there exists y such that

$$y \in \bigcap_{n=1}^{N+1} C_{a_n}.$$

Observing that, $\|x_n^*\| \leq 1$ for $n = 1, \ldots, N+1$ (due to te Lipschitzness of the distance function with the constant 1) and

$$\langle x_n^*, y - x_n \rangle \leq d_{C_{a_n}}(y) - d_{C_{a_n}}(x_n)$$

(due to convexity of $d_{C_{a_n}}$) we have

$$\begin{aligned} \langle x_n^*, y - x \rangle &= \langle x_n^*, y - x_n \rangle + \langle x_n^*, x_n - x \rangle \\ &\leq d_{C_{a_n}}(y) - d_{C_{a_n}}(x_n) + \varepsilon \\ &= -d_{C_{a_n}}(x_n) + \varepsilon < -2r + 2\varepsilon. \end{aligned} \tag{2}$$

Combining (1) and (2) we obtain

$$\begin{aligned} 0 &\leq \sum_{n=1}^{N+1} \lambda_n \langle x_n^*, y - x \rangle + \varepsilon \|y - x\| \\ &\leq -2r + 2\varepsilon + \varepsilon 2R < -r, \end{aligned}$$

a contradiction. □

To implement a similar argument for an N-dimensional Riemannian manifold M of nonpositive curvature we consider the concepts of convex sets and convex functions on such manifolds. A detailed exposition of these concepts can be found in [4, 5, 10, 27]. While an estimate similar to that in (2) still holds, the vector $y - x$ should be replaced by the vector field associated with the geodesic connecting x and y. More subtle fact is that in a setting of the Riemannian manifold the subgradients x_n^* belong to different tangential spaces $T_{x_n}(M)$ and, therefore, cannot be directly compared as in (1). A general framework for handling the representation of subgradients of sup-envelope functions (and other subdifferential calculus on manifolds) was suggested in [18, 19, 20]. In this paper we develop a particular form of such a representation which is needed for implementing the estimate in the proof of Theorem 1.1 in an N-dimensional Riemannian manifold M of nonpositive curvature. Thus, besides deriving Helly's intersection theorem in an N-dimensional Riemannian manifold M of nonpositive curvature, the proof that we present here also serves as a demonstration on how recent developments in nonsmooth analysis on smooth manifolds in combination with convexity can be used to establish generalizations of some classical theorems for convex sets in $\mathbb{R}^N$ for manifolds.

The remainder of the paper is arranged as follows: in the next section we introduce notation and preliminaries on smooth and Riemannian manifolds, elements of nonsmooth analysis on such manifolds and convexity on Riemannian manifolds of nonpositive curvature; in Section 3 we establish a more precise version of the estimate for the Fréchet subdifferential of sup-envelope function; and our main theorems and their proofs are contained in Section 4; some open problems are discussed in the concluding Section 5.

2. Preliminaries

2.1. Smooth manifolds

Let M be an N-dimensional C^∞ manifold (paracompact Hausdorff space) with a C^∞ atlas $\{(U_\alpha, \psi_\alpha)\}_{a\in A}$. For each α, the N components $(x^1_\alpha, ..., x^N_\alpha)$ of ψ_α represent a local coordinate system on (U_α, ψ_α).

A function $g : M \to \mathbb{R}$ is C^r at $m \in M$ if $m \in U_\alpha$ and $g \circ \psi_\alpha^{-1}$ is a C^r function in a neighborhood of $\psi_\alpha(m)$. Here r is a nonnegative integer or ∞. As usual C^0 represents the collection of continuous functions on M. It is well known that this definition is independent of the local coordinate systems. If g is C^r at all $m \in M$, we say g is C^r on M. The collection of all C^r functions on M is denoted by $C^r(M)$. A map $v : C^r(M) \to \mathbb{R}$ is called a tangent vector of M at m provided that, for any $f, g \in C^r(M)$, (1) $v(\lambda f+\mu g) = \lambda v(f) + \mu v(g)$ for all $\lambda, \mu \in R$ and (2) $v(f \cdot g) = v(f)g(m) + f(m)v(g)$.

The collection of all the tangent vectors of M at m forms an (N-dimensional) vector space and is denoted by $T_m(M)$. A mapping $X : M \to \bigcup_{m\in M} T_m(M)$ is called a vector field provided that $X(m) \in T_m(M)$. A vector field X is C^r at $m \in M$ provided so is $X(g)$ for any $g \in C^\infty(M)$. If a vector field X is C^r for all $m \in M$ we say it is C^r on M. The collection of all C^r vector fields on M is denoted by $V^r(M)$.

In particular, if (U, ψ) is a local coordinate neighborhood with $m \in U$ and $(x^1, ..., x^N)$ is the corresponding local coordinate system on (U, ψ) then $(\frac{\partial}{\partial x^n})_m$, $n = 1, ..., N$ defined by $(\frac{\partial}{\partial x^n})_m g = \frac{\partial g\circ\psi^{-1}}{\partial x^n}(\psi(m))$ belong to $V^\infty(M)$ and constitute a basis of $T_m(M)$. Let g be a C^1 function at m, the differential of g at m, $dg(m)$, is an element of $T^*_m(M)$, and is defined by

$$dg(m)(v) = v(g) \quad \forall v \in T_m(M).$$

Let M_1 and M_2 be C^∞ manifolds. Consider a map $\phi : M_1 \to M_2$. Then for every function $g \in C^\infty(M_2)$, ϕ induces a function ϕ^*g on M_1 defined by $\phi^*g = g\circ\phi$. A map $\phi : M_1 \to M_2$ is called C^r at $m \in M_1$ (on $S \subset M_1$) provided that so is ϕ^*g for any $g \in C^\infty(M_2)$. Let $\phi : M_1 \to M_2$ be a C^1 map and let $m \in M_1$ be a fixed element. Define, for $v \in T_m(M_1)$ and $g \in C^\infty(M_2)$, $((\phi_*)_m v)(g) = v(\phi^*g)$. Then $(\phi_*)_m : T_m(M_1) \to T_{\phi(m)}(M_2)$ is a linear map. The dual map of $(\phi_*)_m$ is denoted by ϕ^*_m. It is a map from $T^*_{\phi(m)}(M_2) \to T^*_m(M_1)$ and has the property that, for any $g \in C^1(M_2)$, $\phi^*_m dg(\phi(m)) = d(\phi^*g)(m)$.

2.2. Riemannian manifolds

Let M be a C^∞ manifold. Recall that a mapping $g : T(M) \times T(M) \to \mathbb{R}$ is a C^∞ Riemannian metric if

(1) for each m, $g_m(v, u)$ is an inner product on $T_m(M)$;
(2) if (U, ψ) is a local coordinate neighborhood around m with local coordinate system $(x^1, ..., x^N)$, then $g_{ij}(m) := g_m(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \in C^\infty(M)$.

One can check that (2) is independent of local coordinate systems. A manifold M together with the Riemannian metric g is called a Riemannian manifold.

Let (M, g) be a Riemannian manifold. For each $m \in M$, the Riemannian metric induces an isomorphism between $T_m(M)$ and $T^*_m(M)$ by

$$v^* = g_m(v, \cdot) \quad (\langle v^*, u\rangle = g_m(v, u),\ \forall u \in T_m(M)).$$

Then we define norms on $T_m(M)$ and $T_m^*(M)$ by

$$\|v^*\|^2 = \|v\|^2 = g_m(v, v).$$

The following generalized Cauchy inequality is crucial: for any $v^* \in T_m^*(M)$ and $u \in T_m(M)$,

$$\langle v^*, u\rangle \leq \|v^*\|\|u\|.$$

Let $\gamma : [0, 1] \to M$ be a C^1 curve. The length of γ is

$$l(\gamma) = \int_0^1 \|\gamma'(s)\|ds.$$

Let $m_1, m_2 \in M$. Denote the collection of all C^1 curves joining m_1 and m_2 by $C(m_1, m_2)$. Then the distance between m_1 and m_2 is defined by

$$d(m_1, m_2) := \inf\{l(\gamma) : \gamma \in C(m_1, m_2)\}.$$

The distance between a point $m \in M$ and a set $S \subset M$ is defined by $d_S(m) := \inf\{d(m, m') : m' \in S\}$. It is not hard to check that

$$|d_S(m_1) - d_S(m_2)| \leq d(m_1, m_2). \tag{3}$$

For any two points m_1 and m_2 of a connected Riemannian manifold there exists a smooth curve γ connecting them and satisfying $l(\gamma) = d(m_1, m_2)$. Such minimal length curve is called a *geodesic*. Clearly it can be parametrized in many different ways. We consider only the *canonical parametrization* $\gamma(s)$, $s \in [0, 1]$, such that $\|\gamma'(s)\|$ is constant on $[0, 1]$ and $\gamma(0) = m_1$,$\gamma(1) = m_2$.

It is easy to see that for this canonical parametrization of a geodesic γ we have

$$\|\gamma'(s)\| = d(m_1, m_2) \quad \text{and} \quad d(m_1, \gamma(s)) = sd(m_1, m_2) \quad \text{for any } s \in [0, 1].$$

2.3. Nonsmooth analysis on smooth manifolds

The distance function from points to sets will play an important role in our subsequent discussion. In general, such functions are nonsmooth. Now we briefly discuss the basic concepts of subdifferential calculus of nonsmooth functions on smooth manifolds [18, 19, 20] used in this paper. They are developments of the corresponding theory in Banach spaces (see books [3, 6, 7, 8, 22, 25]). For an extended-valued function $f :\to \mathbb{R} \cup \{+\infty\}$ the domain of f is defined by $\mathrm{dom}(f) := \{m \in M : f(m) < \infty\}$.

Definition 2.1. Let $f : M \to \mathbb{R} \cup \{+\infty\}$ be an extended-valued lower semicontinuous function. We define the Fréchet-subdifferential of f at $m \in \mathrm{dom}(f)$ by

$$\partial_F f(m) := \{dg(m) : g \in C^1(M) \text{ and } f - g \text{ attains a local minimum at } m\}.$$

Elements of the Fréchet-subdifferential are called Fréchet-*subgradients*.

It follows immediately from the subgradient definition that if function f attains a local minimum at the point $\bar{m}$ on M then

$$0 \in \partial_F f(\bar{m}). \tag{4}$$

Later we'll use the following important chain rule. Its proof can be found in [18, 19].

Theorem 2.2. *Let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Suppose that (U, ψ) is a local coordinate neighborhood and $m \in U$. Then*

$$\partial_F f(m) = \psi_m^* \partial_F (f \circ \psi^{-1})(\psi(m)).$$

2.4. Convexity on manifolds of nonpositive curvature

Let M be a Riemannian manifold. We recall that a subset K of M is called *convex* if for any points m_1 and m_2 in K there exists a *unique* geodesic (minimal curve) γ connecting them which is contained in K. Let K be a convex subset of a Riemannian manifold M. A function $f : K \to \mathbb{R}$ is called *convex* if for any points $m_1, m_2 \in K$ and geodesic $\gamma : [0,1] \to K$ connecting them, the function $f \circ \gamma : [0,1] \to \mathbb{R}$ is convex, namely, we have for any $s_1, s_2 \in [0,1]$, $\alpha \in [0,1]$

$$f(\gamma((1-\alpha)s_1 + \alpha s_2)) \leq (1-\alpha) f(\gamma(s_1)) + \alpha f(\gamma(s_2)). \tag{5}$$

For a set $C \subset M$ its closed *convex hull* is defined as the intersection of all closed convex sets containing C. It is clear that, in general, the convex hull of C on the manifold can be empty. It is well known that in the case when $M = \mathbb{R}^N$ the convex hull of any set C is nonempty and it is compact if C is compact (due to Carathéodory's theorem). However, we don't know of any analogue to this result for closed convex hulls on general manifolds and we need to introduce the following property for a convex set K on M.

Definition 2.3. Let K be a convex subset of a Riemannian manifold M. We say that K satisfies the (CC)-condition if for any finite set $\{m_1, m_2, \ldots, m_n\} \subset K$, its convex hull is compact.

Clearly, any compact convex set satisfies the (CC)-condition. An example of a manifold M which convex subsets satisfy the (CC)-condition is given by the Cartan-Hadamard manifold.

Recall that a complete simply connected N-dimensional Riemannian manifold M of nonpositive curvature is called *Cartan-Hadamard*. Due to the Cartan-Hadamard theorem [4, 10], such manifolds are convex and for any convex subset C of M the distance function $d_C(m)$ is convex. In particular, it implies that any closed ball with a finite radius r

$$\overline{B_r(\bar{m})} := \{m \ : \ d(\bar{m}, m) \leq r\}$$

is also a convex compact set. Consider the finite set $C := \{m_1, \ldots, m_K\}$. It is obvious that C is contained in the ball $\overline{B_r(m_1)}$ of finite radius

$$r := \max_{2 \leq n \leq K} d(m_1, m_n).$$

It follows from the convexity of balls in Cartan-Hadamard manifolds that the convex hull of C is compact. Thus Cartan-Hadamard manifolds satisfy the (CC)-condition.

Let $K \subset M$ be convex and let $\gamma_{m_1 m_2} : [0,1] \to M$ denotes the unique geodesic connecting points m_1 and m_2 from K: $\gamma_{m_1 m_2}(0) = m_1$ and $\gamma_{m_1 m_2}(1) = m_2$. For a fixed $\bar{m} \in K$ we consider the vector field

$$v_{\bar{m}}(m) := \gamma'_{\bar{m}m}(1). \tag{6}$$

It is obvious that

$$\|v_{\bar{m}}(m)\| = d(\bar{m}, m).$$

The next Proposition follows from the uniqueness of geodesics in convex sets and from the continuous dependence of solutions of the geodesic equation on initial data.

Proposition 2.4. *Let K be a convex subset of a Riemannian manifold M. Then, for any function $g \in C^\infty(M)$ the function $(m, \bar{m}) \to v_{\bar{m}}(m)(g)$ is continuous at any $m \neq \bar{m}$ on K.*

Our generalization of Helly's theorem is valid for Riemannian manifolds of nonnegative curvature.

It is known [4] that manifolds of nonpositive curvature have the following property: for any convex set K and two geodesics γ_1 and γ_2 from $[0,1]$ to K, the function

$$s \to d(\gamma_1(s), \gamma_2(s))$$

is convex.

This fact is used to show the well-known fact that the distance function to a closed convex set is convex.

Lemma 2.5. *Let M be a Riemannian manifold with nonpositive curvature, $K \subset M$ be convex and C be a closed convex subset of K. Then the distance function $d_C : W \to \mathbb{R}$ is convex on K.*

Proof. Consider $m_1, m_2 \in W$ and let $m_1', m_2' \in C$ be their projections on C, that is, $d(m_n, m_n') = d_C(m_n)$, $n = 1, 2$. Then, for any $s \in [0,1]$,

$$\begin{aligned} d_C \circ \gamma_{m_1m_2}(s) &\leq d(\gamma_{m_1m_2}(s), \gamma_{m_1'm_2'}(s)) \\ &\leq (1-s)d(\gamma_{m_1m_2}(0), \gamma_{m_1'm_2'}(0)) + sd(\gamma_{m_1m_2}(1), \gamma_{m_1'm_2'}(1)) \\ &= (1-s)d_C(m_1) + sd_C(m_2) \\ &= (1-s)d_C \circ \gamma_{m_1m_2}(0) + sd_C \circ \gamma_{m_1m_2}(1). \end{aligned}$$

It is easy to see that this inequality implies convexity of d_C on K. $\square$

For any convex function f the subdifferential of f at $\bar{m} \in \text{int dom } f$ defined by

$$\partial f(\bar{m}) = \{\xi \in T_m^*(M) : f(m) - f(\bar{m}) \geq \langle \xi, \gamma_{\bar{m}m}'(0)\rangle, \forall m \in M\},$$

is nonempty, convex and compact [27]. An element $\xi \in \partial f(m)$ is called a subgradient of f at m. Our next lemma show that for a convex function the subdifferential contains the Fréchet subdifferential.

Lemma 2.6. *Let M be a Riemannian manifold with nonpositive curvature and let f be a convex function. Then for any $m \in$ int dom f we have*

$$\partial_F f(\bar{m}) \subset \partial f(\bar{m}).$$

Proof. Let $\xi \in \partial_F f(\bar{m})$. Then there exists $g \in C^1(M)$ with $dg(\bar{m}) = \xi$ such that $f - g$ attains a minimum 0 at $\bar{m}$. Since f is convex, for any m and $s \in (0,1]$

$$f(m) - f(\bar{m}) \geq \frac{f(\gamma_{\bar{m}m}(s)) - f(\bar{m})}{s} \geq \frac{g(\gamma_{\bar{m}m}(s)) - g(\bar{m})}{s}. \tag{7}$$

Taking limits in (7) as $s \to 0+$ we have

$$f(m) - f(\bar{m}) \geq f'(\bar{m}; \gamma'_{\bar{m}m}(0)) \geq \langle dg(\bar{m}), \gamma'_{\bar{m}m}(0)\rangle = \langle \xi, \gamma'_{\bar{m}m}(0)\rangle.$$

It follows that $\xi \in \partial f(\bar{m})$. □

3. Subdifferentials of sup-envelopes

We now develop an estimate for subgradients of sup-envelopes of Lipschitz functions on smooth Riemannian manifolds. This is an important subject that has been studied extensively [2, 14, 17, 21, 26] in different settings. In Euclidean space $\mathbb{R}^N$ our result below is a combination of the estimate for the subgradient of a sup-envelope function in [21] and Carathéodory's theorem, which asserts that any convex combination can be represented as a convex combination of $N+1$ vectors. The result in [21] actually applies to sup-envelops of lower semicontinuous functions and the corresponding generalization to arbitrary smooth manifolds has been discussed in [18]. Here we trade the more general setting in [18] for a better estimate.

Let $f_a : M \to \mathbb{R}, a \in A$ be a family of Lipschitz functions with the same Lipschitz constant L and let

$$f(m) := \sup_{a \in A} f_a(m)$$

be the sup-envelope of $\{f_a\}_{a\in A}$. We want to establish an estimate for the Fréchet subdifferential of f in terms of the subdifferential of functions f_a. For this we need the following definition.

Definition 3.1. Let M be a C^∞ manifold. We say a set $W \subset V^0(M)$ is locally uniformly bounded and equicontinuous around m if there exists a local coordinate neighborhood (U, ψ) of m such that $\{\psi_* v : v \in W\}$ is uniformly bounded and equicontinuous on U.

It is not hard to see that this definition is independent of the local coordinate neighborhoods.

We will also need the following set

$$\mathcal{G}_{\delta,U}(\bar{m}) := \{(m, a) \in U \times A : f_a(m) \geq f(\bar{m}) - \delta\}$$

defined for positive δ and a neighborhood U of $\bar{m}$. Note that when M is a Riemannian manifold we can consider a neighborhood $U = B_\delta(\bar{m}) := \{m \in M : d(m, \bar{m}) < \delta\}$. The set $\mathcal{G}_{\delta,U}(\bar{m})$ plays a role of a set of approximate solutions in maximizing $a \to f_a(\bar{m})$.

Theorem 3.2. *Let M be a C^∞ Riemannian manifold and let $f_a : M \to \mathbb{R}, a \in A$ be a family of Lipschitz functions with a uniform Lipschitz constant. Define*

$$f(m) := \sup_{a \in A} f_a(m).$$

Suppose that $\xi \in \partial_F f(m)$ and $W \subset V(M)$ is locally uniformly bounded and equicontinuous around m. Then for any $\varepsilon > 0$ there exist convex coefficients $\{\lambda_n\}_{n=1}^{N+1}$, points $(m_n, a_n) \in \mathcal{G}_{\varepsilon, B_\varepsilon(m)}$ and subgradients $\xi_n \in \partial_F f_{a_n}(m_n)$ such that

$$|\langle \xi, v\rangle_m - \sum_{n=1}^{N+1} \lambda_n \langle \xi_n, v\rangle_{m_n}| < \varepsilon, \ \forall v \in W.$$

Proof. Choose a local coordinate system (U, ψ) with $m \in U \subset B_\varepsilon(m)$ such that W is uniformly bounded and equicontinuous on U. Define $h_a = f_a \circ \psi^{-1}$ and $h = f \circ \psi^{-1}$. Then $h = \sup_{a \in A} h_a$. Since ψ is a diffeomorphism and $\{f_a\}_{a \in A}$ and f are Lipschitz with a uniform Lipschitz constant there exists a constant $L > 0$ such that h and $\{h_a\}_{a \in A}$ are Lipschitz with this same Lipschitz constant L. Moreover, $\{\psi_* v = (v_1, \ldots, v_n) : v \in W\}$ is uniformly bounded and equicontinuous on U. Choose $\eta < \varepsilon$ such that $B_\eta(m) \subset U$ and for any $m', m'' \in B_\eta(m)$ and any $v \in W$,

$$\|(\psi_*)_{m'} v - (\psi_*)_{m''} v\| < \frac{\varepsilon}{2L}.$$

Denote $x = \psi(m)$ and $x^* = \psi_m^* \xi$. Then, by Lemma 2.2, $x^* \in \partial_F h(x)$. Choose

$$\varepsilon' < \frac{\varepsilon}{2 \sup\{\|(\psi_*)_m v\| : m \in U\}}$$

such that $B_{\varepsilon'}(x) \subset \psi(B_\eta(m))$. It follows that $(x, a) \in \mathcal{G}_{\varepsilon', B_{\varepsilon'}(x)}$ implies $(\psi^{-1}(x), a) \in \mathcal{G}_{\varepsilon, B_\eta(m)}$. Applying the subdifferential representation of [21] in $\mathbb{R}^N$ combined with Carathéodory's theorem we can conclude that there exist convex coefficients $\{\lambda_n\}_{n=1}^{N+1}$, points $(x_n, a_n) \in \mathcal{G}_{\varepsilon, B_{\varepsilon'}(x)}$ and subgradients $x_n^* \in \partial_F h_{a_n}(x_n)$ such that

$$\|x^* - \sum_{n=1}^{N+1} \lambda_n x_n^*\| < \varepsilon'.$$

Let $m_n = \psi^{-1}(x_n)$ and $\xi_n = \psi_{m_n}^* x_n^*$. Then $m_n \in B_\varepsilon(m)$, $\xi_n \in \partial_F f_{a_n}(m_n)$ and, for any $v \in W$,

$$\begin{aligned}
& |\langle \xi, v \rangle_m - \sum_{n=1}^{N+1} \lambda_n \langle \xi_n, v \rangle_{m_n}| \\
= \; & |\langle \psi_m^* x^*, v \rangle_m - \sum_{n=1}^{N+1} \lambda_n \langle \psi_{m_n}^* x_n^*, v \rangle_{m_n}| \\
= \; & |\langle x^*, (\psi_*)_m v \rangle - \sum_{n=1}^{N+1} \lambda_n \langle x_n^*, (\psi_*)_{m_n} v \rangle| \\
= \; & |\langle x^* - \sum_{n=1}^{N+1} \lambda_n x_n^*, (\psi_*)_m v \rangle + \sum_{n=1}^{N+1} \lambda_n \langle x_n^*, (\psi_*)_m v - (\psi_*)_{m_n} v \rangle| \\
= \; & \|x^* - \sum_{n=1}^{N+1} \lambda_n x_n^*\| \|(\psi_*)_m v\| + \sum_{n=1}^{N+1} \lambda_n \|x_n^*\| \|(\psi_*)_m v - (\psi_*)_{m_n} v\| < \varepsilon.
\end{aligned}$$

□

4. The Main Results

We now state and prove our main theorem that generalizes Helly's intersection theorem on the intersection of convex sets on Riemannian manifolds with nonpositive curvature. We start with a special case.

Theorem 4.1. *Let M be an N-dimensional C^∞ Riemannian manifold of nonpositive curvature. Suppose that there exist an open convex set K, a convex compact set $C \subset K$ and a family of convex closed sets $C_a \subset C$, $a \in A$, such that any intersection of $N+1$ sets C_a*

$$\cap_{n=1}^{N+1} C_{a_n} \tag{8}$$

is nonempty. Then the intersection of all C_a is also nonempty, i.e.,

$$\cap_{a \in A} C_a \neq \emptyset.$$

Proof. Define

$$f = \sup_{a \in A} d_{C_a}.$$

Then f is convex and finite on K. Also f is continuous, in fact, Lipschitz with a Lipschitz constant 1 on M. Consider the problem

$$\text{minimize } f \text{ on the set } C.$$

Due to the compactness of C there exists a minimizer $\bar{m} \in C$.

Using the exact penalization method in [6], we conclude that $\bar{m}$ is also a solution of the problem

$$\text{minimize } f + 2d_C \text{ on the open set } K.$$

If $f(\bar{m}) = 0$ then $\bar{m} \in \bigcap_{a \in A} C_a$. We consider the nontrivial case when $f(\bar{m}) = f(\bar{m}) + 2d_C(\bar{m}) = 2r > 0$.

Choose $\delta > 0$ such that for any $c \in C$, $B_\delta(c) \subset K$ and $m \in B_\delta(\bar{m})$ one has that $f(m) + 2d_C(m) > r > 0$. Further choose $\varepsilon \in (0, \min(\delta/2, r))$. Since $\bar{m}$ minimizes $f + 2d_C$ on K and $\bar{m}$ is interior point of K we have $0 \in \partial_F(f + 2d_C)(\bar{m})$.

For each $c \in C$ define a vector field v_c by $v_c(m) = \gamma'_{cm}(1)$. Note that, for $c \neq m$, $v_c(m)$ is continuous in both c and m by Proposition 2.4.

Since C is compact, $W = \{v_c : c \in C \backslash B_\delta(\bar{m})\} \subset V^0(M)$ is locally uniformly bounded and equicontinuous on $B_\varepsilon(\bar{m})$.

Using the estimate from Theorem 3.2 for the subgradient of the sup-envelope function

$$f + 2d_C = \sup_{a \in A}(d_{C_a} + 2d_C),$$

we conclude that there exist indices $a_1, \ldots, a_{N+1} \in A$, nonnegative numbers $\lambda_1, \ldots, \lambda_{N+1}$ with $\sum_{n=1}^{N+1} \lambda_n = 1$, points $m_n \in B_\varepsilon(\bar{m})$ with $(d_{C_{a_n}} + 2d_C)(m_n) > (f + 2d_C)(\bar{m}) - \varepsilon$ and subgradients

$$\xi_n \in \partial_F(d_{C_{a_n}} + 2d_C)(m_n) \subset \partial(d_{C_{a_n}} + 2d_C)(m_n)$$

satisfying, for any $v \in W$,

$$\| \sum_{n=1}^{N+1} \lambda_n \langle \xi_n, v \rangle \| < \varepsilon. \tag{9}$$

By the assumption of the theorem we have

$$\bigcap_{n=1}^{N+1} C_{a_n} \neq \emptyset.$$

Letting c be an element of this intersection and $v = v_c \in W$, we have

$$\begin{aligned} -\langle \xi_n, v\rangle &\leq (d_{C_{a_n}} + 2d_C)(c) - (d_{C_{a_n}} + 2d_C)(m_n) \\ &= -(d_{C_{a_n}} + 2d_C)(m_n) < -2r + \varepsilon < -r. \end{aligned} \tag{10}$$

Combining (9) and (10), we have

$$-\varepsilon < -\sum_{n=1}^{N+1} \lambda_n \langle \xi_n, v\rangle < -r,$$

a contradiction. □

Now we can prove our main result. It says that the condition that all C_a belong to a convex compact set in Theorem 4.1 can be relaxed to they belong to a convex set satisfying the (CC)-condition and one of them is compact.

Theorem 4.2. *Let M be an N-dimensional C^∞ Riemannian manifold with nonpositive curvature and let K be an open convex subset of M satisfying the (CC)-condition. Let $\{C_a\}_{a\in A}$ be a family of closed convex subsets of K and let at least one of them be compact. Suppose that, for any $N+1$ elements $a_1, \ldots, a_{N+1} \in A$,*

$$\bigcap_{n=1}^{N+1} C_{a_n} \neq \emptyset.$$

Then

$$\bigcap_{a\in A} C_a \neq \emptyset.$$

Proof. Since one of the sets C_a is compact we need only show that for any finite subset $F \subset A$, $\cap_{a\in F} C_a \neq \emptyset$.

Without loss of generality we may assume that A itself is a finite set and $A = \{a_1, a_2, \ldots, a_J\}$ with $J > N+1$. (There is nothing to prove if $J \leq N+1$. For each of the subsets $\{a_{j_1}, \ldots, a_{j_{N+1}}\}$ of A, select

$$m_{j_1,\ldots,j_{N+1}} \in \bigcap_{n=1}^{N+1} C_{a_{j_n}}$$

and consider the convex hull C of the set of all these points

$$C := \overline{conv}\{m_{j_1,\ldots,j_{N+1}} : \{a_{j_1}, \ldots, a_{j_{N+1}}\} \subset A\}.$$

Then the family of convex compact sets

$$\{K_a = C_a \cap C : a \in A\}$$

satisfies the assumption of Theorem 4.1. In fact, $\cup_{a\in A} K_a \subset C$ and for any $\{a_{j_1}, \ldots, a_{j_{N+1}}\} \subset A\}$,

$$m_{j_1,\ldots,j_{N+1}} \in \bigcap_{n=1}^{N+1} K_{a_{j_n}}.$$

It follows from Theorem 4.1 that

$$\emptyset \neq \bigcap_{a\in A} K_a \subset \bigcap_{a\in A} C_a.$$

□

Since any convex subset of a Cartan-Hadamard manifold satisfies the (CC)-condition we have

Corollary 4.3. *Let M be an N-dimensional C^∞ Cartan-Hadamard manifold. Let $\{C_a\}_{a\in A}$ be a family of closed convex subsets of M and at least one of them is compact. Suppose that, for any $N+1$ elements $a_1, \ldots, a_{N+1} \in A$,*

$$\bigcap_{n=1}^{N+1} C_{a_n} \neq \emptyset.$$

Then

$$\bigcap_{a\in A} C_a \neq \emptyset.$$

Proof. Let $K = M$ in Theorem 4.2. □

5. Conclusion

There are two open problems related to the variant of the Helly's intersection theorem in this paper.

The first is related to the (CC)-condition for a convex subset K of M. We recall that an Euclidean space $\mathbb{R}^N$ satisfies this (CC)-condition and that fact follows from Carathédory's theorem.

Let $[m_1, m_2]$ denote the unique geodesic connecting m_1 and m_2 in K. On the set of all subsets $A, B \subset K$ consider the following operator $\mathcal{H}$

$$\mathcal{H}(A, B) := \{m \ : \ m \in [m_1, m_2], m_1 \in A, m_2 \in B\}.$$

Consider the sequence of sets

$$H_0(C) := C, \quad H_n(C) := \mathcal{H}(C, H_{n-1}(C)) \quad n = 1, \ldots \,.$$

Note that when $M = \mathbb{R}^N$ the Carathéodory's theorem is equivalent to the fact that the convex hull of C coincides with $H_N(C)$. Is this true for a general Riemannian manifold with nonpositive curvature?

Conjecture 5.1. *Let X be some subset of the convex set K from the N-dimensional manifold with nonpositive curvature then the convex hull of X coincides with the set $H_N(X)$.*

Note that if Conjecture 5.1 is valid then K satisfies (CC)-condition.

The second open problem concerns Klee's generalization of Helly's theorem when intersection of any $N+1$ sets contains a translate of the compact convex set T. The following conjecture, to some extent, imitates the result by Klee [16].

Conjecture 5.2. *Let M be an N-dimensional simply connected smooth manifold with nonpositive curvature. Suppose there exists a nonnegative r, a convex set K and a family of convex closed sets $C_a \subset K$, $a \in A$ such that for any intersection of $N+1$ sets C_{a_n}, some y,*

$$\bar{B}(y, r) \subset \cap_{n=1}^{N+1} C_{a_n}, \tag{11}$$

and in the case of the infinite index set A let at least one set C_a be compact. If K satisfies (CC)-condition then for some point z

$$\bar{B}(z, r) \subset \cap_{a \in A} C_a.$$

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