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The Fitzpatrick Rate, the Stepanov Rate, the Lipschitz Rate and their Relatives

Jean-Paul Penot
Laboratoire de Mathématiques, CNRS UMR 5142,
Faculté des Sciences, av. de l'Université, 64000 Pau, France

Dedicated to the memory of Simon Fitzpatrick.

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We give some variants of a constant associated to a real-valued function on a Banach space and a point of the space, as introduced by S. Fitzpatrick. Some of them have a one-sided character and enable to give criteria of subdifferentiability (or differentiability) of different types. An application to best approximation is presented.

Keywords: Best approximation, derivative, distance function, Fitzpatrick rate, Lipschitz rate, stability, Stepanov rate, subdifferential

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1. Introduction

It has been shown by S. Fitzpatrick in [15] that a clever modification of the definition of the local Lipschitz constant (or rate or rank) of a real-valued function yields striking consequences in terms of relationship with Fréchet differentiability. A further change introduced by J. Borwein, S. Fitzpatrick and J. Giles in [8] has a close bearing on Gâteaux differentiability. Both studies lead to interesting results about distance functions and continuity of metric projections on closed subsets of Banach spaces.

The present note gives a further twist to this topic. We introduce one-sided variants of the Fitzpatrick's rate which are related to subdifferentiability properties rather than differentiability. We also put in light a connection with the notion of slope which has proved to be a valuable tool for metric estimates in connection with the Ekeland's variational principle. In particular this notion gives a simple approach to a study of conditioning of functions, error bounds, fixed point results: see [3], [4], [16].

We illustrate our results by an application to distance functions and best approximations, an important topic; see [7]-[8], [14], [15], [19]-[24]... for related contributions.

2. Some unilateral rates

Let X be a normed vector space with unit sphere $S_X := \{x \in X : \|x\| = 1\}$ and let $f : X \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$ be a function whose domain $\operatorname{dom} f := \{x \in X : f(x) \in \mathbb{R}\}$ is some neighborhood X_0 of a point $x \in X$. The well-known *Lipschitz rate* of f around x is

$$L_f(x) := \limsup_{y,z \to x} \frac{|f(y) - f(z)|}{\|y - z\|} = \inf_{r>0} L_f(x, r)$$

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with

$$L_f(x,r) := \sup\{\frac{|f(y)-f(z)|}{\|y-z\|} : y,z \in B(x,r)\},$$

where $B(x,r)$ denotes the open ball with center x and radius r and where by convention we set $\frac{0}{0} = 0$ in order to simplify formulas. Given $q \in \mathbb{R}_+$, we define the *quasi-stability rate* of f at x and the *q-quasi-stability rate* of f at x by

$$Q_f(x) := \inf_{q>0} Q_{f,q}(x), \qquad Q_{f,q}(x) := \inf_{r>0} Q_{f,q}(x,r)$$

with

$$Q_{f,q}(x,r) := \sup\{\frac{|f(y)-f(z)|}{\|y-z\|} : y,z \in B(x,r),\ \|z-x\| \le q\,\|y-x\|\}.$$

As observed by an anonymous referee, for $q \ge 1$, one has $Q_{f,q}(x,r) = L_f(x,r)$, $Q_{f,q}(x) = L_f(x)$ by symmetry. For $q = 1/2$ one gets the *Fitzpatrick rate* $N_f(x)$ of f at x. For $q = 0$ one gets the *Stepanov rate*, or stability rate of f at x :

$$S_f(x) := \limsup_{y \to x} \frac{|f(y)-f(x)|}{\|y-x\|}$$

which plays an important role in sensitivity analysis (see [17], [18] for instance and for a pioneering study [13]).

A one-sided version of the preceding rate is the *slope* of f at x given by

$$S_f^-(x) := |\nabla|(f,x) := \limsup_{y \to x} \frac{(f(x)-f(y))_+}{\|x-y\|}$$

(where r_+ stands for $\max(r,0)$ for $r \in \mathbb{R}$), often denoted by $|\nabla f|(x)$ (a confusing notation since f is not required to be differentiable at x). As mentioned above, this notion is an extremely useful tool in nonlinear and nonsmooth analysis. It gives an incentive to introduce

$$Q_f^-(x) := \inf_{q>0} Q_{f,q}^-(x), \quad Q_f^+(x) := \inf_{q>0} Q_{f,q}^+(x),$$

where

$$Q_{f,q}^-(x) := \limsup_{y,z \to x,\ \|z-x\| \le q\|y-x\|} \frac{(f(z)-f(y))_+}{\|y-z\|} := \inf_{r>0} Q_{f,q}^-(x,r),$$

$$Q_{f,q}^+(x) := \limsup_{y,z \to x,\ \|z-x\| \le q\|y-x\|} \frac{(f(y)-f(z))_+}{\|y-z\|} := \inf_{r>0} Q_{f,q}^+(x,r),$$

with

$$Q_{f,q}^-(x,r) := \sup\{\frac{(f(z)-f(y))_+}{\|y-z\|} : y,z \in B(x,r),\ \|z-x\| \le q\,\|y-x\|\},$$

$$Q_{f,q}^+(x,r) := \sup\{\frac{(f(y)-f(z))_+}{\|y-z\|} : y,z \in B(x,r),\ \|z-x\| \le q\,\|y-x\|\}.$$

Clearly $Q_{f,q}^+(x) := Q_{-f,q}^-(x)$, $Q_f^+(x) := Q_{-f}^-(x)$ and $Q_f(x) = \max(Q_f^-(x), Q_f^+(x))$. The upper slope $S_f^+(x) := S_{-f}^-(x) = Q_{f,0}^+(x)$ can play a role for maximization problems but it seems that it has not yet been introduced.

Let us note the following obvious lemma for later use.

Lemma 2.1. *For any $q \in \mathbb{R}_+$, the rate $Q^-_{f,q}(x)$ is the infimum of the set of nonnegative real numbers m for which there exists some $\delta > 0$ such that $f(z) - f(y) \leq m \|y - z\|$ whenever $y, z \in B(x, \delta)$ satisfy $\|z - x\| \leq q \|y - x\|$. Moreover, one has*

$$S^-_f(x) \leq Q^-_f(x) \leq Q^-_{f,q}(x) \leq L_f(x), \quad S^+_f(x) \leq Q^+_f(x) \leq Q^+_{f,q}(x) \leq L_f(x)$$

and if $q \in [0, 1/2]$ one has $Q^-_f(x) \leq Q^-_{f,q}(x) \leq N_f(x) \leq L_f(x)$.

Another comparison can be given for a calm function, i.e. a function f such that $S^-_f(x) < +\infty$.

Lemma 2.2. *Let $f : X \to \mathbb{R}_\infty$ be calm at x. Then $S^+_f(x) = Q^+_f(x)$.*

Proof. Since $S^+_f(x) \leq Q^+_f(x)$, it suffices to prove that assuming $S^+_f(x) < Q^+_f(x)$ leads to a contradiction. Let $b > S^-_f(x)$, so that there exists $\rho > 0$ such that, for every $z \in B(x, \rho)$,

$$f(z) - f(x) \geq -b \|z - x\|.$$

Given $c, c' \in (S^+_f(x), Q^+_f(x))$ with $c' < c$, and a sequence $(q_n) \to 0_+$, we can find sequences (y_n), $(z_n) \to x$ such that $\|z_n - x\| \leq q_n \|y_n - x\|$ and

$$f(y_n) - f(z_n) \geq c \|y_n - z_n\| > 0.$$

Then, for n large enough, we have $c - q_n(b + c) \geq c'$, hence

$$\begin{aligned} f(y_n) - f(x) &\geq (f(y_n) - f(z_n)) + (f(z_n) - f(x)) \\ &\geq c \|y_n - z_n\| - b \|z_n - x\| \\ &\geq c \|y_n - x\| - (b + c) \|z_n - x\| \\ &\geq (c - q_n(b + c)) \|y_n - x\| \geq c' \|y_n - x\|, \end{aligned}$$

a contradiction with the definition of $S^+_f(x)$. □

A comparison with the rate

$$P_f(x) := \sup_{w \in X} \sup_{u \in S_X} \limsup_{t \to 0_+} \frac{f(x + tw + tu) - f(x + tw)}{t}$$

introduced in [8] is not as obvious.

Lemma 2.3. *For any function f and any x in the interior of its domain, one has*

$$P_f(x) \leq \sup_{0<q\leq 1} Q_{f,q}(x) = Q_{f,\infty}(x) := \sup_{q \in \mathbb{R}_+} Q_{f,q}(x) = L_f(x).$$

Proof. The last equality is obvious. The first equality is an easy consequence of the symmetry of the roles of y and z in the definition of $Q_{f,q}(x)$.

Given $w \in X$, $u \in S_X := \{v \in X : \|v\| = 1\}$, with $w + u \neq 0$, let $q > 0$ be such that $\|w\| \leq q \|w + u\|$; then, for each $r > 0$ one has

$$\limsup_{t \to 0_+} \frac{f(x + tw + tu) - f(x + tw)}{t} \leq Q_{f,q}(x, r),$$

hence

$$\limsup_{t\to 0_+} \frac{f(x+tw+tu)-f(x+tw)}{t} \le Q_{f,q}(x) \le Q_{f,\infty}(x).$$

When $w+u=0$, for all $q>0$, we have

$$\limsup_{t\to 0_+} \frac{f(x+tw+tu)-f(x+tw)}{t} \le Q^-_{f,q}(x) \le Q_{f,\infty}(x).$$

Taking the supremum over $w \in W$, $u \in S_X$, we get $P_f(x) \le Q_{f,\infty}(x)$. □

The example of the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(0) = 0$, $f(t) = t\sin(1/t)$ shows that one may have $Q_{f,q}(x) \neq L_f(x)$ for $q = 1/2$, i.e. $N_f(x) \neq L_f(x)$ ([15]). The following example shows that for $p, q \in (0,1)$, $p \neq q$, one may have $Q_{f,p}(x) \neq Q_{f,q}(x)$, what answers a question of a referee.

Example. Let $X := \mathbb{R}$, let $p, q \in (0,1)$, $p < q$, and let (t_n) be a sequence of positive real numbers such that $t_{n+1} \le pt_n$ for all n. Given $c \in (0,+\infty)$, let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(t) := 0$ for $t \in (-\infty, 0] \cup [t_0, +\infty)$ and $t \in (pt_n, qt_n)$ for $n \in \mathbb{N}$ and $f(t) := c(t - qt_n)$ for $t \in [qt_n, t_n]$. Then, for every $r > 0$, one has $Q_{f,q}(0,r) = c$ and $Q_{f,p}(0,r) = c(1-p)^{-1}(1-q) < c$, hence $Q_{f,q}(0) > Q_{f,p}(0)$. One can slightly modify f to make it continuous.

Example. The preceding example also shows that one may have $S_f(0) = c(1-q) < Q_{f,p}(0)$ for $p \in (0,q)$.

3. Relationships with derivatives and subdifferentials

It is the purpose of this section to delineate links between the quasi-stability rates we defined and differentiability properties. More precisely, assuming differentiability properties on the norm of X, we wish to get differentiability of a function f at some $x \in X$ using estimates on its quasi-stability rate at x. In order to make as weak assumptions as possible, we avoid the directional differentiability hypothesis made in [15] by using the (radial) Dini derivatives of f at x which always exist. Recall that the lower and upper *Dini derivatives* of f at x in the direction $u \in X$ are given by

$$D^-f(x,u) := \liminf_{t\to 0_+} \frac{f(x+tu)-f(x)}{t}, \quad D^+f(x,u) := \limsup_{t\to 0_+} \frac{f(x+tu)-f(x)}{t}.$$

When f is locally Lipschitz around x, for every $u \in X$, the lower Dini derivative $D^-f(x,u)$ coincides with the contingent or lower Hadamard derivative of f at x in the direction u given by

$$df(x,u) := \liminf_{(t,v)\to(0_+,u)} \frac{f(x+tv)-f(x)}{t}.$$

Recall that the contingent or *Hadamard subdifferential* of f at x is given by

$$\partial f(x) := \{x^* \in X^* : \forall u \in X \quad df(x,u) \ge \langle x^*, u\rangle\},$$

while the firm or *Fréchet subdifferential* of f at x is defined by

$$\partial^F f(x) := \{x^* \in X^* : \liminf_{u\to 0} \frac{f(x+u)-f(x)-\langle x^*, u\rangle}{\|u\|} \ge 0\}.$$

These two subdifferentials are special instances of subdifferentials associated with bornologies (see [9]). Recall that a *bornology* in X is a covering $\mathcal{B}$ of X by bounded subsets. We may assume that $\mathcal{B}$ is hereditary (for $C \subset B$ with $B \in \mathcal{B}$ one has $C \in \mathcal{B}$) and stable by finite unions. Then one defines the *$\mathcal{B}$-subdifferential* of f at x as the set $\partial^{\mathcal{B}} f(x)$ of $x^* \in X^*$ such that for each $B \in \mathcal{B}$ and each $\varepsilon > 0$ one can find $\tau > 0$ such that

$$\forall t \in (0,\tau), \forall v \in B \quad \langle x^*, tv\rangle - t\varepsilon \leq f(x+tv) - f(x).$$

Taking for $\mathcal{B}$ the whole family of bounded subsets one gets the Fréchet bornology and $\partial^{\mathcal{B}} f(x) = \partial^F f(x)$. Other interesting choices are the families of compact subsets of X (the Hadamard bornology), the family of weakly compact subsets of X and the family of finite subsets of X (the Gâteaux bornology). Recall also that f is said to be *$\mathcal{B}$-differentiable* at x if there exists some $x^* \in X^*$ such that $x^* \in \partial^{\mathcal{B}} f(x) \cap (-\partial^{\mathcal{B}}(-f)(x))$, i.e. if for each $B \in \mathcal{B}$ and each $\varepsilon > 0$ one can find $\tau > 0$ such that

$$\forall t \in (0,\tau), \forall v \in B \quad |f(x+tv) - f(x) - \langle x^*, tv\rangle| \leq t\varepsilon.$$

For the four choices of $\mathcal{B}$ just described, this definition corresponds to Fréchet, Hadamard, weak Hadamard and Gâteaux differentiability respectively.

The following proposition extends [15, Theorem 2.2] which deals with the case $q = 1/2$; the arguments are similar.

Proposition 3.1.

(a) *For every $u \in S_X$ one has $D^+ f(x,u) \leq Q_f^+(x)$ and $-D^- f(x,u) \leq Q_f^-(x)$.*

(b) *For any bornology $\mathcal{B}$ on X, one has*

$$\sup\{\|x^*\| : x^* \in \partial^F f(x)\} \leq \sup\{\|x^*\| : x^* \in \partial^{\mathcal{B}} f(x)\} \leq Q_f^+(x).$$

(c) *Suppose f is Fréchet differentiable at x. Then, for every $q \in (0,1)$, one has*

$$Q_f(x) = Q_{f,q}(x) = Q_{f,q}^+(x) = Q_{f,q}^-(x) = Q_f^+(x) = Q_f^-(x) = \|f'(x)\|.$$

Proof. (a) Given $u \in S_X$, the first inequality follows by taking the infimum over $q > 0$, $r > 0$ in the relation

$$\sup_{0<t<r} \frac{f(x+tu) - f(x)}{t} \leq Q_{f,q}^+(x,r).$$

The second one follows:

$$-D^- f(x,u) = D^+(-f)(x,u) \leq Q_{-f}^+(x) = Q_f^-(x).$$

(b) The first inequality stems from the inclusion $\partial^F f(x) \subset \partial^{\mathcal{B}} f(x)$; the second one is a consequence of (a) and of the relation $\langle x^*, u\rangle \leq D^+ f(x,u) \leq Q_f^+(x)$ for each $u \in S_X$.

(c) Since $\|f'(x)\| = \sup\{f'(x)u : u \in S_X\}$ and $f'(x)u = D^+ f(x,u) \leq Q_f^+(x)$, for every $q \in \mathbb{R}_+$ one has $\|f'(x)\| \leq Q_f^+(x) \leq Q_{f,q}(x)$. In order to prove the reverse inequalities, let us associate with every $\varepsilon > 0$ some $\delta := \delta(\varepsilon) > 0$ such that for every $w \in B(x,\delta)$ one has

$$|f(w) - f(x) - f'(x)(w-x)| \leq \varepsilon \|w - x\|.$$

Then, given $q \in (0,1)$, for every $y, z \in B(x,\delta)$ satisfying $\|z-x\| \le q\|y-x\|$ one has

$$|f(y) - f(z) - f'(x)(y-x) + f'(x)(z-x)| \le \varepsilon\|y-x\| + \varepsilon\|z-x\|. \tag{1}$$

Now the triangle inequality ensures that

$$\|z-x\| \le q\|y-x\| \le q\|y-z\| + q\|z-x\|,$$

hence $\|z-x\| \le (1-q)^{-1}q\|y-z\|$ and

$$\|y-x\| \le \|y-z\| + \|z-x\| \le (1-q)^{-1}\|y-z\|.$$

From (1) it ensues that

$$\begin{aligned}|f(y) - f(z)| &\le |f'(x)(y-z)| + \varepsilon\|y-x\| + \varepsilon\|z-x\| \\ &\le \|f'(x)\|\,\|y-z\| + \varepsilon(1+q)(1-q)^{-1}\|y-z\|.\end{aligned}$$

Therefore $Q_{f,q}(x) \le \|f'(x)\|$ by taking the quotient by $\|y-z\|$, the supremum over y, z and then the infimum over $\varepsilon > 0$. Finally, since $-f$ is also Fréchet differentiable at x, one has $\|f'(x)\| = \|-f'(x)\| = Q^+_{-f}(x) = Q^-_f(x)$. □

The inequality in (b) may be strict: for $f : \mathbb{R} \to \mathbb{R}$ given by $f(0) = 0$ and $f(x) = |x\sin(1/x)|$ for $x \in \mathbb{R}\backslash\{0\}$, one has $\partial f(0) = \{0\}$ and $Q^+_f(0) \ge S_f(0) = 1$.

Corollary 3.2. *Suppose X is the dual of some normed vector space W, f is Fréchet differentiable at x and $f'(x)$ belongs to $W \subset X^*$. Then, there exists some $u \in S_X$ such that $Q_f(x) = f'(x)(u)$.*

Proof. The result follows from the equality $Q_f(x) = \|f'(x)\|$ and from the fact that $f'(x)$ attains its supremum on the unit sphere. □

The theorem which follows is inspired by [15, Thm. 2.4]. However, it has a one-sided character and the directional differentiability assumption on the function f is relaxed. It has been pointed to us by a referee that both these features are present in [6] which also contains a corresponding result in terms of the upper Dini derivative and the upper $\mathcal{B}$-derivative. However, these results are not couched in terms of the Fitzpatrick constant $N_f(x)$ nor of its one-sided variants $Q^+_{f,q}(x)$.

Theorem 3.3. *Suppose that for some $x \in X$ and some $u \in S_X$, $q \in (0,1]$ one has $D^-f(x,u) \ge Q^+_{f,q}(x)$ and the norm is $\mathcal{B}$-differentiable at u with derivative u^*. Then f is $\mathcal{B}$-subdifferentiable at u with $Q^+_{f,q}(x)u^* \in \partial^{\mathcal{B}} f(x)$.*

Proof. Let $m := Q^+_{f,q}(x) \le D^-f(x,u)$. Given $\varepsilon > 0$ and $B \in \mathcal{B}$, let $\beta \ge \sup\{\|b\| : b \in B\}$, $\beta \ge 1$. Since the norm is $\mathcal{B}$-differentiable at u with derivative u^*, we can find $\gamma \in (0, 1/\beta)$ such that

$$\forall v \in B, \forall \lambda \in [-\gamma, \gamma] \quad \|u + \lambda v\| - \|u\| \le \langle u^*, \lambda v\rangle + \varepsilon|\lambda|. \tag{2}$$

Since $m + \varepsilon\gamma q > Q^+_{f,q}(x)$ there exists $r > 0$ such that $m + \varepsilon\gamma q > Q^+_{f,q}(x,r)$: then, for $y, z \in B(x,r)$ with $\|z-x\| \le q\|y-x\|$ one has

$$f(z) - f(y) \ge -(m + \varepsilon\gamma q)\|y-z\|. \tag{3}$$

Let $\delta \in (0, r]$ be such that for $s \in [0, \delta]$ one has

$$f(x + su) - f(x) \geq sD^- f(x, u) - \varepsilon\gamma qs \geq ms - \varepsilon\gamma qs. \tag{4}$$

Then, for $v \in B$, $t \in (0, q\gamma\delta]$, setting $s := q^{-1}\gamma^{-1}t \leq \delta$, one has $\lambda := s^{-1}t = q\gamma \leq \gamma$,

$$\|su - tv\| \leq s(1 + s^{-1}t \|v\|) \leq s(1 + \beta\gamma) \leq 2s, \quad \|tv\| \leq qs\gamma\beta \leq qs = q \|su\| \leq r,$$

hence, for $y := x + su$, $z := x + tv = x + \gamma qsv$, by (2), (3) and (4),

$$\begin{aligned} f(x + tv) - f(x) &\geq f(x + tv) - f(x + su) + f(x + su) - f(x) \\ &\geq -(m + \varepsilon\gamma q) \|su - tv\| + ms - \varepsilon\gamma qs \\ &\geq -ms \left\|u - s^{-1}tv\right\| - \varepsilon\gamma q \|su - tv\| + ms - \varepsilon\gamma qs \\ &\geq -ms(1 - \langle u^*, s^{-1}tv\rangle + \varepsilon s^{-1}t) - 2\varepsilon\gamma qs + ms - \varepsilon\gamma qs \\ &\geq \langle mu^*, tv\rangle - (m + 3)\varepsilon t. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, these inequalities show that $mu^* \in \partial^{\mathcal{B}} f(x)$. □

It would be of interest to know whether one can replace $Q^+_{f,q}(x)$ by $Q^+_f(x)$ in the preceding result.

Let us state two special cases of interest. In the first one we use the fact that since the norm is Lipschitzian, Gâteaux differentiability and Hadamard differentiability coincide for such a function.

Corollary 3.4. *Suppose that for some $x \in X$, $q \in (0, 1]$ and some $u \in S_X$ one has $D^- f(x, u) \geq Q^+_{f,q}(x)$ and the norm is Gâteaux differentiable at u with derivative u^*. Then f is Hadamard subdifferentiable at u with $Q^+_{f,q}(x)u^* \in \partial f(x)$.*

Corollary 3.5. *Suppose that for some $x \in X$, and some $u \in S_X$ one has $D^- f(x, u) \geq Q^+_{f,q}(x)$ and the norm is Fréchet differentiable at u with derivative u^*. Then f is Fréchet subdifferentiable at u with $Q^+_{f,q}(x)u^* \in \partial^F f(x)$.*

A simplified subdifferentiability criterion is given in the following statement. It is a consequence of the two preceding statements since for $q \in (0, 1]$ one has $L_f(x) \geq Q_{f,q}(x) \geq Q^+_{f,q}(x)$.

Corollary 3.6. *Suppose f is Lipschitzian with rate λ around x and there exists $u \in S_X$ such that $D^- f(x, u) = \lambda$. If the norm of X is Fréchet (resp. Gâteaux) differentiable at u, then f is Fréchet (resp. Hadamard) subdifferentiable at u.*

The following corollary includes [15, Thm. 2.4] as the special case $q = 1/2$, $\mathcal{B}$ being the Fréchet bornology. Here no preliminary directional differentiability assumption is imposed.

Corollary 3.7. *Suppose that for some $x \in X$, $q \in (0, 1]$ and some $u, v \in S_X$ one has $D^- f(x, u) \geq Q^+_{f,q}(x)$, $D^+ f(x, v) \leq -Q^-_{f,q}(x)$ and the norm is $\mathcal{B}$-differentiable at u, v. Then f is $\mathcal{B}$-differentiable at x.*

Proof. By the preceding theorem, $\partial^{\mathcal{B}} f(x)$ is nonempty. Moreover, by our assumption, $D^-(-f)(x, v) = -D^+ f(x, v) \geq Q^-_{f,q}(x) = Q^+_{-f,q}(x)$, so that $\partial^{\mathcal{B}}(-f)(x)$ is nonempty. Then, it follows easily that f is $\mathcal{B}$-differentiable at x. □

Remark. The preceding proof shows that if the norm is $\mathcal{B}$-differentiable at $v \in S_X$ and if $D^+ f(x,v) \leq -Q^-_{f,q}(x)$, then f is $\mathcal{B}$-superdifferentiable at x in the sense that $\partial^{\mathcal{B}}(-f)(x)$ is nonempty. □

Corollary 3.8. *Suppose X is reflexive and its norm is Fréchet differentiable at each point of S_X. Then the following assertions are equivalent:*

(a) *f is Fréchet differentiable at x;*
(b) *there exist some $u \in S_X$, $q \in (0,1]$ such that $D^- f(x,u) \geq Q_{f,q}(x)$, $-D^+ f(x,-u) \geq Q_{f,q}(x)$;*
(c) *there exist some $u, v \in S_X$, $q \in (0,1]$ such that $D^- f(x,u) \geq Q^+_{f,q}(x)$, $D^+ f(x,v) \leq -Q^-_{f,q}(x)$.*

Proof. $(a) \Rightarrow (b)$ Suppose f is Fréchet differentiable at x and X is reflexive. Then, there exists some $u \in S_X$ such that $f'(x)(u) = \|f'(x)\|$ and, by Proposition 3.1(c), for every $q \in (0,1]$ one has $D^- f(x,u) = \|f'(x)\| = Q_{f,q}(x)$ and $-D^+ f(x,-u) = \|f'(x)\| = Q_{f,q}(x)$.

$(b) \Rightarrow (c)$ It suffices to pick u as in (b) and $v := -u$, observing that $Q_{f,q}(x) \geq Q^+_{f,q}(x)$ and $Q_{f,q}(x) \geq Q^-_{f,q}(x)$.

$(c) \Rightarrow (a)$ It is given by Corollary 3.7. □

4. Application to distance functions and best approximation

Let us apply the preceding results to the distance function $d_E := \inf_{e \in E} \|\cdot - e\|$ to a nonempty closed subset E of X. The following statement is an immediate consequence of Corollary 3.6, in view of the observations that the Lipschitz rate of d_E at some point $x \in X \backslash E$ is 1 and that the Gâteaux derivative of the norm at any point in $X \backslash \{0\}$ has norm 1.

Proposition 4.1. *Let $x \in X$ be such that there exists some $u \in S_X$ such that $D^- d_E(x,u) = 1$ and the norm is Fréchet (resp. Gâteaux) differentiable at u. Then $\partial^F f(x)$ (resp. $\partial f(x)$) contains an element with norm* 1.

One can deduce from [20, Cor. 3.6] and the preceding result the following criterion for the existence of best approximations. As in [20] we say that $w^* \in X^*$ is *generalized well-posed (on B_X)* if any minimizing sequence of the restriction of w^* to B_X has a convergent subsequence. We denote by W the set of generalized well-posed continuous linear forms on X and we say as in [20] that X is *M-reflexive* (or metrically reflexive) if $X^* = W \cup \{0\}$. It is a consequence of the Šmulian Theorem that any Banach space whose dual norm is Fréchet differentiable at each point of the unit sphere is M-reflexive. We define the *spherical duality map* S by setting for $u \in X \backslash \{0\}$

$$S(u) := \partial \|\cdot\| (u) = \{u^* \in X^* : \|u^*\| = 1, \langle u^*, u \rangle = \|u\|\}.$$

Proposition 4.2. *Suppose that for $x \in X \backslash E$ there exists some $u \in S_X \cap S^{-1}(W)$ such that $D^- d_E(x,u) = 1$ and the norm is Fréchet differentiable at u, with derivative u^*. Then the set $P(E,x)$ of best approximations of x in E is nonempty and $P(E,x) = \{x - d_E(x) S_*(u^*)\} \cap E$, where $S_*(u^*)$ is the set of points in S_X at which u^* attains its maximum.*

Proof. Since $\|\cdot\|$ is differentiable, with $u^* := \|\cdot\|'(u)$, one has $S(u) = \{u^*\}$ and $u^* \in W$ by the Šmulian Theorem. Now Theorem 3.3 ensures that $u^* \in \partial^F d_E(x)$. Since $u^* \in W$, [20, Cor. 3.6] implies that x has a best approximation in E. This can be shown directly, along with the last assertion, following the proof of [14, Thm. 2.6]. Let (e_n) be a sequence of E such that $(\varepsilon_n) \to 0$ for $\varepsilon_n := \|e_n - x\| - d_E(x)$. Let $\alpha : \mathbb{R}_+ \to \mathbb{R}$ be such that $\alpha(r) \to 0$ as $r \to 0$ and

$$d_E(x+v) - d_E(x) \geq \langle u^*, v\rangle - \alpha(v)\|v\| \tag{5}$$

for $\|v\|$ small enough. Taking $v := t_n(e_n - x)$, where $(\varepsilon_n/t_n) \to 0_+$, and observing that for n large enough

$$d_E(x + t_n(e_n - x)) \leq (1 - t_n)\|x - e_n\| = (1 - t_n)d_E(x) + (1 - t_n)\varepsilon_n,$$

one obtains from (5), with $v := t_n(e_n - x)$,

$$\begin{aligned} t_n\langle u^*, e_n - x\rangle - t_n\alpha(t_n(e_n - x))\|e_n - x\| &\leq d_E(x + t_n(e_n - x)) - d_E(x) \\ &\leq -t_n d_E(x) + (1 - t_n)\varepsilon_n. \end{aligned}$$

Dividing by $t_n d_E(x)$ both sides of this relation and taking limits, one gets

$$\limsup_n \frac{1}{\|e_n - x\|}\langle u^*, e_n - x\rangle \leq -1.$$

Since $u^* \in W$, a subsequence of $((x - e_n)/\|e_n - x\|)$ has a limit $w \in S_*(u^*)$. Then (e_n) has a subsequence which converges to $e := x - d_E(x)w$ and $e \in P(E, x)$. Now, for any $e' \in P_E(x)$ one can take $e_n := e'$ in what precedes, so that $w' := (x - e')/\|e' - x\| \in S_*(u^*)$ and we see that $e' = x - d_E(x)w' \in x - d_E(x)S_*(u^*)$. Now for any $w \in S_*(u^*)$ such that $e := x - d_E(x)w \in E$ one has $e \in P(E, x)$ since $\|e - x\| = d_E(x)\|w\| = d_E(x)$. □

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