Journal of Convex Analysis
Volume 14 (2007), No. 2, 413–432

On Moreau-Yosida Approximation and on Stability of Second-Order Subdifferentials of Clarke's Type*

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Received: September 27, 2004
Revised manuscript received: September 9, 2005

For the Gateaux derivative of a $C^{1,1}$ function defined on a reflexive Banach space with Kadec norm $||\cdot||$, we use Moreau-Yosida regularization to show that Clarke's subdifferential of second-order can be weak$^*-$approximated from below. Moreover, in the convex case we can strengthen the inclusion to an equality in the limit.

In another approach for $C^{1,1}$ functions, we establish a weak* stability result for second-order subdifferentials of Clarke's type. We apply the latter result to the continuous behaviour of the Lagrange multipliers in second-order necessary optimality conditions under epi-convergent perturbations and to stability of second-order subdifferentials of Clarke's type of integral functionals and also of the standard type of functionals in calculus of variations.

Since in optimization one frequently has to consider perturbed problems, stability theory is a very important topic [6]. One of the first investigations in this area is due to Mosco. In [13] he uses his concept of set convergence for the analysis of stability of variational inequalities. More recently this stability theory has been extended to a broader class of variational inequalities in [10]. There various applications, including distributed market equilibria and elliptic unilateral boundary value problems are treated. Further in [14] Salvadori has shown that Mosco convergence is inherited to integral functionals from the associated convex normal integrands.

In [16], Zolezzi is concerned with the weak$^*-$ upper convergence of the graphs of Clarke's generalized gradients. We are motivated by this paper to investigate the weak$^*-$ upper convergence of the second-order subdifferentials of Clarke's type of $C^{1,1}$ functions introduced by P. Georgiev and N. Zlateva in [8].

In Section 3, we consider a sequence f_n of $C^{1,1}$ functions defined on a reflexive Banach space X endowed with Kadec norm, which Gateaux derivatives for fixed element $h \in X$, $\langle f_n'(x), h\rangle$, are epi-convergent to $\langle f'(x), h\rangle$ and possess an uniform minorization property. We apply Moreau-Yosida regularization to the functions $\langle f_n'(x), h\rangle$ and under these conditions we obtain Theorem 3.1, which provides a weak$^*-$approximation of the Clarke's

*This research is supported by DFG (German Research Organization) under grant GW 5/2-1.

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

subdifferential $\partial\langle f'(x), h\rangle$ from below. To prove this result we use essentially Theorem 2.1 (Section 2). This latter result summarizes some results obtained by Correa, Jofré and Thibault in [4]. Also in our analysis we use some approximation results due to Attouch and Azé [2] and some generalized differentiability results due to Borwein and Strojwas [3]. In the convex case we can strengthen the inclusion of Theorem 3.1 to an equality by regularization and monotonicity arguments from Attouch [1].

Another approach of this paper (Section 4) introduces conditions on the functions f_n, f that guarantee weak$^*-$upper convergence of the second-order subdifferentials of Clarke's type. In comparison to the work of T. Zolezzi [16], we consider a sequence f_n, f of $C^{1,1}$ functions defined on an arbitrary Banach space, which Gateaux derivatives for fixed h, $\langle f_n'(x), h\rangle$, are locally convergent with respect to the Lipschitz seminorm to $\langle f'(x), h\rangle$. For such functions we extend the main result of Zolezzi [16] to the second-order subdifferentials of Clarke's type.

Further we apply this result not only to the continuous behaviour of the Lagrange multipliers in the second-order necessary optimality conditions but also to the relevant second-order subdifferentials of Clarke's type under epi-convergent perturbations (Section 5). Here, we deal with the second-order necessary optimality conditions for constrained minimization problems with $C^{1,1}$ data obtained by P. Georgiev and N. Zlateva in [8]. As another application we establish the convergence of the second-order subdifferentials of Clarke's type of integral functionals and we show how this result can be extended to the standard type of functionals in calculus of variations (Section 6).

1. Notations and definitions

Let E be a real Banach space with dual E^*. E is said to have a Kadec norm if for any sequence x_n which converges weakly to x (denoted by $x_n \overset{\omega}{\to} x$) with $||x_n|| \to ||x||$, then x_n converges strongly to x (denoted by $x_n \to x$). Analogously, $\omega^* - \lim$ denotes the limit with respect to the weak* topology. Further $\overline{co}^* A$ is the convex topological closure of a set $A \subset E^*$ with respect to the weak* topology.

Throughout the paper we use the definition of epi-convergence: A sequence f_n is called epi-convergent to f (writting $f_n \overset{\text{epi}}{\to} f$) iff $x_n \overset{\omega}{\to} x$ in E implies $\liminf f_n(x_n) \geq f(x)$ and for every $y \in E$ there exists a sequence $y_n \to y$ such that $\limsup f_n(y_n) \leq f(y)$.

We shall denote the value of a linear functional $x^* \in E$ for an element $h \in E$ either by $x^*[h]$ or by $\langle x^*, h\rangle$ and the value of a bilinear functional $L : E \times E \to \mathbb{R}$ for a pair of elements $h_1, h_2 \in E$ by $L[h_1, h_2]$.

Let $\mathcal{L}(E \times E)$ be the Banach space of all bilinear continuous functionals $L : E \times E \to \mathbb{R}$ with the norm

$$||L|| = \sup\left\{|L[h_1, h_2]| \,:\, \|h_1\| = 1\,, \|h_2\| = 1\right\}.$$

In the sequel we shall use the duality map $H : E \to 2^{E^*}$ defined by

$$H(x) = \{x^* \in E^* \,:\, ||x^*|| = ||x|| \quad \text{and} \quad \langle x^*, x\rangle = ||x||^2\}$$

for each $x \in E$. It is known that $H(x)$ is nonempty for every $x \in E$ and also it is the subdifferential of $\phi(x) = \frac{1}{2}||x||_X^2$. If E^* is strictly convex, $H(\cdot)$ is single valued. From the renorming theorem of Troyanski ([15], [11]) every reflexive Banach space can be renormed by an equivalent norm such that both the space and its dual are locally uniformly convex and since locally uniformly convex implies strictly convex, we can assume in the sequel that $H(\cdot)$ is a single valued map.

Let G be an open subset of E. Consider the class $C^{1,1}(G)$ of all functions $f : G \to \mathbb{R}$, whose first Gateaux derivatives $f' : x \in G \to f'(x) \in E^*$ is continuous and locally Lipschitz. We also have that f' is Gateaux differentiable on a dense subset $G(f)$ of G (see [8]), hence f has a twice Gateaux derivative $f''(x)$ at $x \in G(f)$.

For such functions $f \in C^{1,1}(G)$ we consider the Clarke subdifferential $\partial\langle f'(x), h\rangle \subset E^*$ of the function $\langle f'(\cdot), h\rangle$ with respect to $x \in G$ and any fixed $h \in E$. Further we recall the following definition.

Definition 1.1. For every $x \in G$, $h_1, h_2 \in E$ we define

$$\begin{aligned} f^{00}(x; h_1, h_2) &:= \limsup_{\substack{y\to x\\ t\downarrow 0}} \frac{f'(y + th_1)[h_2] - f'(y)[h_2]}{t} \\ &= \langle f'(\cdot), h_2\rangle^0(x; h_1), \end{aligned}$$

$$\partial_c^2 f(x) := \{L \in \mathcal{L}(E \times E) : L[h_1, h_2] \le f^{00}(x; h_1, h_2),\ \forall (h_1, h_2) \in E \times E\}. \tag{1}$$

This gives rise to the image set

$$\partial_c^2 f(x)h = \{\langle Lh, \cdot\rangle \in E^* : L \in \partial_c^2 f(x)\}. \tag{2}$$

and to the multivalued map $h \in E \to \partial_c^2 f(x)h \subset E^*$. In [8] it has been proven that in a real Banach space with separable dual, for every $x \in E$, the set $\partial_c^2 f(x)$ as given by (1) is nonempty convex and ω^*- compact. Also by [8], the multivalued mapping $\partial_c^2 f$ is upper semicontinuous on G and locally norm bounded in $\mathcal{L}(E \times E)$. In [7] all these properties of $\partial_c^2 f(x)$ are extended to arbitrary Banach spaces. In ([7], Theorem 3), it has been shown that if E is an Asplund space, then for every $h_1, h_2 \in E$ and $x \in G$

$$f^{00}(x, h_1, h_2) = \max_{L\in\partial_c^2 f(x)} L[h_1, h_2]. \tag{3}$$

2. Some properties of Moreau-Yosida approximations

In what follows X is a reflexive Banach space. We assume that X is endowed with a Kadec norm such that X and X^* are locally uniformly convex.

Let $f : X \to \overline{\mathbb{R}} = \mathbb{R}\cup\{+\infty\}$ be a proper function, i.e. $dom\, f = \{x \in X : f(x) < \infty\} \ne \emptyset$. Recall that the Frechet subdifferential of f at $x \in dom\, f$ is the set

$$\partial^F f(x) := \{x^* \in X^* : \liminf_{||h||\to 0} \frac{f(x + h) - f(x) - \langle x^*, h\rangle}{||h||} \ge 0\}.$$

Note that due to ([3], Theorem 6.2) any lower semicontinuous function on a reflexive Banach space is densely Frechet subdifferentiable on its domain.

It is easy to see that for a locally Lipschitz function f, we have $\partial^F f(x) \subset \partial f(x)$ for every x, where ∂f stands for the Clarke subdifferential of f.

For any proper $f : X \to \overline{\mathbb{R}}$, the Moreau-Yosida regularization of the index $\lambda > 0$ of f is defined by

$$f_\lambda(x) := \inf_{y \in X} \{f(y) + \frac{1}{2\lambda} ||x - y||^2\}.$$

Assume that f satisfies the growth condition, i.e., there exists a nonnegative constant c such that for all $x \in X$

$$f(x) \geq -\frac{c}{2}(1 + ||x||^2). \tag{4}$$

Then (see [2], Proposition 1.1), provided $\lambda \in (0, \frac{1}{c})$ the function f_λ is a finitely valued function which is Lipschitz continuous on each bounded subset of X. Moreover, the proof of ([2], Proposition 1.2 a)) shows that for $x \in dom f$, for each $\lambda \in (0, \frac{1}{c})$ and for each $\rho > \overline{\rho}$, where

$$\overline{\rho} = \overline{\rho}(x, f, \lambda, c) := \big[\lambda \frac{2f(x) + c(2||x||^2 + 1)}{1 - 2\lambda c}\big]^{1/2}$$

we have

$$f_\lambda(x) = \inf_{||x-y|| \leq \rho} \{f(y) + \frac{1}{2\lambda} ||x - y||^2\}.$$

We now summarize some of the results obtained by Correa, Jofré and Thibault in [4] in the following theorem.

Theorem 2.1. *Let $f : X \to \overline{\mathbb{R}}$ be a lower semicontinuous, proper function that satisfies the growth condition (4). Consider its Moreau-Yosida regularization f_λ for fixed $\lambda \in (0, \frac{1}{2c})$. Then there holds:*

(*A*) *For each point $x_0 \in X$ from the dense subset where f_λ is Frechet subdifferentiable, i.e., $\partial^F f_\lambda(x_0) \neq \emptyset$ there exists a $x_\lambda \in X$ such that*

$$f_\lambda(x_0) = f(x_\lambda) + \frac{1}{2\lambda} ||x_0 - x_\lambda||^2.$$

Moreover, for any $x_\lambda \in argmin_{y \in X} \{f(y) + \frac{1}{2\lambda} ||x_0 - y||^2\}$ there holds:

(*B*) $\partial^F f_\lambda(x_0) = \{\frac{1}{\lambda} H(x_0 - x_\lambda)\}$.

(*C*) $\partial^F f_\lambda(x_0) \subset \partial^F f(x_\lambda)$.

Proof. For the proof of (*A*) see the proof of Lemma 3.7 in [4]. Parts (*B*) and (*C*) follow immediately from

$$\partial^F \frac{1}{2\lambda} || \cdot ||^2 (x_0 - x_\lambda) = \{\frac{1}{\lambda} H(x_0 - x_\lambda)\}$$

and from ([4], Lemma 3.6) which gives

$$\partial^F f_\lambda(x_0) \subset \partial^F f(x_\lambda) \cap \partial^F \frac{1}{2\lambda} || \cdot ||^2 (x_0 - x_\lambda).$$

□

3. An approach by Moreau-Yosida approximation

Let $h \in X$ be fixed. We consider the Moreau-Yosida approximations $\langle f_n'(\cdot), h\rangle_\lambda$, $\langle f'(\cdot), h\rangle_\lambda$ of the functions $\langle f_n'(\cdot), h\rangle$, $\langle f'(\cdot), h\rangle$ respectively.

Theorem 3.1. *Let X be a reflexive Banach space with Kadec norm. Given $f : X \to \mathbb{R}$ and the sequence, $f_n : X \to \mathbb{R}$, $(n \in \mathbb{N})$, we posit the following assumptions:*

(A1) f and every f_n is $C^{1,1}$ on X;
(A2) For every $h \in X$ there exists $r > 0$ such that for every $n \in \mathbb{N}$ and $x \in X$

$$\langle f_n'(x), h\rangle \geq -r\,(1 + ||x||^2);$$

(A3) For every $h \in X$,

$$\langle f_n'(\cdot), h\rangle \overset{\text{epi}}{\to} \langle f'(\cdot), h\rangle.$$

Then for every $h \in X$ and $x \in X$,

$$\overline{co}^*\,\omega^* - \limsup_{\substack{z\to x\\ \lambda\to 0^+}} [\overline{co}^*\{\omega^* - \limsup_{\substack{u\to z\\ n\to\infty}} \partial\langle f_n'(u), h\rangle_\lambda\}] \subseteq \partial\langle f'(x), h\rangle.$$

Proof. We need several lemmas, as follows. In each lemma, the assumptions of Theorem 3.1 are in force.

Let G_λ and G_λ^n be dense subsets, where $\langle f'(\cdot), h\rangle_\lambda$ and $\langle f_n'(\cdot), h\rangle_\lambda$ are Frechet subdifferentiable.

Since $f \in C^{1,1}$ there exists a bounded neighbourhood $V(x)$ of x, on which $\langle f'(\cdot), h\rangle$ is Lipschitz and bounded by the Lipschitz constant $L_{f'}$.

Lemma 3.2. *For every $\lambda \in (0, \frac{1}{4r})$, $y \in G_\lambda$, $h \in X$ and for every $y_n \in G_\lambda^n$ such that $y_n \to y$, it follows that*

$$\exists \tilde{y}_\lambda^n \in argmin_{z\in X}\{\langle f_n'(z), h\rangle + \frac{1}{2\lambda}||z - y_n||^2\}$$

and

$$\exists \hat{y} \in argmin_{z\in X}\{\langle f'(z), h\rangle + \frac{1}{2\lambda}||z - y||^2\}$$

such that

(a) $\tilde{y}_\lambda^n \to \hat{y}$;
(b) $H(y_n - \tilde{y}_\lambda^n) \to H(y - \hat{y})$; and
(c) $\frac{1}{\lambda}H(y - \hat{y}) \in \partial^F\langle f'(y), h\rangle_\lambda$.

Proof. (a) Since $y_n \in G_\lambda^n$, by Theorem 2.1(A) there exists a $\tilde{y}_\lambda^n \in X$ such that

$$\langle f_n'(y_n), h\rangle_\lambda = \langle f_n'(\tilde{y}_\lambda^n), h\rangle + \frac{1}{2\lambda}||\tilde{y}_\lambda^n - y_n||^2 \tag{5}$$

and by (B) and (C)

$$\frac{1}{\lambda}H(y_n - \tilde{y}_\lambda^n) \in \partial^F\langle f_n'(y_n), h\rangle_\lambda \subset \partial^F\langle f_n'(\tilde{y}_\lambda^n), h\rangle \subset \partial\langle f_n'(\tilde{y}_\lambda^n), h\rangle.$$

Let us take some $v \in V(x)$. By assumption $(A3)$, $\exists v_n \to v$ such that $\limsup \langle f_n'(v_n), h\rangle \leq \langle f'(v), h\rangle$ and from (5),

$$\langle f_n'(\tilde{y}_\lambda^n), h\rangle + \frac{1}{2\lambda}||\tilde{y}_\lambda^n - y_n||^2 \leq \langle f_n'(v_n), h\rangle + \frac{1}{2\lambda}||v_n - y_n||^2. \tag{6}$$

Combining $(A2)$ and (6), we calculate

$$\begin{aligned} -r(1+||\tilde{y}_\lambda^n||^2) + \frac{1}{2\lambda}||\tilde{y}_\lambda^n - y_n||^2 &\leq \langle f_n'(\tilde{y}_\lambda^n), h\rangle + \frac{1}{2\lambda}||\tilde{y}_\lambda^n - y_n||^2 \\ &\leq \langle f_n'(v_n), h\rangle + \frac{1}{2\lambda}||v_n - y_n||^2. \end{aligned}$$

Consequently,

$$\frac{1}{2\lambda}||\tilde{y}_\lambda^n - y_n||^2 \leq \langle f_n'(v_n), h\rangle + \frac{1}{2\lambda}||v_n - y_n||^2 + r(1+||\tilde{y}_\lambda^n||^2),$$

which clearly implies that the sequence $\tilde{y}_\lambda^n$ is bounded since $\lambda \in (0, \frac{1}{4r})$. The space X being reflexive, we can extract a weakly convergent subsequence $\tilde{y}_\lambda^n \xrightarrow{\omega} \hat{y}$.

Let us take an arbitrary $z \in X$. By $(A3)$, there exists a sequence $z_n \to z$ such that $\limsup \langle f_n'(z_n), h\rangle \leq \langle f'(z), h\rangle$. From the inequality

$$\langle f_n'(\tilde{y}_\lambda^n), h\rangle + \frac{1}{2\lambda}||\tilde{y}_\lambda^n - y_n||^2 \leq \langle f_n'(z_n), h\rangle + \frac{1}{2\lambda}||z_n - y_n||^2$$

we get

$$\liminf \langle f_n'(\tilde{y}_\lambda^n), h\rangle + \liminf \frac{1}{2\lambda}||\tilde{y}_\lambda^n - y_n||^2 \leq \limsup \langle f_n'(z_n), h\rangle + \frac{1}{2\lambda}||z - y||^2.$$

Using $(A3)$ and the lower semicontinuity of the norm for the weak topology, it follows for every $z \in X$,

$$\langle f'(\hat{y}), h\rangle + \frac{1}{2\lambda}||\hat{y} - y||^2 \leq \langle f'(z), h\rangle + \frac{1}{2\lambda}||z - y||^2,$$

that

$$\hat{y} \in \operatorname{argmin}_{z\in X}\{\langle f'(z), h\rangle + \frac{1}{2\lambda}||z - y||^2\}.$$

In order to obtain the strong convergence of the sequence $\tilde{y}_\lambda^n$, we prove $||\tilde{y}_\lambda^n|| \to ||\hat{y}||$. By $(A3)$, there exists a sequence $\hat{y}_n \to \hat{y}$ such that

$$\limsup \langle f_n'(\hat{y}_n), h\rangle \leq \langle f'(\hat{y}), h\rangle.$$

Hence,

$$\langle f_n'(\tilde{y}_\lambda^n), h\rangle + \frac{1}{2\lambda}||\tilde{y}_\lambda^n - y_n||^2 \leq \langle f_n'(\hat{y}_n), h\rangle + \frac{1}{2\lambda}||\hat{y}_n - y_n||^2,$$

letting $n \to \infty$ leads to

$$\begin{aligned} \limsup \frac{1}{2\lambda}||\tilde{y}_\lambda^n - y_n||^2 &\leq \limsup\,(-\langle f_n'(\tilde{y}_\lambda^n), h\rangle) + \limsup \langle f_n'(\hat{y}_n), h\rangle \\ &\quad + \lim \frac{1}{2\lambda}||\hat{y}_n - y_n||^2 \\ &\leq -\liminf \langle f_n'(\tilde{y}_\lambda^n), h\rangle + \langle f'(\hat{y}), h\rangle + \frac{1}{2\lambda}||\hat{y} - y||^2 \\ &\leq -\langle f'(\hat{y}), h\rangle + \langle f'(\hat{y}), h\rangle + \frac{1}{2\lambda}||\hat{y} - y||^2. \end{aligned}$$

We finally arrive at

$$\limsup ||\tilde{y}^n_\lambda - y_n||^2 \le ||\hat{y} - y||^2$$

and at the strong convergence of the sequence $\tilde{y}^n_\lambda$ to $\hat{y}$. Thus the proof of part (a) is completed.

Due to the Kadec assumption, H is continuous from X to X^* with respect to the strong topology. Therefore part (a) implies part (b).

Since $u^* \in \operatorname{argmin}_{z\in X}\{\langle f'(z), h\rangle + \frac{1}{2\lambda}||z-y||^2\}$, part (c) follows from (C) of Theorem 2.1. $\square$

To prove Lemma 3.4, we need the following lemma ([3], Corollary 7.4) due to Borwein and Strojwas.

Lemma 3.3. *Let f be a locally Lipschitz function on a reflexive Banach space X. Then there holds for every $y \in X$,*

$$\partial f(y) = \overline{co}^*\{\omega^* - \limsup_{z\to y} \partial^F f(z)\}.$$

Lemma 3.4. *For every $\lambda > 0$, $h \in X$ and $x \in X$,*

$$\overline{co}^*\{\omega^* - \limsup_{\substack{y\to x\\ n\to+\infty}} \partial\langle f'_n(y), h\rangle_\lambda\} \subset \partial\langle f'(x), h\rangle_\lambda.$$

Proof. Let $\lambda > 0$, $x \in X$, $h \in X$ be fixed. Since $\partial\langle f'(x), h\rangle_\lambda$ is a convex and ω^*-closed subset of X^*, it suffices to prove that

$$\omega^* - \limsup_{\substack{y\to x\\ n\to+\infty}} \partial\langle f'_n(y), h\rangle_\lambda \subset \partial\langle f'(x), h\rangle_\lambda. \tag{7}$$

Let L belong to the left hand set of (7) i.e. $L = \omega^* - \lim L^*_n$, where $L^*_n \in \partial\langle f'_n(y_n), h\rangle_\lambda$, $y_n \to x$.

By Lemma 3.3,

$$L^*_n \in \overline{\mathrm{co}}^*\{\omega^* - \limsup_{G^n_\lambda \ni z\to y_n} \partial^F\langle f'_n(z), h\rangle_\lambda\}. \tag{8}$$

By Theorem 2.1, for every $n \in \mathbb{N}$ and $z \in G^n_\lambda$ there exist $\tilde{y}^n_\lambda(z) \in argmin_{u\in X}\{\langle f'_n(u), h\rangle + \frac{1}{2\lambda}||u-z||^2\}$ and there holds $\partial^F\langle f'_n(z), h\rangle_\lambda = \{\frac{1}{\lambda}H(z - \tilde{y}^n_\lambda(z))\}$. Consequently, the set $\omega^* - \limsup_{G^n_\lambda \ni z\to y_n} \partial^F\langle f'_n(z), h\rangle_\lambda$ coincides with the following set

$$A_n(y_n) := \omega^* - \limsup_{G^n_\lambda \ni z\to y_n} \{\frac{1}{\lambda}H(z - \tilde{y}^n_\lambda(z))\}.$$

Denote

$$\phi(h_1) = \langle f'(\cdot), h\rangle^0_\lambda(x; h_1).$$

Our aim is to prove that for every $L_n \in A_n(y_n)$ and every $h_1 \in X$ we have

$$\limsup_{n\to\infty} \langle L_n, h_1\rangle \le \phi(h_1). \tag{9}$$

First, we shall show that if (9) holds, then $L \in \partial\langle f'(x), h\rangle_\lambda$, which proves (7). For this purpose we establish that (9) implies the inequality

$$\limsup_{n\to\infty} \sup\{\langle L_n, h_1\rangle \,:\, L_n \in A_n(y_n)\} \le \phi(h_1). \tag{10}$$

Indeed, for every $n \in N$ there exists $\tilde{L}_n \in A_n(y_n)$ such that

$$\langle \tilde{L}_n, h_1\rangle \ge \sup\{\langle L_n, h_1\rangle \,:\, L_n \in A_n(y_n)\} - \frac{1}{n}.$$

Since (9) is assumed, we get

$$\limsup_{n\to\infty} \sup\{\langle L_n, h_1\rangle \,:\, L_n \in A_n(y_n)\} \le \limsup_{n\to\infty} \langle \tilde{L}_n, h_1\rangle + \lim \frac{1}{n} \le \phi(h_1).$$

From (10), for every $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for every $n \ge N_0$ there holds

$$\langle L_n, h_1\rangle \le \phi(h_1) + \varepsilon \quad \text{for every } L_n \in A_n(y_n).$$

Since ϕ is positively homogenous and subadditive on X, the last inequality is also true for every $L_n \in \operatorname{co} A_n(y_n)$. Let $L_n^* \in \overline{\operatorname{co}}^* \{A_n(y_n)\}$. Then for every $\varepsilon > 0$ there exists $L_n \in \operatorname{co} A_n(y_n)$ such that

$$|\langle L_n^* - L_n, h_1\rangle| < \varepsilon.$$

Hence

$$\langle L_n^*, h_1\rangle < \langle L_n, h_1\rangle + \varepsilon \le \phi(h_1) + 2\varepsilon.$$

Consequently

$$\limsup_{n\to\infty} \sup\{\langle L_n^*, h_1\rangle \,:\, L_n^* \in \overline{\operatorname{co}}^* A_n(y_n)\} \le \phi(h_1),$$

which implies that $\langle L, h_1\rangle \le \phi(h_1)$, i.e. $L \in \partial\langle f'(x), h\rangle_\lambda$.

Thus we need only to prove inequality (9). Let $L_n \in A_n(y_n)$. Then $L_n = \omega^* - \lim_{m\to\infty} \frac{1}{\lambda} H(z_{nm} - \tilde{y}^n_\lambda(z_{nm}))$, where for all $m \in \mathbb{N}$ we have $z_{nm} \in G^n_\lambda$ and $z_{nm} \to y_n$.

Hence, for every $\varepsilon > 0$ and $n \in \mathbb{N}$, there exists a $m(n) \in \mathbb{N}$ such that for all n and $z_n := z_{nm(n)}$

$$|\langle L_n, h_1\rangle - \langle \frac{1}{\lambda} H(z_n - \tilde{y}^n_\lambda(z_n)), h_1\rangle| < \varepsilon, \tag{11}$$

where

$$z_n \xrightarrow[n\to\infty]{} x,$$

$$\tilde{y}^n_\lambda(z_n) \xrightarrow[n\to\infty]{} \hat{y} \quad \text{(by Lemma 3.2}(a)),$$

$$\hat{y} \in \operatorname{argmin}_{z\in X}\{\langle f'(z), h\rangle + \frac{1}{2\lambda}||x - z||^2\},$$

and

$$H(z_n - \tilde{y}^n_\lambda(z_n)) \to H(x - \hat{y}) \quad \text{(by Lemma 3.2}(b)).$$

The latter strong convergence gives a constant $N(\varepsilon) > 0$ such that for every $n > N(\varepsilon)$

$$|\langle \frac{1}{\lambda} H(z_n - \tilde{y}^n_\lambda(z_n)), h_1\rangle - \langle \frac{1}{\lambda} H(x - \hat{y}), h_1\rangle| < \varepsilon. \tag{12}$$

By combining inequalities (11) and (12), for every $n > N(\varepsilon)$ we obtain

$$\langle L_n, h_1\rangle < \langle \frac{1}{\lambda} H(x - \hat{y}), h_1\rangle + 2\varepsilon. \tag{13}$$

According to Lemma 3.2(c), $\frac{1}{\lambda} H(x - \hat{y}) \in \partial\langle f'(x), h\rangle_\lambda$. Consequently $\langle \frac{1}{\lambda} H(x - \hat{y}), h_1\rangle \leq \langle f'(\cdot), h\rangle^0_\lambda(x; h_1)$ and after letting $n \to \infty$ in (13) we arrive at the inequality (9). The proof of the lemma is now complete. □

The following lemma extends Theorem 4.4 in [9] from the Hilbert space setting to the setting of a reflexive Banach space.

Lemma 3.5. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz function that satisfies the growth condition* (4)*. Then*

$$\partial f(x) = \overline{co}^* \{\omega^* - \limsup_{\substack{z \to x \\ \lambda \to 0^+}} \partial f_\lambda(z)\}.$$

Proof. Denote

$$C(x) = \omega^* - \limsup_{\substack{z \to x \\ \lambda \to 0^+}} \partial f_\lambda(z).$$

Following the proof of Theorem 4.4 in [9], we obtain, using Frechet subdifferentiability of the functions f_{λ_n} instead of Frechet differentiability and the mean-value theorem of Clarke in [5] instead of the mean-value inequality of Preiss in [9], that $\partial f(x) \subset \overline{co}^* C(x)$.

Let us prove the opposite inclusion. Since $\partial f(x)$ is a convex and ω^*-closed subset of X^*, it is enough to prove that

$$C(x) \subset \partial f(x). \tag{14}$$

If $L^* \in C(x)$, then there exist sequences

$$z_n \to x,\ \lambda_n \to 0^+ \quad \text{and} \quad L^*_n \xrightarrow{\omega^*} L^*.$$

such that

$$L^*_n \in \partial f_{\lambda_n}(z_n).$$

Consequently to prove inclusion (14), it suffices to examine the set $\omega^* - \limsup_{n \to \infty} \partial f_{\lambda_n}(z_n)$ and to show that

$$\omega^* - \limsup_{n \to \infty} \partial f_{\lambda_n}(z_n) \subset \partial f(x). \tag{15}$$

By Lemma 3.3,

$$\partial f_{\lambda_n}(z_n) = \overline{\text{co}}^* \{\omega^* - \limsup_{G_{\lambda_n} \ni z \to z_n} \partial^F (f_{\lambda_n})(z)\}. \tag{16}$$

Denote

$$T_n(z_n) = \omega^* - \limsup_{G_{\lambda_n} \ni z \to z_n} \partial^F (f_{\lambda_n})(z).$$

First, we shall prove that the set $\bigcup_{n\in\mathbb{N}} T_n(z_n)$ is bounded. Let $L_n \in T_n(z_n)$. By Theorem 2.1 there exist $\tilde{y}_{\lambda_n}(z_n)$ such that $\partial^F f_{\lambda_n}(z_n) \subset \partial^F f(\tilde{y}_{\lambda_n}(z_n)) \subset \partial f(\tilde{y}_{\lambda_n}(z_n))$. Since the latter set is contained in $L_f B^*$ (B^* is the ω^*-closed unit ball in X^*) for large n, it follows that L_n are norm bounded. Consequently, we can apply Lemma 2 [16] and obtain

$$\omega^* - \limsup_{n\to\infty} \{\overline{\mathrm{co}}^* \, T_n(z_n)\} \subset \overline{\mathrm{co}}^* \{\omega^* - \limsup_{n\to\infty} T_n(z_n)\}.$$

Moreover by (16),

$$\omega^* - \limsup_{n\to\infty} \partial f_{\lambda_n}(z_n) = \omega^* - \limsup_{n\to\infty} \{\overline{\mathrm{co}}^* \, T_n(z_n)\}.$$

Therefore, to prove (15), it suffices to show that

$$\omega^* - \limsup_{n\to\infty} T_n(z_n) \subset \partial f(x). \tag{17}$$

Let $L \in \omega^* - \limsup\limits_{n\to\infty} T_n(z_n)$. Assume that

$$L \notin \partial f(x). \tag{18}$$

By the separation theorem there exist h_1, $||h_1|| = 1$ and $s > 0$ such that

$$\langle L, h_1\rangle \geq \sup_{a\in\partial f(x)} \langle a, h_1\rangle + s.$$

Using the same notation as in [9], let $U(x)$ be a bounded neighborhood of x, on which f is Lipschitz and $\sup_{z\in U(x)} \overline{\rho}(z, f, \lambda, c) \to 0$ as $\lambda \to 0$. Since ∂f is upper semicontinuous, there exists $\varepsilon > 0$ such that

$$\langle L, h_1\rangle \geq \sup_{a\in\partial f(x)} \langle a, h_1\rangle + s > \sup_{a\in D_\varepsilon(x)} \langle a, h_1\rangle + \frac{s}{2},$$

where $D_\varepsilon(x) = \overline{co}^*\{d \in \partial f(z),\, z \in B(x,\varepsilon) \subset U(x)\}$. On the other hand by definition of L, there exist a subsequence $\{n_k\}_{k\in\mathbb{N}}$ and a sequence $\{L_k\}_{k\in\mathbb{N}}$ such that $L_k \xrightarrow{\omega^*} L$ and $L_k \in T_{n_k}(z_{n_k})$. From the definition of $T_{n_k}(v_k)$, where $v_k := z_{n_k}$, for every $k \in \mathbb{N}$ there exist sequences $\{v_{km}\}_{m\in\mathbb{N}}$ and L_{km} such that $v_{km} \to v_k$, $L_k = \omega^* - \lim L_{km}$ as $m \to \infty$ and $L_{km} \in \partial^F f_{\lambda_{n_k}}(v_{km})$ for every $m \in \mathbb{N}$. Hence, there exist a constant $K(s) > 0$ and a subsequence $\{m_k\}_{k\in\mathbb{N}}$ such that for every $k > K(s)$, one has

$$|\langle L, h_1\rangle - \langle L_k, h_1\rangle| < \frac{s}{8}$$

and

$$|\langle L_k, h_1\rangle - \langle L_{km_k}, h_1\rangle| < \frac{s}{8},$$

where $u_k := v_{km_k}$, $u_k \to x$ and $L_{km_k} \in \partial^F f_{\lambda_{n_k}}(u_k) \subset \partial f(\tilde{y}_{\lambda_{n_k}}(u_k))$.

Hence for sufficiently large k

$$\langle L_{km_k}, h_1\rangle \geq \sup_{a\in D_\varepsilon(x)} \langle a, h_1\rangle + \frac{s}{4}. \tag{19}$$

Since for sufficiently large k, the points $\tilde{y}_{\lambda_{n_k}}(u_k) \in B(x,\varepsilon)$ and $L_{km_k} \in D_\varepsilon(x)$, we obtain a contradiction. $\square$

Now let us give the **Proof of Theorem 3.1.** By Lemma 3.4

$$\begin{aligned}
&\overline{co}^*\omega^* - \limsup_{\substack{z\to x\\ \lambda\to 0^+}} \left[\overline{co}^*\{\omega^* - \limsup_{\substack{u\to z\\ n\to\infty}} \partial\langle f_n'(u), h\rangle_\lambda\}\right] \\
\subseteq\ &\overline{co}^*\omega^* - \limsup_{\substack{z\to x\\ \lambda\to 0^+}} \partial\langle f'(z), h\rangle_\lambda.
\end{aligned}$$

In order to apply Lemma 3.5 for the function $\langle f'(\cdot), h\rangle$ we need to show that this function satisfies the growth condition (4). Indeed, since $\langle f_n'(\cdot), h\rangle$ is epi-convergent to $\langle f'(\cdot), h\rangle$ and $\langle f_n'(\cdot), h\rangle$ satisfy the condition $(A2)$, Corollary 2.67 in [1] implies that

$$\langle f'(x), h\rangle = \sup_{\lambda>0} \limsup_{n} \langle f_n'(x), h\rangle_\lambda.$$

Let $0 < \varepsilon_n \to 0$ be an arbitrary sequence and let $U_n(x)$ be a bounded neighborhood of x, on which $\langle f_n'(\cdot), h\rangle$ is Lipschitz and $\sup_{z\in U_n(x)} \overline{\rho}(z, \langle f_n'(\cdot), h\rangle, \lambda, 2r)$ tends to 0 as $\lambda \to 0$. For every $n \in \mathbb{N}$, let us choose $\lambda_n > 0$ such that $\sup\limits_{z\in U_n(x)} \overline{\rho}(z, \langle f_n'(\cdot), h\rangle, \lambda_n, 2r) \le \varepsilon_n$. Then,

$$||\tilde{y}^n_{\lambda_n}(x) - x|| \le \varepsilon_n.$$

Thus, by Theorem 2.1 and $(A2)$ we have,

$$\begin{aligned}
\langle f_n'(x), h\rangle_{\lambda_n} &= \langle f_n'(\tilde{y}^n_{\lambda_n}(x)), h\rangle + \frac{1}{2\lambda_n}||x - \tilde{y}^n_{\lambda_n}(x)||^2 \\
&\ge -r(1 + ||\tilde{y}^n_{\lambda_n}(x)||^2) + \frac{1}{2\lambda_n}||x - \tilde{y}^n_{\lambda_n}(x)||^2 \\
&\ge -r(1 + ||\tilde{y}^n_{\lambda_n}(x)||^2) \ge -r[1 + (\varepsilon_n + ||x||)^2],
\end{aligned}$$

consequently

$$\langle f'(x), h\rangle \ge \limsup_{n} \langle f_n'(x), h\rangle_{\lambda_n} \ge -r(1 + ||x||^2),$$

which proves the growth condition (4) with $c = 2r$.

Then, by Lemma 3.5,

$$\overline{co}^*\{\omega^* - \limsup_{\substack{z\to x\\ \lambda\to 0^+}} \partial\langle f'(z), h\rangle_\lambda\} = \partial\langle f'(x), h\rangle.$$

This proves Theorem 3.1. □

Remark. Let a sequence of set-valued functions $A_n : E \to E^*$ be given. Then we say that

$$(x, x^*) \in \limsup_{n\to\infty} \operatorname{gph} A_n$$

iff

$$x^* \in \omega^* - \limsup_{\substack{y\to x\\ n\to\infty}} A_n(y),$$

In this sense the result of Theorem 3.1 is equivalent to

$$\limsup_{\lambda\to 0} \big\{\limsup_{n\to\infty} \operatorname{gph} A_{n,\lambda}\big\} \subset \operatorname{gph} \partial\langle f'(\cdot), h\rangle.$$

Here gph $A_{n,\lambda}$ denotes the graph of the set-valued map

$$A_{n,\lambda} : X \to X^*,$$

defined for any $x \in X$ as

$$A_{n,\lambda}(x) := \partial\langle f_n'(x), h\rangle_\lambda.$$

In the convex case we can strengthen the result of Theorem 3.1 in the following way.

Theorem 3.6. *Let X be a reflexive Banach space with a Kadec norm and f_n, f be a sequence of closed convex proper functions defined on X. If the sequence f_n is locally bounded and epi-convergent to f, then for every $x \in X$,*

$$\overline{co}^*\{\omega^* - \limsup_{\substack{u\to x,\ \lambda\to 0^+\\ n\to\infty}} \nabla(f_n)_\lambda(u)\} = \partial f(x).$$

Proof. First we prove the inclusion

$$\omega^* - \limsup_{\substack{u\to x,\ \lambda\to 0^+\\ n\to\infty}} \nabla(f_n)_\lambda(u) \subset \partial f(x).$$

By Theorem 3.24 in [1], $(f_n)_\lambda \in C^1$ and for every $u \in X$ there exists a point $J_\lambda^{f_n}u$ such that $\nabla(f_n)_\lambda(u) = \frac{1}{\lambda}H(u - J_\lambda^{f_n}u) \in \partial f(J_\lambda^{f_n}u)$. Hence it is enough to prove that

$$\omega^* - \limsup_{\substack{u\to x,\ \lambda\to 0^+\\ n\to\infty}} \frac{1}{\lambda}H(u - J_\lambda^{f_n}u) \subset \partial f(x).$$

Let λ_n, z_n and y_n^* be sequences such that $\lambda_n \to 0^+$, $z_n \to x$ and $y_n^* \in \partial f_n(z_n)$. By the local boundedness there exists $\delta > 0$ and a constant $M > 0$ such that $|f_n(y)| \le M$ for every y such that $||y - x|| < \delta$. Hence for all with $||h|| = 1$ we have for sufficiently large n

$$\langle y_n^*, \frac{\delta}{2}h\rangle \le f_n(z_n + \frac{\delta}{2}h) - f_n(z_n) \le 2M.$$

By monotonicity of ∂f_n

$$\langle y_n^* - \frac{1}{\lambda_n}H(z_n - J_{\lambda_n}^{f_n}z_n), z_n - J_{\lambda_n}^{f_n}z_n\rangle \ge 0.$$

Hence

$$\langle\frac{1}{\lambda_n}H(z_n - J_{\lambda_n}^{f_n}z_n), z_n - J_{\lambda_n}^{f_n}z_n\rangle \le \langle y_n^*, z_n - J_{\lambda_n}^{f_n}z_n\rangle$$

$$\frac{1}{\lambda_n}||z_n - J_{\lambda_n}^{f_n}z_n||^2 \le ||y_n^*||\,||z_n - J_{\lambda_n}^{f_n}z_n||$$

$$\frac{1}{\lambda_n}||z_n - J_{\lambda_n}^{f_n}z_n|| \le ||y_n^*|| \le \frac{4M}{\delta}.$$

Consequently the sequence $\{\frac{1}{\lambda_n}H(z_n - J_{\lambda_n}^{f_n}z_n)\}$ is norm bounded and moreover $J_{\lambda_n}^{f_n}z_n \to x$. By boundedness there exists an element $u^* \in X^*$ such that for a subsequence $\frac{1}{\lambda_n}H(z_n -$

$J_{\lambda_n}^{f_n} z_n) \xrightarrow{\omega^*} u^*$. Proposition 3.59 in [1] concerning graph convergence of maximal monotone operators implies that $u^* \in \partial f(x)$.

Now let us prove the opposite inclusion

$$\partial f(x) \subset \overline{co}^*\{\omega^* - \limsup_{\substack{u\to x,\ \lambda\to 0^+ \\ n\to\infty}} \nabla (f_n)_\lambda(u)\}.$$

Note that a closed convex proper function satisfies the growth condition. Therefore Lemma 3.5 applies and for any $L \in \partial f(x)$ there exist sequences λ_k and z_k such that $\lambda_k \to 0^+$, $z_k \to x$ and $L = \omega^* - \lim \nabla f_{\lambda_k}(z_k)$. From definition of graph convergence (Definition 3.58 [1]), it follows that for every $k \in \mathbb{N}$ there exists a sequence z_{kn} such that $z_{kn} \to z_k$ as $n \to \infty$ and $\nabla f_{\lambda_k}(z_k) = \lim_n \nabla(f_n)_{\lambda_k}(z_{kn})$. From the diagonalization formula (Corollary 1.18 in [1]) there exists a mapping $k \to n(k)$ increasing to ∞ such that $L = \omega^* - \lim\ \nabla(f_{n(k)})_{\lambda_k}(z_{kn(k)})$, $z_{kn(k)} \to x$ which proves the inclusion. □

4. A direct approach

Let A be a given subset of a linear normed space X and g be a Lipschitz function on A. We consider the following Lipschitz seminorm:

$$|g|_A := \sup_{\substack{y,z\in A \\ y\neq z}} \frac{g(y)-g(z)}{||y-z||}.$$

Let G be an open subset of X. Let g_n, g be locally Lipschitz functions on G. Then we say that $g_n \to g$ in $C^{0,1}(G)$ iff $\forall x \in G$ $\exists$ neighbourhood $U(x) \subset G$ such that

$$|g_n - g|_{U(x)} \to 0.$$

In this connection we consider the sequence $f, f_n : X \to R$, $n \in \mathbb{N}$ on which we posit the following assumptions:

(i1) f and every f_n is $C^{1,1}$ on G;
(i2) For all $x \in G$ and for all $h \in X$ we have $\langle f_n'(\cdot), h\rangle \to \langle f'(\cdot), h\rangle$ in $C^{0,1}(G)$.

Theorem 4.1. *Assume* (i1) *and* (i2). *Then*

(B1) for every $x \in G$, $x_n \to x$ and $h \in X$

$$\bigcup_{n\in\mathbb{N}} \partial\langle f_n'(x_n), h\rangle \text{ is bounded} \tag{20}$$

and

$$\omega^* - \limsup_{n\to+\infty} \partial\langle f_n'(x_n), h\rangle \subset \partial\langle f'(x), h\rangle; \tag{21}$$

(B2) for every $x \in G$ and $x_n \to x$

$$\bigcup_{n\in\mathbb{N}} \partial_c^2 f_n(x_n) \text{ is bounded} \tag{22}$$

and

$$\omega^* - \limsup_{n\to+\infty} \partial_c^2 f_n(x_n) \subset \partial_c^2 f(x). \tag{23}$$

Proof. Let us prove (*B1*). Let $L_n \in \partial\langle f_n'(x_n), h\rangle$ and h_1 be an arbitrary element of X such that $||h_1|| = 1$. Then

$$L_n[h_1] \le \langle f_n'(\cdot), h\rangle^0(x_n; h_1) = f_n^{00}(x_n; h_1, h)$$

For every $\varepsilon > 0$, by (i2), for given x, $\exists\delta > 0$ such that for every $y, z \in X$ with $||y-x|| < \delta$, $||z-x|| < \delta$ and $\forall n > N(\varepsilon)$ we have $y, z \in G$ and

$$\langle f_n'(y), h\rangle - \langle f'(y), h\rangle - \langle f_n'(z), h\rangle + \langle f'(z), h\rangle < \varepsilon||y-z||.$$

The choice $y = z + th_1$ with $0 < t < \frac{\delta}{2}$ and $||z-x|| < \frac{\delta}{2}$ in this inequality gives

$$\langle f_n'(z+th_1), h\rangle - \langle f'(z+th_1), h\rangle - \langle f_n'(z), h\rangle + \langle f'(z), h\rangle < \varepsilon||th_1||.$$

Hence,

$$\langle f_n'(z+th_1), h\rangle - \langle f_n'(z), h\rangle < \langle f'(z+th_1), h\rangle - \langle f'(z), h\rangle + \varepsilon t.$$

Using the definition of $f_n^{00}(x_n; h_1, h)$ we obtain for large enough $n \in \mathbb{N}$

$$\begin{aligned} f_n^{00}(x_n; h_1, h) &= \limsup_{\substack{z \to x_n \\ t \downarrow 0}} \frac{f_n'(z+th_1)[h] - f_n'(z)[h]}{t} \\ &\le \limsup_{\substack{z \to x_n \\ t \downarrow 0}} \frac{f'(z+th_1)[h] - f'(z)[h]}{t} + \varepsilon \\ &= f^{00}(x_n; h_1, h) + \varepsilon. \end{aligned}$$

Hence for large enough $n \in \mathbb{N}$,

$$L_n[h_1] \le f_n^{00}(x_n; h_1, h) \le f^{00}(x_n; h_1, h) + \varepsilon. \tag{24}$$

Since the set $\partial\langle f'(x_n), h\rangle + \varepsilon B^*$ (B^* is the unit ball in X^*) is ω^*-compact, by an argument using the separation theorem we obtain

$$L_n \in \partial\langle f'(x_n), h\rangle + \varepsilon B^*.$$

Since the set-valued map $x \to \partial\langle f'(x), h\rangle$ is ω^*-upper semicontinuous (see [5], Proposition 2.1.5), for sufficiently large n, we get

$$\partial\langle f'(x_n), h\rangle \subset \partial\langle f'(x), h\rangle + \varepsilon B^*.$$

So, for sufficiently large n, $L_n \in \partial\langle f'(x), h\rangle + 2\varepsilon B^*$. This proves (20). Moreover, since $f^{00}(\cdot; h_1, h)$ is upper semicontinuous in G, we obtain from (24)

$$\limsup_{n\to\infty} L_n[h_1] \le \limsup_{n\to\infty} f^{00}(x_n; h_1, h) \le f^{00}(x; h_1, h). \tag{25}$$

Let us prove (21). Let $L \in \omega^* - \limsup_{n\to+\infty} \partial\langle f_n'(x_n), h\rangle$. The latter set denotes the set of all weak*- limit points of sequences $L_{n_k} \in \partial\langle f_{n_k}'(x_{n_k}), h\rangle$ where $x_{n_k} \to x$ and n_k is a subsequence of $\mathbb{N}$. Consequently $L = \omega^* - \lim L_n$, for a suitable subsequence $L_n \in \partial\langle f_n'(x_n), h\rangle$ denoted by the same index n. Using (25) we obtain

$$L[h_1] = \limsup_n L_n[h_1] \le f^{00}(x; h_1, h) = \langle f'(\cdot), h\rangle^0(x; h_1),$$

i.e. $L \in \partial\langle f'(x), h\rangle$.

To prove part (*B2*), let $\xi_n \in \partial_c^2 f_n(x_n)$. We shall prove that the sequence $\{\xi_n\}_{n\in\mathbb{N}}$ is norm bounded. Let h_1 and h be arbitrary elements of X such that $||h_1|| = 1$ and $||h|| = 1$. Then

$$\begin{aligned}\xi_n[h_1, h] &\leq f_n^{00}(x_n; h_1, h) = \langle f_n'(\cdot), h\rangle^0(x_n; h_1)\\ &= \max_{L\in\partial\langle f_n'(x_n),h\rangle}\langle L, h_1\rangle = \langle L_n, h_1\rangle,\end{aligned}$$

for suitable $L_n \in \partial\langle f_n'(x_n), h\rangle$. Consequently

$$\xi_n[h_1, h] \leq \langle L_n, h_1\rangle. \tag{26}$$

By (*B1*) (20), $\{L_n\}_{n\in\mathbb{N}}$ is norm bounded, and hence $\{\xi_n\}$ is also norm bounded.

Now we prove (23). Let $\xi \in \omega^* - \limsup_{n\to+\infty} \partial_c^2 f_n(x_n)$, i.e. $\xi = \omega^* - \lim \xi_{n_k}$, where $\xi_{n_k} \in \partial_c^2 f_{n_k}(x_{n_k})$, $x_{n_k} \to x$. Consequently $\xi = \omega^* - \lim \xi_n$, for a suitable subsequence $\xi_n \in \partial_c^2 f_n(x_n)$ denoted by the same index n. Using (26) and (25) we obtain

$$\xi[h_1, h] = \lim \xi_n[h_1, h] \leq \limsup\langle L_n, h_1\rangle \leq f^{00}(x; h_1, h).$$

This proves (23) and the proof of Theorem 4.1 is completed. □

5. Continuity of Multipliers

To apply some of the results in [8], we now consider a real Banach space X with separable dual. Let G be an open subset of X and

$$f, f_n; g_i, g_{ni}, \quad i = 1, \ldots, p; \quad h_j, h_{nj}, \quad j = 1, \ldots, q,$$

be sequences in $C^{1,1}(G)$. We consider the following family (P_n) of constrained minimization problems

$$(P_n)\begin{cases} f_n \to \min \\ \text{subject to } x \in G, \quad g_{ni}(x) \leq 0,\ i = \overline{1,p}; \quad h_{nj}(x) = 0,\ j = \overline{1,q}\end{cases}$$

and the analogous problem

$$(P)\begin{cases} f \to \min \\ \text{subject to } x \in G, \quad g_i(x) \leq 0,\ i = \overline{1,p} \quad h_j(x) = 0,\ j = \overline{1,q}.\end{cases}$$

Let us denote the admissible region of (P_n) by C_n and that of (P) by C. Further to state the necessary optimality condition of [8] we have to introduce the following sets

$$K(x) := \left\{h \in X : f'(x)[h] \leq 0,\ g_i'(x)[h] \leq 0,\ i = \overline{1,p},\ h_j'(x)[h] = 0,\ j = \overline{1,q}\right\}$$

and

$$\begin{aligned}\Lambda(x) := \Big\{(\lambda, \mu) \in R^{p+1} \times R^q : \lambda_0 f'(x) + \sum_{i=1}^{p}\lambda_i g_i'(x) + \sum_{j=1}^{q}\mu_j h_j'(x) = 0,\\ \lambda_i g_i(x) = 0, i = \overline{1,p}; \lambda_i \geq 0, i = \overline{0,p}, \sum_{i=0}^{p}\lambda_i + \sum_{j=1}^{q}|\mu_j| = 1\Big\}.\end{aligned}$$

The sets $K_n(x_n)$, $\Lambda_n(x_n)$ are defined analogously.
Denote $F(x) = (h_1(x), \dots, h_q(x))^T$. Then the main result in [8] reads:

If x is a local minimum of (P) and $\operatorname{Im} F'(x) = \mathbb{R}^q$, then $\forall h \in K(x)$ $\exists(\lambda, \mu) \in \Lambda(x)$, $\hat{\xi} \in \partial_c^2 f(x)$, $\xi_i \in \partial_c^2 g_i(x)$, $i = \overline{1,p}$, $\eta_j \in \partial_c^2 h_j(x)$, $j = \overline{1,q}$ such that the bilinear form $\beta = \lambda_0 \hat{\xi} + \sum_{i=1}^{p} \lambda_i \xi_i + \sum_{j=1}^{q} \mu_j \eta_j$ satisfies

$$\beta[h,h] \geq 0. \tag{27}$$

We posit the following assumptions:

1) $f_n \overset{\text{epi}}{\to} f$
2) Each sequence $f, f_n; g_i, g_{ni}, \;\; i = \overline{1,p}; \;\; h_j, h_{nj}, \;\; j = \overline{1,q}$, fulfils the conditions of Theorem 4.1;
3) For every $x \in G$ and every sequence $x_n \to x$

$$\omega^* - \lim g'_{ni}(x_n) = g'_i(x), \; i = \overline{1,p};$$
$$\omega^* - \lim h'_{nj}(x_n) = h'_j(x), \; j = \overline{1,q}.$$

4) $g_{ni} \to g_i, h_{nj} \to h_j$ continuously for every i, j (i.e. for every sequence $x_n \to x$ we have $g_{ni}(x_n) \to g_i(x)$ and $h_{nj}(x_n) \to h_j(x)$)
5) For every $y \in C$ there exists $y_n \in C_n$ such that

$$\limsup f_n(y_n) \leq f(y).$$

Theorem 5.1. *Let the above assumptions* 1)–5) *hold. Let x_n be a local minimum of (P_n) and $x_n \to x$. Then x solves locally (P) and $f_n(x_n) \to f(x)$. Moreover, for any $h \in \cap_n K_n(x_n)$ we have that $h \in K(x)$ and from the sequences $(\lambda_{n0}, \lambda_{ni}, \mu_{nj})$ and $(\hat{\xi}_n, \xi_{ni}, \eta_{nj})$ that satisfy* (27) *with subscript n we can extract subsequences such that*

$$(\lambda_{n0}, \lambda_{ni}, \mu_{nj}) \to (\lambda_0, \lambda_i, \mu_j) \in \Lambda(x) \backslash \{0\}; \tag{28}$$

$$\hat{\xi}_n \overset{\omega^*}{\to} \hat{\xi} \in \partial_c^2 f(x), \; \xi_{ni} \overset{\omega^*}{\to} \xi_i \in \partial_c^2 g_i(x), \; \eta_{nj} \overset{\omega^*}{\to} \eta_j \in \partial_c^2 h_j(x) \tag{29}$$

and $(\lambda_0, \lambda_i, \mu_j)$, $(\hat{\xi}, \xi_i, \eta_j)$ satisfy the necessary optimality conditions (27) *for the given h.*

Proof. Using assumptions 1), 4) and 5), the proof that x is a local minimum of P is exactly as in [16], Theorem 2. Moreover by 1) $f_n(x_n) \to f(x)$.

Let $h \in \cap_n K_n(x_n)$. Since f_n attains a local minimum at x_n, then $0 \in \partial f_n(x_n)$. But $f_n \in C^{1,1}(G)$, consequently $\partial f_n(x_n) = \{f'_n(x_n)\}$ and $f'_n(x_n) = 0$. Analogously $f'(x) = 0$. Hence, it is enough to consider the relations $h'_{nj}(x_n)[h] = 0$, $j = \overline{1,q}$ and $g'_{ni}(x_n)[h] \leq 0$, $i = \overline{1,p}$. For any $i = \overline{1,p}$, by assumption 3) we have

$$\langle g'_i(x), h \rangle = \lim_{n \to \infty} \langle g'_{ni}(x_n), h \rangle \leq 0.$$

Analogously, for any $j = \overline{1,q}$, $\langle h'_j(x), h \rangle = 0$. Hence $h \in K(x)$.

Since $\sum_{i=0}^{p} \lambda_{ni} + \sum_{j=1}^{q} |\mu_{nj}| = 1$, we can obviously assume that the sequences λ_{n0}, λ_{ni}, μ_{nj} are bounded for every i, j and consequently we have (28). Moreover, Theorem 4.1 ($B2$) entails (29).

For any $(\lambda_n, \mu_n) \in \Lambda_n(x_n)$ we have

$$\sum_{i=1}^{p} \lambda_{ni} g'_{ni}(x_n) + \sum_{j=1}^{q} \mu_{nj} h'_{nj}(x_n) = 0$$

and

$$\lambda_{ni} g_{ni}(x_n) = 0.$$

Assumptions 3), 4) and (28) entail that $(\lambda, \mu) \in \Lambda(x)$.

Let $h \in \cap_n K_n(x_n)$. For every n there exist $\hat{\xi}_n \in \partial^2_c f_n(x_n)$, $\xi_{ni} \in \partial^2_c g_{ni}(x_n)$, $i = \overline{1,p}$, $\eta_{nj} \in \partial^2_c h_{nj}(x_n)$, $j = \overline{1,q}$ and $(\tilde{\lambda}_n, \tilde{\mu}_n) \in \Lambda_n(x_n)$ such that $\beta_n[h, h] > 0$, i.e.

$$0 \leq \tilde{\lambda}_{n0} \hat{\xi}_n[h, h] + \sum_{i=1}^{p} \tilde{\lambda}_{ni} \xi_{ni}[h, h] + \sum_{j=1}^{q} \tilde{\mu}_{nj} \eta_{nj}[h, h]. \tag{30}$$

Letting $n \to \infty$ in (30) and using (29), we obtain that

$$0 \leq \tilde{\lambda}_0 \hat{\xi}[h, h] + \sum_{i=1}^{p} \tilde{\lambda}_i \xi_i[h, h] + \sum_{j=1}^{q} \tilde{\mu}_j \eta_j[h, h],$$

so the necessary optimality condition (27) at point x is satisfied. $\square$

Remarks. (1) Theorem 5.1 provides convergence not only for the multipliers but also for the associated second-order Clarke's subdifferentials $\hat{\xi}_n$, ξ_{ni}, η_{nj}.

(2) As it was noted in [16], the assumption of continuous convergence of g_{ni} in 4) can be replaced by the condition

$$\partial^F g_{ni}(x_n) \neq \emptyset \quad \text{for every } n \text{ and } i.$$

6. Stability of the second-order Clarke's subdifferentials of integral functionals

We consider a $\sigma-$ finite positive measure space (T, Σ, dt) and a sequence of functions $g, g_n : T \times \mathbb{R}^m \to \mathbb{R}$. We posit the following assumptions:

(1) For each $y \in \mathbb{R}^m$, every function $g(\cdot, y)$, $g_n(\cdot, y)$ belongs to $L^1(T)$;
(2) For all $t \in T$, $g(t, \cdot)$, $g_n(t, \cdot) \in C^{1,1}(\mathbb{R}^m)$, i.e. for every $x \in \mathbb{R}^m$ there exist a neighborhood $U(x) \subset \mathbb{R}^m$ and a function $k \in L^1(T, \mathbb{R}_+)$ such that, for all $t \in T$, for all $y_1, y_2 \in U(x)$ we have

$$||g'(t, y_1) - g'(t, y_2)||_{\mathbb{R}^m} \leq k(t)\, ||y_1 - y_2||_{\mathbb{R}^m}.$$

(3) For every $y \in \mathbb{R}^m$, the functions $g'(\cdot, y)$ and $g'_n(\cdot, y)$ belong to $L^1(T)$;

(4) For a.e. $t \in T$, $g'_n(t, \cdot) \to g'(t, \cdot)$ in $C^{0,1}(\mathbb{R}^m)$, i.e. for every $x \in \mathbb{R}^m$ there exist $\delta > 0$ and a function $k_1 \in L^1(T, \mathbb{R}_+)$ such that for every $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that for every $y_1, y_2 \in \mathbb{R}^m$ with $||y_1 - x||_{\mathbb{R}^m} < \delta$, $||y_2 - x||_{R^m} < \delta$ and $\forall n > N(\varepsilon)$ we have
$$||g'_n(t, y_1) - g'(t, y_1) - g'_n(t, y_2) + g'(t, y_2)||_{\mathbb{R}^m} < \varepsilon\, k(t)\, ||y_1 - y_2||_{\mathbb{R}^m}$$
for a.e. $t \in T$.

For $u \in L_\infty = L_\infty(T, \mathbb{R}^m)$, we introduce the following integral functionals

$$f(u) = \int_T g(t, u(t))\, dt, \quad f_n(u) = \int_T g_n(t, u(t))\, dt.$$

By (1), f and f_n are well defined.

Since $g(t, \cdot) \in C^{1,1}(\mathbb{R}^m)$, by the mean value theorem $g(t, \cdot)$ is strictly Frechet differentiable and consequently by Proposition 2.2.4 in [5], locally Lipschitz. By ([5], Theorem 2.7.3) for every $u \in L_\infty$, f is locally Lipschitz at u and

$$\partial f(u) = \int_T \partial g(t, u(t))\, dt. \tag{31}$$

Since $\partial g(t, u(t)) = \{g'(t, u(t))\}$, the right hand side of (31) is a singleton and defines a map $\xi[\cdot]$ such that for any $v \in L_\infty$, $\xi[v] = \int_T g'(t, u(t))^T v(t)\, dt$, where by assumption (3) for every $v \in L_\infty$, the function $t \to g'(t, u(t))^T v(t)$ belongs to $L^1(T)$.

The existence of $f'(u; v)$ for any $v \in L_\infty$ and the equality

$$f'(u; v) = \int_T g'(t, u(t); v(t))\, dt$$

follow from the dominated convergence theorem. Hence $f'(u; v) = \xi[v]$ and consequently f admits a Gateaux derivative at $u \in L_\infty$ denoted by $f'(u)$ and $f'(u)[v] = \int_T g'(t, u(t))^T v(t)\, dt$.

Analogously, for every $v \in L_\infty$,

$$f'_n(u)\, [v] = \int_T g'_n(t, u(t))^T\, v(t)\, dt.$$

The fact that the first Gateaux derivatives f' and f'_n are locally Lipschitz on L_∞ follows from assumption (2), as in Theorem 2.7.2 in [5].

By (4), for any $v \in L_\infty$, $||v||_\infty = 1$ we have $\langle f'_n(\cdot), v\rangle \to \langle f'(\cdot), v\rangle$ in $C^{0,1}(L_\infty)$, i.e. $\forall u \in L_\infty$ there exists $\delta > 0$ such that $\forall \varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that for every $n > N(\varepsilon)$, $||u_1 - u||_\infty < \delta$, $||u_2 - u||_\infty < \delta$, we have

$$|\langle f'_n(u_1), v\rangle - \langle f'(u_1), v\rangle - \langle f'_n(u_2), v\rangle + \langle f'(u_2), v\rangle| < \varepsilon ||u_1 - u_2||_\infty.$$

So, we can directly apply Theorem 4.1 within $X = L_\infty$ and obtain the following corollary.

Corollary 6.1. *Let the above assumptions* (1), (2), (3) *and* (4) *hold. Then*

(i) *for every* $u, v \in L_\infty$

$$\omega^* - \limsup_{\substack{w \to u \\ n \to +\infty}} \partial \langle f_n'(w), v \rangle \subset \partial \langle f'(u), v \rangle. \tag{32}$$

(ii) *for every* $u \in L_\infty$

$$\omega^* - \limsup_{\substack{w \to u \\ n \to +\infty}} \partial_c^2 f_n(w) \subset \partial_c^2 f(u). \tag{33}$$

□

Remark. This theorem can be extended to the standard type of functionals in calculus of variations as given by

$$J(\phi) = \int_a^b L(t, \phi(t), \dot{\phi}(t))\, dt, \quad J_n(\phi) = \int_a^b L_n(t, \phi(t), \dot{\phi}(t))\, dt,$$

where ϕ is an absolutely continuous function from $[a, b]$ to $\mathbb{R}^m$.

We claim that for every $\hat{\phi}$

$$\omega^* - \limsup_{\substack{\phi \to \hat{\phi} \\ n \to +\infty}} \partial_c^2 J_n(\phi) \subset \partial_c^2 J(\hat{\phi}). \tag{34}$$

We follow the approach of Clarke in [5], Example 2.7.4. Using the Lebesgue measure λ, we consider the measure space $(T, \Sigma, \mu) = ([a, b], \mathcal{B}([a, b]), \lambda)$ and the linear subspace

$$X = \left\{ (s, v) \in L_\infty(T, \mathbb{R}^{2m}) : \exists c \in \mathbb{R}^m, s(t) = c + \int_a^t v(\tau) d\tau \right\}$$

of $L_\infty(T, \mathbb{R}^{2m})$ and define $g(t; s, v) = L(t, s, v)$.

Note that for any $(s, v) \in X$, we have $f(s, v) = \displaystyle\int_a^b g(t; s, v) dt = J(s)$ and $f_n(s, v) = \displaystyle\int_a^b g_n(t; s, v) dt = J_n(s)$.

With $(\hat{s}, \hat{v})$ a given element of X, (34) follows from (33) by Corollary 6.1 in $L_\infty(T, \mathbb{R}^{2m})$.

Acknowledgements. The authors wish to thank the anonymous referees for their useful comments.

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