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Subgradients of Convex Games and Public Good Games

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The subgradients of a convex game at a given coalition of agents are defined. Given a public good economy, a convex game called public good game is generated. However, not all convex games are public good games. This paper uses subgradients to identify certain classes of public good games. In particular, we find a class of public good games which is a strict subset of convex games and a strict superset of symmetric convex games in the four-agent case. We provide sufficient conditions to identify that class.

Keywords: Convex games, subgradients, public good economy

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1. Introduction

It has long been known that an exchange economy with private goods generates a market game that is a totally balanced game in characteristic function form, and the converse is also true, i.e., a totally balanced game can be generated by an exchange economy (Shapley and Shubik, [6]). The isomorphism between these two classes of games, the market games and the totally balanced games, links economic models of exchange with game theory. One advantage of this connection is that we can apply solution concepts in game theory to models of exchange economy. A well-known example is the *core* concept, which is intimately related to the concept of competitive equilibrium.

In a similar spirit, attempt has been made first by Sprumont [7] to establish a connection between a public good economy and a game in characteristic function form. It has been known that a public good economy generates a convex game, called a public good game (see Moulin, [4]). However, not all convex games are public good games. To characterize the family of convex games that arise from models of public good economy, Sprumont [7] provides a necessary and sufficient condition for a convex game to be a public good game.

Theoretically speaking, the above characterization of public good games provides a complete answer to the question of when a game can arise from a public good economy. However, it is not so easy to use (and/or interpret) the conditions in the characterization to answer whether or not a specific convex game is a public good game. On one hand, it is of mathematical interest to further investigate the underlining structure of public good games. On the other hand, public good economies are important economic models. The game-theoretic approach to public good economies is useful. As in the case of the exchange economy with private goods, the *core* concept in game theory is also very much

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relevant in public good economies along with other competitive equilibrium concepts such as the Lindahl equilibrium (Foley, [2]).

In this paper, we introduce the concept of subgradients for convex games and use this concept to identify certain classes of public good games. We show that in the four-agent case, if a convex game satisfies certain conditions which are weaker than symmetry but stronger than convexity, the game is a public good game (Theorem 3.5). Moreover, using subgradient approach we provide simpler proofs to some important results in Sprumont [7]. In particular, we show that any symmetric convex game is a public good game (Theorem 3.8).

We should point out that our approach to public good games is based on Sprumont [7]. The contribution of this paper is to use the subgradients concept for convex games to refine or extend Sprumont's approach to public good games. The paper shows an interesting application of the concept of subgradients to an important economic model. In particular, we find a class of games, which is a strict subset of convex games and a strict superset of symmetric (convex) games, that are public good games (Theorem 3.5 and Example 3.6).

The paper is organized as follows. Section 2 introduces the model of public good economy and defines the associated public good games. Section 3 discusses the main results. Section 4 concludes the paper with some remarks. The Appendix collects some proofs.

2. Public Good Games

Let $N = \{1, ..., n\}$ be the set of agents. Each agent i, $i \in N$, has a quasilinear utility function $V_i : R_+ \times R \to R$, which is continuous, nondecreasing in the public good y, linear in the private good (e.g., money) t, and $V_i(0, 0) = 0$. Specifically, assume that

$$V_i(y, t) = U_i(y) + t, \ i \in N, \tag{1}$$

where $U_i (i \in N)$ is agent i's utility (benefit) function of the public good, and is continuous, nondecreasing, and $U_i(0) = 0$.

Let C be the cost function of the public good. It costs $C(y)$ (units of private good) to produce y units of public good. Assume that $C : R_+ \to R_+$ is continuous, nondecreasing, and $C(0) = 0$.

Definition 2.1. A public good economy is a pair $(U; C)$, where $U = (U_1, ..., U_n)$ and $U_i (i = 1, ..., n)$ are agents' utility functions of the public good and C is the cost function of the public good.

Assume that there exists a $\overline{y} > 0$ such that

$$\sum_{i=1}^{n} U_i(y) < C(y), \ \forall y \geq \overline{y}.$$

This condition says that there is an upper bound beyond which it is inefficient to provide more public good. It guarantees that there exists an optimal level of public good for all agents together as well as an optimal level for each subset of agents.

A *game* is a pair $(N; v)$, where $N = \{1, ..., n\}$ is the set of agents and v is a function from the subsets of N to the real numbers satisfying $v(\emptyset) = 0$, called the characteristic function. The subsets of N are called coalitions. For a coalition S, $v(S)$ is called the value (or worth) of the coalition.

Definition 2.2. A game $(N; v)$ is generated by a public good economy $(U; C)$ and called a public good game if

$$v(S) = \max_{y \geq 0} \{ \sum_{i \in S} U_i(y) - C(y) \}, \; S \subseteq N. \tag{2}$$

A central question we ask in this paper is the following: which games can arise from public good economies? In other words, which games are public good games?

It is helpful to consider first the class of convex games. A game $(N; v)$ is convex if

$$v(S) + v(T) \leq v(S \cap T) + v(S \cup T), \; \forall S, T \subseteq N, \tag{3}$$

or equivalently

$$v(S \cup i) - v(S) \leq v(T \cup i) - v(T), \; \forall S \subseteq T \subseteq N, i \notin T. \tag{4}$$

In words, for a convex game, the larger the coalition an agent joins, the larger his marginal contribution to the coalition.

We focus on convex games because of the following well-known fact: If $(N; v)$ is a public good game, then $(N; v)$ is convex (see Moulin, [4]).

Now a natural question follows: Is any convex game a public good game? More precisely, given a convex game $(N; v)$, are there a list of utility functions $U_i, i = 1, ..., n$, and a cost function C, such that the game $(N; v)$ is generated by the public good economy (U, C), where $U = (U_1, ..., U_n)$?

The answers are Yes and No: Yes, certain convex games are public good games (Theorems 3.5 and 3.8). No, there are convex games that are not public good games as demonstrated below by Example 2.3.

For notational simplicity, hereafter we write, for example, $v(\{1, 2\})$ as $v(12)$.

Example 2.3. Sprumont [7] shows that the (convex) game defined below cannot be generated by a public good economy. Let $(N; v)$ be a four-agent game defined by:

$$v(\emptyset) = v(i) = v(13) = v(14) = v(23) = v(24) = 0, \; i \in N = \{1, 2, 3, 4\},$$

$$v(12) = v(34) = v(123) = v(124) = 3,$$

$$v(134) = v(234) = 4, \; v(1234) = 8.$$

To know which convex games are public good games, let $\preceq$ be a *preordering* on $2^N = \{S | S \subseteq N\}$, which is a complete and transitive binary relation. The preordering $\preceq$ is called *inclusion-compatible* if, for all $S, T \subseteq N$, $S \subseteq T$ implies $S \preceq T$. We now state Sprumont's characterization for public good games below. We refer reader to Sprumont [7] for the proof.

Lemma 2.4 (Sprumont, [7]). *A game $(N; v)$ is a public good game if and only if,* (*1*) *for $n \leq 3$, v is convex;*

(*2*) *for $n \geq 4$, there exist a vector $u = (u_i(S))_{i \in N, S \subseteq N}$ and an inclusion-compatible preordering $\preceq$ on 2^N satisfying the following system of inequalities:*

$$\sum_{i \in T} u_i(S) - \sum_{i \in S} u_i(S) \leq v(T) - v(S), \ \forall S, T \subseteq N,$$

$$u_i(S) \leq u_i(T), \ \forall i \in N, \forall S, T \subseteq N \text{ such that } S \preceq T. \tag{5}$$

In theory, Lemma 2.4 provides a complete answer to our central question. In practice, the necessary and sufficient conditions (5) in Lemma 2.4 are not so easy to use to answer whether or not a particular game is a public good game because the vector u and the preordering $\preceq$ in Lemma 2.4 are not easy to interpret (and construct). An analogue of Lemma 2.4 is the well-known Bondareva [1] theorem, which states a necessary and sufficient condition for a game to have nonempty core. The latter theorem has similar problem of how to interpret the so-called *balanced weights* (see below for detail). For the sake of comparison, recall the following statement of the Bondareva theorem (see Moulin, [4]):

(1) Given the set N of agents, a balanced family of coalitions is a subset $\mathcal{B}$ of 2^N such that there exists, for each S in $\mathcal{B}$, a weight $\delta_S, 0 \leq \delta_S \leq 1$, satisfying

$$\forall i \in N : \sum_{S \in \mathcal{B}_i} \delta_S = 1, \text{where } \mathcal{B}_i = \{S \in \mathcal{B} | i \in S\}.$$

A mapping δ satisfying the above equations is called a vector of balanced weights.

(2) Given a game $(N; v)$, a vector $x \in R^N$ is in the *core* (of the game) if we have

$$\sum_{i \in N} x_i = v(N), \text{ and for all } S \subseteq N: \ v(S) \leq \sum_{i \in S} x_i.$$

(3) A game $(N; v)$ is called *balanced* if for every vector of balanced weights δ, we have

$$\sum_{S \subset N} \delta_S \cdot v(S) \leq v(N).$$

(4) A game has a core if and only if it is balanced (Bondareva, [1]).

A game is called *totally balanced* if all of its subgames are balanced. By a subgame of $(N; v)$ we mean a game $(T; v)$, $T \subseteq N$ where v is the same function but restricted to the domain consisting of the subsets of T.

A market or an exchange economy is a list $(N, \{U_i\}_{i \in N}, \{\omega_i\}_{i \in N}, X)$ where N is the set of agents, $U_i (i \in N) : R_+^N \to R$ is agent i's utility function, $\omega_i \in R_+^N (i \in N)$ is agent i's initial endowment, and $X \subseteq R_+^N$ is the consumption set. An exchange economy generates a game $(N; v)$, called a *market game*, in the following way:

$$v(S) = \max_{X^S} \sum_{i \in S} U_i(x^i), \ S \subseteq N, \ v(\emptyset) = 0, \tag{6}$$

where

$$X^S = \{(x^i)_{i \in S} | x^i \in X, i \in S, \text{ and } \sum_{i \in S} x^i = \sum_{i \in S} \omega_i\}.$$

Proposition 2.5 (Shapley and Shubik, [6]). *A game is a market game if and only if it is totally balanced.*

This important result tells us precisely which games can arise from economic models of exchange.

Similarly, it is important to know which games can arise from models of public good economy. As we have known from the above, the set of public good games is a *strict* subset of convex games (and thus a strict subset of the set of totally balanced games since a convex game is totally balanced but the converse is not true). The purpose of this paper is to identify certain classes of convex games that are public good games.

For that purpose, we introduce the concept of subgradients for convex games in the next section.

3. Subgradients of Convex Games and Public Good Games

Let $(N;v)$ be a convex game.

Definition 3.1. A vector $x \in R^N$ is called a *subgradient* of $(N;v)$ at $S(\subseteq N)$ if

$$v(T) - v(S) \geq x(T) - x(S),\ \forall T \subseteq N \tag{7}$$

where $x(T) = \sum_{i \in T} x_i$.

Denote $\partial v(S)$ the set of all *subgradients* of $(N;v)$ at S and call it the subdifferential of $(N;v)$ at S.

Example 3.2. Let $N = \{1,2\}$. Define $(N;v)$ by letting $v(\emptyset) = 0$, $v(1) = 1$, $v(2) = 1.5$, $v(12) = 3$. Obviously, $(N;v)$ is convex. It is easy to check that

$$\partial v(\emptyset) \cap R^2_+ \neq \emptyset, \partial v(1) \cap \partial v(2) \neq \emptyset, \partial v(12) \neq \emptyset, \tag{8}$$

and moreover,

$$\forall x \in \partial v(\emptyset), y \in \partial v(1) \cap \partial v(2), z \in \partial v(12) \quad x \leq y \leq z. \tag{9}$$

(We denote $x \leq y$ if $x_i \leq y_i, \forall i \in N$.)

Lemma 3.3. *Consider a convex game $(N;v)$ with three agents $N = \{1,2,3\}$. We have*

$$\begin{aligned}
&\partial(0) \equiv \partial v(\emptyset) \cap R^3_+ \neq \emptyset, \\
&\partial(1) \equiv \partial v(1) \cap \partial v(2) \cap \partial v(3) \neq \emptyset, \\
&\partial(2) \equiv \partial v(12) \cap \partial v(13) \cap \partial v(23) \neq \emptyset, \\
&\partial(3) \equiv \partial v(123) \neq \emptyset.
\end{aligned}$$

Moreover,

$$\forall x \in \partial(0), y \in \partial(1), z \in \partial(2), w \in \partial(3) \quad x \leq y \leq z \leq w.$$

The proof of Lemma 3.3 can be found in the Appendix.

Let $(N;v)$, $N=\{1,2,3\}$, be a three-agent convex game. Define below an inclusion-compatible preordering $\preceq$ on 2^N:

$$S \preceq T \Longleftrightarrow |S| \leq |T| \tag{10}$$

where $|S|$ denotes the number of elements in the set S.

With this preordering we can define a vector u using the subgradients in Lemma 3.3. By definition, they satisfy the conditions in (5). Invoking Lemma 2.4 shows that the convex game $(N;v)$ is a public good game.

More concretely, for each $S \subseteq N$, if $|S|=i$, choose a vector $u(S) \in \partial(i)$ as defined in Lemma 3.3. Choose the preordering $\preceq$ defined by (10). It is apparent that these u and $\preceq$ satisfy the requirements (5) in Lemma 2.4. We thus have the following result.

Proposition 3.4 (Sprumont, [7]). *Given a game $(N;v)$, $N=\{1,...,n\}$. If $n \leq 3$ and $(N;v)$ is convex, then $(N;v)$ is a public good game.*

The formal proof is in the Appendix.

Now we state our main result of the paper.

Theorem 3.5. *If a convex game $(N;v)$ with four agents, $N=\{1,2,3,4\}$, satisfies the following conditions, then $(N;v)$ is a public good game:*

$$v(12)+v(34)=v(13)+v(24)=v(14)+v(23) \tag{11}$$

$$v(S\cup i)-v(S) \leq v(T\cup i)-v(T),\ \forall S,T \subseteq N\setminus i \text{ such that } \mid S \mid < \mid T \mid . \tag{12}$$

Proof. Let $(N;v)$ be a convex game where $N=\{1,2,3,4\}$.

1) It is easy to see that $\partial v(\emptyset) \cap R^4_+ \neq \emptyset$.

2) We show that $\partial(1) \equiv \cap_i \partial v(i) \neq \emptyset$. We only compute $\partial v(1)$. We leave it to the reader to calculate others. Note that $x \in \partial v(1)$ if and only if

$$v(T)-v(1) \geq x(T)-x_1,\ \forall T \subseteq N$$

i.e.,

$$\begin{aligned}
x_1 &\geq v(1)\\
v(2)-v(1) &\geq x_2-x_1\\
v(3)-v(1) &\geq x_3-x_1\\
v(4)-v(1) &\geq x_4-x_1\\
v(12)-v(1) &\geq x_2\\
v(13)-v(1) &\geq x_3\\
v(14)-v(1) &\geq x_4\\
v(23)-v(1) &\geq x_2+x_3-x_1\\
v(24)-v(1) &\geq x_2+x_4-x_1\\
v(34)-v(1) &\geq x_3+x_4-x_1\\
v(123)-v(1) &\geq x_2+x_3\\
v(124)-v(1) &\geq x_2+x_4
\end{aligned}$$

$$\begin{aligned} v(234) - v(1) &\geq x_2 + x_3 + x_4 - x_1 \\ v(1234) - v(1) &\geq x_2 + x_3 + x_4. \end{aligned}$$

Taking into account of other subdifferentials at other singleton coalition, we can summarize the above conditions for a subgradient $x \in \partial(1)$ as follows:

$$\begin{aligned} v(i) \leq x_i &\leq \min_{j \neq i}\{v(ij) - v(j)\},\ i = 1, ..., 4 \\ x_i - x_j &= v(i) - v(j),\ \forall i, j = 1, ..., 4 \\ x_i + x_j &\leq v(ijk) - v(k),\ \forall i \neq j \neq k \\ v(ij) - v(k) &\geq x_i + x_j - x_k,\ \forall i \neq j \neq k \\ x_i + x_j + x_k &\leq v(1234) - v(l),\ i \neq j \neq k \neq l \\ x_i + x_j + x_k - x_l &\leq v(ijk) - v(l). \end{aligned}$$

Choose $x_1 = v(1), x_2 = v(2), x_3 = v(3), x_4 = v(4)$. It is easy to see that $(x_1, x_2, x_3, x_4) \in \partial(1)$, and moreover $\forall x \in \partial(1), y \in \partial(0)$,

$$x \geq y.$$

3) Denote $\partial(2) \equiv \partial v(12) \cap \partial v(13) \cap \partial v(14) \cap \partial v(23) \cap \partial v(24) \cap \partial v(34)$. We now find sufficient conditions on v so that $\partial(2) \neq \emptyset$.

Compute $\partial v(12)$. By definition $x \in \partial v(12)$ if and only if

$$v(T) - v(12) \geq x(T) - x_1 - x_2, \forall T \subseteq N,$$

i.e.,

$$\begin{aligned} x_1 + x_2 &\geq v(12) \\ x_2 &\geq v(12) - v(1) \\ x_1 &\geq v(12) - v(2) \\ x_1 + x_2 - x_3 &\geq v(12) - v(3) \\ x_1 + x_2 - x_4 &\geq v(12) - v(4) \\ v(13) - v(12) &\geq x_3 - x_2 \\ v(14) - v(12) &\geq x_4 - x_2 \\ v(23) - v(12) &\geq x_3 - x_1 \\ v(24) - v(12) &\geq x_4 - x_1 \\ v(34) - v(12) &\geq x_3 + x_4 - x_1 - x_2 \\ v(123) - v(12) &\geq x_3 \\ v(124) - v(12) &\geq x_4 \\ v(134) - v(12) &\geq x_3 + x_4 - x_2 \\ v(234) - v(12) &\geq x_3 + x_4 - x_1 \\ v(1234) - v(12) &\geq x_3 + x_4. \end{aligned}$$

Thus $x \in \partial(2)$ if and only if the following inequalities hold:

$$\begin{aligned} \max\{v(S \cup i) - v(S) \mid S \subseteq N \setminus i, \mid S \mid = 1\} \leq x_i \quad \text{and} \\ x_i \leq \min\{v(S \cup i) - v(S) \mid S \subseteq N \setminus i, \mid S \mid = 2\},\ i = 1, ..., 4 \end{aligned}$$

$$\begin{aligned} v(ij) \le x_i + x_j \le v(ijkl) - v(kl),\ \forall i \ne j \ne k \ne l \\ x_i - x_j = v(ik) - v(jk),\ \forall i \ne j \ne k \\ v(ij) - v(k) \le x_i + x_j - x_k \le v(ijl) - v(lk),\ \forall i \ne j \ne k \ne l. \end{aligned}$$

Now we require that

$$v(12) + v(34) = v(13) + v(24) = v(14) + v(23)$$

$$v(S \cup i) - v(S) \le v(T \cup i) - v(T),\ \forall S, T \subseteq N \setminus i \text{ such that } \mid S \mid < \mid T \mid .$$

Let

$$t = \min\{v(123) - v(23), v(124) - v(24), v(134) - v(34)\},$$

and $x_1 = t$, then

$$\begin{aligned} x_2 &= t + v(23) - v(13) \\ x_3 &= t + v(23) - v(12) \\ x_4 &= t + v(24) - v(12). \end{aligned}$$

It is easy to check that they satisfy the above inequalities and therefore $(x_1, x_2, x_3, x_4) \in \partial(2)$.

4) $\partial(3) \equiv \partial v(123) \cap \partial v(124) \cap \partial v(134) \cap \partial v(234) \ne \emptyset$.

Consider $\partial v(123)$. By definition $x \in \partial v(123)$ if and only if

$$v(T) - v(123) \ge x(T) - x_1 - x_2 - x_3,\ \forall T \subseteq N$$

i.e.,

$$\begin{aligned} x_1 + x_2 + x_3 &\ge v(123) \\ x_2 + x_3 &\ge v(123) - v(1) \\ x_1 + x_3 &\ge v(123) - v(2) \\ x_1 + x_2 &\ge v(123) - v(3) \\ x_1 + x_2 + x_3 - x_4 &\ge v(123) - v(4) \\ x_3 &\ge v(123) - v(12) \\ x_2 &\ge v(123) - v(13) \\ x_1 &\ge v(123) - v(23) \\ x_2 + x_3 - x_4 &\ge v(123) - v(14) \\ x_1 + x_3 - x_4 &\ge v(123) - v(24) \\ x_1 + x_2 - x_4 &\ge v(123) - v(34) \\ v(124) - v(123) &\ge x_4 - x_3 \\ v(134) - v(123) &\ge x_4 - x_2 \\ v(234) - v(123) &\ge x_4 - x_1 \\ v(1234) - v(123) &\ge x_4. \end{aligned}$$

Therefore if $x \in \partial(3)$, we must have the following inequalities:

$$\begin{aligned}
\max\{v(S \cup i) - v(S) \mid S \subseteq N \setminus i, \mid S \mid = 2\} \le x_i \le v(1234) - v(1234 \setminus i),\ \forall i = 1, ..., 4 \\
x_i - x_j = v(ilk) - v(jlk),\ \forall i \neq j \neq k \neq l \\
x_i + x_j \geq v(ijk) - v(k),\ \forall i \neq j \neq k \\
x_i + x_j - x_k \geq v(ijl) - v(lk),\ \forall i \neq j \neq k \neq l \\
x_i + x_j + x_k \geq v(ijk),\ \forall i \neq j \neq k \\
x_i + x_j + x_k - x_l \geq v(ijk) - v(l),\ \forall i \neq j \neq k \neq l
\end{aligned}$$

Consider

$$\begin{aligned}
x_1 - x_2 = v(134) - v(234) \\
x_1 - x_3 = v(124) - v(234) \\
x_1 - x_4 = v(123) - v(234).
\end{aligned}$$

Let $x_1 = v(1234) - v(234)$, then

$$\begin{aligned}
x_2 = v(1234) - v(134) \\
x_3 = v(1234) - v(124) \\
x_4 = v(1234) - v(123).
\end{aligned}$$

It is easy to check that $(x_1, x_2, x_3, x_4) \in \partial(3)$. Moreover $\forall x \in \partial(3), y \in \partial(2)$

$$x \geq y.$$

5) $\partial(4) \equiv \partial v(1234) \neq \emptyset$.

$x \in \partial v(1234)$ if and only if

$$v(T) - v(1234) \geq x(T) - x_1 - x_2 - x_3 - x_4,\ \forall T \subseteq N.$$

Choose $x_i, i = 1, 2, 3, 4$ sufficiently large we see that $\partial(4) \neq \emptyset$.

Since $x_i \geq v(1234) - v(1234 \setminus i)$ we have $\forall x \in \partial(4), y \in \partial(3)$

$$x \geq y.$$

6) Now, we define an inclusion-compatible preordering $\preceq$ on 2^N by

$$S \preceq T \iff |S| \leq |T|,$$

and define a vector $u = (u_i(S))_{i \in N, S \subseteq N}$ using the subgradients of $\partial(|S|)$ as in Lemma 3,3. Invoking Lemma 2.4 proves the theorem. □

Note that the game in Example 2.3 fails to satisfy the sufficient condition in Theorem 3.5.

A game $(N; v)$ is *symmetric* if the value of any coalition S depends only on its size, $s = |S|$: there is a mapping $\tilde{v}$ defined on $\{0, 1, ..., n\}$ satisfying $v(S) = \tilde{v}(s)$ for each $S \subseteq N$ and $\tilde{v}(0) = 0$.

In the following, we provide an example of a four-agent public good game which is non-symmetric.

Example 3.6. Let $(N;v)$ be a four-agent game defined by:

$$\begin{aligned} v(\emptyset) &= v(i) = 0,\ i \in N = \{1,2,3,4\}, \\ v(12) &= v(23) = 1,\ v(13) = v(24) = 2,\ v(14) = v(34) = 3, \\ v(ijk) &= 6,\ \forall i \neq j \neq k, \\ v(1234) &= 11. \end{aligned}$$

It is easy to check that $(N;v)$ satisfies the sufficient condition in Theorem 3.5 and thus $(N;v)$ is a public good game.

At this point, it is appropriate to ask: "Are symmetric convex games public good games?" The answer is affirmative as Sprumont [7] has shown that every symmetric convex game is a public good game. In the following we provide an alternative proof of this important result by using subgradients.

Lemma 3.7. *Let $(N;v)$ be a symmetric game. Then $(N;v)$ is convex if and only if for any $s = 0,1,...,n$, $\tilde{v}'(s+1) \geq \tilde{v}'(s)$, where $\tilde{v}'(s) = \tilde{v}(s+1) - \tilde{v}(s)$.*

Proof. By definition.

Theorem 3.8 (Sprumont, [7]). *Let $(N;v)$ where $N = \{1,...,n\}$ and $n \geq 5$ be a symmetric convex game. Then, $(N;v)$ is a public good game.*

Proof. It is easy to see that $\partial v(\emptyset) \cap R^N_+ \neq \emptyset$.

For each given $s = 1,2,...,n$, we show that

$$\cap_{|S|=s} \partial v(S) \neq \emptyset.$$

By definition

$$x \in \partial v(S),\ S \subseteq N \Longleftrightarrow v(T) - v(S) \geq x(T) - x(S), \forall T \subseteq N.$$

We will show that there exists a $\overline{x} \in R$ such that $x = \overline{x} \cdot \mathbf{1} \in \partial v(S)$, where $\mathbf{1} = (1,...,1) \in R^N_+$. By symmetry,

$$\overline{x} \cdot \mathbf{1} \in \partial v(S)$$

if and only if

$$v(T) - v(S) \geq \overline{x}(t-s),\ \forall T \subseteq N$$

(where $t = |T|, s = |S|$) i.e.,

$$\frac{\tilde{v}(t) - \tilde{v}(s)}{t-s} \geq \overline{x} \ \text{ if } t > s$$

and

$$\frac{\tilde{v}(t) - \tilde{v}(s)}{t-s} \leq \overline{x} \ \text{ if } t < s.$$

Thus, it suffices to choose $\overline{x} = \tilde{v}'(s)$. By Lemma 3.7, we have

$$\overline{x} \cdot \mathbf{1} \in \cap_{|S|=s} \partial v(S),$$

and

$$\tilde{v}'(s) \cdot \mathbf{1} \leq \tilde{v}'(t) \cdot \mathbf{1}, \ s \leq t.$$

Define an inclusion-compatible preordering $\preceq$ on 2^N by

$$S \preceq T \Longleftrightarrow |S| \leq |T|.$$

Then it is easy to see that $u = (\tilde{v}'(s) \cdot \mathbf{1})_{S:|S|=s, s=0,1,\ldots,n}$ and $\preceq$ satisfy (5) in Lemma 2.4. The theorem is thus proved. □

4. Concluding Remarks

We have identified in Theorem 3.5 a class of convex games that are public good games in the four-agent case. Using the subgradient technique, we also have provided a simple proof of an important result in Sprumont [7] that all symmetric convex games are public good games in Theorem 3.8.

We must point out that the result in Theorem 3.5 is very restrictive in that it only deals with convex games with no more than four agents. We ask the following open question: Are all games meeting the conditions in (12) public good games?

This question is important in the theory of cooperative games. We have known in Proposition 2.5 that all totally balanced games are games generated by exchange economies (Shapley and Shubik, [6]). Similarly, we attempt to identify what games are generated by public good economies.

5. Appendix

Proof of Lemma 3.3 and Proposition 3.4. It only needs to show that for each $i = 0, 1, 2, 3$, $\partial v(i) \neq \emptyset$ and $x \in \partial v(i), y \in \partial v(j), i < j$ implies $x \leq y$.

1)

$$\partial v(\emptyset) = \{x \in R^3 \mid x(T) \leq v(T), \forall T\} \neq \emptyset$$

is obvious since $(N; v)$ is convex. Further $\partial v(\emptyset) \cap R^3_+ \neq \emptyset$ is true because $(N; v)$ is nonnegative and convex (choosing $x(i) = v(i), i = 1, \ldots, n$ suffices).

2) $\partial(1) \neq \emptyset$.

Compute $\partial v(1)$ first. By definition $x \in \partial v(1)$ if and only if

$$v(T) - v(1) \geq x(T) - x_1, \ \forall T \subseteq N.$$

It consists of the following inequalities:

$$\begin{aligned}
x_1 &\geq v(1) \\
v(12) - v(1) &\geq x_2 \\
v(2) - v(1) &\geq x_2 - x_1 \\
v(3) - v(1) &\geq x_3 - x_1 \\
v(13) - v(1) &\geq x_3 \\
v(23) - v(1) &\geq x_2 + x_3 - x_1 \\
v(123) - v(1) &\geq x_2 + x_3.
\end{aligned}$$

We can similarly compute $\partial v(2)$ and $\partial v(3)$. In summary, if $x \in \partial v(1) \cap \partial v(2) \cap \partial v(3)$ then x must satisfy the following inequalities:

$$\begin{aligned}
v(1) \le x_1 &\le \min\{v(12) - v(2), v(13) - v(3)\} \\
v(2) \le x_2 &\le \min\{v(12) - v(1), v(23) - v(3)\} \\
v(3) \le x_3 &\le \min\{v(13) - v(1), v(23) - v(2)\} \\
v(2) - v(1) &= x_2 - x_1 \\
v(3) - v(1) &= x_3 - x_1 \\
v(3) - v(2) &= x_3 - x_2 \\
v(123) - v(1) &\ge x_2 + x_3 \\
v(123) - v(2) &\ge x_1 + x_3 \\
v(123) - v(3) &\ge x_1 + x_2 \\
v(23) - v(1) &\ge x_2 + x_3 - x_1 \\
v(13) - v(2) &\ge x_1 + x_3 - x_2 \\
v(12) - v(3) &\ge x_1 + x_2 - x_3.
\end{aligned}$$

Choose $x_1 = v(1), x_2 = v(2), x_3 = v(3)$. Then it is easy to check that (x_1, x_2, x_3) satisfies the above inequalities and thus $(x_1, x_2, x_3) \in \partial(1)$. Moreover, we have

$$\forall x \in \partial(0), y \in \partial(1) \quad x \le y.$$

3) $\partial(2) \neq \emptyset$.

First compute $\partial v(12)$. By definition $x \in \partial v(12)$ if and only if

$$v(T) - v(12) \ge x(T) - x_1 - x_2, \ \forall T \subseteq N,$$

i.e.,

$$\begin{aligned}
x_1 + x_2 &\ge v(12) \\
x_2 &\ge v(12) - v(1) \\
x_1 &\ge v(12) - v(2) \\
x_1 + x_2 - x_3 &\ge v(12) - v(3) \\
v(13) - v(12) &\ge x_3 - x_2 \\
v(23) - v(12) &\ge x_3 - x_1 \\
v(123) - v(12) &\ge x_3.
\end{aligned}$$

In the same way we can compute $\partial v(13)$, $\partial v(23)$ and we see their common solutions must satisfy the following inequalities:

$$\begin{aligned}
\max\{v(12) - v(2), v(13) - v(3)\} &\le x_1 \le v(123) - v(23) \\
\max\{v(23) - v(3), v(12) - v(1)\} &\le x_2 \le v(123) - v(13)
\end{aligned}$$

$$\begin{aligned}
\max\{v(23)-v(2), v(13)-v(1)\} \le x_3 &\le v(123)-v(12)\\
x_1 - x_2 &= v(13)-v(23)\\
x_3 - x_2 &= v(13)-v(12)\\
v(12) &\le x_1+x_2\\
v(13) &\le x_1+x_3\\
v(23) &\le x_2+x_3\\
v(12)-v(3) &\le x_1+x_2-x_3\\
v(13)-v(2) &\le x_1-x_2+x_3\\
v(23)-v(1) &\le -x_1+x_2+x_3.
\end{aligned}$$

These inequalities have the following solution if we choose $x_1 = v(123)-v(23)$, $x_2 = v(123)-v(13)$, $x_3 = v(123)-v(12)$. We also see that from the first group of inequalities $\forall x \in \partial(1), y \in \partial(2)$

$$x \le y.$$

4) $\partial(3) \neq \emptyset$.

By definition $x \in \partial v(123)$ if and only if

$$v(T) - v(123) \ge x(T) - x_1 - x_2 - x_3,\ \forall T \subseteq N.$$

They are

$$\begin{aligned}
x_1+x_2+x_3 &\ge v(123)\\
x_2+x_3 &\ge v(123)-v(1)\\
x_1+x_3 &\ge v(123)-v(2)\\
x_1+x_2 &\ge v(123)-v(3)\\
x_3 &\ge v(123)-v(12)\\
x_2 &\ge v(123)-v(13)\\
x_1 &\ge v(123)-v(23).
\end{aligned}$$

That $\partial(3)$ is nonempty is obvious since we can choose x_1, x_2, x_3 sufficiently large. From the last three inequalities we see that

$$\forall x \in \partial(2), y \in \partial(3) \quad x \le y.$$

5) Finally, consider the preordering $\preceq$ defined by (10) in Section 3. It is inclusion-compatible. Then choose a vector $u(S) \in \partial(|S|)$ for each $S \subseteq N$. It is easy to check that they satisfy the conditions in Lemma 2.4. This completes our proof. □

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References

[1] O. N. Bondareva: Some applications of linear programming methods to the theory of cooperative games, Probl. Kibern. 10 (1963) 119–139 (in Russian).

[2] D. K. Foley: Lindahl's solution and the core of an economy with public goods, Econometrica 38 (1970) 66-72.

[3] S. Fujishige: Submodular systems and related topics, Math. Program. Study 22 (1984) 113–131.

[4] H. Moulin: Cooperative Microeconomics: A Game-Theoretic Introduction, Princeton University Press, Princeton (1995).

[5] R. T. Rockafellar: Convex Analysis, Princeton University Press, Princeton (1970).

[6] L. Shapley, M. Shubik: On market games, J. Econ. Theory 1 (1969) 9–25.

[7] Y. Sprumont: On the game-theoretic structure of public-good economies, Int. J. Game Theory 133 (1997) 1–18.