Journal of Convex Analysis
Volume 13 (2006), No. 3+4, 525–559

Characterizations of Prox-Regular Sets in Uniformly Convex Banach Spaces

Frédéric Bernard
*Université Montpellier II, Département de Mathématiques,
CC 051, Place Eugène Bataillon, 34095 Montpellier, France
bernard@math.univ-montp2.fr*

Lionel Thibault
*Université Montpellier II, Département de Mathématiques,
CC 051, Place Eugène Bataillon, 34095 Montpellier, France
thibault@math.univ-montp2.fr*

Nadia Zlateva*
*Institute of Mathematics and Informatics, Bulgarian Academy of Sciences,
Acad. G. Bonchev Str., Bl. 8, 1113 Sofia, Bulgaria
zlateva@math.bas.bg*

Dedicated to the memory of Simon Fitzpatrick.

Received: June 1, 2005
Revised manuscript received: November 15, 2005

The aim of the paper is to extend to the setting of uniformly convex Banach spaces the results obtained for prox-regular sets in Hilbert spaces. Prox-regularity of a set C at a point $x \in C$ is a variational condition related to normal vectors and which is common to many types of sets. In the context of uniformly convex Banach spaces, the prox-regularity of a closed set C at x is shown to be still equivalent to the property of the distance function d_C to be continuously differentiable outside of C on some neighbourhood of x. Additional characterizations are provided in terms of metric projection mapping. We also examine the global level of prox-regularity corresponding to the continuous differentiability of the distance function d_C over an open tube of uniform thickness around the set C.

Keywords: Distance function, metric projection mapping, uniformly convex Banach space, variational analysis, proximal normal, prox-regular set

2000 Mathematics Subject Classification: Primary 49J52, 58C06, 58C20; Secondary 90C30

The differentiability of the distance function d_C to a nonempty closed subset C of a Banach space X and the single valuedness of the metric projection mapping P_C are longstanding subjects of study. For a convex set C, the differentiability of d_C^2 and the single valuedness and continuity of P_C on the whole space X are well known in smooth Banach spaces. In the finite dimensional Euclidean case, Motzkin [36] seems to be the first to prove that a nonempty closed set C is convex if and only if its metric projection mapping P_C is single-valued everywhere. In the Hilbert setting, Klee [29] proved that for weakly closed sets C, the convexity of C is also characterized in this way. The characterization in the

*The research has been supported by a Marie Curie fellowship of the European Community programme Improving Human Potential under contract No. HPMF-CT-2001-01345.

ISSN 0944-6532 / $ 2.50

Hilbert setting of the convexity of a norm closed set C by the metric projection mapping P_C being single-valued and norm-to-weak continuous is due to Asplund [1]. When the dual space of X is rotund, Vlasov [46] extended in some sense Asplund's result by showing that a norm closed set C of X is convex if and only if the metric projection mapping is single-valued and norm-to-norm continuous. See also [45] for the case of approximately compact sets.

Using the original result by S. Fitzpatrick [27] reducing the differentiability of d_C to its Gâteaux directional derivability in a certain key direction, Borwein, Fitzpatrick and Giles [6] characterized closed convex sets C of a Banach space X with rotund dual as closed sets C for which the distance function d_C is Gâteaux differentiable on $X \setminus C$ with $\|\nabla^G d_C(x)\| = 1$ for all $x \in X \setminus C$. This result of Fitzpatrick will be stated below as a theorem (see Theorem 4.4) because of its importance.

In order to extend the Steiner polynomial formula concerning the n-dimensional measure of the r-neighbourhood (with respect to the Euclidean norm) of a closed convex subset or a compact C^2-submanifold of $\mathbb{R}^n$ to a much larger class of sets, Federer [26] introduced the concept of subsets of $\mathbb{R}^n$ with *positive reach*. For a nonempty closed set $C \subset \mathbb{R}^n$, denoting by $\mathrm{Unp}(C)$ the set of all points $x \in \mathbb{R}^n$ for which C contains a unique nearest point to x, Federer defined its *reach* (that he denoted by $\mathrm{reach}(C)$) as the largest r (possibly $+\infty$) such that $\{x \in \mathbb{R}^n : 0 < d_C(x) < r\} \subset \mathrm{Unp}(C)$. Then, he declared C to be positively reached whenever $\mathrm{reach}(C) > 0$ and established, among other results, that d_C is continuously differentiable on the set $\{x \in \mathbb{R}^n : 0 < d_C(x) < \mathrm{reach}(C)\}$. Note that Federer also worked, for a fixed point $\bar{x} \in C$ with $\mathrm{reach}(C, \bar{x})$, i.e., the supremum of all $r > 0$ such that the open ball centered at $\bar{x}$ with radius r is included in $\mathrm{Unp}(C)$. Considering, in Hilbert space, the concept of p-convex set C of Degiovanni, Marino and Tosques [21], Canino [12] established that on a suitable open neigbourhood of C the metric projection mapping P_C is single-valued and locally Lipschitz continuous. Staying on the global level in Hilbert space, Clarke, Stern and Wolenski [16] introduced and studied the *proximally smooth sets*. Such sets correspond to closed sets C for which the distance function d_C is continuously differentiable on an open tube around C of the type

$$U_C(r) := \{u \in X : 0 < d_C(u) < r\}$$

for some $r > 0$. In view of Federer's result recalled above, in finite dimensions those sets are positively reached and vice versa. Clarke, Stern and Wolenski characterized, in Hilbert space, proximal smooth sets in several interesting ways, in particular in terms of proximal normals and proximal mapping P_C. They also provided (in finite dimensions) a detailed analysis of locally Lipschitz continuous functions for which the epigraph is proximally smooth. Another previous interesting result was obtained by Shapiro [42] on the local level, in the Hilbert setting. He proved, for a closed set C and a point $\overline{x} \in C$, that the metric projection mapping P_C is single-valued on a neighbourhood of $\overline{x}$ whenever the distance to the general Boulingand contingent cone to C satisfies a property refered to as the *Shapiro property* by Poliquin, Rockafellar and Thibault [39].

On the local level, in the study of sets C for which d_C is *locally* differentiable and its consequences for the metric projection mapping P_C, Poliquin, Rockafellar and Thibault [39] recently made advance in the Hilbert setting with a different point of view, by making the link with the local property of C called *prox-regularity*. This property has been introduced as a new important regularity in variational analysis by Poliquin and Rockafellar [38]. They defined this concept for functions and for sets, and studied it in the finite dimensional setting. Rich geometric implications and characterizations of such a concept were obtained by Poliquin, Rockafellar and Thibault. In their work [39], they relied, in Hilbert space, on the prox-regularity of a closed set C at $\overline{x} \in C$ and characterized it in terms of d_C and P_C, e.g. d_C being continuously differentiable outside of C on a neighbourhood of $\overline{x}$ or P_C being single-valued and norm-to-weak continuous on this same neighbourhood. They also gave a subdifferential characterization of such sets with the normal cone to C. Coming back to the *global level*, they showed that proximally smooth sets are exactly *uniformly prox-regular* sets and provided new insights on those sets.

Our main aim is to extend the scope of those results to the more general setting of uniformly convex Banach spaces (e.g. l_p, L^p and W^p_m with $1 < p < \infty$) and to find their analogues in the context of such spaces. We will rely on a property that we introduce, analogous to the prox-regularity.

The paper is organized as follows. In Section 2 we give the necessary notation and preliminaries. In Section 3 we consider the definition and the first properties of prox-regular sets on a uniformly convex Banach space X. We also introduce related definitions for functions and for set-valued mappings. Section 4 is devoted to the study of properties of local Moreau envelopes of functions on X. In Section 5 we establish several characterizations of prox-regular sets in X (see Theorem 4.9) extending in this way the results of [39]. In the final Section 6 we use some techniques developed in the previous sections to obtain in Theorem 5.2 various results similar to the characteristic Theorem 4.9 but on the global level of proximally smooth sets. So, with Theorems 4.9 and 5.2 we extend several results of [26], [12], [16], [39], and [17].

1. Notation and preliminaries

We begin by recalling some of the properties of uniformly convex Banach spaces which can be found in [23, 11, 2, 22].

For a Banach space X the following are equivalent:

(X1) X has an equivalent *uniformly convex* norm $\|.\|$, i.e., such that its *modulus of convexity*

$$\delta_{\|.\|}(\varepsilon) := \inf\left\{1-\left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1, \|x-y\|\geq\varepsilon\right\}$$

satisfies $\delta_{\|.\|}(\varepsilon)>0$ for all $\varepsilon\in]0,2]$.

(X2) X has an equivalent uniformly convex norm $\|.\|$ with modulus of convexity of *power type* q, i.e., for some $k > 0$ one has $\delta_{\|.\|}(\varepsilon) \geq k\varepsilon^q$, for all $\varepsilon \in]0,2]$. From Dvoretzky's theorem necessarily $q \geq 2$.

(X3) X has an equivalent *uniformly smooth* norm $\|.\|$, i.e., such that its *modulus of smoothness*

$$\rho_{\|.\|}(\tau) := \frac{1}{2}\sup\{\|x+y\| + \|x-y\| - 2 : \|x\| = 1, \|y\| \leq \tau\}$$

satisfies $\lim_{\tau\downarrow 0} \frac{\rho_{\|.\|}(\tau)}{\tau} = 0$.

(X4) X has an equivalent uniformly smooth norm $\|.\|$ with modulus of smoothness of *power type* s, i.e., such that for some $c > 0$ one has $\rho_{\|.\|}(\tau) \leq c\tau^s$ for all $\tau \geq 0$. From Dvoretzky's theorem necessarily $1 < s \leq 2$.

(X5) X has an equivalent norm which is both uniformly convex and uniformly smooth and which has moduli of convexity and smoothness of power type.

It is well-known (see [23, 32, 33]) that all Hilbert spaces H and the Banach spaces l^p, L^p, and W^p_m $(1 < p < \infty)$ all are (for their usual norms) uniformly convex and uniformly smooth with moduli of convexity and smoothness of power type. More precisely, for $\varepsilon \in]0, 2]$ and $\tau \geq 0$

$$\delta_H(\varepsilon) = 1 - \sqrt{1 - (1/4)\varepsilon^2} \geq \varepsilon^2/4,$$

$$\delta_{l^p}(\varepsilon) = \delta_{L^p}(\varepsilon) = \delta_{W^p_m}(\varepsilon) = \begin{cases} \frac{p-1}{8}\varepsilon^2 + o(\varepsilon^2) > \frac{p-1}{8}\varepsilon^2, & 1 < p < 2, \\ 1 - \left[1 - \left(\frac{\varepsilon}{2}\right)^p\right]^{1/p} > \frac{1}{p}\left(\frac{\varepsilon}{2}\right)^p, & p \geq 2, \end{cases}$$

$$\rho_H(\tau) = (1 + \tau^2)^{1/2} - 1 < \tau,$$

$$\rho_{l^p}(\tau) = \rho_{L^p}(\tau) = \rho_{W^p_m}(\tau) = \begin{cases} (1 + \tau^p)^{1/p} - 1 < \frac{1}{p}\tau^p, & 1 < p < 2, \\ \frac{p-1}{2}\tau^2 + o(\tau^2) < \frac{p-1}{2}\tau^2, & p \geq 2. \end{cases}$$

Throughout all the paper we will work in a *uniformly convex* Banach space X which is equipped with an *equivalent norm* $\|.\|$ *that satisfies* (X5). Such a norm is a *Kadec norm*, i.e., it satisfies the property that whenever $x_n \xrightarrow[n\to\infty]{w} x$ with $\|x_n\| \xrightarrow[n\to\infty]{} \|x\|$, then $x_n \xrightarrow[n\to\infty]{\|.\|} x$. Let q and s be the power types of moduli of convexity and smoothness of $\|.\|$, respectively. Then X^* is also uniformly convex and its dual norm has modulus of convexity of power type $q^* = s(s-1)^{-1}$ and modulus of smoothness of power type $s^* = q(q-1)^{-1}$. We will denote by $\mathbb{B}$ the closed unit ball of X, and by $B[x, r]$ (resp. $B(x, r)$) the closed (resp. open) ball with center x and radius r. The mapping $J : X \to X^*$ defined by

$$J(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x^*\|.\|x\|, \quad \|x^*\| = \|x\|\}$$

is generally called the *normalized duality mapping*. Let us put together, in the context of uniformly convex Banach space X satisfying (X5), some of its properties that we will use throughout the paper (see [13]):

(J) The mapping $J : X \to X^*$ is single-valued, bijective and norm-to-norm uniformly continuous on bounded sets, $J(\lambda x) = \lambda J(x)$ for all $\lambda \in \mathbb{R}$, $\|J(x)\| = \|x\|$, and $J(x) = \nabla \frac{1}{2}\|.\|^2(x)$ for all $x \in X$.

An analogous property (J^*) holds for the normalized duality mapping $J^* : X^* \to X$. Moreover, $J^* = J^{-1}$.

It is known (see e.g. [47]), that for $r > 0$ there exist positive constants K_r, K'_r such that

$$\langle J(x) - J(y), x - y\rangle \geq K_r\|x - y\|^q, \quad \forall x, y \in r\mathbb{B}, \tag{1}$$

$$\|J(x) - J(y)\| \leq K'_r\|x - y\|^{s-1}, \quad \forall x, y \in r\mathbb{B}. \tag{2}$$

The space $X \times \mathbb{R}$ will be endowed with the norm $|\!|\!| \cdot |\!|\!|$ given by $|\!|\!|(x, r)|\!|\!| = \sqrt{\|x\|^2 + r^2}$. So, for the normalized duality mapping $J_{X\times\mathbb{R}} : X \times \mathbb{R} \to X^* \times \mathbb{R}$ associated with the norm $|\!|\!| \cdot |\!|\!|$, one has the equality

$$J_{X\times\mathbb{R}}(x, r) = (J(x), r). \tag{3}$$

Recall that a normed vector space $(Y, \|\cdot\|)$ is *rotund* or *strictly convex* provided for any $y, y' \in Y$ with $\|y\| = \|y'\| = 1$ and $y \neq y'$ one has $\|\frac{1}{2}(y + y')\| < 1$. According to (X1), the uniform convexity holds when this inequality is fulfilled in some uniform way. Recall (see for example [22]) that the strict convexity of the norm $\|\cdot\|$ is equivalent to require for any non zero $y, y' \in Y$, $y \neq y'$ the equality

$$\|y + y'\| = \|y'\| + \|y\| \tag{4}$$

to entail $y' = \mu y$ for some $\mu > 0$.

We will need the following elementary result concerning nearest points of a closed subset C to a point in Y. It can be found in Hilbert space for example in [15, p. 4]. It must also be known in the general strictly convex setting but we did not find it in the literature. Recall that for any $u \in Y$ the notation $P_C(u)$ means the set of all nearest points of C to the point u.

Lemma 1.1. *Let $(Y, \|\cdot\|)$ be a strictly convex normed vector space, C be a closed subset of Y and $u \notin C$. Assume that $P_C(u) \neq \emptyset$. Then for any $p \in P_C(u)$ and any $t \in]0, 1]$, one has $P_C(u + t(p - u)) = \{p\}$.*

Proof. The case $t = 1$ being obvious, we may suppose $t \in]0, 1[$. Putting $u_t := u + t(p - u)$ we have

$$\|u_t - p\| = \|u + t(p - u) - p\| = (1 - t)\|u - p\| = (1 - t)d_C(u).$$

Further, for any $y \in C$,

$$\begin{aligned} \|u_t - y\| = \|u + t(p - u) - y\| &\geq \|u - y\| - t\|u - p\| \\ &\geq d_C(u) - t\|u - p\| \\ &= (1 - t)d_C(u) \\ &= \|u_t - p\|, \end{aligned}$$

and hence $p \in P_C(u_t)$.
Suppose that there exists $p_t \neq p$ with $p_t \in P_C(u_t)$. Then setting $y = p_t$ in the above sequence of inequalities we obtain

$$\begin{aligned} d_C(u_t) = \|u_t - p_t\| = \|u - p_t + t(p - u)\| &\geq \|u - p_t\| - t\|u - p\| \\ &\geq d_C(u) - t\|u - p\| \\ &= \|u_t - p\| = d_C(u_t). \end{aligned}$$

All the last inequalities are then equalities and hence

$$\|u - p_t\| = d_C(u) \tag{5}$$

and

$$\|u - p_t\| = \|u - p_t + t(p - u)\| + \|t(u - p)\|.$$

Further, obviously $t(u - p) \neq 0$, and one also has $u - p_t + t(p - u) \neq 0$ since $\|u - p_t + t(p - u)\| \geq (1 - t)d_C(u)$. So, because of the rotundity, the last equality above entails (see (4)) that there exists $\mu > 0$ with

$$u - p_t + t(p - u) = \mu t(u - p),$$

that is,

$$u - p_t = t(\mu + 1)(u - p). \tag{6}$$

Using (5) and taking the norm of both members of (6) yield $d_C(u) = t(\mu+1)d_C(u)$ and since $d_C(u) > 0$ we have $t(\mu+1) = 1$. Putting this value in (6) gives $p_t = p$, which completes the proof. □

For a set $C \subset X$ we will denote by $\mathrm{cl}\, C$ its norm closure in X. A vector $p \in X$ is said to be a *proximal normal vector* to C at $x \in \mathrm{cl}\, C$ (see [8]) if there are $u \notin \mathrm{cl}\, C$ and $r > 0$ such that $p = r^{-1}(u - x)$ and $\|u - x\| = d_C(u)$. It is known according to Lau theorem [30] that in any reflexive Banach space endowed with a Kadec norm, the set of those points which have a nearest point to any fixed closed subset is a dense set. In the appropriately renormed space X we are working in, the above property holds and hence there are proximal normal vectors at any point of some dense subset of the boundary of C. Observe that the proximal normality of a non-zero $p \in X$ to C at $x \in \mathrm{cl}\, C$ corresponds to the existence of some $r > 0$ such that $x \in P_{\mathrm{cl}\, C}(x + rp)$. The cone of all such vectors p, together with the origin, will be denoted by $N_C(x)$.

The concept is local in the sense that for any $u \notin \mathrm{cl}\, C$ and any closed ball $V := B[x, \beta]$ centered at $x \in \mathrm{cl}\, C$ such that $\|u - x\| = d_{C\cap V}(u)$ one has $u - x \in N_C(x)$. Indeed, put $\rho := d_{C\cap V}(u) > 0$ and $u_t := x + t(u - x)$ for any fixed positive $t < \min(1, \frac{\beta}{2\rho})$. Observe first that $u_t \in \mathrm{int} V$ according to the inequality $t < \frac{\beta}{2\rho}$ and hence $u_t \notin \mathrm{cl}\, C$ because otherwise one would get $u_t \in \mathrm{cl}\,(C \cap V)$ and

$$\|u_t - u\| = (1 - t)\|u - x\| < d_{C\cap V}(u)$$

which would be a contradiction. Further $\|u_t - x\| = t\rho$ and, on the one hand, for any $y \in C \cap V$ one can write

$$\|u_t - y\| = \|u + (1 - t)(x - u) - y\| \geq \|u - y\| - (1 - t)\|u - x\| \geq t\rho = \|u_t - x\|.$$

On the other hand, for any $y \in C \setminus V$ one has

$$\|u_t - y\| \geq \|y - x\| - \|u_t - x\| > \beta - t\|u - x\| = \beta - t\rho > t\rho = \|u_t - x\|,$$

the last inequality being due to the choice of t. So $\|u_t - x\| = d_C(u_t)$ and hence by the definition of $N_C(x)$ we have

$$u - x = t^{-1}(u_t - x) \in N_C(x). \tag{7}$$

A functional $p^* \in X^*$ is said to be a *proximal normal functional* to C at $x \in \mathrm{cl}\, C$ (see [8]) if there are $u \notin \mathrm{cl}\, C$, $r > 0$ such that $p^* = r^{-1}J(u - x)$ and $\|u - x\| = d_C(u)$. Or, equivalently, a non-zero $p^* \in X^*$ is a proximal normal functional to C at $x \in \mathrm{cl}\, C$ if there exists $r > 0$ such that $x \in P_{\mathrm{cl}\, C}(x + rJ^*(p^*))$. The cone of all such functionals p^*, together with the origin, will be denoted by $N^*_C(x)$. One easily verifies that if $p \in N_C(x)$, then $J(p) \in N^*_C(x)$, and that if $p^* \in N^*_C(x)$, then $J^*(p^*) \in N_C(x)$. Hence, $N_C(x)$ and $N^*_C(x)$ completely determine each other.

A functional $x^* \in X^*$ is said to be a *Fréchet normal functional* (see [8]) to C at x if for any $\varepsilon > 0$ there exists a neighbourhood U_ε of x such that the inequality $\langle x^*, x' - x\rangle - \varepsilon\|x' - x\| \leq 0$ holds for all $x' \in C \cap U_\varepsilon$.

As the norm of our space X is Fréchet differentiable away from the origin, it is not difficult to verify that for any closed subset $C \subset X$ and any $x \in \operatorname{cl} C$, any proximal normal functional to C at x is also a Fréchet normal functional to C at x (see [8], Corollary 3.1).

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous (lsc) function. By definition, the effective *domain* of f is the set $\operatorname{dom} f := \{x \in X : f(x) < +\infty\}$ and the *epigraph* of f is the set $\operatorname{epi} f := \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$. Let $x \in \operatorname{dom} f$. We say that $p^* \in X^*$ is a *proximal subgradient* of f at x if $(p^*, -1)$ is a proximal normal functional to the epigraph of f at $(x, f(x))$. The *proximal subdifferential* of f at x, denoted by $\partial_p f(x)$, consists of all such functionals. Thus, we have $p^* \in \partial_p f(x)$, if and only if, $(p^*, -1) \in N^*_{\operatorname{epi} f}(x, f(x))$. The functional $x^* \in X^*$ is said to be a *Fréchet subgradient* of f at x if $(x^*, -1)$ is a Fréchet normal functional to the epigraph of f at $(x, f(x))$. The *Fréchet subdifferential* of f at x, denoted by $\partial_F f(x)$, consists of all such functionals. If $x \notin \operatorname{dom} f$ then all subdifferentials of f at x are empty, by convention. It is known that for a lsc function f on a reflexive Banach space with a Kadec and Fréchet differentiable norm (in particular, on X), the set $\operatorname{dom} \partial_p f$ is dense in $\operatorname{dom} f$ (see [9], Theorem 7.1). Moreover, from what we saw above, $\partial_p f(x) \subset \partial_F f(x)$ for all $x \in X$. The Fréchet subgradients are known (see [28]) to have an analytical characterization in the sense that $x \in \partial_F f(x)$, if and only if, $\liminf\limits_{y \to x} \dfrac{f(y) - f(x) - \langle x^*, y - x\rangle}{\|y - x\|} \geq 0$. When $\partial_F f(x) \neq \emptyset$, one says that f is Fréchet subdifferentiable at the point x.

As usual, we will denote by ψ_C the *indicator function* of a closed set $C \subset X$, i.e., $\psi_C(y) = 0$ if $y \in C$ and $\psi_C(y) = +\infty$ otherwise. It is easily checked that $\partial_p \psi_C(x) = N^*_C(x)$ for any $x \in C$.

Like for the proximal normal cone in Hilbert space (see [16] and [10]) one can express, in our uniformly convex space X, the proximal normal functional cone to C in terms of the proximal subdifferential of d_C. We denote by $\mathbb{B}^*$ the closed unit ball of X^* and by $C(\rho)$ the ρ-enlargement of the set C, i.e., $C(\rho) := \{u \in X : d_C(u) \leq \rho\}$.

Proposition 1.2. *For any closed subset C of X and any $x \in C$,*

$$\partial_p d_C(x) = N^*_C(x) \cap \mathbb{B}^*.$$

Proof. The inclusion $x^* \in \partial_p d_C(x)$ means $(x^*, -1) \in N^*_{\operatorname{epi} d_C}(x, 0)$, or equivalently, for any $t > 0$ small enough,

$$\inf_{(y,\lambda)\in \operatorname{epi} d_C} \{\|x + tv - y\|^2 + (-t - \lambda)^2\} = t^2\|v\|^2 + t^2, \tag{8}$$

where $v = J^*(x^*)$. This entails that

$$\inf_{y \in C} \{\|x + tv - y\|^2\} = t^2\|v\|^2,$$

hence $v \in N_C(x)$. Further (8) ensures for all $y \in X$ that

$$\|x + tv - y\|^2 + 2td_C(y) + d^2_C(y) \geq t^2\|v\|^2$$

and since $-2tJ(v) = \nabla(-\|\cdot\|^2)(tv)$, for each $\epsilon > 0$ there exists some positive number $r < \varepsilon$ such that for all $y \in B[x, r]$

$$2t\langle -J(v), x - y\rangle \leq \epsilon\|x - y\| + 2td_C(y) + d^2_C(y)$$

and taking the inequality $d_C(y) \le \|x-y\|$ into account we see that

$$2t\langle -J(v), x-y\rangle \le (\varepsilon + 2t + \|x-y\|)\|x-y\| \le (2\varepsilon + 2t)\|x-y\|.$$

This easily yields $\|v\| = \|-J(v)\| \le 1$ and hence $x^* \in N_C^*(x) \cap \mathbb{B}^*$.

Take conversely $x^* \in N_C^*(x) \cap \mathbb{B}^*$. Put $v = J^*(x^*)$ and choose $t > 0$ small enough that $d_C^2(x+tv) = t^2\|v\|^2$. Then

$$\inf_{y\in X}\{\|x+tv-y\|^2 + (t+d_C(y))^2\} = \inf_{\rho\ge 0} h(\rho),$$

where $h(\rho) := \inf_{y\in C(\rho)}\{\|x+tv-y\|^2 + (t+\rho)^2\}$. Obviously we have the equality $h(\rho) = d_{C(\rho)}^2(x+tv) + (t+\rho)^2$. Consider two cases:

- If $d_C(x+tv) \le \rho$, then $h(\rho) = (t+\rho)^2 \ge t^2 + d_C^2(x+tv)$.
- If $d_C(x+tv) > \rho$, using Lemma 3.1 in [10] or Lemma 5.3 in Section 6, we have $d_{C(\rho)}(x+tv) = d_C(x+tv) - \rho$ and thus,

$$h(\rho) = d_C^2(x+tv) + t^2 + 2\rho[\rho + t - d_C(x+tv)] \ge d_C^2(x+tv) + t^2.$$

The last estimation comes from the fact that $\|v\| \le 1$.
Finally, making use of the above inequalities and of the equality $d_C^2(x+tv) = t^2\|v\|^2$ we obtain

$$\inf_{y\in X}\{\|x+tv-y\|^2 + (t+d_C(y))^2\} = t^2\|v\|^2 + t^2,$$

which entails that $(v,-1)$ is in $N_{\mathrm{epi}\, d_C}(x,0)$ or, in other words, that $x^* \in \partial_p d_C(x)$. $\square$

2. Prox-regular sets

The concept of prox-regularity was introduced for functions from $\mathbb{R}^n$ into $\mathbb{R} \cup \{+\infty\}$ by Poliquin and Rockafellar in [38], extending the class of primal lower nice (pln) functions previously considered by Poliquin in [37]. The class of pln functions considerably enlarged the scope of functions that possess, like convex functions, good properties as regards regularization, integrability or subdifferential determination, generalized second-order behavior etc, see [37, 31, 5, 34] and the references therein. The introduction of prox-regular functions in [38] has been motivated by the study of second-order properties of some non convex functions. A subset of $\mathbb{R}^n$ is defined to be prox-regular in [38] when its indicator function is prox-regular. The concept of prox-regularity of sets has then been studied and developed in Hilbert space in [39] by Poliquin, Rockafellar and Thibault who showed in particular its rich geometric implications.

The prox-regularity property for a closed set C is, like for functions, *local* and *directional*, concerning a point $\overline{x} \in C$ and a direction $\overline{p} \in N_C(\overline{x})$. If the property holds for all possible proximal normal vectors to C at $\overline{x}$, the set is said to be prox-regular at $\overline{x}$. Considering the prox-regularity at a point, Poliquin, Rockafellar and Thibault [39] showed it to be a localization of Federer's positive reach concept (see [26]) or proximal smoothness property of Clarke, Stern and Wolenski [16]. The localization of the mentioned behavior is made clear by the fact that the authors showed in [39] the equivalence of the prox-regularity of a set C, of the local single valuedness and continuity of the metric projection mapping P_C, and of the local C^1 regularity of the square distance function d_C^2 to C among other characterizations. Their approach allowed them to retreive also the important global level results of [16].

Before extending the above concepts to uniformly convex spaces, we must point out that, in addition to Federer's study of Steiner polynomial formula (see [26]) and Canino's work related to geodesics, the first strong applications of proximal smoothness of sets to Control theory has been provided by Clarke, Ledyaev, Stern and Wolenski [15]. For other recent applications to evolution problems with moving sets putting in light the amenability of prox-regular and proximally smooth sets, we refer to [24, 43].

Our extension of the definition of a prox-regular set to our setting will use the duality mapping as follows, in order to find striking characterizations of prox-regularity like in the Hilbert space setting.

Definition 2.1. A closed set $C \subset X$ is called prox-regular at $\overline{x} \in C$ for $\overline{p}^* \in N_C^*(\overline{x})$ if there exist $\varepsilon > 0$ and $r > 0$ such that for all $x \in C$ and for all $p^* \in N_C^*(x)$ with $\|x - \overline{x}\| < \varepsilon$ and $\|p^* - \overline{p}^*\| < \varepsilon$ the point x is a nearest point of $\{x' \in C : \|x' - \overline{x}\| < \varepsilon\}$ to $x + rJ^*(p^*)$. The set C is prox-regular at $\overline{x}$ if this property holds for all $\overline{p}^* \in N_C^*(\overline{x})$.

The following proposition shows that the prox-regularity concept for subsets of X in fact does not depend on any direction. The first part of its proof reproduces ideas of the proof of Proposition 1.2 in [39].

Proposition 2.2. *A closed set $C \subset X$ is prox-regular at $\overline{x}$, if and only if, it is prox-regular at $\overline{x}$ for $\overline{p}^* = 0$. If the closed set C is prox-regular at $\overline{x}$ for $\overline{p}^* = 0$ with ε and r, then for all $x \in C$ with $\|x - \overline{x}\| < \varepsilon$ and for all $p^* \in N_C^*(x)$ with $\|p^*\| \leq \varepsilon$*

$$0 \geq \langle J[J^*(p^*) - r^{-1}(x' - x)], x' - x\rangle, \quad \forall x' \in C \text{ with } \|x' - \overline{x}\| < \varepsilon. \tag{9}$$

Proof. Obviously, if C is prox-regular at $\overline{x}$ for all $\overline{p}^* \in N_C^*(\overline{x})$ then it is so for $\overline{p}^* = 0$. To establish the converse, let us assume that C is prox-regular at $\overline{x}$ for $\overline{p}^* = 0$ with $\varepsilon > 0$ and $r > 0$. Take any $\overline{p}^* \in N_C^*(\overline{x})$ with $\overline{p}^* \neq 0$, and set $\varepsilon' := \min\{\varepsilon/2, \|\overline{p}^*\|/2\}$. For $x \in C$ and $p^* \in N_C^*(x)$ with $\|x - \overline{x}\| < \varepsilon'$ and $\|p^* - \overline{p}^*\| < \varepsilon'$ we have that

$$\frac{\varepsilon}{2\|\overline{p}^*\|}\|p^*\| \leq \frac{\varepsilon}{2\|\overline{p}^*\|}[\|\overline{p}^* - p^*\| + \|\overline{p}^*\|] \leq \frac{\varepsilon}{2\|\overline{p}^*\|}\varepsilon' + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{2} < \varepsilon.$$

We may rewrite the latter as $\|\frac{\varepsilon p^*}{2\|\overline{p}^*\|} - 0\| < \varepsilon$. By prox-regularity of C at $\overline{x}$ for 0, we have that x is a nearest point of $\{x' \in C : \|x' - \overline{x}\| < \varepsilon\}$ to $x + rJ^*\left(\frac{\varepsilon p^*}{2\|\overline{p}^*\|}\right) = x + \frac{r\varepsilon}{2\|\overline{p}^*\|}J^*(p^*)$. This means that C is prox-regular at $\overline{x}$ for $\overline{p}^*$ with constants ε' and $r' = \frac{r\varepsilon}{2\|\overline{p}^*\|}$.

To prove the second claim, we suppose that C is prox-regular at $\overline{x}$ for $\overline{p}^* = 0$ with ε and r. Fix any $x \in C$ with $\|x - \overline{x}\| < \varepsilon$ and any $p^* \in N_C^*(x)$ with $0 < \|p^*\| < \varepsilon$. By definition, x is a nearest point of $\{x' \in C : \|x' - \overline{x}\| < \varepsilon\}$ to $x + rJ^*(p^*)$, that is,

$$\|x + rJ^*(p^*) - x\| \leq \|x + rJ^*(p^*) - x'\|, \quad \forall x' \in C \text{ with } \|x' - \overline{x}\| < \varepsilon.$$

Setting $p := J^*(p^*)$ we rewrite the latter as

$$r\|p\| \leq \|x - x' + rp\|, \quad \forall x' \in C \text{ with } \|x' - \overline{x}\| < \varepsilon. \tag{10}$$

Since $J(u)$ is the derivative of $\frac{1}{2}\|.\|^2$ at u, we have for all $t > 0$ and all x' that

$$\begin{aligned}\gamma &:= \langle J[J^*(p^*) - r^{-1}(x' - x)], x' - x\rangle = \langle J(p - r^{-1}(x' - x)), x' - x\rangle \\ &\leq [2t]^{-1}\{\|p - r^{-1}(x' - x) + t(x' - x)\|^2 - \|p - r^{-1}(x' - x)\|^2\}.\end{aligned}$$

In particular for $t = r^{-1}$

$$\begin{aligned}\gamma &\leq \frac{r}{2}\{\|p\|^2 - \|p - r^{-1}(x' - x)\|^2\} = \frac{r}{2}\{\|p\|^2 - r^{-2}\|x - x' + rp\|^2\} \\ &= \frac{1}{2r}\left\{(r\|p\|)^2 - \|x - x' + rp\|^2\right\}.\end{aligned}$$

Taking (10) into account we deduce that $\gamma \leq 0$ for all $x' \in C$ with $\|x' - \overline{x}\| < \varepsilon$, which is (9). The case $\|p^*\| = \varepsilon$ is obtained via a limit process and hence the proof is complete. $\square$

In what follows, saying that C is prox-regular at $\overline{x}$ with ε and r we will mean that the constants ε and r are taken from prox-regularity of C at $\overline{x}$ for $\overline{p}^* = 0$. It is clear that if the closed set C is prox-regular at $\overline{x}$ for $\overline{p}^* = 0$ with some positive constants ε and r then it is so for any constants $0 < \varepsilon' \leq \varepsilon$ and $0 < r' \leq r$.

The notion corresponding to the inequality (9) in the case of functions is introduced in the following Definition 2.3. Let us note that another definition is considered in [4], using the "proximal-type" estimation with the square of the norm as in Poliquin-Rockafellar [38] instead of (11). In the Hilbert setting, the definition given below is equivalent to that in [38] or [4] for the large subclass of (pln) functions. The concept of pln functions has been developed in Poliquin [37] in $\mathbb{R}^n$ and the case of Hilbert space has been studied in [31]. See also the references in [4], where it was shown that in uniformly convex Banach spaces endowed with a smooth norm with modulus of convexity of power type $q = 2$, the Moreau envelopes of prox-regular functions enjoy various remarkable properties.

We will see in the next section that the J-plr concept introduced in Definition 2.3 below for functions also yields, concerning their Moreau envelopes, various important properties which have their own interest. Recall that the context here is broader than in [4] since the power of the modulus of convexity is any $q \geq 2$. These properties applied to the indicator functions of sets will be among the keys of the development of our study.

Definition 2.3. A lsc function $f : X \to \mathbb{R} \cup \{+\infty\}$ is *J-primal lower regular* (*J-plr*) at $\overline{x} \in \operatorname{dom} f$ if there exist positive constants ε and r such that

$$f(y) \geq f(x) + \langle J[J^*(p^*) - t(y - x)], y - x\rangle \tag{11}$$

for all $x, y \in B(\overline{x}, \varepsilon)$, all $p^* \in \partial_p f(x)$, and all t such that $\|p^*\| \leq \varepsilon r t$.

It is easily seen that if f is J-plr at $\overline{x}$ with some positive constants ε and r then it is so for any constants $0 < \varepsilon' \leq \varepsilon$ and $0 < r' \leq r$. If the lsc function f is J-plr at $\overline{x} \in \operatorname{dom} f$ with positive constants ε and r, one can derive that

$$\langle J[J^*(p^*) - t(y - x)] - J[J^*(q^*) - t(x - y)], y - x\rangle \leq 0 \tag{12}$$

for all $x, y \in B(\overline{x}, \varepsilon)$, for all $p^* \in \partial_p f(x)$, $q^* \in \partial_p f(y)$, and all t such that $\max\{\|p^*\|, \|q^*\|\}$ $\leq \varepsilon r t$. This is the analog of the hypomonotonicity of certain truncations of $\partial_p f$ that characterizes pln functions in Hilbert spaces: see [37], [31], [5] and the references therein. The hypomonotonicity is no more appropriate in our setting and therefore we introduce the following closely related concept, that we call J-hypomonotonicity.

Definition 2.4. A set-valued mapping $T : X \rightrightarrows X^*$ is said to be J-hypomonotone of degree $t \geq 0$ if for any $(x_i, x_i^*) \in \operatorname{gph} T := \{(x, x^*) \in X \times X^* : x^* \in T(x)\}$, $i = 1, 2$, one has

$$\langle J[J^*(x_1^*) - t(x_2 - x_1)] - J[J^*(x_2^*) - t(x_1 - x_2)], x_2 - x_1\rangle \leq 0.$$

Throughout, we will denote by $\operatorname{Dom} T$ the domain of the set-valued mapping $T : X \rightrightarrows X^*$, i.e., $\operatorname{Dom} T := \{x \in X : T(x) \neq \emptyset\}$. We will also use the following concept of truncation of a set-valued mapping, as in [5].

Definition 2.5. Let $T : X \rightrightarrows X^*$ be a set-valued mapping and let $\varepsilon > 0$ and $t \geq 0$. Then its ε, t-truncation at a point $\overline{x} \in X$ is the set-valued mapping $T_{\overline{x},\varepsilon,t}$ defined by

$$\operatorname{gph} T_{\overline{x},\varepsilon,t} := \{(x, x^*) \in \operatorname{gph} T : \|x - \overline{x}\| < \varepsilon,\ \|x^*\| \leq t\}.$$

Without ambiguity, $T_{\overline{x},\varepsilon,t}$ will be denoted simply by T_t.

So, by (12) we see that if f is J-plr at $\overline{x}$ with ε and r, then $(\partial_p f)_{\overline{x},\varepsilon,\varepsilon rt}$ is J-hypomonotone of degree t for any $t \geq 0$. We are not far from sets since the next proposition shows that the prox-regularity of a set C entails the J-plr property of its indicator function ψ_C. The equivalence will be obtained later in Theorem 4.9, as well as the equivalence with the J-hypomonotonicity of a certain truncation of the normal cone N_C^*.

It will be convenient, for any $\sigma \geq 0$, to denote below by $N_C^{*\sigma}$ the set-valued mapping N_C^* truncated with $\sigma\mathbb{B}^*$, i.e.,

$$N_C^{*\sigma}(x) := N_C^*(x) \cap \sigma\mathbb{B}^* \quad \text{for all } x. \tag{13}$$

Proposition 2.6. *If the closed set $C \subset X$ is prox-regular at $\overline{x} \in C$ with ε and r, then the indicator function ψ_C of C is J-plr at $\overline{x}$, and hence* (12) *yields, for any $t \geq 0$, the J-hypomonotonicity of degree t on $B(\bar{x}, \varepsilon)$ of the set-valued mapping $N_C^{*\sigma}$, where $\sigma := \varepsilon rt$.*

Proof. As C is prox-regular at $\overline{x}$ for $\overline{p}^* = 0$ with $\varepsilon > 0$ and $r > 0$, from Proposition 2.2 we have

$$\psi_C(x') \geq \psi_C(x) + \langle J[J^*(p^*) - r^{-1}(x' - x)], x' - x\rangle,$$

whenever $x' \in B(\overline{x}, \varepsilon)$, $x \in C \cap B(\overline{x}, \varepsilon)$, and $\|p^*\| \leq \varepsilon$ with $p^* \in N_C^*(x)$. We have already seen in Section 2 that $p^* \in N_C^*(x)$ if and only if $p^* \in \partial_p \psi_C(x)$. If $p^* \in N_C^*(x)$ and $\|p^*\| \leq \varepsilon rt$, then $r^{-1}t^{-1}p^* \in N_C^*(x)$ with $\|r^{-1}t^{-1}p^*\| \leq \varepsilon$, hence

$$\psi_C(x') \geq \psi_C(x) + \langle J[J^*(r^{-1}t^{-1}p^*) - r^{-1}(x' - x)], x' - x\rangle,$$
$$\psi_C(x') \geq \psi_C(x) + \langle J[J^*(p^*) - t(x' - x)], x' - x\rangle,$$

which means that the function ψ_C is J-plr at $\overline{x}$. The proof is complete. □

We will need another result concerning J-hypomonotone set-valued mappings. It will be one of the key steps of our development of the proof of Theorem 3.5. First recall that a set-valued mapping $T : X \rightrightarrows X^*$ is bounded when its range $T(X) := \cup_{x\in X} T(x)$ is a bounded set in X^*.

Lemma 2.7. *Let $T : X \rightrightarrows X^*$ be a bounded set-valued mapping which is J-hypomonotone of degree $\overline{r}$. Then for any $r > 2\overline{r}$ we have that $(I + r^{-1}J^* \circ T)^{-1}$ is a single-valued mapping on its domain which is $\frac{1}{q}$-Hölder continuous on its intersection with any bounded subset.*

Proof. Let us denote by μ any upper bound of $\{\|z\| : z \in T(x), x \in \operatorname{Dom} T\}$. For any $r > 2\overline{r}$ and $\rho > 0$, take $x_i \in \operatorname{Dom}(I + r^{-1}J^* \circ T)^{-1}$ with $\|x_i\| \le \rho$, $i = 1, 2$. Choose any $y_i \in (I + r^{-1}J^* \circ T)^{-1}(x_i)$, i.e., $J[r(x_i - y_i)] \in T(y_i)$, $i = 1, 2$. Hence $\|x_i - y_i\| \le \mu/r$. By assumption, for any $t \ge \overline{r}$,

$$0 \ge \langle J\{J^*(J[r(x_1 - y_1)]) - t(y_2 - y_1)\} - J\{J^*(J[r(x_2 - y_2)]) - t(y_1 - y_2)\}, y_2 - y_1\rangle$$
$$0 \ge \langle J(rx_1 - ty_2 + (t - r)y_1) - J(rx_2 - ty_1 + (t - r)y_2), y_2 - y_1\rangle.$$

Now for any $\lambda \in]0, 1[$ such that $\frac{\lambda r}{2} > \overline{r}$, replacing t in the above inequality by rt_λ where $t_\lambda := \lambda/2$ we obtain

$$\begin{aligned} 0 \ge\; & \langle J(x_1 - t_\lambda y_2 - (1 - t_\lambda)y_1) - J(x_2 - t_\lambda y_1 - (1 - t_\lambda)y_2), x_1 - x_2 + (1 - 2t_\lambda)(y_2 - y_1)\rangle \\ & + \langle J(x_1 - t_\lambda y_2 - (1 - t_\lambda)y_1) - J(x_2 - t_\lambda y_1 - (1 - t_\lambda)y_2), x_2 - x_1\rangle := (\mathrm{I}) + (\mathrm{II}). \end{aligned}$$

Note also that

$$\begin{aligned} \|x_1 - t_\lambda y_2 - (1 - t_\lambda)y_1\| &\le (1 - t_\lambda)\|x_1 - y_1\| + t_\lambda\|x_1 - y_2\| \\ &\le (1 - t_\lambda)\|x_1 - y_1\| + t_\lambda(\|x_1 - x_2\| + \|x_2 - y_2\|) \\ &\le (1 - t_\lambda)\tfrac{\mu}{r} + t_\lambda(2\rho + \tfrac{\mu}{r}) \le \gamma, \end{aligned}$$

where $\gamma := \rho + \frac{\mu}{r}$, and similarly $\|x_2 - t_\lambda y_1 - (1 - t_\lambda)y_2\| \le \gamma$. A first estimation of (I) is obtained by using (1), that is,

$$(\mathrm{I}) \ge K_\gamma\|x_1 - x_2 + (1 - 2t_\lambda)(y_2 - y_1)\|^q.$$

To proceed further in the estimation, we need to consider two cases.

The *first case* is when $(1-2t_\lambda)\|y_1-y_2\| > \|x_1-x_2\|$. In that case, we need to estimate below $\|a-b\|^q$ when $\|a\| > \|b\|$. Since $\|a-b\| \ge \|a\|-\|b\| > 0$, we derive $\|a-b\|^q \ge [\|a\|-\|b\|]^q = \|a\|^q \left[1 - \frac{\|b\|}{\|a\|}\right]^q$. This leads us to consider the real-valued function $g(s) = [1 - s]^q + qs$ on the interval $s \in [0, 1[$. As the derivative $g'(s) = -q[1 - s]^{q-1} + q$ is non-negative on this interval, the function g is non-decreasing on $[0, 1[$ and then $g(s) \ge g(0) = 1$ for all $s \in [0, 1[$. Finally, $[1 - s]^q \ge 1 - qs$ for $s \in [0, 1[$. We conclude that

$$\|a - b\|^q \ge \|a\|^q \left[1 - \frac{\|b\|}{\|a\|}\right]^q \ge \|a\|^q \left[1 - q\frac{\|b\|}{\|a\|}\right] = \|a\|^q - q\|a\|^{q-1}\|b\|.$$

Using the latter we continue to estimate (I) by

$$\begin{aligned} (\mathrm{I}) &\ge K_\gamma[(1 - 2t_\lambda)^q\|y_1 - y_2\|^q - q(1 - 2t_\lambda)^{q-1}\|y_1 - y_2\|^{q-1}\|x_1 - x_2\|] \\ &\ge \gamma_1\|y_1 - y_2\|^q - \gamma_2\|x_1 - x_2\|, \end{aligned}$$

where γ_1, γ_2 are some nonnegative constants, depending on λ. On the other hand,

$$\begin{aligned} (\mathrm{II}) &\ge -\|J(x_1 - t_\lambda y_2 - (1 - t_\lambda)y_1) - J(x_2 - t_\lambda y_1 - (1 - t_\lambda)y_2)\|.\|x_2 - x_1\| \\ &\ge -K'_\gamma\|x_1 - x_2 + (1 - 2t_\lambda)(y_2 - y_1)\|^{s-1}.\|x_2 - x_1\| \\ &\ge -\gamma_3\|x_2 - x_1\|, \end{aligned}$$

where we used (2) for the second estimation, and γ_3 is some nonnegative constant depending on λ. Finally, $0 \geq \text{(I)} + \text{(II)} \geq \gamma_1 \|y_1 - y_2\|^q - (\gamma_2 + \gamma_3)\|x_1 - x_2\|$ and hence for some constant $\gamma' > 0$ that depends on λ,

$$\|y_1 - y_2\| \leq \gamma' \|x_1 - x_2\|^{1/q}.$$

The *second case* is when $(1 - 2t_\lambda)\|y_1 - y_2\| \leq \|x_1 - x_2\|$. Observing that $\|x_1 - x_2\| \leq (2\rho)^{1-\frac{1}{q}} \|x_1 - x_2\|^{\frac{1}{q}}$, we see that in both cases we have that $\|y_1 - y_2\| \leq \gamma'' \|x_1 - x_2\|^{1/q}$ for some constant $\gamma'' > 0$, so $\left(I + r^{-1} J^* \circ T\right)^{-1}$ is a single-valued mapping on its domain and it is $\frac{1}{q}$-Hölder continuous on the intersection of its domain with the set $\rho\mathbb{B}$. □

3. Local Moreau envelopes

Several properties of d_C^2 and P_C will be derived from corresponding ones (with their own interest) concerning the so-called local Moreau envelope of a function. Here we will give the definition and properties of local Moreau envelopes. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lsc function and $W \subset X$ be a nonempty closed subset where f is bounded from below and finite at some point. The *local Moreau envelope of index* $\lambda > 0$ of f (relative to W), is defined as

$$e_{\lambda,W} f(x) := \inf_{y \in W} \left\{ f(y) + \frac{1}{2\lambda} \|x - y\|^2 \right\}. \tag{14}$$

We fix W with the above property and we will write, when there is no risk of confusion, $e_\lambda f$ instead of $e_{\lambda,W} f$. Note that the infimum in (14) may be seen as taken over all X for the function $\tilde{f}$ given by $\tilde{f}(x) = f(x)$ if $x \in W$ and $\tilde{f}(x) = +\infty$ otherwise. It is easy to see that the functions $e_\lambda f$ are everywhere defined and Lipschitz on bounded subsets. As usual we will consider the set

$$P_\lambda f(x) := \{y \in W : e_\lambda f(x) = f(y) + \frac{1}{2\lambda}\|x - y\|^2\}.$$

Whenever there exists some $p_\lambda(x) \in P_\lambda f(x)$ one has by [19]

$$\partial_F e_\lambda f(x) \subset \{\lambda^{-1} J(x - p_\lambda(x))\} \cap \partial_F \tilde{f}(p_\lambda(x)), \tag{15}$$

hence $P_\lambda f(x)$ is either empty or a singleton thanks to the one-to-one property of the mapping J. We know by Theorem 11 of [7] that the infimum is attained whenever x is a point of Fréchet subdifferentiability of $e_\lambda f$. Denoting by G_λ the subset of X where $e_\lambda f$ is Fréchet subdifferentiable, we obtain for any $x \in G_\lambda$ that $P_\lambda f(x) = \{p_\lambda(x)\}$ and $\partial_F e_\lambda f(x) = \{\lambda^{-1} J(x - p_\lambda(x))\}$. Note that G_λ is dense in X according to the result in [35] and [40] concerning the density of subdifferentiability points.
When $P_\lambda f(x) = \{p_\lambda(x)\}$ is a singleton, we will write sometimes $P_\lambda f(x)$ in the place of $p_\lambda(x)$.

In the proof of the next lemma we follow an idea due to Borwein and Giles from [7]. The lemma will be used in the proof of Proposition 4.3.

Lemma 3.1. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lsc function bounded from below on W and with $W \cap \operatorname{dom} f \neq \emptyset$. Then the following assertions are equivalent:*

(a) $\partial_F e_\lambda f(x) \neq \emptyset$;

(b) $\quad$ *$e_\lambda f$ is Fréchet differentiable at x.*

Further, in the case of (a) or (b), $\nabla^F e_\lambda f(x) = \lambda^{-1} J(x - p_\lambda(x))$.

Proof. Obviously (b) implies (a). Now suppose that (a) holds. As we have seen above, $P_\lambda f(x)$ is single-valued with a unique element $p_\lambda(x)$ and $\partial_F e_\lambda f(x) = \{\lambda^{-1} J(x - p_\lambda(x))\}$. Then, for any $\varepsilon > 0$ there exists some $\delta > 0$ such that for any $t \in]0, \delta[$ and for any $y \in \mathbb{B}$

$$\langle \lambda^{-1} J(x - p_\lambda(x)), ty\rangle \le e_\lambda f(x + ty) - e_\lambda f(x) + \varepsilon t,$$

hence,

$$t^{-1}[e_\lambda f(x + ty) - e_\lambda f(x)] - \langle \lambda^{-1} J(x - p_\lambda(x)), y\rangle \ge -\varepsilon. \tag{16}$$

At the same time, taking δ smaller if necessary and using the definition of $e_\lambda f$ and the fact that the function $\frac{1}{2}\|\cdot\|^2$ is Fréchet differentiable with

$$J(x - p_\lambda(x)) = \nabla(\frac{1}{2}\|\cdot\|^2)(x - p_\lambda(x)),$$

we have for any $y \in \mathbb{B}$

$$\begin{aligned} e_\lambda f(x + ty) - e_\lambda f(x) \ &\le\ f(p_\lambda(x)) + \frac{1}{2\lambda}\|x + ty - p_\lambda(x)\|^2 - f(p_\lambda(x)) - \frac{1}{2\lambda}\|x - p_\lambda(x)\|^2 \\ &\le\ \langle \lambda^{-1} J(x - p_\lambda(x)), ty\rangle + \varepsilon t, \end{aligned}$$

i.e.,

$$t^{-1}[e_\lambda f(x + ty) - e_\lambda f(x)] - \langle \lambda^{-1} J(x - p_\lambda(x)), y\rangle \le \varepsilon. \tag{17}$$

Combining (16) and (17) we obtain the Fréchet differentiability of $e_\lambda f$ at x as well as the equality $\nabla^F e_\lambda f(x) = \lambda^{-1} J(x - p_\lambda(x))$. $\square$

The differentiability of the Moreau envelopes is also related to their regularity. This connection given by the equivalence between assertions (a) and (b) of the next lemma will be needed in Proposition 4.3. The lemma also establishes in view of Theorem 3.5 the differentiability of $e_\lambda f$ under the single valuedness and continuity of $P_\lambda f$. Before giving its statement, let us recall that a function f is Fréchet regular at x provided $\partial_F f(x) = \partial_C f(x)$, where ∂_C stands for the Clarke subdifferential.

Lemma 3.2. *Under the assumptions of Lemma 3.1, for any open subset U of X, the equivalences $(a) \Leftrightarrow (b)$ and $(c) \Leftrightarrow (d)$ hold for the following properties:*

(a) $\quad$ *$e_\lambda f$ is Fréchet regular on U;*

(b) $\quad$ *$e_\lambda f$ is Fréchet differentiable on U and its Fréchet derivative $\nabla^F e_\lambda f : U \to X^*$ is norm-to-weak* continuous;*

(c) $\quad$ *$e_\lambda f$ is continuously Fréchet differentiable on U (and hence (a) and (b) hold);*

(d) $\quad$ *$P_\lambda f$ is a single-valued norm-to-norm continuous mapping on U.*

In any one of these cases, $e_\lambda f$ is Fréchet differentiable on U with $\nabla^F e_\lambda f(x) = \lambda^{-1} J(x - P_\lambda f(x))$.

Proof. $(a) \Rightarrow (b)$: If $\partial_F e_\lambda f = \partial_C e_\lambda f$ on U, then we have that $\partial_F e_\lambda f(x) \neq \emptyset$ for any $x \in U$ and, hence, $e_\lambda f$ is Fréchet differentiable on U from Lemma 3.1. Moreover, $\partial_C e_\lambda f(x) =$

$\{\nabla^F e_\lambda f(x)\}$ for any $x \in U$, and by the norm-to-weak* upper semicontinuity of $\partial_C e_\lambda f$ we have that $\nabla^F e_\lambda f$ is norm-to-weak* continuous.

$(b) \Rightarrow (a)$: Conversely, if $e_\lambda f$ is Fréchet differentiable and norm-to-weak* continuous on U, we have that $\partial_F e_\lambda f(x) = \{\nabla^F e_\lambda f(x)\} = \partial_L e_\lambda f(x)$ for any $x \in U$, where for a locally Lipschitz continuous function $g : X \to \mathbb{R}$ the *limiting subdifferential* $\partial_L g$ is defined as the weak* sequential outer limit

$$w^* - \limsup_{y \to x} \partial_F g(y) := \{w^* - \lim x_n^* : x_n^* \in \partial_F g(x_n), x_n \to x\}. \tag{18}$$

By Mordukhovich and Shao [35], we know that $\partial_C g(x) = \overline{\mathrm{co}}^* \partial_L g(x)$, where $\overline{\mathrm{co}}^*$ denotes the weak* closed convex hull in X^*. Thus, we obtain that $\partial_F e_\lambda f(x) = \partial_C e_\lambda f(x)$ for $x \in U$, which is the assertion (a).
$(c) \Rightarrow (d)$: The continuous differentiability of $e_\lambda f$ on U implies via Lemma 3.1 the single valuedness and norm-to-norm continuity of $P_\lambda f$, i.e., the implication holds.
$(d) \Rightarrow (c)$: Assume now that $P_\lambda f$ is a single-valued norm-to-norm continuous mapping on U. This continuity property along with (15) and (18) gives $\partial_C e_\lambda f(x) = \{\lambda^{-1} J(x - P_\lambda f(x))\}$ which entails that $e_\lambda f$ is Gâteaux differentiable on U with $\nabla^G e_\lambda f(x) = \lambda^{-1} J(x - P_\lambda f(x))$. The norm-to-norm continuity of $P_\lambda f$ once again yields the existence of $\nabla^F e_\lambda f$ as well as its norm-to-norm continuity on U. The proof of the lemma is then complete. $\square$

In the remainder of this section, we fix a point $\overline{x} \in \mathrm{dom}\, f$ and $\rho > 0$ such that f is *bounded from below* over $B[\overline{x}, 4\rho]$ and hence we also fix $W = B[\overline{x}, 4\rho]$. Note that according to the lower semicontinuity of f one always has some $\rho > 0$ with the desired property. So it is natural to write $e_{\lambda,\rho,\overline{x}} f(x)$ in place of $e_{\lambda,W} f(x)$ and when $\overline{x}$ and ρ with the above mentioned properties are fixed, it will be convenient to keep as above for index only λ since this will not cause any confusion.

By the useful localization lemma (see [44], Lemma 4.2), there exists some $\lambda_0 > 0$ such that for all $\lambda \in]0, \lambda_0]$

$$P_\lambda f(x) \subset B(\overline{x}, 3\rho) \quad \text{for all } x \in U := B(\overline{x}, \rho). \tag{19}$$

So, for any $x \in U \cap G_\lambda$ the unique element $p_\lambda(x)$ of $P_\lambda f(x)$ belongs to $B(\overline{x}, 3\rho)$ and then by (15)

$$\nabla^F e_\lambda f(x) = \lambda^{-1} J(x - p_\lambda(x)) \in \partial_F f(p_\lambda(x)) \quad \forall x \in U \cap G_\lambda \tag{20}$$

and moreover

$$\|x - p_\lambda(x)\| \le \|x - \overline{x}\| + \|\overline{x} - p_\lambda(x)\| < \rho + 3\rho = 4\rho. \tag{21}$$

In fact, we can make (20) more precise by proving in the following lemma that the stronger inclusion $\nabla^F e_\lambda f(x) \in \partial_p f(p_\lambda(x))$ holds for $x \in U \cap G_\lambda$.

Lemma 3.3. *For any $\lambda \in]0, \lambda_0]$, $x \in U \cap \mathrm{Dom}\, P_\lambda$, and $p_\lambda(x) \in P_\lambda f(x)$, we have that $\lambda^{-1} J(x - p_\lambda(x)) \in \partial_p f(p_\lambda(x))$. In other words, for any $x \in U$ and any $\lambda \in]0, \lambda_0]$,*

$$P_\lambda f(x) \subset (I + \lambda J^* \circ \partial_p f)^{-1}(x).$$

Proof. Fix any $0 < \lambda \leq \lambda_0$, $x \in U \cap \mathrm{Dom}\, P_\lambda$, and $p_\lambda(x) \in P_\lambda f(x)$. Then,

$$f(p_\lambda(x)) + (2\lambda)^{-1}\|x - p_\lambda(x)\|^2 \leq f(y) + (2\lambda)^{-1}\|x - y\|^2, \quad \forall y \in W,$$

that is,

$$(2\lambda)^{-1}\|x - p_\lambda(x)\|^2 - (2\lambda)^{-1}\|x - y\|^2 \leq f(y) - f(p_\lambda(x)), \quad \forall y \in W.$$

Since $p_\lambda(x) \in B[\overline{x}, 3\rho]$ the last inequality holds true in particular for all $y \in B[p_\lambda(x), \rho]$. Let us set $p := \lambda^{-1}(x - p_\lambda(x))$. From the last inequality we have

$$2^{-1}\lambda\|p\|^2 - (2\lambda)^{-1}\|x - y\|^2 \leq f(y) - f(p_\lambda(x)), \quad \forall y \in B[p_\lambda(x), \rho],$$

which entails

$$\frac{\lambda^2}{2}\|p\|^2 - \frac{1}{2}\|\lambda p + p_\lambda(x) - y\|^2 \leq \lambda[f(y) - f(p_\lambda(x))], \quad \forall y \in B[p_\lambda(x), \rho],$$

or,

$$\lambda^2\|p\|^2 - \|\lambda p + p_\lambda(x) - y\|^2 \leq 2\lambda[\beta - f(p_\lambda(x))],$$
$$\forall (y, \beta) \in \mathrm{epi}\, f \text{ with } y \in B[p_\lambda(x), \rho].$$

Adding λ^2 to both sides yields

$$\lambda^2\|p\|^2 - \|\lambda p + p_\lambda(x) - y\|^2 + \lambda^2 \leq 2\lambda[\beta - f(p_\lambda(x))] + \lambda^2,$$
$$\forall (y, \beta) \in \mathrm{epi}\, f \text{ with } y \in B[p_\lambda(x), \rho],$$

and using the inequality $2\lambda[\beta - f(p_\lambda(x))] + \lambda^2 \leq [\beta - f(p_\lambda(x)) + \lambda]^2$ we obtain that

$$\lambda^2\|p\|^2 + \lambda^2 \leq \|\lambda p + p_\lambda(x) - y\|^2 + [\beta - f(p_\lambda(x)) + \lambda]^2,$$
$$\forall (y, \beta) \in \mathrm{epi}\, f \text{ with } y \in B[p_\lambda(x), \rho].$$

So, we obtain that for all $(y, \beta) \in \mathrm{epi}\, f$ with $y \in B[p_\lambda(x), \rho]$

$$|\!|\!|\lambda(p, -1)|\!|\!| \leq |\!|\!|(p_\lambda(x), f(p_\lambda(x))) + \lambda(p, -1) - (y, \beta)|\!|\!|.$$

By (7) this inequality entails that $(p, -1) \in N_{\mathrm{epi}\, f}(p_\lambda(x), f(p_\lambda(x)))$, which gives (see Section 2) that $J(p) \in \partial_p f(p_\lambda(x))$. This means that

$$\lambda^{-1}J(x - p_\lambda(x)) \in \partial_p f(p_\lambda(x)),$$

which entails the inclusion of the lemma. □

Remark 3.4. (a) In the case when the lsc function f is the indicator function ψ_C of a non-empty closed set C, the above conclusions hold for $W = X$ and any $\lambda > 0$ or for $\rho = +\infty$, any $\overline{x} \in C$, and any $\lambda > 0$. Further with $\rho = +\infty$ one has $e_\lambda f(x) = \frac{1}{2\lambda} d_C^2(x)$ and $P_\lambda f(x) = P_C(x)$ for all $x \in X$.
(b) Still with $f = \psi_C$, for any $\overline{x} \in C$, any $\rho \in]0, +\infty[$, and any $\lambda > 0$ one has $e_\lambda f(x) = \frac{1}{2\lambda} d_{C\cap W}^2(x)$ and $P_\lambda f(x) = P_{C\cap W}(x)$ for all $x \in X$.

But for any $x \in B[\overline{x}, 2\rho]$ and any $y \in C \setminus W$,

$$\|x - y\| \geq \|y - \overline{x}\| - \|x - \overline{x}\| > 4\rho - 2\rho = 2\rho \geq \|x - \overline{x}\| \geq d_{C\cap W}(x).$$

Hence, for $x \in B[\overline{x}, 2\rho]$, $d_{C\cap W}(x) = d_C(x)$ and $P_{C\cap W}(x) = P_C(x)$. Therefore, $e_\lambda f(x) = \frac{1}{2\lambda} d_C^2(x)$ and $P_\lambda f(x) = P_C(x)$ for any $x \in U := B(\overline{x}, \rho)$.

Recall that a function g is of class $C^{1,\alpha}$ on an open set $U \subset X$ when it is differentiable on U and the derivative ∇g is locally α-Hölder continuous on U.

Theorem 3.5. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lsc function which is J-plr at $\overline{x} \in \operatorname{dom} f$ with positive real numbers ε and r, such that $\varepsilon < 1 < \frac{1}{r}$. Let $\rho \in]0, \frac{r\varepsilon}{16}]$ be fixed in such a way that f is bounded from below on $B[\overline{x}, 4\rho]$. Then there exists $\lambda_0 > 0$ such that for any $\lambda \in]0, \lambda_0]$ the map $x \mapsto P_\lambda f(x)$ is single-valued on $U := B(\overline{x}, \rho)$ with, for some constant $\gamma \geq 0$,*

$$\|p_\lambda(x) - p_\lambda(x')\| \leq \gamma \|x - x'\|^{\frac{1}{q}}, \quad \forall x, x' \in U, \tag{22}$$

where p_λ is given by $P_\lambda f(y) = \{p_\lambda(y)\}$ for any $y \in U$. Moreover, for each $\lambda \in]0, \lambda_0]$ the function $e_\lambda f$ is of class $C^{1,\alpha}$ on U with $\alpha := q^{-1}(s-1)$ and $\nabla e_\lambda f(x) = \lambda^{-1} J(x - p_\lambda(x))$ for all $x \in U$.

Proof. Let λ_0 be given by the analysis of (19) above for ρ fixed as in the statement of the theorem. We will work with arbitrary fixed $\lambda \in]0, \lambda_0]$. Put $c := r\varepsilon$ and $T_{ct} := (\partial_p f)_{\overline{x},\varepsilon,ct}$ for any $t \geq 0$. The proof is divided in three steps.

Step 1. Let us prove that $P_\lambda f$ is $\frac{1}{q}$-Hölder continuous on $U \cap \operatorname{Dom} P_\lambda f$.
We have from Definition 2.4 and from (12) that the set-valued mapping T_{ct} is J-hypomonotone of degree t for any $t \geq 0$. Hence, for $t_\lambda := c/(4\lambda)$, the set-valued mapping T_{t_λ} is J-hypomonotone of degree $\overline{r} := 1/(4\lambda)$. As $\lambda^{-1} > 2\overline{r}$, Lemma 2.7 entails that $(I + \lambda J^* \circ T_{t_\lambda})^{-1}$ is a single-valued mapping on its domain and this mapping is $\frac{1}{q}$-Hölder continuous on the intersection of its domain with any bounded subset of X. From Lemma 3.3, we have that $P_\lambda f(x) \subset (I + \lambda \partial_p f)^{-1}(x)$ for any $x \in U$. We claim that we even have

$$P_\lambda f(x) \subset (I + \lambda J^* \circ T_{t_\lambda})^{-1}(x) \quad \text{for any } x \in U. \tag{23}$$

Indeed, fixing any $x \in U \cap \operatorname{Dom} P_\lambda f$ and $p_\lambda(x) \in P_\lambda f(x)$, we know that $p_\lambda(x) \in B[\overline{x}; 3\rho]$. So $\|p_\lambda(x) - \overline{x}\| < \varepsilon$, and

$$\begin{aligned}&\|\lambda^{-1} J(x - p_\lambda(x))\| \\ &\leq \lambda^{-1}\|x - p_\lambda(x)\| \leq \lambda^{-1}(\|x - \overline{x}\| + \|\overline{x} - p_\lambda(x)\|) \leq 4\rho\lambda^{-1} \leq (c\lambda^{-1})/4 = t_\lambda.\end{aligned}$$

Hence, as also $\lambda^{-1} J(x - p_\lambda(x)) \in \partial_p f(p_\lambda(x))$, we have that $\lambda^{-1} J(x - p_\lambda(x)) \in T_{t_\lambda}(p_\lambda(x))$, which proves the claim. Therefore we obtain that $P_\lambda f$ is a single-valued $\frac{1}{q}$-Hölder continuous mapping on $U \cap \operatorname{Dom} P_\lambda f$, that is, there exists some constant $\gamma \geq 0$ such that for all $x, x' \in U \cap \operatorname{Dom} P_\lambda f$

$$\|p_\lambda(x) - p_\lambda(x')\| \leq \gamma \|x - x'\|^{\frac{1}{q}}. \tag{24}$$

Step 2. Let us prove that $U \subset \operatorname{Dom} P_\lambda f$.
The proof now is similar to that of [3] or [4]. Take any $x \in U$ and fix some integer $k \geq 1$ with $B(x, 1/k) \subset U$. According to the density of the Fréchet subdifferentiability points of $e_\lambda f$, for any integer $n \geq k$ there exists some point $x_n \in G_\lambda \cap B(x, 1/n)$. By (24), for any integers $n, m \geq k$,

$$\|p_\lambda(x_n) - p_\lambda(x_m)\| \leq \gamma \|x_n - x_m\|^{\frac{1}{q}}.$$

Hence, $(p_\lambda(x_n))_n$ is a Cauchy sequence. Denote by z_λ its limit. By the definition of $p_\lambda(x_n)$ we have

$$f(p_\lambda(x_n)) + \frac{1}{2\lambda}\|x_n - p_\lambda(x_n)\|^2 \leq f(y) + \frac{1}{2\lambda}\|x_n - y\|^2, \quad \forall y \in W.$$

Since f is lsc, the latter implies

$$f(z_\lambda) + \frac{1}{2\lambda}\|x - z_\lambda\|^2 \le f(y) + \frac{1}{2\lambda}\|x - y\|^2, \quad \forall y \in W.$$

This means that $z_\lambda \in P_\lambda f(x)$, which yields $U \cap \operatorname{Dom} P_\lambda f = U$. Hence (24) holds for all $x, x' \in U$ and, through step 1, $P_\lambda f$ is a single-valued $\frac{1}{q}$-Hölder continuous mapping on U. Then by Lemma 3.2, the envelope $e_\lambda f$ is continuously Fréchet differentiable on U with $\nabla^F e_\lambda f(x) = \lambda^{-1} J(x' - p_\lambda(x))$ for any $x \in U$.

Step 3. We will prove that $e_\lambda f$ is of class $C^{1,\alpha}$ on U with $\alpha = q^{-1}(s-1)$. Let us take any $x, x' \in U$ and, for $r := 4\rho$, use (2) to estimate

$$\begin{aligned} & \|\nabla^F e_\lambda f(x) - \nabla^F e_\lambda f(x')\| = \lambda^{-1}\|J(x - p_\lambda(x)) - J(x' - p_\lambda(x'))\| \\ \le\ & \lambda^{-1} K'_r \|x - p_\lambda(x) - x' + p_\lambda(x')\|^{s-1} \le \lambda^{-1} K'_r [\|x - x'\| + \|p_\lambda(x) - p_\lambda(x')\|]^{s-1} \\ \le\ & \lambda^{-1} K'_r [\|x - x'\| + \gamma\|x - x'\|^{\frac{1}{q}}]^{s-1} \le \lambda^{-1} K'_r (1+\gamma)^{s-1} \|x - x'\|^{\frac{s-1}{q}}, \end{aligned}$$

where the third inequality is due to (22) and the last one to the fact that $\|x - x'\| < 1$. The proof of the theorem is then complete. □

We now state in the next proposition the relation obtained between the proximal mappings and some truncations of the subdifferential of a J-plr function.

Proposition 3.6. *Under the assumptions of Theorem 3.5, one has for all $\lambda \in]0, \lambda_0]$ and $x \in U$*

$$P_\lambda f(x) = (I + \lambda J^* \circ T_{t_\lambda})^{-1}(x),$$

where $t_\lambda := \varepsilon r/(4\lambda)$ and $T_{t_\lambda} := (\partial_p f)_{\overline{x}, \varepsilon, t_\lambda}$.

Proof. The inclusion of the first member in the second one is (23) of Theorem 3.5, and the reverse one follows from the inclusion $U \subset \operatorname{Dom} P_\lambda f$ established in step 2 of the same proof since $(I + \lambda J^* \circ T_{t_\lambda})^{-1}$ is at most single-valued as we saw in step 1 above. □

In the next corollary and throughout the paper (like for $P_\lambda f(x)$), when $P_C(x) = \{p(x)\}$ is a singleton, we will make no distinction between $P_C(x)$ and $p(x)$.

Corollary 3.7. *Let $C \subset X$ be a non-empty closed set such that its indicator function ψ_C is J-plr at $\overline{x} \in C$ with positive real numbers ε and r satisfying $\varepsilon < 1 < \frac{1}{r}$. Then for $\rho = \frac{\varepsilon r}{16}$ the mapping $x \mapsto P_C(x)$ is single-valued on $U = B(\overline{x}, \rho)$ and for some constant $\gamma \ge 0$*

$$\|P_C(x) - P_C(x')\| \le \gamma \|x - x'\|^{\frac{1}{q}}, \quad \forall x, x' \in U. \tag{25}$$

Further, the function d_C^2 is of class $C^{1,\alpha}$ on U with $\alpha = q^{-1}(s-1)$ and $\nabla d_C^2(x) = 2J(x - P_C(x))$ for all $x \in U$.

Corollary 3.8. *Under the assumptions of Corollary 3.7, we have that*

$$P_C(x) = (I + J^* \circ N_C^{*\sigma})^{-1}(x) \quad \forall x \in U,$$

where $\sigma := \varepsilon r/4$ and the set-valued mapping $N_C^{*\sigma}$ is defined by (13).

Proof. Taking $\rho = \frac{\varepsilon r}{16}$ as in the statement of Corollary 3.7, since $r < 1$ it is easily checked for $T := \partial_p \psi_C$ that for all $x \in U = B(\overline{x}, \rho)$

$$(I + J^* \circ T_{\overline{x},\varepsilon,\frac{\varepsilon r}{4}})^{-1}(x) = (I + J^* \circ N_C^{*\varepsilon r/4})^{-1}(x).$$

So it suffices to apply Proposition 3.6 with $\lambda = 1$ to $f = \psi_C$, keeping in mind Remark 3.4(b). □

4. Characterizations of prox-regular sets

In this section we will give different characterizations of prox-regularity of a set.

Lemma 4.1. *Let $C \subset X$ be a closed subset. Then the single valuedness and norm-to-weak continuity of the projection mapping P_C over an open set U imply its norm-to-norm continuity on U.*

Proof. Let $u_n \underset{n\to\infty}{\xrightarrow{\|\cdot\|}} u$ and $P_C(u_n) \underset{n\to\infty}{\xrightarrow{w}} P_C(u)$. From the Lipschitz continuity of the distance function, $\|u_n - P_C(u_n)\| = d_C(u_n) \underset{n\to\infty}{\longrightarrow} d_C(u) = \|u - P_C(u)\|$. By the Kadec property of the norm, $P_C(u_n) \underset{n\to\infty}{\xrightarrow{\|\cdot\|}} P_C(u)$. □

The following proposition establishes that the continuity of the metric projection mapping to a closed set C is equivalent to the continuous differentiability of the distance function d_C, as shown in the Hilbert setting in [39], where it is proved that those properties characterize the prox-regularity of a set in a Hilbert space. Its proof follows directly from Lemma 3.2 with $f = \psi_C$.

Proposition 4.2. *Let $C \subset X$ be a closed set and $U \subset X$ be an open set. Then the following are equivalent:*

(a) *P_C is single-valued and norm-to-norm continuous on U;*
(b) *d_C^2 is of class C^1 on U.*

In fact these properties are equivalent to the only Fréchet subdifferentiability of the distance function as we can see from the following proposition.

Proposition 4.3. *For any closed set $C \subset X$ and any open set U of X the following are equivalent:*

(a) *d_C is continuously differentiable on $U \setminus C$;*
(b) *$\partial_F d_C(x)$ is non-empty for all $x \in U$;*
(c) *$\partial_F d_C^2(x)$ is non-empty at all points x in U;*
(d) *d_C is Fréchet differentiable on $U \setminus C$;*
(e) *d_C is Fréchet regular on $U \setminus C$;*
(f) *d_C is Gâteaux differentiable on $U \setminus C$ with $\|\nabla^G d_C(x)\| = 1$ for all $x \in U \setminus C$.*

Proof. (a) ⇒ (b) is obvious since one always has $0 \in \partial_F d_C(u)$ for any $u \in C$.
(b) ⇒ (c) follows from the fact that for any $x^* \in \partial_F d_C(x)$, one has $2d_C(x)x^* \in \partial_F d_C^2(x)$ according to Lemma 3.9 in [39].

$(c) \Rightarrow (d)$: By Lemma 3.1 and Remark 3.4(a) we have that d_C^2 is Fréchet differentiable on U, hence so is d_C on $U \setminus C$.
$(d) \Rightarrow (a)$: It is clear that (d) entails (b) and hence (c). From Lemma 3.1 and Remark 3.4(a) we get the Fréchet differentiability of d_C^2 on U. The latter implies that P_C is single-valued on U and that

$$\nabla^F d_C^2(x) = 2J(x - P_C(x)) \quad \text{for any } x \in U. \tag{26}$$

So, it remains to prove the norm-to-norm continuity of P_C over U and we will obtain that of $\nabla^F d_C^2$. Take any $x_0 \in U$, and $U \ni x_n \xrightarrow[n\to\infty]{\|\cdot\|} x_0$. Then we also have that

$$\|x_n - P_C(x_n)\| = d_C(x_n) \xrightarrow[n\to\infty]{} d_C(x_0) = \|x_0 - P_C(x_0)\|,$$

which entails that the sequence $(P_C(x_n))_n$ is bounded. Taking if necessary a subsequence, we may suppose that $P_C(x_n) \xrightarrow[n\to\infty]{w} z$ for some $z \in X$. As $\|x_0 - P_C(x_n)\| \xrightarrow[n\to\infty]{} \|x_0 - P_C(x_0)\|$, having in mind the Kadec property of the norm, it suffices to prove that

$$\|x_0 - z\| = \|x_0 - P_C(x_0)\| \tag{27}$$

to get that $P_C(x_n) \xrightarrow[n\to\infty]{\|\cdot\|} z$, and hence that $z \in C$. Then, using (27) and the fact that $P_C(x_0)$ is single-valued, we will have that $P_C(x_0) = \{z\}$, so P_C is norm-to-norm continuous at x_0. To this end, following an idea of Borwein and Giles from [7], let us set $t_n^2 := \|x_0 - P_C(x_n)\|^2 - d_C^2(x_0)$. If $t_n = 0$ then $P_C(x_n) = P_C(x_0)$ due to the single-valuedness of P_C, and there would be nothing to prove if this equality holds for infinitely many n, so we may suppose that $t_n > 0$ for any integer n. Fix any $\varepsilon > 0$. From the Fréchet differentiability of d_C^2 at x_0 it follows that

$$\begin{aligned}
\langle \nabla^F d_C^2(x_0), P_C(x_n) - x_0\rangle &\le \frac{d_C^2(x_0 + t_n(P_C(x_n) - x_0)) - d_C^2(x_0)}{t_n} + \frac{\varepsilon}{4} \\
&\le \frac{\|x_0 + t_n(P_C(x_n) - x_0) - P_C(x_n)\|^2 - d_C^2(x_0)}{t_n} + \frac{\varepsilon}{4} \\
&\le \frac{(1-t_n)^2\|x_0 - P_C(x_n)\|^2 - d_C^2(x_0)}{t_n} + \frac{\varepsilon}{4}
\end{aligned}$$

and hence for n large enough

$$\begin{aligned}
\langle \nabla^F d_C^2(x_0), P_C(x_n) - x_0\rangle &\le \frac{\|x_0 - P_C(x_n)\|^2 - d_C^2(x_0)}{t_n} - 2\|x_0 - P_C(x_n)\|^2 + \frac{\varepsilon}{2} \\
&= t_n - 2\|x_0 - P_C(x_n)\|^2 + \frac{\varepsilon}{2} \\
&\le -2d_C^2(x_0) + \varepsilon.
\end{aligned}$$

Passing to the limit, we obtain $\langle \nabla^F d_C^2(x_0), x_0 - z\rangle \ge 2d_C^2(x_0)$, or,

$$\langle 2J(x_0 - P_C(x_0)), x_0 - z\rangle \ge 2\|x_0 - P_C(x_0)\|^2,$$

which entails that $\|x_0 - z\| \geq \|x_0 - P_C(x_0)\|$. Since $\|x_n - P_C(x_n)\| \leq \|x_n - P_C(x_0)\|$, one has that $\liminf_{n\to\infty} \|x_n - P_C(x_n)\| \leq \lim_{n\to\infty} \|x_n - P_C(x_0)\|$, and hence, by the weak lower semicontinuity of the norm, one gets $\|x_0 - z\| \leq \|x_0 - P_C(x_0)\|$. Finally, $\|x_0 - z\| = \|x_0 - P_C(x_0)\|$, that is, (27), and hence the implication $(d) \Rightarrow (a)$ holds.
The implications $(e) \Rightarrow (d)$ and $(a) \Rightarrow (e)$ follow from Lemma 3.2 and the equivalence $(d) \Leftrightarrow (f)$ is a direct consequence of Theorem 2.4 of Fitzpatrick [27]. The proof of the proposition is then complete. □

In addition to Proposition 4.3 we state the following theorem providing a weak derivability condition on d_C under which P_C is continuous, supposing it is nonempty-valued. It is a direct consequence of Theorem 2.4 in Fitzpatrick [27] as observed in Corollary 2 of Borwein, Fitzpatrick and Giles [6]. Note that in those works, the framework is beyond the uniform convexity. Further, Fitzpatrick's important condition of Fréchet differentiability concerns general functions and not merely the distance functions. The result of the theorem will be used in the proof of Theorem 4.9.

Recall that, for $v \in X$, a function $f : X \to \mathbb{R}$ has a Gâteaux directional derivative at a point x in the full direction v provided that the limit $\lim_{t\to 0} t^{-1}[f(x + tv) - f(x)]$ exists and is finite.

Theorem 4.4. *Let $C \subset X$ be a closed set and $x \in X \backslash C$ be such that $P_C(x) \neq \emptyset$. If d_C has a Gâteaux directional derivative at x in the full direction $x - p(x)$ for some $p(x) \in P_C(x)$, then d_C is Fréchet differentiable at x.*

Another interest of this result will appear in the proof of Theorem 5.10.

We now proceed to establish two lemmas. The first one is a key result proved in [39, Lemma 3.3] in the Hilbert context. The proof is valid in our setting, and we sketch the main parts below.

Lemma 4.5. *Let C be a closed subset of X. Assume that d_C is Fréchet differentiable on a neighbourhood of a point $\overline{u} \notin C$. Then there exists $\delta > 0$ such that whenever $u \in B(\overline{u}, \delta)$ and $P_C(u) = x$, there exists some $t > 0$ such that the point $u_t := u + t(u - x)$ likewise has $P_C(u_t) = x$.*

Proof. By Propositions 4.3 and 4.2, there exists $\varepsilon > 0$ such that P_C is single-valued and norm-to-norm continuous on $B(\overline{u}, 2\varepsilon)$, with d_C continuously Fréchet differentiable on this ball as well. For each $u \in B(\overline{u}, \varepsilon)$ and each $t > 0$ put $u_t := u + t(u - P_C(u))$. Following the proof of Lemma 3.3 in [39], we find out some positive numbers $\delta < \varepsilon$ and $s < 1$ such that for all $u \in B(\overline{u}, \delta)$ one has $d_C(u) \geq \delta$, $sd_C(u) < \delta$ and $d_C(u_s) > d_C(u)$. Fix now $u \in B(\overline{u}, \delta)$ and consider the closed set $D := \{w \in X : d_C(w) \geq d_C(u_s)\}$. As $u \notin D$, according to Lau's theorem (see [30]) there is a sequence $D \not\ni y_n \xrightarrow[n\to\infty]{\|\cdot\|} u$ with $P_D(y_n) \neq \emptyset$. Choosing $w_n \in P_D(y_n)$ we have $d_C(w_n) = d_C(u_s)$ (because w_n is a boundary point of D). For all n large enough, $w_n \in B(\overline{u}, 2\delta)$ since

$$\|y_n - w_n\| = d_D(y_n) \leq \|y_n - u_s\| \underset{n\to\infty}{\longrightarrow} \|u - u_s\| = s\|u - P_C(u)\| = sd_C(u) < \delta. \quad (28)$$

Consequently d_C is Fréchet differentiable at w_n and by (26) we have

$$\nabla^F d_C(w_n) = J(w_n - P_C(w_n))/d_C(w_n) \quad \text{and} \quad \|\nabla^F d_C(w_n)\| = 1.$$

Therefore, the half-space $E := \{v \in X : \langle -\nabla^F d_C(w_n), v\rangle \leq 0\}$ gives the Clarke tangent cone to D at w_n (see [14]) and hence its negative polar cone $-[0,\infty[\,\nabla^F d_C(w_n)$ is the Clarke normal cone to D at w_n. The nonzero functional $J(y_n - w_n)$ being a proximal normal functional to D at w_n, it belongs to the Clarke normal cone to D at w_n. Consequently, there exists some $\lambda_n > 0$ such that $J(y_n - w_n) = -\lambda_n \nabla^F d_C(w_n)$ which entails

$$y_n - w_n = -\lambda_n (w_n - P_C(w_n))/d_C(w_n) \quad \text{and} \quad \lambda_n = \|y_n - w_n\|.$$

For n large enough, we have by (28) that $\lambda_n < \delta$ and hence

$$\lambda_n < \delta \leq d_C(u) < d_C(u_s) = d_C(w_n).$$

It follows that for $\alpha_n := \lambda_n/d_C(w_n)$ we have $\alpha_n \in]0,1[$ and $y_n = (1-\alpha_n)w_n + \alpha_n P_C(w_n)$. Hence, $P_C(y_n) = P_C(w_n)$ and

$$\lambda_n = \|y_n - w_n\| = d_C(w_n) - d_C(y_n) = d_C(u_s) - d_C(y_n).$$

Putting $t_n := \dfrac{\alpha_n}{1-\alpha_n} = \dfrac{d_C(u_s) - d_C(y_n)}{d_C(y_n)}$, we obtain $w_n = y_n + t_n(y_n - P_C(y_n))$. As $(t_n)_n$ converges to $t := (d_C(u_s) - d_C(u))/d_C(u) > 0$, we have $w_n \xrightarrow[n\to\infty]{\|\cdot\|} u_t$ and $u_t \in B(\overline{u}, 2\delta)$ by (28) and by the inclusion $u \in B(\overline{u}, \delta)$. So by continuity of P_C over $B(\overline{u}, 2\delta)$ we get $P_C(w_n) \xrightarrow[n\to\infty]{\|\cdot\|} P_C(u_t)$. But we also have $P_C(w_n) = P_C(y_n) \xrightarrow[n\to\infty]{\|\cdot\|} P_C(u)$. Finally, for this number t we have $P_C(u_t) = P_C(u)$ and hence the proof is complete. □

The next lemma follows from the previous one.

Lemma 4.6. *Let $C \subset X$ be a closed subset of the space X and $\overline{x} \in C$. If the mapping P_C is single-valued and norm-to-norm continuous in a neighbourhood U of $\overline{x}$, then there exists some $\varepsilon > 0$ such that for all $x \in C \cap B(\overline{x}, \varepsilon)$ and all $p \in N_C(x)$ with $p \neq 0$ the equality $P_C(x + \varepsilon \frac{p}{\|p\|}) = x$ holds.*

Proof. Let P_C be single-valued and norm-to-norm continuous in $B(\overline{x}, \delta)$. Take $\varepsilon < \delta/2$ and consider any non-zero $p \in N_C(x)$ with $\|x - \overline{x}\| < \varepsilon$. By definition of the proximal normal cone and by Lemma 1.1, there exists $\lambda > 0$ such that $P_C(x + \lambda p) = x$. Set $\lambda_s := \sup\{\lambda \leq \varepsilon : P_C(x + \lambda \frac{p}{\|p\|}) = x\}$. By the continuity of P_C on $B(\overline{x}, \delta)$ we have that $P_C(x + \lambda_s \frac{p}{\|p\|}) = x$. Suppose that $\lambda_s < \varepsilon$. As $x + \lambda_s \frac{p}{\|p\|}$ belongs to the open set $B(\overline{x}, \delta)$ where d_C is Fréchet differentiable according to Proposition 4.2, by Lemma 4.5 there exists $\eta > 0$ with $\lambda_s + \eta \leq \varepsilon$ such that $P_C(x + (\lambda_s + \eta)\frac{p}{\|p\|}) = x$. This gives a contradiction with the definition of λ_s. Hence $\lambda_s = \varepsilon$. □

Lemma 4.6 allows us to establish the following proposition which prepares the theorem on characterizations of prox-regularity.

Proposition 4.7. *Let $C \subset X$ be a closed subset of the space X. The following assertions are equivalent:*

(*a*) *C is prox-regular at $\overline{x} \in C$;*

(b) *there exists* $\varepsilon>0$ *such that the condition* $\left.\begin{array}{l} x=P_C(u), x\neq u \\ 0<\|u-\overline{x}\|<\varepsilon \end{array}\right\}$ *implies that* $x=P_C(u')$ *for* $u':=x+\varepsilon\frac{u-x}{\|u-x\|}$*;*

(c) *there exists* $\varepsilon>0$ *such that* $p\in N_C(x)$ *with* $x\in B(\overline{x},\varepsilon)$ *and* $p\neq 0$ *imply that*

$$P_C(x+\varepsilon\frac{p}{\|p\|})=x.$$

Proof. $(a)\Rightarrow(b)$: If C is prox-regular at $\overline{x}$, then P_C is single valued and Hölder continuous in a neighbourhood U of $\overline{x}$ according to Corollary 3.7. From the continuity of P_C, for the positive number ε of Lemma 4.6, there exists $\varepsilon'\in]0,\varepsilon[$ such that $\left.\begin{array}{l} x=P_C(u), x\neq u \\ 0<\|u-\overline{x}\|<\varepsilon' \end{array}\right\}$ implies that $\|x-\overline{x}\|=\|P_C(u)-P_C(\overline{x})\|<\varepsilon$. Lemmas 4.6 and 1.1 ensure for $p=u-x$ that $P_C(x+\varepsilon'\frac{u-x}{\|u-x\|})=x$.

$(b)\Rightarrow(c)$: Let us suppose that (b) holds with some $\varepsilon>0$. If $\|x-\overline{x}\|<\varepsilon/2$ and $p\in N_C(x)$ with $p\neq 0$, then by definition of $N_C(x)$ and by Lemma 1.1 there exists some $\eta\in]0,\varepsilon/2[$ such that $x=P_C(u)$, where $u:=x+\eta\frac{p}{\|p\|}$. We have

$$\|u-\overline{x}\|\leq\|u-x\|+\|x-\overline{x}\|<\varepsilon/2+\varepsilon/2,$$

and from (b), $P_C(x+\varepsilon\frac{u-x}{\|u-x\|})=x$. Hence, one obtains (c) with $\varepsilon/2$.

$(c)\Rightarrow(a)$: We suppose that (c) holds with some $\varepsilon>0$. Let $0\neq p^*\in N_C^*(x)$ with $\|x-\overline{x}\|<\varepsilon$ and $\|p^*\|<\varepsilon$. There exists $u\notin C$ such that $\langle\frac{p^*}{\|p^*\|},u-x\rangle=\|u-x\|=d_C(u)$. Thus, $u-x\in N_C(x)$ and $\frac{p^*}{\|p^*\|}=J(\frac{u-x}{\|u-x\|})$. We have by (c) that $P_C(x+\varepsilon\frac{u-x}{\|u-x\|})=x$, so $P_C(x+\frac{\varepsilon}{\|p^*\|}J^*(p^*))=x$. Now, for all $s\leq 1$ (since $1\leq\frac{\varepsilon}{\|p^*\|}$), we have that $P_C(x+sJ^*(p^*))=x$. By Definition 2.1 and by Proposition 2.2, the set C is prox-regular at $\overline{x}$. □

Now, following [38] and [39] we give a subdifferential characterization of the prox-regularity of a set in terms of the truncated cone (see (13)) of proximal normal functionals.

Proposition 4.8. *A set* $C\subset X$ *is prox-regular at* $\overline{x}\in C$*, if and only if, for some* $\varepsilon,\rho>0$*, the set-valued mapping* $N_C^{*\varepsilon}:X\rightrightarrows X^*$ *that assigns to each* $x\in X$ *the truncated cone of proximal normal functionals* $N_C^{*\varepsilon}(x)$ *is* J*-hypomonotone of degree* ρ *on* $B(\overline{x},\varepsilon)$*.*

Proof. By Proposition 2.6, if C is prox-regular at $\overline{x}$ then, for some $\varepsilon>0$ and $\rho>0$, the truncated normal functional cone mapping $N_C^{*\varepsilon}$ is J-hypomonotone of degree ρ on $B(\overline{x},\varepsilon)$. Conversely, suppose that $N_C^{*\varepsilon}$ is J-hypomonotone of degree ρ. Then the argument of Theorem 3.5 or Corollary 3.7 works as well (since it only makes use of the J-hypomonotonicity of the truncation of $\partial_p f$), to get that P_C is single-valued and continuous on a neighbourhood of $\overline{x}$. It just remains to invoke Lemma 4.6 and Proposition 4.7 to conclude. □

Now we can state the theorem giving several characterizations of the prox-regularity of a set. Recall first that the (lower) Dini subdifferential of a locally Lipschitz continuous function $f:X\to\mathbb{R}$ at a point x is defined by

$$\partial^- f(x):=\{x^*\in X^*:\langle x^*,h\rangle\leq\liminf_{t\downarrow 0}t^{-1}[f(x+th)-f(x)],\ \forall h\in X\}.$$

Theorem 4.9. *Let $C \subset X$ be a closed set. The following are equivalent:*

(a) *C is prox-regular at $\overline{x}$;*

(b) *P_C is single-valued and norm-to-norm $\frac{1}{q}$-Hölder continuous on some neighbourhood U of the point $\overline{x}$;*

(c) *P_C is single-valued and norm-to-weak continuous on some neighbourhood U of $\overline{x}$;*

(d) *there exists $\varepsilon > 0$ such that $p \in N_C(x)$ with $x \in B(\overline{x}, \varepsilon)$ and $p \neq 0$ implies that $P_C(x + \varepsilon \frac{p}{\|p\|}) = x$;*

(e) *there exists $\varepsilon>0$ such that the condition $\left.\begin{array}{l} x=P_C(u), x \neq u \\ 0 < \|u-\overline{x}\| < \varepsilon \end{array}\right\}$ implies that $x=P_C(u')$ for $u'=x+\varepsilon\frac{u-x}{\|u-x\|}$;*

(f) *d_C^2 is of class $C^{1,\alpha}$ on some neighbourhood U of $\overline{x}$ with $\alpha = q^{-1}(s-1)$;*

(g) *d_C is Fréchet differentiable on $U \setminus C$ for some neighbourhood U of $\overline{x}$;*

(h) *for some neighbourhood U of $\overline{x}$, the function d_C is Gâteaux differentiable on $U \setminus C$ with $\|\nabla^G d_C(x)\| = 1$ for all $x \in U \setminus C$;*

(i) *d_C is Fréchet subdifferentiable on U for some neighbourhood U of $\overline{x}$;*

(j) *d_C is Fréchet regular on $U \setminus C$ for some neighbourhood U of $\overline{x}$;*

(k) *P_C is nonempty-valued on U and d_C is Dini subdifferentiable on U for some neighbourhood U of $\overline{x}$;*

(l) *the indicator function ψ_C is J-plr at $\overline{x}$;*

(m) *there exist $\varepsilon, \rho > 0$ such that the truncated normal functional cone mapping $N_C^{*\varepsilon}$ is J-hypomonotone of degree ρ on $B(\overline{x}, \varepsilon)$.*

If C is weakly closed, one has one more equivalent condition

(n) *P_C is single-valued on some neighbourhood U of $\overline{x}$.*

Proof. First we will establish all the equivalences without specifying the Hölder character of the continuity in (b) and (f). The proof follows the scheme:

$$\begin{array}{ccccccccccccccccccc}
(m) & \Leftrightarrow & (a) & \Rightarrow & (l) & \Rightarrow & (f) & \Leftrightarrow & (j) & \Leftrightarrow & (h) & \Leftrightarrow & (i) & \Leftrightarrow & (g) & \Leftrightarrow & (k) \\
 & & & & & & \Updownarrow & & & & & & & & & & \\
 & & & & (c) & \Leftrightarrow & (b) & \Rightarrow & (d) & \Leftrightarrow & (e) & \Leftrightarrow & (a). & & & &
\end{array}$$

$(m) \Leftrightarrow (a)$ is Proposition 4.8.
$(a) \Rightarrow (l)$ is established in Proposition 2.6.
$(l) \Rightarrow (f)$ follows from Corollary 3.7.
$(f) \Leftrightarrow (j) \Leftrightarrow (h) \Leftrightarrow (i) \Leftrightarrow (g)$ is Proposition 4.3.
$(g) \Rightarrow (k)$: Assume that (g) holds. This obviously ensures the Dini subdifferentiability of d_C on $U \setminus C$ and since one always has $0 \in \partial^- d_C(x)$ for all $\in C$, we obtain that d_C is Dini subdifferentiable on U. The nonvacuity of P_C on U follows from the above implication (g) $\Rightarrow (f)$ and from Proposition 4.2.
$(k) \Rightarrow (g)$: For any $x \in U \setminus C$, there exists some x^* in the Dini subdifferential $\partial^- d_C(x)$ of d_C at x. Choose $p(x) \in P_C(x)$. By the definition of Dini subdifferential, for any $\varepsilon > 0$, there is some $\delta > 0$ such that, for any $t \in]0, \delta[$, one has

$$\langle x^*, p(x) - x \rangle \leq t^{-1}[d_C(x + t(p(x) - x)) - d_C(x)] + \varepsilon \tag{29}$$

and hence

$$\begin{aligned}\langle x^*, x - p(x)\rangle &\geq t^{-1}[d_C(x) - d_C(x + t(p(x) - x))] - \varepsilon \\ &\geq t^{-1}[\|x - p(x)\| - \|x + t(p(x) - x) - p(x)\|] - \varepsilon.\end{aligned}$$

Then, for any $\varepsilon > 0$, using the equality $\nabla(\|\cdot\|^2)(x - p(x)) = 2J(x - p(x))$ and taking some $t > 0$ small enough, we obtain

$$\begin{aligned}\langle x^*, x - p(x)\rangle &\geq \langle \frac{J(x - p(x))}{\|x - p(x)\|}, x - p(x)\rangle - 2\varepsilon \\ &= \|x - p(x)\| - 2\varepsilon,\end{aligned}$$

the last inequality being due the fact that $\langle J(y), y\rangle = \|y\|^2$. Therefore,

$$\begin{aligned}&\|x - p(x)\| \leq \langle x^*, x - p(x)\rangle \leq \liminf_{t\downarrow 0} t^{-1}[d_C(x + t(x - p(x)) - d_C(x)] \\ \leq\ &\limsup_{t\downarrow 0} t^{-1}[d_C(x + t(x - p(x)) - d_C(x)] \leq \|x - p(x)\|\end{aligned}$$

(the second inequality coming from $x^* \in \partial^- d_C(x)$), so

$$\lim_{t\downarrow 0} t^{-1}[d_C(x + t(x - p(x)) - d_C(x)] = \|x - p(x)\|.$$

Further observe (as in the proof of Corollary 2 in [6]) that for each $t \in [-1, 0[$ one has $p(x) \in P_C(x + t(x - p(x))$ and hence

$$t^{-1}[d_C(x + t(x - p(x))) - d_C(x)] = \|x - p(x)\|.$$

So $\lim_{t\to 0} t^{-1}[d_C(x + t(x - p(x)) - d_C(x)] = \|x - p(x)\|$, that is, d_C has a Gâteaux directional derivative in the full direction $x - p(x)$. Consequently, (g) follows from Theorem 4.4.
$(f) \Leftrightarrow (b)$ is Proposition 4.2.
$(b) \Leftrightarrow (c)$ is Lemma 4.1.
$(b) \Rightarrow (d)$ is Lemma 4.6.
$(d) \Leftrightarrow (e) \Leftrightarrow (a)$ is Proposition 4.7.

Under the additional assumption, to see that we have $(n) \Leftrightarrow (c)$ we need to prove the implication $(n) \Rightarrow (c)$.

Let us take any $u \in U$ and $u_n \xrightarrow[n\to\infty]{\|\cdot\|} u$. By weak compactness, taking if necessary a subsequence, we may suppose that $P_C(u_n) \xrightarrow[n\to\infty]{w} v$. From the weak lower semicontinuity of the norm, we have $\|v - u\| \leq \liminf_{n\to\infty} \|u_n - P_C(u_n)\|$ and hence $\|v - u\| \leq d_C(u)$. Therefore, as C is weakly closed, $v \in C$ and $P_C(u) = v$. Thus, $P_C(u_n) \xrightarrow[n\to\infty]{w} P_C(u)$, which gives the norm-to-weak continuity of P_C on U.

To conclude, we apply Corollary 3.7 to obtain $(a) \Leftrightarrow (b) \Leftrightarrow (f)$ but now with the Hölder character of the continuity that holds on (possibly smaller) neighbourhood of $\overline{x}$. The proof is then complete. $\square$

5. Characterizations of uniformly prox-regular sets

In this final section we proceed to the study of the global setting of positively reached or proximally smooth set C, corresponding (see [16]) to the continuous Fréchet differentiability of the distance function d_C over all an open tube of uniform thickness around the set C. We still have most of the characterizations for these sets given in [26] for finite dimensional space and in [16] and [39] in Hilbert space. We also add the characterizations (c), (e), and (g).

Definition 5.1. Following our definition of prox-regular set (see Definition 2.1) and modifying slightly Definition 2.4 of [39], we will say that the closed set C is *uniformly r-prox-regular* if whenever $x \in C$ and $p^* \in N_C^*(x)$ with $\|p^*\| < 1$, then x is the unique nearest point of C to $x + rJ^*(p^*)$.

For the subset C of X, let us first recall the definitions of
the r-enlargement of C

$$C(r) := \{x \in X : d_C(x) \leq r\},$$

the open r-tube around C

$$U_C(r) := \{x \in X : 0 < d_C(x) < r\},$$

and let us define the set of r-distance points to C

$$D_C(r) := \{x \in X : d_C(x) = r\}.$$

Theorem 5.2. *Let $C \subset X$ be a closed set and $r > 0$. The following are equivalent:*

(a) *C is uniformly r-prox-regular;*
(b) *d_C is continuously differentiable on $U_C(r) \setminus C$;*
(c) *d_C is Fréchet regular on $U_C(r) \setminus C$;*
(d) *d_C is Fréchet differentiable on $U_C(r) \setminus C$;*
(e) *d_C is Gâteaux differentiable on $U_C(r) \setminus C$ with $\|\nabla^G d_C(x)\| = 1$ for all $x \in U_C(r) \setminus C$;*
(f) *$\partial_F d_C$ is nonempty-valued at all points in $U_C(r)$;*
(g) *P_C and $\partial^- d_C$ are nonempty-valued at all points in $U_C(r)$;*
(h) *d_C^2 is C^1 on $U_C(r)$ with locally Hölder continuous derivative mapping;*
(i) *P_C is single-valued and locally Hölder continuous on $U_C(r)$;*
(j) *P_C is single-valued and norm-to-weak continuous on $U_C(r)$;*
(k) *For any non-zero $p \in N_C(x)$ with $x \in C$ one has $x \in P_C(x + r\frac{p}{\|p\|})$;*
(l) *If $u \in U_C(r)$ and $x = P_C(u)$, then $x \in P_C(u')$ for $u' = x + r\frac{u-x}{\|u-x\|}$.*

If C is weakly closed, one has the additional equivalent condition

(m) *P_C is single-valued on $U_C(r)$.*

Proof. We will follow the scheme:

$$\begin{array}{ccccccccccccc}
(a) & \Leftrightarrow & (k) & \Leftrightarrow & (l) & \Rightarrow & (i) & \Rightarrow & (h) & & & & \\
 & & & & \Uparrow & & & & \Downarrow & & & & \\
 & & (j) & \Leftrightarrow & (b) & \Leftrightarrow & (c) & \Leftrightarrow & (d) & \Leftrightarrow & (e) & \Leftrightarrow & (f) \\
 & & & & & & & & \Updownarrow & & & & \\
 & & & & & & & & (g) & & & & .
\end{array}$$

$(k) \Leftrightarrow (l)$ is obvious and here below each one will be considered.
$(k) \Rightarrow (a)$ follows from Lemma 1.1.
$(a) \Rightarrow (k)$: Fix any nonzero $p \in N_C(x)$ and take $p_n := p/(\|p\| + \frac{1}{n})$. For any $x' \in C$, (a) entails $\|x' - (x + rp_n)\| \geq r\|p_n\|$ which yields after taking the limit $\|x' - (x + r\frac{p}{\|p\|})\| \geq r$. The latter means that $x \in P_C(x + r\frac{p}{\|p\|})$, which is (k).
$(b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e) \Leftrightarrow (f)$ follows from Proposition 4.3.
$(d) \Leftrightarrow (g)$ results from Theorem 4.4 like in the proof of the similar equivalence in Theorem 4.9.
$(b) \Leftrightarrow (j)$ is a consequence of Proposition 4.2 and Lemma 4.1.
$(b) \Rightarrow (l)$: Let $u \in U_C(r)$ and $x = P_C(u)$. Since (b) holds, according to Lemma 4.5 there exists some $t_0 > 0$ such that $P_C(u_t) = x$ for all $u_t := u + t(u - x)/\|u - x\|$ with $0 < t < t_0$. As in [39] one may consider the number λ_0 given by the supremum over all $t \in [0, r - d_C(u)]$ such that $x \in P_C(u_t)$. Using the equivalence (note that $x \in C$ and $\|x - u_t\| = d_C(u) + t$)

$$x \in P_C(u_t) \Leftrightarrow \forall x' \in C,\ \|x' - u_t\| \geq d_C(u) + t,$$

it is easily seen that the supremum λ_0 is attained. We now claim that $\lambda_0 = r - d_C(u)$. Assume the contrary, i.e., $\lambda_0 < r - d_C(u)$. Then one would have on the one hand $x \in U_C(r)$ and on the other hand $x = P_C(u_{\lambda_0})$ because of the assumptions (b) and Proposition 4.2. Applying Lemma 4.5 again one would obtain a contradiction with the supremum property of λ_0. So the equality $\lambda_0 = r - d_C(u)$ holds. As u_t can be written in the form $u_t = x + (d_C(u) + t)(u - x)/\|u - x\|$, taking $t = \lambda_0$ gives (l).
$(l) \Rightarrow (i)$: We will proceed in five steps.

Step 1. For any $x^* \in N_C^{*r}(x) = N_C^*(x) \cap r\mathbb{B}^*$ with $x \in C$ and any $\alpha \in]0, 1]$, from $(k) \Leftrightarrow (l)$ one has $P_C(x + r\alpha\frac{J^*x^*}{\|x^*\|}) \ni x$. This means that, for any $x' \in C$,

$$\|x + r\alpha\frac{J^*x^*}{\|x^*\|} - x\| \leq \|x + r\alpha\frac{J^*x^*}{\|x^*\|} - x'\|.$$

Besides, as $J = \nabla(\frac{1}{2}\|\cdot\|^2)$, one also has

$$\frac{1}{2}\|x + r\alpha\frac{J^*x^*}{\|x^*\|} - x'\|^2 + \langle J(x - x' + r\alpha\frac{J^*x^*}{\|x^*\|}), x' - x\rangle \leq \frac{1}{2}\|r\alpha\frac{J^*x^*}{\|x^*\|}\|^2.$$

So $\langle J(J^*(x^*) - \frac{\|x^*\|}{r\alpha}(x' - x)), x' - x\rangle \leq 0$. It is possible to take any α in $]0, \frac{\|x^*\|}{r}]$. Hence, whenever $x_i \in C, x_i^* \in N_C^{*r}(x_i)$, $i = 1, 2$, and $t \geq 1$, one has

$$\begin{aligned} &\langle J(J^*(x_1^*) - t(x_2 - x_1)), x_2 - x_1\rangle \leq 0 \\ \text{and}\quad &\langle J(J^*(x_2^*) - t(x_1 - x_2)), x_1 - x_2\rangle \leq 0. \end{aligned}$$

By adding, one obtains

$$\langle J(J^*(x_1^*) - t(x_2 - x_1)) - J(J^*(x_2^*) - t(x_1 - x_2)), x_1 - x_2\rangle \leq 0$$

which is the J-hypomonotonicity of N_C^{*r} of degree t for any $t \geq 1$.

Step 2. For any $\alpha \in]0, 1/2[$, we have by step 1 and by Lemma 2.7 that the set-valued mapping $(I + J^* \circ N_C^{*\alpha r})^{-1}$ is $\frac{1}{q}$-Hölder continuous on the intersection of its domain with any bounded subset. We also have, for any $r' > 0$,

$$P_C(x) \subset (I + J^* \circ N_C^{*r'})^{-1}(x) \quad \text{for any } x \in U_C(r'). \tag{30}$$

Indeed, for any $x \in U_C(r')$, the inclusion $y \in P_C(x)$ entails that $J(x-y) \in N_C^*(y)$, and $\|y - x\| < r'$ so $J(x-y) \in N_C^{*r'}(y)$. So we have that for any $\alpha \in]0, 1/2[$, P_C is $\frac{1}{q}$-Hölder continuous on the intersection of any bounded set with $\operatorname{Dom} P_C \cap U_C(\alpha r)$. Then by the arguments of step 2 in the proof of Theorem 3.5, P_C is also nonempty, single-valued on $U_C(\alpha r)$. As α can be made as close as one wants to $\frac{1}{2}$, P_C is nonempty, single-valued, locally $\frac{1}{q}$-Hölder continuous on $U_C(r/2)$.

Step 3. Step 3 corresponds to the two following lemmas. The first one completes the result of Lemma 3.1 of Bounkhel-Thibault [10]. As usual the line segment between two points $u, v \in X$ will be denoted by $[u, v]$, that is, $[u, v] := \{tu + (1-t)v \,:\, t \in [0, 1]\}$.

Lemma 5.3. *Let C be a nonempty closed subset of a normed vector space $(Y, \|\cdot\|)$. Let $\rho > 0$ and $u \notin C(\rho)$. Then the following hold:*

(a) $d_C(u) = \rho + d_{C(\rho)}(u) = \rho + d_{D_C(\rho)}(u)$;
(b) *If $u_0 \in P_C(u)$ and $y_0 \in [u_0, u] \cap D_C(\rho)$, then $y_0 \in P_{C(\rho)}(u)$;*
(c) *If $y \in P_{C(\rho)}(u)$ and $z \in P_C(y)$, then $z \in P_C(u)$. Further, if $P_{C(\rho)}(u) = \{y\}$ and $z \in P_C(y)$, then $y \in [z, u]$ and $P_C(u) = \{z\}$.*

Proof. (a) For all $y \in C(\rho)$

$$d_C(u) \le d_C(y) + \|u - y\| \le \rho + \|u - y\|$$

hence

$$d_C(u) \le \rho + d_{C(\rho)}(u) \le \rho + d_{D_C(\rho)}(u). \tag{31}$$

Fix any $\varepsilon > 0$ and choose $u_\varepsilon \in C$ with $\|u - u_\varepsilon\| \le d_C(u) + \varepsilon$. Since $d_C(u_\varepsilon) = 0$ and $d_C(u) > \rho$ we may choose $y_\varepsilon \in [u_\varepsilon, u] \cap D_C(\rho)$ and hence

$$d_C(u) + \varepsilon \ge \|u - u_\varepsilon\| = \|u - y_\varepsilon\| + \|y_\varepsilon - u_\varepsilon\| \ge d_{D_C(\rho)}(u) + d_C(y_\varepsilon) = d_{D_C(\rho)}(u) + \rho.$$

Combining this with (31) we obtain

$$d_C(u) = \rho + d_{C(\rho)}(u) = \rho + d_{D_C(\rho)}(u). \tag{32}$$

(b) Assume now that $u_0 \in P_C(u)$ and $y_0 \in [u_0, u] \cap D_C(\rho)$. Then $u_0 \in P_C(y_0)$ and by (32)

$$\|u - y_0\| + \rho = \|u - y_0\| + d_C(y_0) = \|u - y_0\| + \|y_0 - u_0\| = \|u - u_0\| = \rho + d_{C(\rho)}(u)$$

and hence $\|u - y_0\| = d_{C(\rho)}(u)$, i.e., $y_0 \in P_{C(\rho)}(u)$.
(c) Assume that $y \in P_{C(\rho)}(u)$ and $z \in P_C(y)$. Then

$$\|u - z\| \le \|u - y\| + \|y - z\| = d_{C(\rho)}(u) + d_C(y) \le d_{C(\rho)}(u) + \rho = d_C(u)$$

(the last equality being due to (32)) and hence $z \in P_C(u)$.

Assume now that $P_{C(\rho)}(u) = \{y\}$. Taking $y' \in [z, u] \cap D_C(\rho) \neq \emptyset$, we have by (b) that $y' \in P_{C(\rho)}(u)$ and hence $y = y' \in [z, u]$. If there exists $z' \neq z$ with $z' \in P_C(u)$, then one sees that $z' \notin u + [0, +\infty[(z - u)$ and by (b) for $y'' \in [z', u] \cap D_C(\rho) \neq \emptyset$ (hence $y'' \neq u$) one would have $y'' \in P_{C(\rho)}(u)$ and hence $y'' = y \in [z, u]$ which would contradict $z' \notin u + [0, +\infty[(z - u)$. This completes the proof of the lemma. $\square$

Lemma 5.4. *If C satisfies the assertion (l) of Theorem 5.2 with parameter r and P_C is nonempty, single-valued on $U_C(\alpha r)$ for some $\alpha \in]0,1]$, then for any $\alpha' \in]0,\alpha[$, the set $C(\alpha' r) := \{x \in X : d_C(x) \le \alpha' r\}$ satisfies (l) with parameter $r(1-\alpha')$.*

Proof. Take $u \in U_{C(\alpha' r)}(r(1-\alpha'))$ and put $r' := d_C(u)$. Note that $0 < d_{C(\alpha' r)}(u) < r(1-\alpha')$ and hence by (a) of Lemma 5.3 one has $\alpha' r < d_C(u) < r$, which implies in particular $u \in U_C(r)$. Suppose that $P_{C(\alpha' r)}(u) = \{y\}$. We have to prove that $y \in P_{C(\alpha' r)}(y + r(1-\alpha')\frac{u-y}{\|u-y\|})$. Observing that $y \in U_C(\alpha r)$, we may put $z := P_C(y)$ according to the assumption on P_C over $U_C(\alpha r)$. By (c) of Lemma 5.3, we have $z \in P_C(u)$. Since $d_C(z) = 0$ and $d_C(u) > \alpha' r$, we may take $y_1 \in [z,u] \cap D_C(\alpha' r) \neq \emptyset$. The assertion (b) of Lemma 5.3 says that $y_1 \in P_C(\alpha' r)(u)$ and hence $y_1 = y$. Now, since $y_1 \in U_C(r)$ and $z = P_C(y_1)$ we have by (l) that $P_C(u') \ni z$ for

$$u' := z + r\frac{y_1 - z}{\|y_1 - z\|} = z + r\frac{u-z}{\|u-z\|}.$$

This entails by (b) of Lemma 5.3 again that $P_{C(\alpha' r)}(u') \ni y$ since (recall that $y = y_1$)

$$y \in [z,u] \cap D_C(\alpha' r) \subset [z,u'] \cap D_C(\alpha' r).$$

Further, since $\|u' - y\| = \|u' - z\| - \|y - z\| = r - \alpha' r$ (the second equality being due to the definition of u' and to the inclusion $y \in D_C(\alpha' r)$) we see that

$$u' = y + (r - \alpha' r)\frac{u-y}{\|u-y\|}.$$

So, we obtain that $C(\alpha' r)$ satisfies (l) with parameter $r(1-\alpha')$, and the proof of the lemma is complete. □

Step 4. For $\alpha \in]0,1]$, let us consider the property

$$\mathcal{P}(\alpha) \begin{cases} C \text{ satisfies } (l) \text{ with parameter } r \text{ and} \\ P_C \text{ is single-valued, locally Hölder continuous on } U_C(\alpha r). \end{cases}$$

We claim that $\mathcal{P}(\alpha) \Rightarrow \mathcal{P}(\frac{\alpha+1}{2})$.
Suppose that $\mathcal{P}(\alpha)$ holds. From step 3 and steps 1-2, we have that for any $\alpha' \in]0,\alpha[$,

$$P_{C(\alpha' r)} \text{ is single-valued, locally Hölder continuous on } U_{C(\alpha' r)}(\frac{r(1-\alpha')}{2}). \tag{33}$$

Take any $u \in U_C(\alpha' r + \frac{r(1-\alpha')}{2})$ such that $r' := d_C(u) > \alpha' r$. By (a) of Lemma 5.3 we have $d_{C(\alpha' r)}(u) = r' - \alpha' r$, and so

$$u \in U_{C(\alpha' r)}(\frac{r(1-\alpha')}{2}) \tag{34}$$

since $r' - \alpha' r < \frac{r(1-\alpha')}{2}$ because of the inclusion $u \in U_C(\alpha' r + \frac{r(1-\alpha')}{2})$. We may then put $y := P_{C(\alpha' r)}(u)$ according to (33) and put $z := P_C(y)$ according to the second assumption in $\mathcal{P}(\alpha)$. Then by (c) of Lemma 5.3 we have $z = P_C(u)$ and so $P_C(u) = z = P_C \circ P_{C(\alpha' r)}(u)$. So for any $\alpha' \in]0, \alpha[$, (33), (34), and $\mathcal{P}(\alpha)$ ensure that P_C is single-valued and locally Hölder continuous on $U_C(\alpha' r + \frac{r(1-\alpha')}{2}) \setminus C(\alpha' r)$. By assumption it is also locally Hölder continuous on $U_C(\alpha r)$. So P_C is single valued, locally Hölder continuous on $U_C(\alpha' r + \frac{r(1-\alpha')}{2})$ for any $\alpha' \in]0, \alpha[$ and hence also on $U_C(\alpha r + \frac{r(1-\alpha)}{2}) = U_C(\frac{\alpha+1}{2} r)$. This establishes the claim and finishes the proof of step 4.

Step 5. Define (α_n) by $\alpha_0 = 1/2$, $\alpha_{n+1} = (\alpha_n + 1)/2$. We have $\alpha_n \to 1$ and by step 4, $\mathcal{P}(\alpha_n) \Rightarrow \mathcal{P}(\alpha_{n+1})$. As $\mathcal{P}(\alpha_0)$ is true by step 1, we have $\mathcal{P}(1)$, that entails (i).

$(i) \Rightarrow (h)$: This implication follows from Lemma 3.2 and Remark 3.4.
$(h) \Rightarrow (d)$ is obvious.

Since the additional equivalence (m) can be established like in Theorem 4.9, the proof is now complete. □

The following proposition gives the expression of the metric projection mapping in terms of the normal cone.

Proposition 5.5. *Under the assumptions of Theorem 5.2, one has for all $x \in U_C(r)$*

$$\nabla^F d_C(x) = J(x - P_C(x))/d_C(x) \quad \textit{and} \quad P_C(x) = (I + J^* \circ N_C^{*r})^{-1}(x).$$

Proof. The equality concerning the derivative follows easily from the theorem above and from the expression of $\nabla e_\lambda f(x)$ in Lemma 3.1. Let us prove the equality concerning the metric projection mapping. We know by (30) that $P_C(x) \subset (I + J^* \circ N_C^{*r})^{-1}(x)$ for all $x \in U_C(r)$, so it is enough to check that $(I + J^* \circ N_C^{*r})^{-1}$ is at most single-valued on $U_C(r)$. Suppose that y_1, y_2 are in $(I + J^* \circ N_C^{*r})^{-1}(x)$ with $x \in U_C(r)$, that is, $J(x - y_i) \in N_C^{*r}(y_i)$, $i = 1, 2$. Then $P_C(x) = y_1 = y_2$ by (k). □

The uniform prox-regularity also entails the J-hypomonotonicity of the truncated normal functional cone.

Proposition 5.6. *Under the assumptions of Theorem 5.2, we also have*

(n) *The truncated normal functional cone mapping N_C^{*r} is J-hypomonotone of degree t for any $t \geq 1$.*

Conversely, (n) entails the assertions of Theorem 5.2 with parameter $r/2$ instead of r.

Proof. See steps 1 and 2 in the proof of Theorem 5.2. □

The following corollary gives several characterizations of convex sets. The condition (k) in the corollary is Vlasov's extension (see [46]) to Banach space with rotund dual of the known result proved by Asplund in Hilbert space (see [1]). The characterization (f) is exactly Theorem 18 of Borwein, Fitzpatrick and Giles [6]. Theorem 18 in [6] was established in the more general case where the dual space is merely rotund and it was, in some sense, a generalization of Theorem 3.6 of Fitzpatrick [27] where both the norm and the dual norm were assumed to be Fréchet differentiable away from the origins. Note also that the characterization (h) was explicitely given by Borwein, Fitzpatrick and Giles [6, Theorem 17] (with directional Gâteaux derivability as in Theorem 4.4 instead of the Dini subdifferentiability), in the larger context of Banach space Y with rotund dual Y^*. Both

Theorems 17 and 18 in [6] are derived by the authors by proving that any closed subset of a Banach space Y such that

$$\limsup_{\|y\|\to 0} \frac{d_C(x+y)-d_C(x)}{\|y\|} = 1 \quad \text{for all } x \in Y \setminus C$$

is almost convex and hence, according to a result due to Vlasov [46], convex whenever Y^* is rotund for its dual norm.

In our corollary, the requirement (X5) is imposed for a large part because of characterizations (b), (i), (j), (m), (n).

Corollary 5.7. *Let $C \subset X$ be a closed set. The following are equivalent:*

(a) *C is convex;*
(b) *C is uniformly ∞-prox-regular, i.e., uniformly r-prox-regular for any real number $r > 0$;*
(c) *d_C is continuously differentiable on $X \setminus C$;*
(d) *d_C is Fréchet regular on $X \setminus C$;*
(e) *d_C is Fréchet differentiable on $X \setminus C$;*
(f) *d_C is Gâteaux differentiable on $X \setminus C$ with $\|\nabla^G d_C(x)\| = 1$ for all $x \in X \setminus C$;*
(g) *$\partial_F d_C(x) \neq \emptyset$ for all $x \in X$;*
(h) *$P_C(x) \neq \emptyset$ and $\partial^- d_C(x) \neq \emptyset$ for all $x \in X$;*
(i) *d_C^2 is C^1 on X with locally Hölder continuous derivative mapping on X;*
(j) *P_C is single-valued and locally Hölder continuous on X;*
(k) *P_C is single-valued and norm-to-norm continuous on X;*
(l) *P_C is single-valued and norm-to-weak continuous on X;*
(m) *For any $p \in N_C(x)$ with $x \in C$ one has $x \in P_C(x+p)$;*
(n) *If $u \in X \setminus C$ and $x = P_C(u)$, then $x \in P_C(u')$ for $u' = x + r(u-x)$ and any $r > 0$.*

Proof. The equivalence between all the assertions from (b) to (n) is easily seen to follow from Theorem 5.2. The implication $(a) \Rightarrow (g)$ is obvious according to the convexity of the continuous function d_C under (a). By Proposition 5.6, the condition (g) entails that N_C^{*r} is J-hypomonotone of degree 1 on $U_C(r)$ for any $r > 0$. Fix any $x, y \in C$ and any $x^* \in N_C^*(x)$, $y^* \in N_C^*(y)$. By definition of J-hypomonotonicity of degree 1 for all $s > 0$ large enough

$$\langle J[J^*(sx^*) - (y-x)] - J[J^*(sy^*) - (x-y)], x-y \rangle \geq 0,$$

or

$$\langle J[J^*(x^*) - \frac{1}{s}(y-x)] - J[J^*(y^*) - \frac{1}{s}(x-y), x-y \rangle \geq 0.$$

Using the continuity of J and J^* and passing to the limit when s goes to $+\infty$ we obtain

$$\langle x^* - y^*, x-y \rangle \geq 0.$$

This means that N_C^* is monotone and hence by [18] (see also [19, 20]) the set C is convex, that is, (a). The proof of the corollary is then complete. $\square$

In the remainder of the paper, we will give, for an r-prox-regular set C, some properties of the ρ-enlargement $C(\rho)$ with $\rho \in]0, r[$ and of the set of ρ-external points to C

$$E_C(\rho) := \{u \in X \,:\, d_C(u) \geq \rho\}.$$

The first result concerns the proximal normal cone to the ρ-enlargement of C. In its statement we will denote by N_S^{Cl} the Clarke normal cone to a set S and by $\mathbb{R}_+$ the set of all non negative real numbers.

Proposition 5.8. *Assume that C is uniformly r-prox-regular. Then, for any $\rho \in]0, r[$ and any $y \in D_C(\rho)$,*

$$N^*_{C(\rho)}(y) = N^{Cl}_{C(\rho)}(y) = \mathbb{R}_+ \nabla^F d_C(y) \subset N^*_C(P_C(y)).$$

Proof. Fix $y \in D_C(\rho)$. From Lemma 5.4 and Theorem 5.2, the set $C(\rho)$ is uniformly $(r - \rho)$-prox-regular and $P_{C(\rho)}$ is single-valued on $U_C(r)$. To see that $\nabla^F d_C(y)$ actually belongs to $N^*_{C(\rho)}(y)$, put

$$u := y + \varepsilon(y - P_C(y)), \tag{35}$$

where ε is small enough that $d_C(u) < r$. As $P_C(u) = P_C(y)$ by Theorem 5.2(l), we have

$$d_C(u) = \|u - P_C(y)\| = \|y + \varepsilon(y - P_C(y)) - P_C(y)\| = (1 + \varepsilon)\|y - P_C(y)\| = (1 + \varepsilon)\rho > \rho.$$

Then by Lemma 5.3(b), since $P_{C(\rho)}$ is single-valued on $U_C(r)$, one has $y = P_{C(\rho)}(u)$. So $u - y \in N_{C(\rho)}(y)$ and hence $y - P_C(y) \in N_{C(\rho)}(y)$ which entails $\nabla^F d_C(y) \in N^*_{C(\rho)}(y)$. Further, as $C(\rho) = \{x \in X \,:\, d_C(x) \leq d_C(y)\}$ because $y \in D_C(\rho)$, we know by Clarke [14, Theorem 2.4.7 and Corollary 1] that $N^{Cl}_{C(\rho)} = \mathbb{R}_+ \nabla^F d_C(y)$ and hence

$$\mathbb{R}_+ \nabla^F d_C(y) \subset N^*_{C(\rho)}(y) \subset N^{Cl}_{C(\rho)}(y) = \mathbb{R}_+ \nabla^F d_C(y).$$

So it remains to see that the inclusion $\mathbb{R}_+ \nabla^F d_C(y) \subset N^*_C(P_C(y))$ follows from $\nabla^F d_C(y) = \dfrac{y - P_C(y)}{\|y - P_C(y)\|} \in N^*_C(P_C(y))$. □

Before establishing the second result, let us prove the following lemma which has its own interest.

Lemma 5.9. *Assume that C is uniformly r-prox-regular. Then for any $\rho \in]0, r]$ and $y \in U_C(\rho)$ one has*

$$d_C(y) + d_{E_C(\rho)}(y) = \rho. \tag{36}$$

Proof. Fix $y \in U_C(\rho)$ and put $x := P_C(y)$ according to (i) of Theorem 5.2. Then for $u := x + \rho\frac{y - x}{\|y - x\|}$ one has $x \in P_C(u)$ by Theorem 5.2(k) and hence $u \in D_C(\rho) \subset E_C(\rho)$ and

$$d_C(y) + d_{E_C(\rho)}(y) \leq \|y - x\| + \|y - u\| = \|u - x\| = \rho. \tag{37}$$

For any $z \in E_C(\rho)$ we have

$$\|z - y\| \geq \|z - P_C(y)\| - \|y - P_C(y)\| \geq d_C(z) - d_C(y) \geq \rho - d_C(y)$$

hence

$$d_{E_C(\rho)}(y) \geq \rho - d_C(y). \tag{38}$$

It follows from (37) and (38) that $d_C(y) + d_{E_C(\rho)}(y) = \rho$. □

From this lemma we see, on the one hand, through Theorem 5.2(d) that if C is uniformly r-prox-regular, then for any $\rho \in]0, r[$, the set $E_{C(\rho)}$ is *uniformly ρ-prox-regular*, because $d_{E_C(\rho)}(.) = \rho - d_C(.)$ and $U_C(\rho) = U_{E_C(\rho)}(\rho)$. On the other hand, the lemma allows us to retreive the following characterization of uniform r-prox-regularity given by Clarke, Stern and Wolenski [16] in the context of Hilbert space. Their proof is different and it relies on the analysis of an appropriate infimum value function. In our characterization below, the additional nonvacuity of $P_C(y)$ is not required (compare with [16, Theorem 4.1(c)]).

Theorem 5.10. *A closed set C is uniformly r-prox-regular for some $r > 0$ if and only if*

$$d_C(y) + d_{E_C(r)}(y) = r \quad \text{for all } y \in U_C(r). \tag{39}$$

Proof. The fact that (39) is implied by the uniform r-prox-regularity of C follows from Lemma 5.9 with $\rho = r$. Assume now that (39) holds and consider any $y \in U_C(r)$ for which $P_C(y)$ is a singleton, say $P_C(y) = x$. Then, for any $y' \in]x, y[$, one has $P_C(y') = x$. This yields for any non zero $t \in] - 1, \frac{\|y-y'\|}{\|y'-x\|}[$

$$\begin{aligned} &t^{-1}[d_C(y' + t(y' - x))) - d_C(y')] \\ &= t^{-1}[\|y' + t(y' - x) - x\| - \|y' - x\|] = \|y' - x\|, \end{aligned}$$

which entails that d_C has a Gâteaux directional derivative in the full direction $y' - x$ and hence by Theorem 4.4 the function d_C is Fréchet differentiable at y'. Thus by (39) the function $d_{E_C(r)}$ is also differentiable at y'. We then deduce (see the formula for $\nabla e_\lambda f$ in Lemma 3.1) that $P_{E_C(r)}(y')$ is a singleton that will be denoted by u'. Using successively the inclusion $u' \in D_C(r)$ and (39) we obtain

$$r \le \|x - u'\| \le \|y' - x\| + \|y' - u'\| = r,$$

so $y' \in]x, u'[$ (see (4)) and $P_C(u') \ni x$. By Theorem 5.2(l), this means that C is uniformly r-prox-regular. □

References

[1] E. Asplund: Čebyšev sets in Hilbert spaces, Trans. Amer. Math. Soc. 144 (1969) 235–240.

[2] B. Beauzamy: Introduction to Banach Spaces and their Geometry, 2nd Edition, North-Holland, Amsterdam (1985).

[3] F. Bernard, L. Thibault: Prox-regular functions in Hilbert spaces, J. Math. Anal. Appl. 303 (2005) 1–14.

[4] F. Bernard, L. Thibault: Prox-regular functions and sets in Banach spaces, Set-Valued Anal. 12 (2004) 25–47.

[5] F. Bernard, L. Thibault, D. Zagrodny: Integration of primal lower nice functions in Hilbert spaces, J. Optimization Theory Appl. 124 (2005) 561–579.

[6] J. M. Borwein, S. P. Fitzpatrick, J. R. Giles: The differentiability of real functions on normed linear space using generalized subgradients, J. Math. Anal. Appl. 128 (1987) 512–534.

[7] J. M. Borwein, J. R. Giles: The proximal normal formula in Banach space, Trans. Amer. Math. Soc. 302 (1987) 371–381.

[8] J. M. Borwein, H. M. Strojwas: Proximal analysis and boundaries of closed sets in Banach space. I: Theory, Can. J. Math. 38 (1986) 431–452.

[9] J. M. Borwein, H. M. Strojwas: Proximal analysis and boundaries of closed sets in Banach space. II: Applications, Canad. J. Math. 39 (1987) 428–472.

[10] M. Bounkhel, L. Thibault: On various notions of regularity of sets in nonsmooth analysis, Nonlinear Anal. 48 (2002) 223–246.

[11] H. Brezis: Analyse Fonctionnelle, Masson, Paris (1983), (in French).

[12] A. Canino: On p-convex sets and geodesics, J. Differential Equations 75 (1988) 118–157.

[13] I. Cioranescu: Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, Mathematics and its Applications 62, Kluwer, Dordrecht (1990).

[14] F. H. Clarke: Optimization and Nonsmooth Analysis, John Wiley & Sons, New York (1983).

[15] F. H. Clarke, Y. Ledyaev, J. R. Stern, R. P. Wolenski: Nonsmooth Analysis and Control Theory, Springer, New York (1997).

[16] F. H. Clarke, J. R. Stern, R. P. Wolenski: Proximal smoothness and the lower-C^2 property, J. Convex Analysis 2 (1995) 117–144.

[17] G. Colombo, V. V. Goncharov: Variational inequalities and regularity properties of sets in Hilbert spaces, J. Convex Analysis 8 (2001) 197–221.

[18] R. Correa, P. Gajardo, L. Thibault: Subdifferential representation formula and subdifferential criteria for the behavior of nonsmooth functions, Nonlinear Anal., to appear.

[19] R. Correa, A. Jofre, L. Thibault: Characterization of lower semicontinuous convex functions, Proc. Amer. Math. Soc. 116 (1992) 61–72.

[20] R. Correa, A. Jofre, L. Thibault: Subdifferential monotonicity as characterization of convex functions, Numer. Funct. Anal. Optimization 15 (1994) 531–536.

[21] M. Degiovanni, A. Marino, M. Toques: General properties of (p, q)-convex functions and operators, Ricerche Mat. 32 (1983), 285–319.

[22] R. Deville, G. Godefroy, V. Zizler: Smoothness and Renormings in Banach Spaces, Pitman Monographs and Surveys in Pure and Applied Mathematics 64, John Wiley & Sons, New York (1993).

[23] J. Diestel: Geometry of Banach Spaces. Selected Topics, Lecture Notes in Mathematics 485, Springer, Berlin (1975).

[24] J. F. Edmond, L. Thibault: BV solutions of nonconvex sweeping process differential inclusion with perturbation, J. Differential Equations, to appear.

[25] I. Ekeland, G. Lebourg: Generic Fréchet differentiability and perturbed optimization problems in Banach spaces, Trans. Amer. Math. Soc. 224 (1976) 193–216.

[26] H. Federer: Curvature measures, Trans. Amer. Math. Soc. 93 (1959) 418–491.

[27] S. Fitzpatrick: Differentiation of real-valued functions and continuity of metric projections, Proc. Amer. Math. Soc. 91 (1984) 544–548.

[28] A. D. Ioffe: Proximal analysis and approximate subdifferentials, J. London Math. Soc. 41 (1990) 175–192.

[29] V. Klee: Convexity of Chebyshev sets, Math. Annalen 142 (1961) 292–304.

[30] K. S. Lau: Almost Chebyshev subsets in reflexive Banach spaces, Indiana Univ. Math. J. 27(5) (1978) 791–795.

[31] A. B. Levy, R. A. Poliquin, L. Thibault: Partial extension of Attouch's theorem with applications to proto-derivatives on subgradient mappings, Trans. Amer. Math. Soc. 347 (1995) 1269–1294.

[32] J. Lindenstrauss, L. Tzafriri: Classical Banach Spaces I. Sequence Spaces, Springer, Berlin (1977).

[33] J. Lindenstrauss, L. Tzafriri: Classical Banach Spaces II. Function Spaces, Springer, Berlin (1979).

[34] S. Marcellin, L. Thibault: Evolution problems associated with primal lower nice functions, J. Convex Analysis 13 (2006) 385–421.

[35] B. S. Mordukhovich, Y. Shao: Nonsmooth sequential analysis in Asplund spaces, Trans. Amer. Math. Soc. 124 (1996) 1235–1280.

[36] T. S. Motzkin: Sur quelques propriétés caractéristiques des ensembles convexes, Atti Accad. Naz. Lincei, Rend., VI. Ser. 21 (1935) 562–567.

[37] R. A. Poliquin: Integration of subdifferentials of nonconvex functions, Nonlinear Anal. 17 (1991) 385–398.

[38] R. A. Poliquin, R. T. Rockafellar: Prox-regular functions in variational analysis, Trans. Amer. Math. Soc. 348 (1996) 1805–1838.

[39] R. A. Poliquin, R. T. Rockafellar, L. Thibault: Local differentiability of distance functions, Trans. Amer. Math. Soc. 352 (2000) 5231–5249.

[40] D. Preiss: Differentiability of Lipschitz functions on Banach spaces, J. Funct. Anal. 91 (1990) 312–345.

[41] R. T. Rockafellar, R. J.-B. Wets: Variational Analysis, Springer, Berlin (1998).

[42] A. S. Shapiro: Existence and differentiability of metric projections in Hilbert spaces, SIAM J. Optimization 4 (1994) 130–141.

[43] L. Thibault: Sweeping process with regular and nonregular sets, J. Differential Equations 193 (2003) 1–26.

[44] L. Thibault, D. Zagrodny: Integration of subdifferentials of lower semicontinuous functions on Banach spaces, J. Math. Anal. Appl. 189 (1995) 33–58.

[45] L. P. Vlasov: On Chebyshev sets, Soviet Math. Dokl. 8 (1967) 401–404.

[46] L. P. Vlasov: Almost convex and Chebyshev sets, Math. Notes 8 (1970) 776–779.

[47] Z. B. Xu, G. F. Roach: Characteristic inequalities of uniformly convex and uniformly smooth Banach spaces, J. Math. Anal. Appl. 157 (1991) 189–210.