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Hausdorff Distance and Convex Sets

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If A and B are two bounded, closed, non-empty, and convex subsets of a normed space X, then the Hausdorff distance between A and B is the same as the Hausdorff distance between the boundary of A and the boundary of B.

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The following definition is taken from [4, p. 274].

Definition 1. Let (X, d) be a metric space, with A and B nonempty subsets of X. Define $d(A, B) = \inf\{d(\mathbf{a}, \mathbf{b}) \mid \mathbf{a} \in A, \mathbf{b} \in B\}$. For $\gamma > 0$, let us define $A_\gamma = \{\ \mathbf{x}\ \in X \mid d(\{\mathbf{x}\}, A) < \gamma\}$ and $B_\gamma = \{\mathbf{x} \in X \mid d(\{\mathbf{x}\}, B) < \gamma\}$. Define $d_H(A, B) = \inf\{\gamma > 0 \mid A \subset B_\gamma, B \subset A_\gamma\}$. Then $d_H(A, B)$ is the **Hausdorff distance** between A and B.

The Hausdorff distance as defined above is not strictly speaking a metric on the subsets of X. If A and B are not required to be closed, then d_H is not positive definite. If A and B are not required to be bounded, then $d_H(A, B)$ may not be finite. We note that d_H as a map on ordered pairs of subsets of X is well defined for any A and B, nonempty subsets of X, provided that we allow the range of d_H to be $[0, \infty]$. Further, the triangle inequality always holds. d_H is a metric when A and B are required to be closed, bounded, and non-empty [4, p. 274].

We now introduce notation that we will use throughout the paper.

Notation 2. The pair $(X, ||.||)$ is a (real or complex) normed linear space, and d_H is the Hausdorff distance induced by the norm. In the examples (except Example 21), $X = \mathbb{R}^2$ with the Euclidian norm. For $B \subset X$, B° denotes the interior of B, $Cl(B)$ denotes the closure of B, $B^c = X \backslash B$ is the complement of B in X, and ∂B denotes the topological boundary of B. Unless otherwise stated, all subsets of X under consideration will have nonempty boundary. For $\mathbf{a}, \mathbf{b} \in X$, we say that $L(\mathbf{a}, \mathbf{b}) := \{(1 - t)\mathbf{a} + t\mathbf{b} \mid 0 \le t \le 1\}$ is the **line segment** connecting $\mathbf{a}$ and $\mathbf{b}$. $D_t(\mathbf{a})$ denotes the open ball of radius $t > 0$ centered at $\mathbf{a}$.

The following lemma is well known, but we give details for completeness.

Lemma 3. *Let B be a nonempty subset of X and $\mathbf{a} \notin B^\circ$. Let $\mathbf{b} \in Cl(B)$. Then there exists $\mathbf{s} \in L(\mathbf{a}, \mathbf{b}) \cap \partial B$ and $||\mathbf{a} - \mathbf{s}|| \le ||\mathbf{a} - \mathbf{b}||$.*

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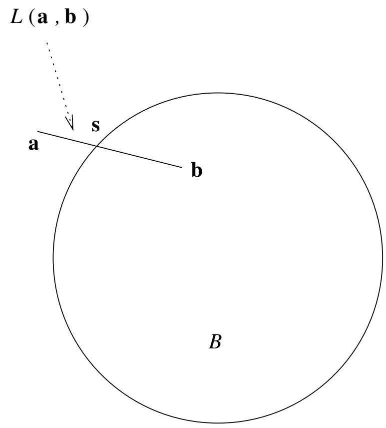

Figure 1: Illustration for Lemma 3

Proof. If $\mathbf{b} \in \partial B$ we can take $\mathbf{s} = \mathbf{b}$. We may therefore assume that $\mathbf{b} \in B^\circ$. If $\mathbf{a} \in Cl(B)$, then $\mathbf{a} \in Cl(B) \backslash B^\circ = \partial B$, and so we may take $\mathbf{s} = \mathbf{a}$, and the inequality is immediate. We may therefore assume that $\mathbf{a} \notin Cl(B)$, and so $\mathbf{a} \in X \backslash Cl(B) = (B^c)^\circ$.

We claim that the map $\phi : [0,1] \to L(\mathbf{a},\mathbf{b})$, given by $\phi(t) = (1-t)\mathbf{a} + t\mathbf{b}$, is a homeomorphism. To see this, we note that ϕ is a bijection on the specified range and domain. It is immediate that ϕ is continuous.

Now ϕ is bijective and continuous, $[0,1]$ is compact, and $L(\mathbf{a},\mathbf{b})$ is Hausdorff. Therefore ϕ^{-1} is also continuous [6, pp. 83–89], and $L(\mathbf{a},\mathbf{b})$ is connected [5, p. 182].

If $\partial B \cap L(\mathbf{a},\mathbf{b}) = \emptyset$, then

$$(B^\circ \cap L(\mathbf{a},\mathbf{b})) \cup ((B^c)^\circ \cap L(\mathbf{a},\mathbf{b})) = L(\mathbf{a},\mathbf{b}).$$

Since $\mathbf{b} \in L(\mathbf{a},\mathbf{b}) \cap B^\circ$, and $\mathbf{a} \in L(\mathbf{a},\mathbf{b}) \cap (B^c)^\circ$, this contradicts the connectedness of $L(\mathbf{a},\mathbf{b})$. Therefore $\partial B \cap L(\mathbf{a},\mathbf{b}) \neq \emptyset$.

It is immediate that for $\mathbf{s} \in \partial B \cap L(\mathbf{a},\mathbf{b})$, we have that $||\mathbf{a}-\mathbf{s}|| \leq ||\mathbf{a}-\mathbf{b}||$. □

Lemma 4. *Suppose A and B are subsets of X. Let $d_H(A,B) = \epsilon \in [0,\infty)$. If $\mathbf{a} \in Cl(A) \backslash B^\circ$ then $\mathbf{a} \in (\partial B)_\gamma$ for all $\gamma > \epsilon$.*

Proof. Fix $\gamma > \epsilon$. We have by hypothesis that $A \subset B_\gamma$ and $B \subset A_\gamma$. Let $\mathbf{a} \in Cl(A) \backslash B^\circ$. Let $\{\mathbf{a}_n\}_{n\in\mathbb{N}}$ be a sequence in A converging to $\mathbf{a}$. Without loss of generality $||\mathbf{a}_n - \mathbf{a}|| \leq \frac{1}{n}$. For each $\mathbf{a}_n$, there exists $\mathbf{b}_n \in B$ such that $||\mathbf{a}_n - \mathbf{b}_n|| < \epsilon + \frac{1}{n}$. This is because $A \subset B_{\epsilon+\frac{1}{n}}$. Then

$$||\mathbf{a}-\mathbf{b}_n|| \leq ||\mathbf{a}-\mathbf{a}_n|| + ||\mathbf{a}_n - \mathbf{b}_n|| \leq \epsilon + \frac{2}{n}.$$

For sufficiently large n, $\epsilon + \frac{2}{n} < \gamma$. Fix such an n. If $\mathbf{b}_n \in \partial B$ we are done.

Otherwise, $\mathbf{b}_n \in B^\circ$. In this situation, we argue in the following way: since $\mathbf{a}$ is not in B°, we may apply Lemma 3 to the line segment $L(\mathbf{a},\mathbf{b}_n)$. There exists $\mathbf{s}_n$ in $L(\mathbf{a},\mathbf{b}_n) \cap \partial B$ and

$$||\mathbf{a}-\mathbf{s}_n|| \leq ||\mathbf{a}-\mathbf{b}_n|| \leq \epsilon + \frac{2}{n} < \gamma.$$

□

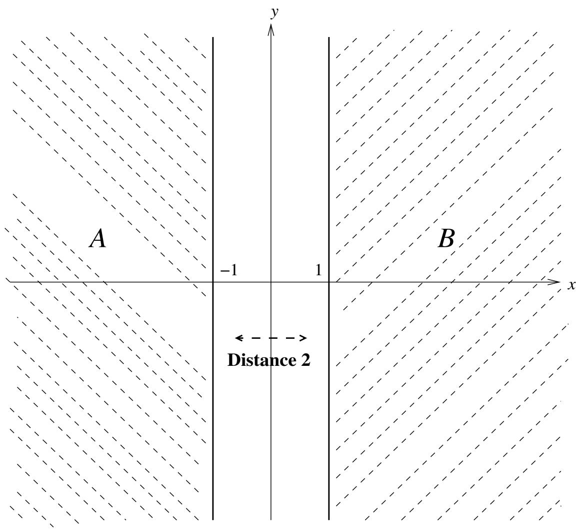

Figure 2: Illustration for Example 6

Corollary 5. *If A and B are subsets of X, $\partial A \cap B^\circ = \emptyset$ and $\partial B \cap A^\circ = \emptyset$, then $d_H(A, B) \geq d_H(\partial A, \partial B)$.*

Proof. By hypothesis, no $\mathbf{a} \in \partial A$ can be in B° and hence, by Lemma 4, $\mathbf{a} \in (\partial B)_\gamma$ for any $\gamma > \epsilon := d_H(A, B)$, and every $\mathbf{a} \in \partial A$. A symmetrical argument holds for ∂B and so

$$d_H(\partial A, \partial B) \leq \inf_{\gamma > \epsilon} \gamma = \epsilon.$$

□

Example 6. We note that the inequality in Corollary 5 may be strict.

Let $X = \mathbb{R}^2$, $A = \{(x, y) \mid x \leq -1\}$ and $B = \{(x, y) \mid x \geq 1\}$. Then $\partial A = \{(x, y) \mid x = -1\}$ and $\partial B = \{(x, y) \mid x = 1\}$. We have

$$d_H(A, B) = \infty > 2 = d_H(\partial A, \partial B).$$

Example 7. Corollary 5 is false if we do not require $\partial B \cap A^\circ$ and $\partial A \cap B^\circ$ to be empty.

Proof. Let $X = \mathbb{R}^2$. Let $\mathbf{a} = (0, 0)$, $\mathbf{b} = (1, 1)$, $\mathbf{c} = (\frac{1}{2}, 0)$, $\mathbf{d} = (1, -1)$, and $\mathbf{e} = (\frac{2}{3}, 0)$. Let A be the quadrilateral **abcd**, and let B be the quadrilateral **abed**. Here we include the interior of the quadrilaterals. Note that $A \subset B$, so $A \subset B_\epsilon$ for any $\epsilon > 0$. The point in B that is farthest away from any point in A is $\mathbf{e}$ and it is easy to check that $d(\{\mathbf{e}\}, \partial A) = d(\{\mathbf{e}\}, A) = \frac{\sqrt{5}}{15}$. Hence, $d_H(A, B) = \frac{\sqrt{5}}{15}$.

On the other hand, the point on ∂A farthest from any point on ∂B is $\mathbf{c}$, and we find that the Hausdorff distance between ∂A and ∂B is $\frac{1}{6}$. Thus, we have that

$$d_H(\partial A, \partial B) = \frac{1}{6} > \frac{\sqrt{5}}{15} = d_H(A, B).$$

□

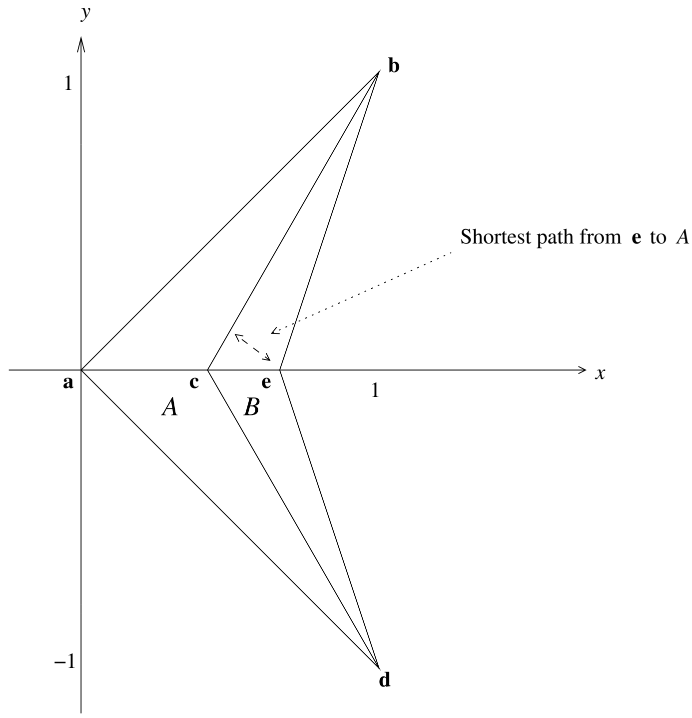

Figure 3: Illustration for Example 7

Notation 8. For $A \subset X$ non-empty and bounded, and ϕ a non-zero real continuous linear functional on X, define:

$$\begin{aligned} s(\phi, A) &= \sup\{\phi(\mathbf{x}) \mid \mathbf{x} \in A\} \\ H_{A,\phi} &= \{\mathbf{x} \in X \mid \phi(\mathbf{x}) = s(\phi, A)\} \\ H^{-}_{A,\phi} &= \{\mathbf{x} \in X \mid \phi(\mathbf{x}) \leq s(\phi, A)\}. \end{aligned}$$

We recall the Ascoli formula [1, p. 5]: For $c \in \mathbb{R}$, define $H = \{\mathbf{x} \in X \mid \phi(x) = c\}$. Then for $\mathbf{a} \in X$,

$$d(\mathbf{a}, H)||\phi|| = |\phi(\mathbf{a}) - c|.$$

Remark 9. Suppose A and B are nonempty bounded convex subsets of X, $||\phi|| = 1$. If there exists $\mathbf{a} \in \partial A \cap B^{\circ}$ such that $\phi(\mathbf{a}) = s(\phi, A)$, then by Corollary 3.2.8 in [1],

$$d_H(A, B) \geq s(\phi, B) - s(\phi, A) = s(\phi, B) - \phi(\mathbf{a}).$$

Proposition 10. *Suppose that $A, B \subset X$ are bounded, convex and have nonempty interiors. Let $\epsilon = d_H(A, B)$. Then for $\mathbf{a} \in \partial A \cap B$ we have $\mathbf{a} \in (\partial B)_{\gamma}$ for all $\gamma > \epsilon$.*

Proof. Fix $\gamma > \epsilon$. Suppose that $\mathbf{a} \in \partial A \cap B$. If $\mathbf{a} \in \partial B$, then $\mathbf{a} \in (\partial B)_{\gamma}$. Therefore we may assume that $\mathbf{a} \in B^{\circ}$. Further, via the translation $\mathbf{x} \to \mathbf{x} - \mathbf{a}$ we may assume that $\mathbf{a} = \mathbf{0}$. Since A is convex, A° is convex [3, pp. 110–112]. Then $\{\mathbf{0}\}$ and A° are disjoint convex sets. By the Hahn-Banach separation theorem [2, pp. 4–8], there exists a real-valued continuous linear functional, ϕ (with $||\phi|| = 1$), on X such that $0 = \phi(\mathbf{0}) = s(\phi, A)$. By the Ascoli formula, $d(\mathbf{0}, H_{B,\phi}) = s(\phi, B)$. For $n \in \mathbb{N}$, there exists $\mathbf{h}_n \in H_{B,\phi}$ such that

$||\mathbf{h}_n|| \leq s(\phi, B) + \frac{1}{n}$. Now $\mathbf{0} \in B^\circ$ and $B \subset H^-_{B,\phi}$. Hence, by Lemma 3 there exists $\mathbf{c}_n \in \partial B \cap L(\mathbf{0}, \mathbf{h}_n)$. By Remark 9,

$$||\mathbf{c}_n|| \leq ||\mathbf{h}_n|| \leq s(\phi, B) + \frac{1}{n} \leq d_H(A, B) + \frac{2}{n}.$$

Hence for sufficiently large n,

$$d(\mathbf{0}, \partial B) \leq ||\mathbf{c}_n|| \leq d_H(A, B) + \frac{2}{n} < \gamma.$$

□

The argument of Proposition 10 relies on the usual topology of the ambient space. For example, if a convex set $A \subset \mathbb{R}^n$ has dimension $n-1$, we automatically have that $Cl(A) = \partial A \supset A$, and hence $d_H(A, \partial A) = 0$. Suppose A and B are convex subsets of $\mathbb{R}^n$, $m, k \in \mathbb{N}$, $\inf(m, k) < n$, A is locally homeomorphic to $\mathbb{R}^m$, and B is locally homeomorphic to $\mathbb{R}^k$. The convexity of A and B guarantees that they will lie in hyperplanes in $\mathbb{R}^n$ of dimension m and k respectively. Thus, we may think of A and B as subsets of $\mathbb{R}^m$ and $\mathbb{R}^k$ respectively. Then one may ask whether Proposition 10 holds if we consider the boundaries of A and B with respect to the usual topologies on $\mathbb{R}^m$ and $\mathbb{R}^k$ respectively. The answer to that question is no as the following example makes clear.

Example 11. Let $n = 2$, $m = k = 1$. Let $A = \{0\} \times [-1, 1]$ and $B = [-1, 1] \times \{0\}$. Then using the $\mathbb{R}$ topology for the boundaries,

$$d_H(\partial A, \partial B) = \sqrt{2} > 1 = d_H(A, B).$$

Proposition 12. *If $A, B \subset X$ are non-empty, bounded and convex then $d_H(A, B) \leq d_H(\partial A, \partial B)$.*

Proof. Let $d_H(\partial A, \partial B) = \epsilon$. Let $\mathbf{a} \in A$. It is enough to consider the case when $a \in A^\circ$. We will show that for every $n \in \mathbb{N}$, $\mathbf{a} \in B_{\epsilon+\frac{2}{n}}$. If $\mathbf{a} \in Cl(B)$ then $\mathbf{a} \in B_\epsilon \subset B_{\epsilon+\frac{2}{n}}$. Suppose that $\mathbf{a}$ is not in $Cl(B)$. Let $\delta = \inf\{||\mathbf{a} - \mathbf{b}|| \mid \mathbf{b} \in B\}$. Since $\mathbf{a} \in A^\circ \cap (X \backslash Cl(B))$, an open set, we have that $0 < \delta$. Since B is non-empty, δ is finite. Given a fixed $n \in \mathbb{N}$, there exists $\mathbf{b}_n \in B$ such that $||\mathbf{a} - \mathbf{b}_n|| \leq \delta + \frac{1}{n}$. Since $\mathbf{a} \in A^\circ$ and A is bounded, there exists $\mathbf{f}_n \in \partial A$ such that $\mathbf{a} \in L(\mathbf{b}_n, \mathbf{f}_n)$. Now $\mathbf{f}_n \in \partial A$ implies there exists $\mathbf{c}_n \in \partial B$ such that $||\mathbf{f}_n - \mathbf{c}_n|| \leq \epsilon + \frac{1}{n}$. Since $Cl(B)$ is convex, $L(\mathbf{b}_n, \mathbf{c}_n) \subset Cl(B)$. There exists $t \in [0, 1]$ such that $\mathbf{a} = (1-t)\mathbf{b}_n + t\mathbf{f}_n$. Define $\mathbf{d}_n = ((1-t)\mathbf{b}_n + t\mathbf{c}_n) \in Cl(B) \cap L(\mathbf{b}_n, \mathbf{c}_n)$.

Then

$$||\mathbf{a} - \mathbf{d}_n|| = ||((1-t)\mathbf{b}_n + t\mathbf{f}_n) - ((1-t)\mathbf{b}_n + t\mathbf{c}_n)|| = ||t(\mathbf{f}_n - \mathbf{c}_n)|| \leq t||\mathbf{f}_n - \mathbf{c}_n|| \leq \epsilon + \frac{1}{n}.$$

Since $\mathbf{d}_n \in Cl(B)$ there exists $\mathbf{e}_n \in B$ such that $||\mathbf{d}_n - \mathbf{e}_n|| \leq \frac{1}{n}$. By the triangle inequality

$$||\mathbf{a} - \mathbf{e}_n|| \leq ||\mathbf{a} - \mathbf{d}_n|| + ||\mathbf{d}_n - \mathbf{e}_n|| \leq \epsilon + \frac{2}{n}.$$

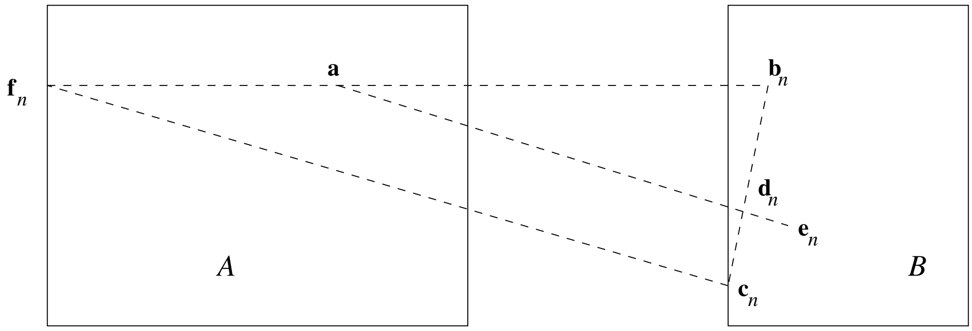

Figure 4: Illustration for Proposition 12

Therefore $\mathbf{a} \in B_{\epsilon+\frac{2}{n}}$ for every $n \in \mathbb{N}$, $\mathbf{a} \in A$. Hence, $A \subset B_{\epsilon+\frac{2}{n}}$. Reversing the rôles of A and B, we get $B \subset A_{\epsilon+\frac{2}{n}}$. Therefore,

$$d_H(A,B) \leq \inf_{n\in\mathbb{N}} \left(\epsilon + \frac{2}{n}\right) = \epsilon.$$

□

Proposition 12 is false if we relax convexity. The next example shows why.

Example 13. Let $X = \mathbb{R}^2$. Let $A = \{(x,y) \mid x^2+y^2 \leq 1\}$. Let $B = \{(x,y) \mid x^2+y^2 = 4\}$. Note that B is not convex. We have that $\partial B = B$ and $\partial A = \{(x,y) \mid x^2+y^2 = 1\}$. Thus

$$d_H(A,B) = 2 > 1 = d_H(\partial A, \partial B).$$

Theorem 14. *If $A, B \subset X$ are bounded, convex and have non-empty interior, then $d_H(A,B) = d_H(\partial A, \partial B)$.*

Proof. Set $\epsilon = d_H(A,B)$. Let $\mathbf{a} \in \partial A$. If $\mathbf{a} \notin B^\circ$, then Lemma 4 applies. Otherwise Proposition 10 applies and either way $\mathbf{a} \in (\partial B)_\gamma$ for all $\gamma > \epsilon$. Hence $\partial A \subset (\partial B)_\gamma$ for every $\gamma > \epsilon$. Similarly, $\partial B \subset (\partial A)_\gamma$ for every $\gamma > \epsilon$. Hence,

$$d_H(\partial A, \partial B) \leq \inf_{\gamma>\epsilon} \gamma = \epsilon.$$

By Proposition 12, $\epsilon \leq d_H(\partial A, \partial B)$. □

We next state a result which is immediate.

Proposition 15. *Suppose that $\gamma > 0$, and A is a non-empty, bounded and convex set. Then A_γ has non-empty interior, and is bounded and convex.*

There is one remaining question left. Does $d_H(A,B) = d_H(\partial A, \partial B)$ when A, B are bounded, non-empty and convex, B has non-empty interior, and A has empty interior? The problem is that if A has empty interior, we cannot apply the Hahn-Banach separation theorem. Unfortunately, the answer to our question is no, as Example 21 will show. We therefore try to find sufficient conditions that will give the desired equality.

Proposition 16. *Let $A, B \subset X$ be bounded and non-empty. If $A^\circ = \emptyset = B^\circ$ then $d_H(A,B) = d_H(\partial A, \partial B)$.*

Proof. Note that in this situation, $A \subset \partial A = Cl(A)$, $B \subset \partial B = Cl(B)$ and $d_H(A, \partial A) = d_H(B, \partial B) = 0$. Hence,

$$\begin{aligned} d_H(A, B) &\leq d_H(A, \partial A) + d_H(\partial A, \partial B) + d_H(B, \partial B) \\ &= d_H(\partial A, \partial B) \\ &\leq d_H(\partial A, A) + d_H(A, B) + (\partial B, B) \\ &= d_H(A, B). \end{aligned}$$

Since both ends of this string of inequalities are the same, equality must hold throughout, giving us the desired result. □

Notation 17. Suppose $A \subset X$ is bounded. For $\mathbf{a} \in \partial A$, Let $\mu_n(\mathbf{a}) = \inf\{||\mathbf{c} - \mathbf{a}|| \mid \mathbf{c} \in \partial(A_{\frac{1}{n}})\}$. Now $\mathbf{a} \in A_{\frac{1}{n+1}} \subset A_{\frac{1}{n}}$ and $A_{\frac{1}{n}}$ is bounded for every n. Therefore $0 \leq \mu_{n+1}(\mathbf{a}) \leq \mu_n(\mathbf{a}) < \infty$. Hence, for each $\mathbf{a} \in \partial A$ the sequence $\{\mu_n(\mathbf{a})\}_{n=1}^{\infty}$ is a bounded, monotonic, non-increasing sequence and therefore converges, to $\mu(\mathbf{a})$, say.

Theorem 18. *Let $A, B \subset X$ be bounded, non-empty and convex. Suppose $A^\circ = \emptyset \neq B^\circ$. If $\mu(\mathbf{a}) = 0$ for every $\mathbf{a} \in \partial A$, then $d_H(A, B) = d_H(\partial A, \partial B)$.*

Proof. By Proposition 12, $d_H(A, B) \leq d_H(\partial A, \partial B)$. Hence, all we need to show is that $d_H(\partial A, \partial B) \leq d_H(A, B) := \epsilon$.

Let $\mathbf{a} \in \partial A$. If $\mathbf{a} \notin B^\circ$ then by Lemma 4, $\mathbf{a} \in (\partial B)_\gamma$ for every $\gamma > \epsilon$. We may therefore assume that $\mathbf{a} \in B^\circ$. Without loss of generality, $\mathbf{a} = \mathbf{0}$. Fix $\gamma > \epsilon$. We show that $\mathbf{0} \in (\partial B)_\gamma$.

For each $n \in \mathbb{N}$, there exists $\mathbf{a}_n \in \partial(A_{\frac{1}{n}})$ such that $||\mathbf{a}_n|| \leq \mu_n(\mathbf{0}) + \frac{1}{n}$. By hypothesis, $\mu(\mathbf{0}) = 0$, and so $\mathbf{a}_n \to \mathbf{0}$ as $n \to \infty$. By choosing n sufficiently large, $\mathbf{a}_n \in B^\circ$, since $\mathbf{0} \in B^\circ$.

Let $\beta = \gamma - \epsilon > 0$. Choose n such that $\frac{2}{n} < \frac{\beta}{8}$ and $||\mathbf{a}_n|| < \frac{\beta}{4}$. Since $A_{\frac{1}{n}}$ has nonempty interior, we may apply Proposition 10 to $A_{\frac{1}{n}}$ and B to get $\mathbf{b}_n \in \partial B$ such that

$$\begin{aligned} ||\mathbf{a}_n - \mathbf{b}_n|| &\leq d_H(A_{\frac{1}{n}}, B) + \frac{2}{n} \\ &\leq d_H(A, B) + d_H(A, A_{\frac{1}{n}}) + \frac{\beta}{8} \\ &\leq d_H(A, B) + \frac{1}{n} + \frac{\beta}{8} \\ &\leq d_H(A, B) + \frac{\beta}{4}. \end{aligned}$$

Hence,

$$||\mathbf{b}_n|| \leq ||\mathbf{a}_n - \mathbf{b}_n|| + ||\mathbf{a}_n|| \leq d_H(A, B) + \frac{\beta}{4} + \frac{\beta}{4} = d_H(A, B) + \frac{\beta}{2} = \frac{\epsilon + \gamma}{2} < \gamma.$$

Therefore, $d(\mathbf{0}, \partial B) < \gamma$.

Next, let $\mathbf{b} \in \partial B$. Since $A^\circ = \emptyset$, $\mathbf{b} \in Cl(B) \backslash A^\circ$. By Lemma 4, $\mathbf{b} \in (\partial A)_\gamma$ for every $\gamma > \epsilon$. Since $\mathbf{b}$ was arbitrary, $\partial B \subset (\partial A)_\gamma$ for every $\gamma > \epsilon$.

Since $\gamma > \epsilon$ was arbitrary,

$$d_H(\partial A, \partial B) \leq \inf_{\gamma > \epsilon} \gamma = \epsilon.$$

□

Requiring $\mu(\mathbf{a}) = 0$ for every $\mathbf{a} \in \partial A$ is an unenlightening technical condition. We now search for a more intuitive condition that implies $\mu(\mathbf{a}) = 0$ for every $\mathbf{a} \in \partial A$.

Proposition 19. *Let $A \subset X$ be bounded and convex. If $(Cl(A))^\circ = A^\circ$ then $\mu(\mathbf{a}) = 0$ for all $\mathbf{a} \in \partial A$.*

Proof. Fix $\mathbf{a} \in \partial A$. Without loss of generality $\mathbf{a} = \mathbf{0}$. We would like to show that $\mu(\mathbf{0}) = 0$. Suppose not. Then $\mu(\mathbf{0}) > 0$.

Suppose $Cl(D_{\mu(\mathbf{0})}(\mathbf{0})) \subset Cl(A)$. Then $D_{\mu(\mathbf{0})}(\mathbf{0}) \subset (Cl(A))^\circ = A^\circ$ which, since $D_{\mu(\mathbf{0})}(\mathbf{0})$ is open and nonempty, contradicts the fact that $\mathbf{0} \in \partial A$. Hence there exists $\mathbf{d} \in X$ such that $||\mathbf{d}|| = \mu(\mathbf{0})$ and $\mathbf{d} \notin Cl(A)$.

Suppose $\mathbf{d} \in X \backslash Cl(A_{\frac{1}{n}})$. By Lemma 3, there exists $t \in [0,1]$ such that $\mathbf{e} := (1-t)\mathbf{d} \in \partial(A_{\frac{1}{n}})$. Since $\mathbf{0}$ and (by assumption) $\mathbf{d}$ are not in $\partial(A_{\frac{1}{n}})$, $t \in (0,1)$. But

$$||\mathbf{e}|| = ||(1-t)\mathbf{d}|| \leq (1-t)||\mathbf{d}|| < ||\mathbf{d}|| = \mu(\mathbf{0}) \leq \mu_n(\mathbf{0})$$

contradicting the definition of $\mu_n(\mathbf{0})$. Hence $\mathbf{d} \in Cl(A_{\frac{1}{n}})$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, there exists $\mathbf{f}_n \in A_{\frac{1}{n}}$ such that $||\mathbf{f}_n - \mathbf{d}|| \leq \frac{1}{n}$. Therefore, there exists $\mathbf{e}_n \in A$ such that $||\mathbf{e}_n - \mathbf{f}_n|| \leq \frac{1}{n}$ and so

$$||\mathbf{e}_n - \mathbf{d}|| \leq ||\mathbf{e}_n - \mathbf{f}_n|| + ||\mathbf{f}_n - \mathbf{d}|| \leq \frac{2}{n}.$$

Thus $\mathbf{e}_n \to \mathbf{d}$. Hence $\mathbf{d} \in Cl(A)$ which is a contradiction and so our assumption that $\mu(\mathbf{0}) > 0$ is false. Therefore $\mu(\mathbf{0}) = 0$. Since $\mathbf{0} = \mathbf{a} \in \partial A$ was arbitrary, the result follows. □

Theorem 20. *If A and B are non-empty, closed, bounded convex sets, $d_H(A, B) = d_H(\partial A, \partial B)$.*

Proof. If $A^\circ \neq \emptyset \neq B^\circ$, this follows from Theorem 14. If $A^\circ = \emptyset = B^\circ$, this follows from Proposition 16. If $A^\circ = \emptyset \neq B^\circ$ we observe that $A = Cl(A)$. Hence by Proposition 19, $\mu(\mathbf{a}) = 0$ for every $\mathbf{a} \in A$, and so Theorem 18 applies. □

We note that in Theorem 18 we do need $\mu(\mathbf{a}) = 0$ for every $\mathbf{a} \in \partial A$.

Example 21. Let $X = l_2(\mathbb{N})$ (equipped with the usual norm, with entries in $\mathbb{C}$). Let $\{\mathbf{e}_i\}_{i \in \mathbb{N}}$ be the standard basis for X. Let C be the set of all finite linear combinations of the $\mathbf{e}_i$'s. Then C is convex, since finite linear combinations of finite linear combinations are again finite linear combinations. Let $B = D_1(\mathbf{0})$. Let $A = C \cap B$. Then A is bounded since B is, and convex since both C and B are [3, p. 14]. We claim that A is dense in B. To see this, let $\mathbf{b} = \sum_{i=1}^{\infty} b_i \mathbf{e}_i \in B$. Then $\sum_{i=1}^{\infty} |b_i|^2 < 1$. Hence, for $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\sum_{i=N}^{\infty} |b_i|^2 < \epsilon^2$, and so $\sum_{i=1}^{N} b_i \mathbf{e}_i \in A \cap D_\epsilon(\mathbf{b})$. Since ϵ can be arbitrarily small, and the argument holds for every $\mathbf{b} \in B$ the claim is verified. Therefore $Cl(A) = Cl(B)$ and $d_H(A, B) = 0$.

We also claim that $A^\circ = \emptyset$. To see this, fix $\mathbf{a} = \sum_{i=1}^{N} a_i \mathbf{e}_i \in A$. Choose $\epsilon > 0$. Since $\sum_{i=1}^{\infty} \frac{1}{i^2} < \infty$ there exists $M \in \mathbb{N}$ such that $k \geq M \Rightarrow \sum_{i=k}^{\infty} \frac{1}{i^2} < \epsilon^2$. Choosing $k > \sup\{N, M\}$, define $\mathbf{b} = \mathbf{a} + \sum_{i=k}^{\infty} \frac{1}{i} \mathbf{e}_i$. Then

$$||\mathbf{b} - \mathbf{a}|| = ||\sum_{i=k}^{\infty} \frac{1}{i} \mathbf{e}_i|| = \big(\sum_{i=k}^{\infty} \frac{1}{i^2}\big)^{\frac{1}{2}} < \epsilon.$$

Then $\mathbf{b}$ is not in A but is in $D_\epsilon(\mathbf{a})$. Since ϵ can be arbitrarily small, and the argument holds for every $\mathbf{a} \in A$, the claim is verified.

Therefore, $A \subset \partial A$, and hence $\mathbf{0} \in \partial A$. Since $A_{\frac{1}{n}} = D_{1+\frac{1}{n}}(\mathbf{0}) \supset Cl(A)$ for every $n \in \mathbb{N}$, $\mu_n(\mathbf{0}) \geq 1$ for every n, and so $\mu(\mathbf{0}) \geq 1$. On the other hand, $\partial B = \{\mathbf{x} \in X \mid ||\mathbf{x}|| = 1\}$. Hence

$$d_H(A, B) = 0 < 1 \leq d_H(\partial A, \partial B).$$

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