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# Differentiability of Lipschitz Functions on a Space with Uniformly Gâteaux Differentiable Norm

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*Dedicated to the memory of Simon Fitzpatrick.*

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On a normed linear space with uniformly Gâteaux differentiable norm a locally Lipschitz function which satisfies a property relating the Dini derivatives to a constant determined by the Michel-Penot derivative possesses significant differentiability properties. This generalises previous joint work with Simon Fitzpatrick.

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Zajiček [13, Theorem 3, p. 300] showed that distance functions on a normed linear space with uniformly Gâteaux differentiable norm have particular differentiability properties. Vlasov [11, 12] showed that distance functions on a Banach space with rotund dual satisfying a certain differentiability condition are convex. So we are led to ask whether more general Lipschitz functions on a normed linear space with uniformly Gâteaux differentiable norm and satisfying a Vlasov type condition have particular differentiability properties. Our results come from the generalisation of a Vlasov type condition explored by Fitzpatrick [6], [3, Theorem 1, p. 516].

The outstanding global result given by Preiss [9] is that a Lipschitz function on a Banach space with Gâteaux differentiable norm is Gâteaux differentiable on a dense subset of its domain. Recently, Preiss and Zajiček [10] developed further dense differentiability properties for Lipschitz functions on separable Banach spaces. But here we are interested in exploring local differentiability properties.

## 1. A differentiability property related to a Michel-Penot constant

A real valued function $\psi$ on a nonempty open subset $A$ of a normed linear space $X$ is *locally Lipschitz* on $A$ if for every $x_0 \in A$ there exists a $K_0 > 0$ and a $\delta_0 > 0$ such that

$$|\psi(x) - \psi(z)| \leq K_0\|x - z\| \quad \text{for all } x, z \in B(x_0; \delta).$$

For $\psi$, the *upper Dini derivative* at $x \in A$ in the direction $y \in X$ is

$$\psi^+(x)(y) \equiv \limsup_{t \to 0+} \frac{\psi(x + ty) - \psi(x)}{t}$$

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and the *lower Dini derivative* at $x \in A$ in the direction $y \in X$ is

$$\psi^{-}(x)(y) \equiv \liminf_{t \to 0+} \frac{\psi(x+ty) - \psi(x)}{t}.$$

$\psi$ has a *right hand derivative* at $x \in A$ in the direction $y \in X$ if

$$\psi'_{+}(x)(y) \equiv \lim_{t \to 0+} \frac{\psi(x+ty) - \psi(x)}{t} \quad \text{exists,}$$

$\psi$ has a *Gâteaux derivative* at $x \in A$ in the direction $y \in X$ if

$$\psi'(x)(y) \equiv \lim_{t \to 0} \frac{\psi(x+ty) - \psi(x)}{t} \quad \text{exists}$$

and $\psi$ is *Gâteaux differentiable* at $x \in A$ if $\psi'(x)(y)$ exists for all $y \in X$ and is linear in $y$.

As a tool in nonsmooth analysis, Michel and Penot [8] introduced the *Michel-Penot generalised derivative* at $x \in A$ in the direction $y \in X$ defined by

$$\psi^{\diamond}(x)(y) \equiv \sup_{z \in X} \limsup_{t \to 0+} \frac{\psi(x+tz+ty) - \psi(x+tz)}{t}.$$

The function $y \mapsto \psi^{\diamond}(x)(y)$ is continuous and sublinear. Clearly in general $\psi^{+}(x)(y) \leq \psi^{\diamond}(x)(y)$ for all $x \in A$ and $y \in X$. If $\psi$ is Gâteaux differentiable at $x \in A$ then $\psi'(x)(y) = \psi^{\diamond}(x)(y)$ for all $y \in X$. If $\psi$ is a convex function on a nonempty open convex subset $A$ then $\psi'_{+}(x)(y) = \psi^{\diamond}(x)(y)$ for all $y \in X$. For given $x \in A$ we define the constant

$$P_{\psi}(x) \equiv \sup_{\|y\|=1} \psi^{\diamond}(x)(y).$$

Then

$$-P_{\psi}(x) = \inf_{\|y\|=1} \inf_{z \in X} \liminf_{t \to 0+} \frac{\psi(x+tz+ty) - \psi(x+tz)}{t}.$$

Fitzpatrick *et al.* [3, Theorem 1, p. 516] explored differentiability properties of $\psi$ at a point where a directional derivative attains such a constant. Here we generalise that result on a normed linear space with uniformly Gâteaux differentiable norm.

A normed linear space $X$ has *uniformly Gâteaux differentiable* norm if given $y \in X, \|y\| = 1$ and $\epsilon > 0$ there exists $\delta(\epsilon, y) > 0$ such that

$$\left| \frac{\|x+ty\| - \|x\|}{t} - \|x\|'(y) \right| < \epsilon \quad \text{for all } 0 < |t| < \delta \quad \text{for all } x \in X, \|x\| = 1.$$

**Theorem 1.1.** *Consider a locally Lipschitz function $\psi$ on a nonempty open subset $A$ of a normed linear space $X$ with uniformly Gâteaux differentiable norm. If for $x \in A$,*

$$\sup_{\|y\|=1} \psi^{+}(x)(y) = P_{\psi}(x)$$

*then there exists an $f_{+} \in X^{\star}, \|f_{+}\| \leq 1$ such that*

$$\psi^{+}(x)(y) \geq P_{\psi}(x) f_{+}(y) \quad \textit{for all } y \in X.$$

**Proof.** Since $X$ has uniformly Gâteaux differentiable norm, given $y \in X, \|y\| = 1$ and $\epsilon > 0$ there exists $\gamma(\epsilon, y) > 0$ such that for all $y_0 \in X, \|y_0\| = 1$

$$\left| \frac{\|y_0 + ty\| - \|y_0\|}{t} - \|y_0\|'(y) \right| < \epsilon \quad \text{for all } 0 < |t| \leq \gamma.$$

Now there exists $\{y_n\}, \|y_n\| = 1$ such that

$$\lim_{n\to\infty} \psi^+(x)(y_n) = P_\psi(x)$$

so there exists $\nu(\epsilon, y) \in \mathbb{N}$ such that

$$\left|\psi^+(x)(y_n) - P_\psi(x)\right| \leq \frac{\epsilon}{2}\gamma \quad \text{for all } n \geq \nu.$$

But also for each $n \geq \nu$ there exists $\{t_{nk}\}$, $t_{nk} \to 0+$ as $k \to \infty$ such that

$$\lim_{n\to\infty} \frac{\psi(x + t_{nk}y_n) - \psi(x)}{t_{nk}} = \psi^+(x)(y_n)$$

so for each $n \geq \nu$ there exists $\nu_n'(\epsilon, y_n) \in \mathbb{N}$ such that

$$\left| \frac{\psi(x + t_{nk}y_n) - \psi(x)}{t_{nk}} - \psi^+(x)(y_n) \right| \leq \frac{\epsilon}{2}\gamma \quad \text{for all } k > \nu_n'.$$

Then

$$\left| \frac{\psi(x + t_{nk}y_n) - \psi(x)}{t_{nk}} - P_\psi(x) \right| \leq \epsilon\gamma \quad \text{for all } n > \nu \quad \text{and all } k > \nu_n'. \qquad \star$$

For each $n \in \mathbb{N}$, $\psi\,(x + t\gamma y) - \psi(x) = \psi(x + ty_n + t(-y_n + \gamma y)) - \psi(x+ty_n) + \psi(x+ty_n) - \psi(x)$ for all $t > 0$. From the definition of $-P_\psi(x)$ there exists $\delta_n(\epsilon, x, y_n, -y_n + \gamma y) > 0$ such that

$$\psi\,(x + ty_n + t(-y_n + \gamma y)) - \psi(x + ty_n) > -P_\psi(x)\|y_n - \gamma y\|t - \epsilon\gamma t \quad \text{for all } 0 < t < \delta_n.$$

So

$$\begin{aligned} \psi(x + t_{nk}\gamma y) - \psi(x) \;&>\; -P_\psi(x)\left(\|y_n - \gamma y\| - \|y_n\|\right) t_{nk} - 2\epsilon\gamma t_{nk} \\ &\qquad\qquad \text{for } n > \nu,\ k > \nu_n' \text{ and } 0 < t_{nk} < \delta_n \\ &>\; P_\psi(x)\left(\|y_n\|'(y) - \epsilon\right) t_{nk}\gamma - 2\epsilon\gamma t_{nk}. \end{aligned}$$

Since $B(X^\star)$ is weak$^\star$ compact there exists $f_+ \in X^\star, \|f_+\| \leq 1$ such that

$$\psi^+(x)(y) \geq P_\psi(x) f_+(y) \quad \text{for all } y \in X.$$

$\square$

We should note that on any normed linear space $X$ if a locally Lipschitz function $\psi$ on a nonempty open subset $A$ has the property that at $x \in A$

$$\psi^+(x)(y) \geq P_\psi(x) f_+(y) \quad \text{where } \|f_+\| = 1$$

then $\sup\limits_{\|y\|=1} \psi^+(x)(y) = P_\psi(x)$.

Theorem 1.1 has a simpler proof if there is actually a $y_0 \in X$, $\|y_0\| = 1$ such that

$$\psi^+(x)(y_0) = P_\psi(x).$$

Then we can significantly reduce the differentiability condition on the space to having a norm Gâteaux differentiable at $y_0$. The conclusion is then that

$$\psi^+(x)(y) \geq P_\psi(x)\|y_0\|'(y) \quad \text{for all } y \in X.$$

**Corollary 1.2.** *If $X$ has uniformly Gâteaux differentiable norm and for $x \in A$*

$$\inf_{\|y\|=1} \psi^-(x)(y) = -P_\psi(x)$$

*then there exists $f_- \in X^\star$, $\|f_-\| \leq 1$ such that*

$$\psi^-(x)(y) \leq -P_\psi(x) f_-(y) \quad \textit{for all } y \in X.$$

**Proof.** Now $\psi^-(x)(y) = -(-\psi)^+(x)(y)$ so $\sup_{\|y\|=1} (-\psi)^+(x)(y) = P_\psi(x) = P_{-\psi}(x)$. Then there exists $f_- \in X^\star$, $\|f_-\| = 1$ such that

$$(-\psi)^+(x)(y) \geq P_\psi(x) f_-(y)$$

so $\psi^-(x)(y) \leq -P_\psi(x) f_-(y)$ for all $y \in X$. □

**Corollary 1.3.** *If $X$ has uniformly Gâteaux differentiable norm and for $x \in A$ both*

$$\sup_{\|y\|=1} \psi^+(x)(y) = P_\psi(x)$$

*and*

$$\inf_{\|y\|=1} \psi^-(x)(y) = -P_\psi(x)$$

*then $\psi$ is intermediately differentiable at $x$, that is, there exists $f_0 \in X^\star$ such that*

$$\psi^-(x)(y) \leq f_0(y) \leq \psi^+(x)(y) \quad \textit{for all } y \in X.$$

**Proof.** From Theorem 1.1 and Corollary 1.2 we deduce that the closed cones $\overline{\text{co}}\,\{(y, \psi^+(x)(y)) \in X \times \mathbb{R} : y \in X\}$ and $\overline{\text{co}}\,\{(y, \psi^-(x)(y)) \in X \times \mathbb{R} : y \in X\}$ can be separated by a hyperplane generated by an $f_0 \in X^\star$. □

A locally Lipschitz function $\psi$ on a nonempty open subset $A$ of a normed linear space $X$ is said to be *M-P pseudo-regular* at $x \in A$ if

$$\psi^+(x)(y) = \psi^\diamond(x)(y) \quad \text{for all } y \in X.$$

Then such a function satisfies

$$\sup_{\|y\|=1} \psi^+(x)(y) = P_\psi(x) \qquad \text{for all equivalent norms for } X.$$

On a separable Banach space, every locally Lipschitz function on a nonempty open subset is M-P pseudo-regular at the points of a dense $G_\delta$ subset of its domain, [7, Theorem, p. 208]. So if $X$ is a separable Banach space, both $\psi$ and $-\psi$ are M-P pseudo-regular at the points of a dense $G_\delta$ subset of $A$. Further, a separable Banach space can be equivalently renormed to have uniformly Gâteaux differentiable norm, [4, p. 66]. so then we make the following deduction from Corollary 1.3.

**Corollary 1.4** [7, Cor. 1, p. 209]**.** *A locally Lipschitz function $\psi$ on a nonempty open subset $A$ of a separable Banach space $X$ is intermediately differentiable at the points of a dense $G_\delta$ subset of $A$.*

We can recapture the Fitzpatrick result using the comment after Theorem 1.1.

**Corollary 1.5** [3, Theorem 1, p. 516]**.** *Suppose there exists $y_0 \in X, \|y_0\| = 1$ such that the Gâteaux derivative of $\psi$ in the direction $y_0$ exists at $x \in A$ and*

$$\psi'(x)(y_0) = P_\psi(x).$$

*If the normed linear space $X$ has norm Gâteaux differentiable at $y_0$ then $\psi$ is Gâteaux differentiable at $x$ and*

$$\psi'(x)(y) = P_\psi(x)\|y_0\|'(y) \quad \textit{for all } y \in X.$$

**Proof.** Since $\psi'(x)(y_0) = P_\psi(x)$ we have

$$\psi^+(x)(y_0) = P_\psi(x) \quad \text{and} \quad \psi^-(x)(-y_0) = -P_\psi(x).$$

So

$$\psi^-(x)(y) \leq P_\psi(x)\|y_0\|'(y) \leq \psi^+(x)(y) \quad \text{for all } y \in X.$$

The stronger differentiability condition replacing $\star$ in Theorem 1.1 enables us to conclude as in [3, pp. 517-8] that

$$\psi'_+(x)(y) \leq P_\psi(x)\|y_0\|'(y) \leq \psi'_+(x)(y) \quad \text{for all } y \in X$$

and our conclusion holds. □

## 2. Differentiability properties of distance functions

On a real normed linear space $X$, a nonempty subset $K$ generates a *distance function $d$* on $X$ where

$$d(x) = \inf\{\|x - y\| : y \in K\}.$$

The distance function $d$ always satisfies the Lipschitz condition

$$|d(x) - d(y)| \leq \|x - y\| \quad \text{for all } x, y \in X.$$

The set $K$ is said to be *proximinal (Chebyshev)* if, for each $x \in X\backslash K$ there exists a (unique) point $p(x) \in K$ such that

$$d(x) = \|x - p(x)\|.$$

A proximinal (Chebyshev) set is necessarily closed. For $x \in X\backslash K$ where $K$ is a proximinal set,

$$d^\diamondsuit(x)\,(x - p(x)) \geq \|x - p(x)\|, \quad [3, \text{Theorem 5, p. 520}].$$

and so $P_d(x) = 1$.

The most general conditions under which a subset is convex was given by Vlasov [11, 12] which may be presented in the following form.

**Proposition 2.1.** *In a Banach space $X$ with rotund dual $X^\star$, a nonempty closed subset $K$ which generates a distance function $d$ satisfying*

$$\limsup_{\|y\|\to 0} \frac{d(x+y)-d(x)}{\|y\|} = 1 \quad \textit{for all } x \in X\backslash K,$$

*is convex.*

For a proximinal set we see that this condition is precisely

$$\sup_{\|y\|=1} d^+(x)(y) = P_d(x)$$

and from Theorem 1.1 we have that if the space $X$ has uniformly Gâteaux differentiable norm then there exists an $f_+ \in X^\star$, $\|f_+\| \le 1$ such that

$$d^+(x)(y) \ge f_+(y) \quad \text{for all } y \in X.$$

We present Zajiček's argument again so that we can see the fuller implications of differentiability properties of the distance function.

The following characterisations of uniform Gâteaux differentiability will be needed:
The normed linear space $X$ has uniformly Gâteaux differentiable norm if and only if for any $r > 0$ and each $y \in X$, given $\epsilon > 0$ there exists a $\delta(\epsilon, y) > 0$ such that

$$\left| \frac{\|x+ty\| - \|x\|}{t} - \|x\|'(y) \right| < \epsilon \quad \text{for all } 0 < |t| < \delta \quad \text{and } x \in X, \|x\| > r.$$

Equivalently, if and only if for any $r > 0$ and each $y \in X$ the mapping $x \mapsto \|x\|'(y)$ is uniformly continuous on $\{x \in X; \|x\| > r\}$, [13. Proposition 7, p. 299].

With the form of the Michel-Penot generalised derivative in mind, we note the following elementary properties of distance functions.

**Lemma 2.2.** *In a normed linear space $X$ with nonempty closed subset $K$, for each $x \in X\backslash K$ and $t > 0$ and $y, z \in X$, choose $p_t(x+tz+ty) \in K$ such that*

$$\|(x+tz+ty) - p_t(x+tz+ty)\| < d(x+tz+ty) + t^2.$$

*Then*

$$\|x - p_t(x+tz+ty)\| \to d(x) \quad \textit{as } t \to 0+.$$

**Proof.**

$$\begin{aligned} d(x) &\le \|x - p_t(x+tz+ty)\| \le \|x+tz+ty - p_t(x+tz+ty)\| + t(\|y\|+\|z\|) \\ &\le d(x+tz+ty) + t^2 + t(\|y\|+\|z\|) \le d(x) + t^2 + 2t(\|y\|+\|z\|). \end{aligned}$$

□

**Theorem 2.3.** *On a normed linear space $X$ with uniformly Gâteaux differentiable norm, the distance function $d$ generated by a nonempty closed subset $K$ has a right hand derivative at each $x \in X\backslash K$ and for any $y \in X$,*

$$d'_+(x)(y) = \inf_{z\in X} \liminf_{t\to 0+} \frac{d(x+tz+ty) - d(x+tz)}{t}.$$

**Proof.** Given $z \in X$,

$$\begin{aligned}
&\liminf_{t\to 0+} \frac{d(x+tz+ty)-d(x+tz)}{t} \\
\geq\ & \liminf_{t\to 0+} \frac{1}{t}\left(\|x+tz+ty-p_t(x+tz+ty)\| - t^2 - \|x+tz-p_t(x+tz+ty)\|\right) \\
\geq\ & \liminf_{t\to 0+} \|x+tz-p_t(x+tz+ty)\|'(y) \\
\geq\ & \liminf_{t\to 0+} \|x-p_t(x+tz+ty)\|'(y) \quad \text{from the uniform continuity characterisation} \\
\geq\ & \liminf\{\|x-v\|'(y) : \|x-v\| \to d(x), v \in K\} \quad \text{from Lemma 2.2.}
\end{aligned}$$

Now there exists $v_n \in K$ where $\|x-v_n\| \to d(x)$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \|x-v_n\|'(y) = \liminf\{\|x-v\|'(y) : \|x-v\| \to d(x), v \in K\}.$$

Since the norm is uniformly Gâteaux differentiable, given $\epsilon > 0$ there exists a $\delta(\epsilon, y) > 0$ such that

$$\left|\frac{\|x+sy-v_n\| - \|x-v_n\|}{s} - \|x-v_n\|'(y)\right| < \epsilon$$

for all $0 < |s| < \delta$ and $n$ sufficiently large. Then

$$\|x-v_n\|'(y) + \epsilon \geq \frac{d(x+sy)-d(x)}{s} + \frac{d(x)-\|x-v_n\|}{s}.$$

So $\liminf\{\|x-v\|'(y) : \|x-v\| \to d(x), v \in K\} + \epsilon \geq \dfrac{d(x+sy)-d(x)}{s}$ for all $0 < |s| < \delta$, and

$$d^+(x)(y) \quad \leq \quad \inf_{z\in X} \liminf_{t\to 0+} \frac{d(x+tz+ty)-d(x+tz)}{t}$$

but

$$\leq \quad \liminf_{t\to 0+} \frac{d(x+ty)-d(x)}{t} = d^-(x)(y).$$

Therefore, $d'_+(x)(y)$ exists and our conclusion follows. □

We can now deduce a differentiability result for distance functions.

**Corollary 2.4.** *On a normed linear space $X$ with uniformly Gâteaux differentiable norm, the distance function $d$ generated by a nonempty closed set $K$ has the property that for $x \in X\backslash K$,*

$$\sup_{\|y\|=1} d^+(x)(y) = P_d(x)$$

*if and only if $d$ is Gâteaux differentiable at $x$.*

**Proof.** If $d$ is Gâteaux differentiable at $x$ then

$$P_d(x) = \|d'(x)\|.$$

Conversely, from Theorem 2.3, $d'_+(x)(y) = -(-d)^\diamond(x)(y)$ for all $y \in X$ so the mapping $y \mapsto d'_+(x)(y)$ is superlinear. From Theorem 1.1 this mapping $y \mapsto d'_+(x)(y)$ dominates a linear functional. So we conclude that the mapping $y \mapsto d'_+(x)(y)$ is linear and that $d$ is Gâteaux differentiable at $x$. □

For a Chebyshev set $K$ in a normed linear space $X$, the mapping $x \mapsto p(x)$ on $X$ is called the *metric projection*. Early attempts to determine the convexity of $K$ invoked the continuity of the metric projection. Now if $K$ has a continuous metric projection then the distance function $d$ satisfies Vlasov's differentiability condition of Proposition 2.1.

Asplund [1] had proved that in a Hilbert space a Chebyshev set with continuous metric projection is convex. Vlasov [12] had shown that in a Banach space with rotund dual a Chebyshev set with continuous metric projection is convex. However, it was known that the continuity of the metric projection is not in general necessary for the convexity of Chebyshev sets [2].

Now of course, if $K$ is convex on a normed linear space with uniformly Gâteaux differentiable norm then the distance function $d$ is Gâteaux differentiable. However, even in Hilbert space the Gâteaux differentiability of a distance function $d$ at a point $x \in X \backslash K$ does not necessarily imply that $\|d'(x)\| = 1$, [5, Example 5.1, p. 308]. Nevertheless, on any normed linear space $X$ if the distance function $d$ generated by $K$ is Gâteaux differentiable at $x \in X \backslash K$ then

$$\sup_{\|y\|=1} d'(x)(y) = P_d(x)$$

and if $K$ is proximinal then $P_d(x) = 1$.

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