Journal of Convex Analysis
Volume 14 (2007), No. 1, 1–11

Convex Bodies with Sheafs of Elliptic Sections II*

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Received: September 29, 2004

A convex body $D \subset E^d$ is an ellipsoid if all the sections given by hyperplanes are elliptic. We study whether we can restrict the hyperplanes to those that contain one of two fixed linear varieties. In a previous paper we considered the case where one of the varieties cuts the interior of D, and now we assume that the varieties support D.

Keywords: Convex body, elliptic section, ellipsoid

2000 Mathematics Subject Classification: 52A20, 46C15

1. Introduction

Let E^d ($d \geq 3$) be the d-dimensional Euclidean space and let $D \subset E^d$ be a convex body (i.e., a compact convex set with non-empty interior) with boundary S. It is well known that S is an ellipsoid if and only if all the sections of S given by hyperplanes are ellipsoidal. This characterization of ellipsoids can be improved by limiting the hyperplanes we consider. For example, we can assume that all the hyperplanes go through a fixed point p. Busemann [4] proved this result for $p \in \operatorname{Int} D$, Burton [3] extended it for $p \in E^d$, and Petty [5] pointed out that one can consider p in the d-dimensional projective space P^d. With the aim of expressing a number of characterizations of this kind in a unified language, in [2] we gave the following definition:

For a linear variety l with $0 \leq \dim l \leq d-2$ we call D (or, equivalently, S) *elliptic through* l if for every hyperplane P such that $l \subset P$ and $P \cap \operatorname{Int} D \neq \emptyset$ the section $P \cap S$ is ellipsoidal.

The Busemann-Burton-Petty result can then be re-formulated by saying that S is an ellipsoid if and only if it is elliptic through a point $p \in P^d$. In [1] we showed that S is an ellipsoid if and only if it is centrally symmetric and elliptic through three $(d-2)$-dimensional linear varieties of the hyperplane at infinity. E.g., a centrally symmetric convex body in E^3 is an ellipsoid if and only if all the sections by planes parallel to three

*Partially supported by MCYT (Spain) and FEDER (BFM2001-0849).

ISSN 0944-6532 / $ 2.50

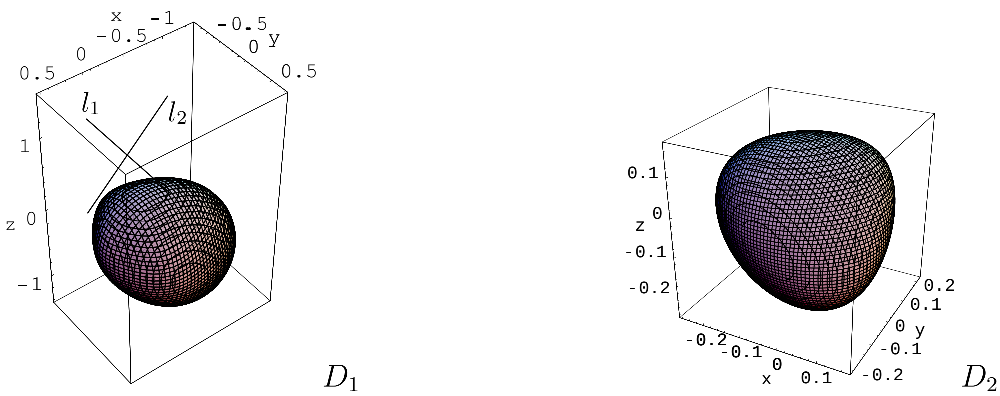


Figure 2.1:

fixed planes are elliptic. This result is not true either if S is not centrally symmetric or if only two planes are considered. In [2], we obtained the following result:

Theorem 1.1 ([2]). *Let S be the boundary of a convex body $D \subset E^d$ $(d \geq 3)$, and let l_1 and l_2 be two linear varieties such that $1 \leq \dim l_i \leq d-2$, $i = 1, 2$, $l_1 \not\subset l_2$, $l_2 \not\subset l_1$, and $l_1 \cap \operatorname{Int} D \neq \emptyset$. If S is elliptic through l_1 and l_2, then S is an ellipsoid.*

Straightforward counterexamples show that Theorem 1.1 is false when either the hypothesis $l_1 \cap \operatorname{Int} D \neq \emptyset$ is eliminated or only one variety is considered. We here extend the result in Theorem 1.1 by considering cases where no variety cuts the interior of D but with some new hypotheses added.

2. Results

For l_1 and l_2 linear varieties in E^d, let $l_1 + l_2$ denote the lowest-dimensional linear variety that contains l_1 and l_2. If $D \subset E^d$ is a convex body with boundary S and l is a linear variety, then l supports D (or, equivalently, S) if $l \cap D = l \cap S \neq \emptyset$. It is easy to see that if l supports S and S is elliptic through l then $l \cap S$ is reduced to one point. Our main result is the following theorem which we have stated so as to include Theorem 1.1.

Theorem 2.1. *Let S be the boundary of a convex body $D \subset E^d$ $(d \geq 3)$, and let l_1 and l_2 be two linear varieties such that $1 \leq \dim l_i \leq d-2$, $i = 1, 2$, and $l_1 \not\subset l_2$, $l_2 \not\subset l_1$. Assume that S is elliptic through l_1 and l_2 and that one of the following properties holds:*

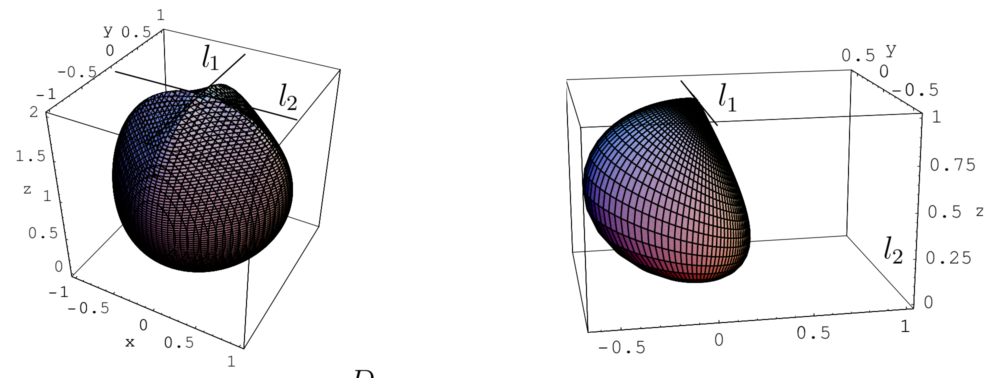
(i) *Either l_1 or l_2 cuts the interior of D.*

(ii) *l_1 and l_2 support S, $l_1 \cap S = l_2 \cap S$, and $\dim(l_1 + l_2) < d$.*

(iii) *l_1 and l_2 support S, $l_1 \cap l_2 = \emptyset$, and $\dim(l_1 + l_2) = d$.*

Then S is an ellipsoid.

Example 2.2 below shows that one can not reduce the hypotheses in Theorem 2.1 (*i*) and (*ii*). The lines l_i, $i = 1, 2, 3$, are coplanar and none cut either D_1 or D_2. None of the bodies is an ellipsoid. Nevertheless, D_1 is elliptic through l_1 and l_2, and D_2 is elliptic through the three lines. The body D_3 in Example 2.3 satisfies all the hypotheses in (*ii*), but is non-convex. The convex bodies D_5 and D_6 in Example 2.5 are elliptic through any line in a plane that does not cut their interior, even if the line supports it. This shows



D3 D4

Figure 2.2:

that the hypothesis $l_1 \cap S = l_2 \cap S$ is necessary in (ii) and that $\dim(l_1 + l_2) = d$ is necessary in (iii). The sets D_5 and D_6 are made by glueing together two pieces of quadrics. The convex body D_4 in Example 2.4 is elliptic through two lines. As also is the case for the bodies D_1, D_2, and D_3, no piece of D_4 is contained in a quadric. The particularity of this body is that while l_1 supports it, the plane $l_1 + l_2$ does not cut its interior.

Example 2.2. The sets of points $(x, y, z) \in \mathbb{R}^3$ defined by

$$D_1 \equiv \left\{ \begin{matrix} 4(x^2 + y^2 + z^2)(x + z - 1)^2 - (x + z - 1)^4 + 16x^2y^2 \leq 0 \\ x + z - 1 \neq 0 \end{matrix} \right\}$$

$$D_2 \equiv \left\{ \begin{matrix} 25(x + z - 1)(x^2 + y^2 + z^2) - (x + z - 1)^3 + 125\, xyz \geq 0 \\ -\frac{1}{3} \leq x \leq \frac{1}{5}, \quad -\frac{1}{3} \leq y \leq \frac{1}{3}, \quad -\frac{1}{3} \leq z \leq \frac{1}{5} \end{matrix} \right\}$$

are convex bodies elliptic through the lines l_i ($i = 1, 2$ for D_1; $i = 1, 2, 3$ for D_2), where $l_1 \equiv \{x = 0,\ z = 1\}$, $l_2 \equiv \{y = 0,\ x + z = 1\}$ and $l_3 \equiv \{x = 1,\ z = 0\}$. Neither of them is an ellipsoid (see Figure 2.1).

Example 2.3. The set

$$D_3 = \{(x, y, z) \in \mathbb{R}^3 :\ (x^2 + y^2)(z - 2)^2 + x^2y^2 + z(z - 2)^3 \leq 0,\ z < 2\} \cup \{(0, 0, 2)\}$$

is a non-convex body in $\mathbb{R}^3$ elliptic through the lines $l_1 \equiv \{x = 0,\ z = 2\}$ and $l_2 \equiv \{y = 0,\ z = 2\}$. Both lines support D_3 at the point $(0, 0, 2)$ (see Figure 2.2).

Example 2.4. The set

$$D_4 = \{(x, y, z) \in \mathbb{R}^3 :\ x^2(x - 1) + y^2(x + z - 1) + z(x + z - 1)^2 \geq 0, \\ z < 1,\ 2x + z < 1\} \cup \{(0, 0, 1)\}$$

is a convex body in $\mathbb{R}^3$ elliptic through the lines $l_1 \equiv \{x = 0,\ z = 1\}$ and $l_2 \equiv \{x = 1,\ z = 0\}$. It is not an ellipsoid (see Figure 2.2).

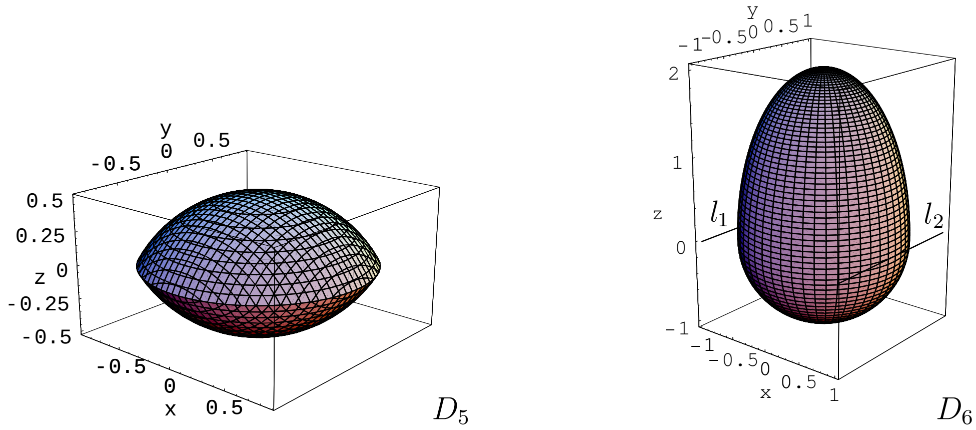


Figure 2.3:

Example 2.5. The sets of points $(x, y, z) \in \mathbb{R}^3$ defined by

$$D_5 \equiv \left\{ x^2 + y^2 + (|z| + \tfrac{1}{2})^2 \leq 1 \right\}, \quad D_6 \equiv \left\{ \begin{matrix} x^2 + y^2 + z^2 \leq 1 & \text{if } z \leq 0 \\ x^2 + y^2 + \dfrac{z^2}{4} \leq 1 & \text{if } z \geq 0 \end{matrix} \right\}$$

are convex bodies elliptic through any line in the plane $z = 0$ that does not cut the circle $\{x^2 + y^2 < 1, \ z = 0\}$. Neither of them is an ellipsoid (see Figure 2.3).

In the proof of Proposition 2.7 below we shall use the following lemma which shows that if we remove the hypothesis $l_1 \cap S = l_2 \cap S$ from Theorem 2.1 (ii), then the only possible counterexamples are of the type described in Example 2.5.

Lemma 2.6. *Let S be the boundary of a convex body $D \subset E^3$. Let l_1 and l_2 be two lines that support S such that $l_1 \cap S \neq l_2 \cap S$ and* $\dim(l_1 + l_2) = 2$. *If D is elliptic through l_1 and l_2, then S is made by glueing together two pieces of quadrics along $(l_1 + l_2) \cap S$.*

Consider now the bodies in Example 2.5 (see Figure 2.3). One observes that D_5 is symmetric about the plane where the two pieces of quadric are glued, but there is no uniqueness of supporting hyperplanes at the points in that plane. In contrast, D_6 has the opposite properties, i.e., it lacks the symmetry but has a unique supporting hyperplane at any point. These two bodies inspired Proposition 2.7.

Proposition 2.7. *Let S be the boundary of a convex body $D \subset E^d$ $(d \geq 3)$, and let l_1 and l_2 be two linear varieties such that $1 \leq \dim l_i \leq d - 2$, $i = 1, 2$. Assume that*

(a) *S is elliptic through l_1 and l_2,*

(b) *l_1 and l_2 support S, and $l_1 \cap S \neq l_2 \cap S$,*

(c) *there exists a hyperplane P such that $l_1 + l_2 \subset P$ and one of the following properties holds:*

  (i) *S is symmetric about P and there exists only one supporting hyperplane at every point of $P \cap S$,*

  (ii) *there exists a hyperplane Q, $Q \neq P$, such that $Q \cap P \cap \operatorname{Int} D \neq \emptyset$ and $Q \cap S$ is ellipsoidal.*

Then S is an ellipsoid.

Finally, Corollary 2.8 shows that if we require S to be centrally symmetric, then only one linear variety is necessary in Proposition 2.7.

Corollary 2.8. *Let S be the boundary of a centrally symmetric convex body $D \subset E^d$ $(d \geq 3)$, and let l be a linear variety with $1 \leq \dim l \leq d-2$, that supports S. Let P be a hyperplane that contains l and the centre of S, and assume that one of the properties* (*i*) *or* (*ii*) *in Proposition* 2.7 *holds. If S is elliptic through l, then S is an ellipsoid.*

3. Proofs and remarks

Proof of Theorem 2.1. Case (i) reduces to Theorem 1.1, whose proof can be found in [2]. For cases (ii) and (iii), we shall consider first the proof for $d = 3$ and then for $d > 3$. For a line l, we shall denote the sheaf of planes of axis l by *l-sheaf.*

$(d = 3)$ Case (ii). Let p be the point where the lines l_1 and l_2 meet, and let P be the plane defined by l_1 and l_2. Since S is elliptic through l_1 and l_2, P supports S at p. Let P' be the plane parallel to P that supports S at some other point denoted by p'. Let l_1' and l_2' be the lines parallel to l_1 and l_2, respectively, that meet at p'. For $i = 1, 2$, let E_i be the ellipse that defines in S the plane of the l_i-sheaf that contains p'. For every $x \in P'$, the plane of the l_1-sheaf that contains x cuts E_2 at some point other than p, and consequently it cuts S in an ellipse. The same is the case with the plane of the l_2-sheaf that contains x.

Let us now consider a projective transformation that moves P to the plane at infinity. We continue using the same notation for the images of the points, lines, and planes as described in the above paragraph. Since l_i, $i = 1, 2$, is now in the plane at infinity, the planes of the l_i-sheaf are parallel, and the section of S by any of these planes is a parabola with p as the point at infinity. The plane P' still supports S at p'. Let l_3' be the line that goes through p' and has the direction defined by p. For every $x \in P'$, the plane through x parallel to l_i' and l_3' cuts S in a parabola with p as the point at infinity.

From the above it follows that, taking p' as the origin of the space and considering a coordinate system defined by the lines l_1', l_2', and l_3', we can define a function

$$F : \, (x_1, x_2) \in \mathbb{R}^2 \to F(x_1, x_2) \geq 0,$$

such that $(x_1, x_2, F(x_1, x_2)) \in S$ for every $(x_1, x_2) \in \mathbb{R}^2$. Moreover, we know that with either x_1 or x_2 fixed, the resulting function is parabolic. Therefore, there exist real functions a_i, b_i, $i = 0, 1, 2$, such that

$$\begin{aligned} F(x_1, x_2) = & \; a_0(x_1) + a_1(x_1)x_2 + a_2(x_1)x_2^2 \\ = & \; b_0(x_2) + b_1(x_2)x_1 + b_2(x_2)x_1^2, \end{aligned} \tag{1}$$

for every $(x_1, x_2) \in \mathbb{R}^2$. The restriction of F to any of the coordinate axes is a parabola tangent at the origin. Hence, taking an appropriate basis, we can assume that

$$a_0(x_1) = F(x_1, 0) = x_1^2, \qquad b_0(x_2) = F(0, x_2) = x_2^2\,,$$

so that

$$x_1^2 + a_1(x_1) + a_2(x_1) = F(x_1, 1) = 1 + b_1(1)x_1 + b_2(1)x_1^2,$$
$$x_1^2 - a_1(x_1) + a_2(x_1) = F(x_1, -1) = 1 + b_1(-1)x_1 + b_2(-1)x_1^2.$$

By solving the above two equations for $a_1(x_1)$ and $a_2(x_1)$ and substituting these values into (1), we get

$$F(x_1, x_2) = x_1^2 + x_2^2 + c_1x_1x_2 + c_2x_1^2x_2 + c_3x_1x_2^2 + c_4x_1^2x_2^2,$$

for certain constants $c_1, \dots, c_4$.

We next have to show that F is a quadratic function, i.e., $c_2 = c_3 = c_4 = 0$. At this point, the role played by the convexity of D is crucial. By hypothesis, our original D is a convex body. Therefore, the sections of S by planes parallel to P' enclose convex regions. This means that, after the projective transformation, for every $t > 0$ the closed curve $F(x_1, x_2) = t$ encloses a convex region. As will be seen below, it is this fact that forces F to be a quadratic function. For every $t > 0$, the points

$$(\sqrt{t}, 0, t), \quad (0, \sqrt{t}, t), \quad (0, -\sqrt{t}, t)$$

are in the section of S defined by $F(x_1, x_2) = t$. Let us consider the line defined by the points $(\sqrt{t}, 0, t)$ and $(0, \sqrt{t}, t)$, i.e.,

$$\big(\lambda\sqrt{t},\ (1-\lambda)\sqrt{t},\ t\big), \quad \lambda \in \mathbb{R}. \tag{2}$$

Obviously, a point of this line (identified by λ) is in the section defined by $F(x_1, x_2) = t$ if and only if

$$G_t(\lambda) := F\big(\lambda\sqrt{t},\ (1-\lambda)\sqrt{t}\big) - t = 0.$$

For every $t > 0$, $G_t(\lambda)$ is a polynomial of fourth degree in λ such that $G_t(0) = G_t(1) = 0$. Therefore $G_t(\lambda)$ is divisible by $\lambda(\lambda - 1)$, and we get

$$G_t(\lambda) = \lambda(\lambda - 1)\, t\, H_t(\lambda),$$

where

$$H_t(\lambda) = c_4t\lambda^2 + (c_3\sqrt{t} - c_2\sqrt{t} - c_4t)\lambda + (2 - c_1 - c_3\sqrt{t}\,).$$

Each real root of $H_t(\lambda)$ defines a point where the line (2) cuts the curve $F(x_1, x_2) = t$. Then, since the region enclosed by $F(x_1, x_2) = t$ is convex, the only possible real roots of $H_t(\lambda)$ are 0 and 1.

Recall that we want to show that $c_2 = c_3 = c_4 = 0$. Assume first that $c_4 \neq 0$. Since

$$\frac{(c_3\sqrt{t} - c_2\sqrt{t} - c_4t)^2 - 4c_4t(2 - c_1 - c_3\sqrt{t})}{t^2} \xrightarrow[t \to +\infty]{} c_4^2 > 0,$$

it follows that there exists $t_0 > 0$ such that if $t > t_0$ then the discriminant of $H_t(\lambda)$ is positive. Hence, its roots $\lambda_1(t)$, $\lambda_2(t)$ can only take the values 0 and 1. From the identity $H_t(\lambda) = c_4t(\lambda - \lambda_1(t))(\lambda - \lambda_2(t))$, it follows that

$$c_4\lambda_1(t)\lambda_2(t) = \frac{2 - c_1}{t} - \frac{c_3}{\sqrt{t}}.$$

For $t > t_0$, the left-hand member of the above identity can only take the values c_4 or 0, which implies $c_1 = 2$ and $c_3 = 0$.

Considering now the line through the points $(\sqrt{t}, 0, t)$ and $(0, -\sqrt{t}, t)$ and repeating the previous arguments, we have that for t large enough the polynomial

$$\bar{H}_t(\lambda) = c_4 t\lambda^2 + (c_3\sqrt{t} + c_2\sqrt{t} - c_4 t)\lambda + (2 + c_1 - c_3\sqrt{t})$$

has the roots 0 and 1 only. This implies $c_1 = -2$ and $c_3 = 0$, in contradiction with the previous result. Therefore $c_4 = 0$, and

$$H_t(\lambda) = \sqrt{t}\,(c_3 - c_2)\lambda + (2 - c_1 - c_3\sqrt{t}),$$

$$\bar{H}_t(\lambda) = \sqrt{t}\,(c_3 + c_2)\lambda + (2 + c_1 - c_3\sqrt{t}).$$

Assume now that either c_2 or c_3 is non-zero. Then, either $c_3 - c_2 \neq 0$ or $c_3 + c_2 \neq 0$. Assume $c_3 - c_2 \neq 0$. In that case the root of $H_t(\lambda)$ is

$$\lambda(t) = \frac{c_1 - 2 + c_3\sqrt{t}}{(c_3 - c_2)\sqrt{t}}$$

for every $t > 0$. If $\lambda(t) = 0$, we get $c_1 = 2$ and $c_3 = 0$. Therefore, $c_2 \neq 0$ and $\bar{H}_t(\lambda)$ has the root $-4(c_2\sqrt{t})^{-1}$, which is an absurdity. On the other hand, if $\lambda(t) = 1$ then $c_1 = 2$ and $c_2 = 0$, and we again get an absurdity by considering the root of $\bar{H}_t(\lambda)$. The alternative assumption that $c_3 + c_2 \neq 0$ leads to the same conclusion. Hence, $c_2 = c_3 = 0$.

We have thus proved that F is a quadratic function. The corresponding quadric then comes (via the projective transformation) from a quadric contained in the original S. But S is bounded, which implies that S has to be an ellipsoid.

$(d = 3)$ Case (iii). Let p_1 and p_2 be the points where the lines l_1 and l_2, respectively, support D. Necessarily, $p_1 \neq p_2$. Let P_1 and $\bar{P}_1$ be two planes of the l_1-sheaf that cut the interior of D but do not contain p_2. Then, $E_1 = P_1 \cap S$ and $\bar{E}_1 = \bar{P}_1 \cap S$ are ellipses. Let P_2 be a plane of the l_2-sheaf that does not contain p_1 and cuts E_1 at two points a, b, and $\bar{E}_1$ also at two points $\bar{a}$, $\bar{b}$. Let $E_2 = P_2 \cap S$. Then $E_1 \cap E_2 = \{a, b\}$ and $\bar{E}_1 \cap E_2 = \{\bar{a}, \bar{b}\}$.

Let $q \in \bar{E}_1$ be a point different from p_1, $\bar{a}$, and $\bar{b}$. There then exists a unique quadric C that contains E_1, E_2, and q. The conic $\bar{P}_1 \cap C$ goes through the points p_1, $\bar{a}$, $\bar{b}$, and q. Moreover, l_1 is tangent to $\bar{P}_1 \cap C$ at p_1 because l_1 is tangent to E_1 at that point, $E_1 \subset C$, and $l_1 = P_1 \cap \bar{P}_1$. Therefore, $\bar{P}_1 \cap C = \bar{E}_1$.

Let $\bar{P}_2$ be the plane of the l_2-sheaf that contains p_1. Then $l_1 \not\subset \bar{P}_2$, and $\bar{P}_2$ cuts each ellipse defined in S by planes of the l_1-sheaf at another point besides p_1. This forces $\bar{P}_2$ to cut the interior of D, so that $\bar{P}_2 \cap S$ is an ellipse. This ellipse has the points p_1, p_2 and the other two points where $\bar{P}_2$ cuts E_1 and $\bar{E}_1$ in common with the conic $\bar{P}_2 \cap C$. Moreover, l_2 is tangent to $\bar{P}_2 \cap S$ at p_2, and it is also tangent to $\bar{P}_2 \cap C$ at that point because it is tangent to $E_2 \subset C$. We conclude that $\bar{P}_2 \cap S = \bar{P}_2 \cap C$.

The planes P_2 and $\bar{P}_2$ divide the space into four quadrants. Let M be the quadrant that contains the triangle of vertices a, b, and p_1, and let Q be any plane of the l_2-sheaf that goes through M. The ellipse $Q \cap S$ and the conic $Q \cap C$ coincide at p_2 (with the same tangent) and at the four points where they cut E_1 and $\bar{E}_1$. Therefore, $Q \cap S = Q \cap C$, so that, in the region M, the quadric C coincides with S.

Each plane of the l_1-sheaf that goes through the interior of D cuts S in an ellipse with tangent l_1 at p_1. But l_1 passes through $\bar{P}_2$ at p_1. Hence, part of that ellipse is in M, and from the above paragraph it follows that the entire ellipse is in C. Similarly, we get that any plane of the l_2-sheaf that goes through the interior of D cuts S in an ellipse that also is in C.

Finally, let $x \in S$, $x \neq p_1$. If the plane of the l_1-sheaf that contains x cuts the interior of D, then x is in an ellipse contained in C. On the other hand, assume that this plane does not cut the interior of D. In this case the segment $\overline{xp_1}$ is in S. Let (x_n), $x_n \neq x$, be a sequence in $\overline{xp_1}$ such that $x_n \to x$. For each n, the plane of the l_2-sheaf that contains x_n cuts the interior of D. Then $x_n \in C$, which implies $x \in C$. We therefore have that $S \subset C$, and consequently C is an ellipsoid, and $S = C$.

($d > 3$) In both cases, (ii) and (iii), we can assume that $\dim l_1 = \dim l_2 = d-2$, since otherwise if, for example, $\dim l_1 < d-2$, then we can take a $(d-2)$-dimensional linear variety L_1 such that $l_1 \subset L_1$, $l_2 \not\subset L_1$ and L_1 cuts the interior of D. The varieties L_1 and l_2 satisfy the hypothesis of Case (i), so that S is an ellipsoid.

Having proved the two cases for $d = 3$, we can proceed by induction on d assuming that they are true for dimension $d-1$. The proof will consist in showing that from the hypotheses of Cases (ii) and (iii) it follows that there exists a plane L such that $L \not\subset l_1$, $l_1 \not\subset L$, $L \cap \operatorname{Int} D \neq \emptyset$, and S is elliptic through L. Then, from Case (i) we get that S is an ellipsoid.

Let p_1 and p_2 be the points where l_1 and l_2, respectively, support S.

Case (ii). In this case $p_1 = p_2$. Let L be a plane that contains p_1 and a point $p \in l_1 \setminus l_2$, and that cuts the interior of D. To show that S is elliptic through L, let $\widehat{E}$ be a hyperplane in E^d such that $L \subset \widehat{E}$ and $\widehat{E} \cap \operatorname{Int} D \neq \emptyset$. If $l_i \subset \widehat{E}$ for $i = 1$ or 2, then, from the hypothesis, we have that $\widehat{E} \cap S$ is an ellipsoid. On the other hand, if $l_1 \not\subset \widehat{E}$ and $l_2 \not\subset \widehat{E}$, then, from the hypothesis on l_1 and l_2, it follows easily that the $(d-1)$-dimensional convex body $\widehat{E} \cap S$ and the $(d-3)$-dimensional linear varieties $\widehat{E} \cap l_1$ and $\widehat{E} \cap l_2$ satisfy the hypotheses of Case (ii). Therefore, from the hypothesis of induction, we get that $\widehat{E} \cap S$ is an ellipsoid.

Case (iii). Now $p_1 \neq p_2$. Let $\overrightarrow{l_1}$ and $\overrightarrow{l_2}$ be the subspaces such that $l_1 = p_1 + \overrightarrow{l_1}$ and $l_2 = p_2 + \overrightarrow{l_2}$. Since $\dim(l_1 + l_2) = d$, there exists $\overrightarrow{v} \in \overrightarrow{l_1} \setminus \overrightarrow{l_2}$. Let L be a plane such that $p_1, p_2 \in L$ and $\overrightarrow{v}$ is in the subspace of directions of L. Since S is elliptic through l_1 and l_2, the line containing p_1 and p_2 cuts the interior of D and so does L. The proof follows as in Case (ii), but changing "hypotheses of Case (ii)" to "hypotheses of Case (iii)". □

Proof of Lemma 2.6. Let $P = l_1 + l_2$, and let p_1 and p_2 be the points where l_1 and l_2, respectively, support S. For $i = 1, 2$, the line l_i is tangent to all the ellipses defined in S by planes of the l_i-sheaf.

Let us first assume that l_1 and l_2 are parallel. Consider $u \in \operatorname{Int} D$, $u \notin P$, and let E_i $(i = 1, 2)$ be the ellipse defined in S by the plane of the l_i-sheaf that contains u. The ellipses E_1 and E_2 meet in two points, and the line defined by these points is parallel to l_1 and l_2. Let $\bar{E}_2$ be the ellipse defined in S by another plane of the l_2-sheaf that cuts E_1

in two points. It is a simple matter to check that in the above situation there exists a unique quadric C that contains the three ellipses.

The plane P and the plane that contains E_1 divide the space into four quadrants. Let us denote by M and N the quadrants that contain u, and assume that $p_2 \in M$. Each plane of the l_1-sheaf that cuts the interior of M cuts S in an ellipse and cuts C in a conic. But the two curves share the point p_1 and the four points where they cut E_2 and $\bar{E}_2$. Therefore, S and C coincide in the interior of M.

Now, let $x \in S$. If $x \in \operatorname{Int} M$, then $x \in C$. Assume that $x \in N$ and let P_x be the plane of the l_2-sheaf that contains x. If P_x cuts the interior of D then $P_x \cap S$ is an ellipse that is partially contained in $\operatorname{Int} M$. Therefore, $P_x \cap S = P_x \cap C$ and $x \in C$. If P_x does not cut the interior of D, then the segment $\overline{xp_2}$ is in S and is partially contained in $\operatorname{Int} M$. But in $\operatorname{Int} M$ the quadric C coincides with S. Therefore the entire line defined by x and p_2 is in C. In particular, $x \in C$.

To summarize, if D has interior points in one of the two open semi-spaces defined by P then, in that semi-space, S coincides with a bounded piece of quadric. In this semi-space, therefore, the sections of S by planes parallel to P must be ellipses, and the limit of these ellipses as the planes converge to P must be a bounded conic, i.e., an ellipse. If one of the semi-spaces has no interior points of D, then $S \cap P$ is the disc defined by the above limit ellipse. Such is the case, for example, when D is a cone.

Finally, assume that l_1 and l_2 meet in a point q. It is obvious that $q \notin D$. By considering a plane that goes through q but does not touch D, and a projective transformation that moves this plane into the plane at infinity, we return to the first case. □

Proof of Proposition 2.7. (d=3) In this case, $P = l_1 + l_2$. From Lemma 2.6 we know that S is made up of two pieces of quadrics C and C'.

(i) From the symmetry about P, it follows that D has interior points in the two semi-spaces defined by P, and also that C and C' are not planes. Let m be a line that defines the symmetry about P, and let Q be any plane parallel to m that cuts D. Then $Q \cap S$ is symmetric about $Q \cap P$. The section $Q \cap S$ is made up of two pieces of the conics $Q \cap C$ and $Q \cap C'$ that meet at the points $\{p, p'\} = P \cap Q \cap S$. From the hypothesis of symmetry and supporting hyperplane uniqueness, it follows that at p (respectively, p') the tangents to $Q \cap C$ and $Q \cap C'$ coincide and are parallel to m. This forces $Q \cap C$ and $Q \cap C'$ to be ellipses contained in S, and $Q \cap C = Q \cap C'$. Varying the plane Q, we get that $C = C' = S$, and consequently S is an ellipsoid.

(ii) Since $Q \cap S$ is an ellipse, there are interior points of D in the two semi-spaces defined by P. Moreover, since there is a piece of the ellipse $Q \cap S$ in both of the quadrics C and C', the entire ellipse is in both quadrics, i.e., $Q \cap S = Q \cap C = Q \cap C'$. The quadrics C and C' also meet at the ellipse $P \cap S$. Therefore, to show that $C = C'$, and consequently that $S = C = C'$ is an ellipsoid, we only need to show that they share another point.

Let $q_1, q_2 \in Q \cap S$ be two points such that q_1 is in one of the semi-spaces defined by P and q_2 in the other. Let m be the line through q_1 and q_2. Since Q cuts C and C' in an ellipse, we can take another plane R in the m-sheaf, "close enough" to Q for $R \cap C$ and $R \cap C'$ to still be ellipses. Let $\{p_1, p_2\} = R \cap P \cap S$. The four points p_1, p_2, q_1, and q_2 are in the ellipses $R \cap C$ and $R \cap C'$. We shall now show that the two ellipses coincide, which allows us to get the new point (in fact, an entire ellipse) we are looking for in $C \cap C'$.

The section $R \cap S$ is a closed convex curve composed of two arcs of ellipses, one in $R \cap C$ and the other in $R \cap C'$, that meet in p_1 and p_2. Let us assume that $\widehat{p_1q_1p_2}$ is the arc in $S \cap C$ and $\widehat{p_1q_2p_2}$ is the arc in $S \cap C'$. If the two ellipses coincide at some point other than p_1, p_2, q_1, and q_2, then they are the same. They must also be the same if at some of those points the tangents to each ellipse coincide. Let us assume that the two ellipses are different, and that, for example, the arc $\widehat{q_1p_1}$ in $R \cap C$ is outside $R \cap C'$. Then the arc $\widehat{p_1q_2}$ in $R \cap C$ is inside $R \cap C'$, which implies that in p_1 the curve $R \cap S$ is not convex. Other possible situations lead to the same absurdity.

($d > 3$) As in the proof of Theorem 1.1, we can assume that $\dim l_1 = \dim l_2 = d - 2$, and the proof is again based on showing that S is elliptic through a plane L that cuts the interior of D.

Let $\{p_i\} = l_i \cap S$, $i = 1, 2$, and let l be the line through p_1 and p_2. In both cases, (i) and (ii), it follows easily that l cuts the interior of D.

Case (i). Let l' be the line through the point midway between p_1 and p_2 that has the direction defined by the symmetry about P.

Case (ii). If $l \subset Q$, let $q \in Q \setminus P$. If $l \not\subset Q$, let $q \in Q \cap P \cap \operatorname{Int} D$, $q \notin l$. Let l' be the line through p_1 and q.

The plane $L = l + l'$ cuts the interior of D. That S is elliptic through L follows as in Theorem 1.1. □

Proof of Corollary 2.8. The proof follows from Proposition 2.7 by taking $l_1 = l$ and considering as l_2 the linear variety symmetric to l about the centre of S. □

Remarks on Example 2.2. The set D_1 follows from the set $D_1' = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 + x^2y^2 \leq 1\}$ by applying the projectivity that fixes the plane $x + z + 2 = 0$ (which does not intersect D_1') as the plane at infinity. In [1] it is proved that D_1' is a convex body.

In [1] we showed that the set

$$\bar{D}_2 = \left\{(\bar{x}, \bar{y}, \bar{z}) \in \mathbb{R}^3 : \bar{x}^2 + \bar{y}^2 + \bar{z}^2 + \bar{x}\bar{y}\bar{z} \leq 1,\ \max\{|\bar{x}|, |\bar{y}|, |\bar{z}|\} \leq 1\right\}$$

is a convex body, and that the sections by planes parallel to any of the planes $\bar{x} = 0$, $\bar{y} = 0$, or $\bar{z} = 0$ are ellipses; i.e., $\bar{D}_2$ is elliptic through the lines at infinity, $\bar{l}_1$, $\bar{l}_2$, and $\bar{l}_3$, of those planes. If we apply to $\bar{D}_2$ the projective transformation

$$x = \frac{\bar{x}}{\bar{x} + \bar{z} - 5}, \qquad y = \frac{\bar{y}}{\bar{x} + \bar{z} - 5}, \qquad z = \frac{\bar{z}}{\bar{x} + \bar{z} - 5},$$

we get D_2, which is still a convex body because the plane $\bar{x} + \bar{z} - 5 = 0$ does not touch $\bar{D}_2$. The lines $\bar{l}_1$, $\bar{l}_2$, and $\bar{l}_3$ are transformed into the lines l_1, l_2, and l_3, respectively.

Remarks on Example 2.3. The set D_3 is not convex because the points $(\frac{8}{17}, 0, \frac{32}{17})$ and $(0, \frac{8}{17}, \frac{32}{17})$ are in D_3, but the midpoint is not.

Let $(x, y, z) \in D_3$, $z < 2$. Since $z(z-2)^3 \leq 0$, it follows that $z \geq 0$. From $x^2(z-2)^2 \leq (x^2+y^2)(z-2)^2+x^2y^2 \leq z(2-z)^3$, we get $x^2 \leq z(2-z) \leq 1$. Similarly, $y^2 \leq 1$. Therefore D_3 is bounded.

To see that D_3 is closed, let $(x_n, y_n, z_n) \in B$ be such that $x_n \to x$, $y_n \to y$, $z_n \to z$. Then $(x^2+y^2)(z-2)^2+x^2y^2+z(z-2)^3 \leq 0$, and $z \leq 2$. If $z < 2$, then $(x, y, z) \in D_3$. On the other hand, if $z = 2$ then it follows from the inequality $x_n^2 \leq z_n(2-z_n)$ that $x^2 \leq z(2-z) = 0$, so that $x = 0$; similarly, $y = 0$.

It may be interesting to remark on how D_3 was conceived. If one considers

$$\bar{x} = \frac{x}{2-z}, \qquad \bar{y} = \frac{y}{2-z}, \qquad \bar{z} = \frac{z}{2-z},$$

then it is easy to see that

$$\bar{x}^2 + \bar{y}^2 + \bar{x}^2\bar{y}^2 - \bar{z} \sim 0$$

if and only if

$$(x^2+y^2)(z-2)^2 + x^2y^2 + z(z-2)^3 \sim 0,\ z < 2,$$

where $\sim$ denotes either "$\leq$" or "$=$". I.e., the boundary of D_3 comes from a projective transformation of the graph of the function $F(\bar{x}, \bar{y}) = \bar{x}^2 + \bar{y}^2 + \bar{x}^2\bar{y}^2$, adding the point $(0, 0, 2)$ that was originally at infinity. The restriction of F to the lines $\bar{x} = \text{const}$, $\bar{y} = \text{const}$, gives rise to parabolas that all meet at the same point at infinity. These parabolas are transformed into the ellipses defined by the l_1 and l_2 sheafs in D_3. The sections of the graph of F by planes $z = \text{const}$, for large enough z, are not convex.

Remarks on Example 2.4. The set D_4 was studied in detail in [2], where we showed that it follows from the set

$$D_4' = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2(1-z) - z(1-z)^2 \leq 0,\ 0 \leq z < 1\} \cup \{(0, 0, 1)\}$$

by applying the projectivity that sends the plane $x + 1 = 0$ (which does nor intersect D_4') to infinity. It is easier to see that D_4' is a convex body than D_4. All the sections of D_4' by planes parallel to the plane $z = 0$ or that contain the line $\{x = 0,\ z = 1\}$ (which is tangent to D_4') are ellipses.

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