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Relaxation in BV of Integral Functionals Defined on Sobolev Functions with Values in the Unit Sphere

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In this paper we study the relaxation with respect to the L^1 norm of integral functionals of the type

$$F(u) = \int_\Omega f(x, u, \nabla u)\, dx, \quad u \in W^{1,1}(\Omega; S^{d-1}),$$

where Ω is a bounded open set of R^N, S^{d-1} denotes the unite sphere in R^d, N and d being any positive integers, and f satisfies linear growth conditions in the gradient variable. In analogy with the unconstrained case, we show that, if, in addition, f is quasiconvex in the gradient variable and satisfies some technical continuity hypotheses, then the relaxed functional $\overline{F}$ has an integral representation on $BV(\Omega; S^{d-1})$ of the type

$$\bar{F}(u) = \int_\Omega f(x, u, \nabla u)\, dx + \int_{S(u)} K(x, u^-, u^+, \nu_u)\, d\mathcal{H}^{N-1} + \int_\Omega f^\infty(x, u, dC(u)),$$

where the suface energy density K is defined by a suitable Dirichlet-type problem.

Keywords: Relaxation, unit sphere, BV-functions

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1. Introduction

In this paper we study the relaxation with respect to the L^1 norm of integral functionals of the type

$$F(u) = \int_\Omega f(x, u, \nabla u)\, dx, \quad u \in W^{1,1}(\Omega; S^{d-1}), \tag{1}$$

where Ω is a bounded open set of $\mathbb{R}^N$, S^{d-1} denotes the unite sphere in $\mathbb{R}^d$, N and d being any positive integers, and f satisfies linear growth conditions in the gradient variable.

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The relaxed functional $\overline{F}$ is defined by

$$\overline{F}(u) := \inf\{\liminf_{n\to+\infty} F(u_n) : \ u_n \in W^{1,1}(\Omega; S^{d-1}), \ u_n \to u \text{ in } L^1\}. \tag{2}$$

The main motivation for this analysis is the use of constrained energy functionals of this kind in variational models for the study of equilibria of magnetostrictive materials (see [7]).

A wide literature is available for analogous relaxation problems in the non constrained case. In particular, we refer to the work by Fonseca and Müller [12], where they study the relaxation with respect to the L^1 norm of integral functional of the type (1) but defined for $u \in W^{1,1}(\Omega; \mathbb{R}^d)$, under the same growth conditions on f. They prove that, if, in addition, f is quasiconvex in the gradient variable and satisfies some technical continuity hypotheses (see Theorem 2.8), then the relaxed functional $\overline{F}$ has an integral representation on $BV(\Omega; \mathbb{R}^d)$ of the type

$$\overline{F}(u) = \int_\Omega f(x,u,\nabla u)\,dx + \int_{S(u)} H(x,u^-,u^+,\nu_u)\,d\mathcal{H}^{N-1} + \int_\Omega f^\infty(x,u,dC(u)), \tag{3}$$

(see Section 2.2 for the definition of BV function and all the quantities above), where f^∞ is the recession function of f (see (9)) and the surface energy density $H(x,a,b,\nu)$ is defined by a Dirichlet-type problem on a unit cell in the direction ν (see (14)).

In the case the admissible functions are constrained to take values on a manifold, the problem of relaxation with respect to stronger topologies of functional of the type (1) has been faced by Dacorogna, Fonseca, Malý and Trivisa in [9]. More precisely, they study the relaxation with respect to the weak topology in $W^{1,p}(\Omega)$, $p \geq 1$, of functional of the form

$$\mathcal{F}(u) = \int_\Omega f(\nabla u)\,dx, \quad u \in W^{1,p}(\Omega; \mathcal{M}),$$

where f is a positive continuous function, $\mathcal{M}$ is any C^1 manifold in $\mathbb{R}^d$, including in particular the case $\mathcal{M} = S^{d-1}$. It turns out that the relaxed functional $\overline{\mathcal{F}}$ is of the form

$$\overline{\mathcal{F}}(u) = \int_\Omega Q_T f(u,\nabla u)\,dx, \quad u \in W^{1,p}(\Omega; \mathcal{M}), \tag{4}$$

where $Q_T f$ is the so called tangential quasiconvexification of f, defined by

$$Q_T f(y,\xi) := \inf\left\{ \int_{(0,1)^N} f(\nabla\varphi)\,dx, \ \varphi \in W_0^{1,\infty}((0,1)^N; T_y(\mathcal{M})) \right\},$$

for $y \in \mathcal{M}$ and $\xi \in [T_y(\mathcal{M})]^N$, where $T_y(\mathcal{M})$ denotes the tangent space to $\mathcal{M}$ at y.

In [2] it was shown that, if F is of the form

$$F(u) = \int_\Omega f(u,\nabla u)\,dx, \quad u \in W^{1,1}(\Omega; S^{d-1}),$$

with f satisfying some technical continuity conditions and having linear growth in the gradient variable, then the relaxation of F with respect to the L^1 norm is still of the form (4) on $W^{1,1}(\Omega; S^{d-1})$. In particular, if f is tangentially quasiconvex, that is $f =$

$Q_T f$, then F is lower semicontinuous with respect to the L^1 norm on $W^{1,1}(\Omega; S^{d-1})$. Anyway, this result is not satisfactory as far as we are concerned with minimum problems involving functionals of this kind, since sequences with bounded energies are not compact in $W^{1,1}(\Omega; S^{d-1})$.

In the case $d = 2$ and under convexity hypotheses in the gradient variable, the problem of relaxation in $BV(\Omega; S^1)$ of integral functionals with linear growth defined on $C^1(\Omega; S^1)$ has been faced by Giaquinta, Modica and Souček in [14] (see also Demengel and Hadiji [11]).

In this paper, we consider an integral functional of the type (1) and we give an integral representation in $BV(\Omega; S^{d-1})$ of the relaxed functional $\overline{F}$ defined by (2). More precisely, we prove that, under hypotheses on f and f^∞ analogous with those given in [12] (see (H1)-(H5) in Section 3) and if in addition f is tangentially quasiconvex, then $\overline{F}$ is still of the form (3) on $BV(\Omega; S^{d-1})$ (see Theorem 3.1). The main difference with respect to the result proved in [12] is that in the Dirichlet problem defining the surface energy density K the test functions take values on S^{d-1} instead of all the space $\mathbb{R}^d$ (see definition (18)). It turns out, for example, that if we consider the isotropic case $f(x, y, z) = |z|$, we obtain $K(x, a, b, \nu) = d_g(a, b)$, d_g denoting the geodetic distance on the unit sphere, while $H(x, a, b, \nu) = |a - b|$ in the non constrained case considered in [12] (see Remark 4.3).

It is worth noting that, thanks to a strong density result of smooth functions between manifolds in Sobolev Spaces proved by Bethuel and Zheng in [6] and in a more general version by Bethuel in [5], in (2) we can restrict to approximating sequences u_n belonging to a class of S^{d-1}-valued smooth functions. This class is $C^\infty(\Omega; S^{d-1})$ if $d \neq 2$, while, if $N > 1$ and $d = 2$ is given by all the S^1-valued functions which are C^∞ except at most on sets of codimension 2 (see Theorem 2.2 and Remark 3.3). On the other hand, in [14] it was shown that, in the case $d = 2$, if one restricts to approximating sequences in $C^1(\Omega; S^1)$, nonlocal terms appear in the relaxed functionals.

The proof of our result closely follows the outline of the proof of the integral representation result in [12], based on blow-up techniques and a localization argument. The main difficulty to adapt the proof of [12] to our setting is that convolution and cut-off arguments cannot be applied in a standard way, due to the fact that the admissible functions are constrained to take values on the non convex set S^{d-1}. We overcome this difficulty by using a construction analogous with that used in the proof of the density results in [6] (see also [16] and [15]). It consists in suitably projecting a smooth function taking values on the unit ball onto S^{d-1}, without increasing too much its energy (see the proof of Lemmas 5.2 and 6.4 and Proposition 6.2).

We eventually derive a relaxation result for functionals of the type

$$\mathcal{G}(u) = \int_\Omega f(x, u, \nabla u) + \int_{\mathbb{R}^N} |h_u|^2 \, dx - \int_\Omega \langle h_{ext}, u \rangle \, dx, \quad u \in W^{1,1}(\Omega; S^{N-1}),$$

subject to the constraint

$$\begin{cases} \text{curl } h_u = 0 \\ \text{div } (h_u + u\chi_\Omega) = 0, \end{cases} \tag{5}$$

with f satisfying the hypotheses described above. Functionals of this kind generalize those involved in variational models for micromagnetics, where u represents the magnetization of a ferromagnetic material subject to an external magnetic field h_{ext} and h_u is the induced magnetic field related to u through the Maxwell's equations (5) (see [7], [17] for a detailed explanation of the model). We note that the additional terms are continuous and so they do not affect the form of the relaxed functional $\overline{\mathcal{G}}$, which is given by

$$\overline{\mathcal{G}}(u) = \overline{F}(u) + \int_{\mathbb{R}^N} |h_u|^2\,dx - \int_\Omega \langle h_{ext}, u\rangle\,dx, \quad u \in BV(\Omega; S^{N-1}),$$

with h_u satisfying (5) and $\overline{F}$ given by (3) (see Theorem 7.2).

2. Preliminaries and notation

We denote by $\langle\cdot,\cdot\rangle$ the scalar product in $\mathbb{R}^N$ and with $|\cdot|$ the usual euclidean norm, without specifying the dimension N when there is no risk of confusion. For every $t \in \mathbb{R}$, $[t]$ denotes its integer part. Given $\nu \in S^{N-1}$, Q_ν is an open unit cube centered at the origin with two of its faces normal to ν.

If Ω is a bounded open subset of $\mathbb{R}^N$, $\mathcal{A}(\Omega)$ and $\mathcal{B}(\Omega)$ are the families of open and Borel subsets of Ω, respectively. We denote by $\mathcal{X}_B$ the characteristic function of the set $B \in \mathcal{B}(\Omega)$.

If μ is a Borel measure and B is a Borel set, then the measure $\mu \llcorner B$ is defined as $\mu \llcorner B(A) = \mu(A \cap B)$. We denote by $\mathcal{L}^N$ the Lebesgue measure in $\mathbb{R}^N$ and by $\mathcal{H}^k$ the k-dimensional Hausdorff measure, $k \geq 0$. The notation *a.e.* stands for almost everywhere with respect to the Lebesgue measure, unless otherwise specified. We use standard notations for Lebesgue and Sobolev spaces.

2.1. Strong density result in $W^{1,1}(\Omega; S^{d-1})$

In this section we recall a result about the density of S^{d-1}-valued smooth functions in $W^{1,1}(\Omega; S^{d-1})$, which has been proved in [6] and, in a more general version, in [5]. We write it in a form which is suitable to our purposes.

Definition 2.1. Given $N \in \mathbb{N}$, $N > 1$, we will denote by $\mathcal{G}$ the family of all closed subsets of C^∞ $(N-2)$-dimensional manifolds of $\mathbb{R}^N$.

Theorem 2.2. *Let $N, d \in \mathbb{N}$, $\Omega \subset \mathbb{R}^N$ be an open set and let $\mathcal{D}(\Omega; S^{d-1}) \subset W^{1,1}(\Omega; S^{d-1})$ be defined by*

$$\mathcal{D}(\Omega; S^{d-1}) := C^\infty(\Omega; S^{d-1}) \cap W^{1,1}(\Omega; S^{d-1}), \tag{6}$$

for $d \neq 2$, if $N > 1$, and for any $d \in \mathbb{N}$, if $N = 1$,

$$\mathcal{D}(\Omega; S^1) := \{u \in W^{1,1}(\Omega; S^1) : \exists k \in \mathbb{N}, \Gamma_i \in \mathcal{G}, i = 1, \dots, k : \ u \in C^\infty(\Omega \setminus \bigcup_{i=1}^k \Gamma_i; S^1)\}, \tag{7}$$

for $N > 1$. Then $\mathcal{D}(\Omega; S^{d-1})$ is dense in $W^{1,1}(\Omega; S^{d-1})$ for the $W^{1,1}$ norm.

2.2. Functions of bounded variation

We recall some definitions and basic results on functions with bounded variation. Our main reference is the book [3].

Definition 2.3. Let $u \in L^1\left(\Omega;\mathbb{R}^d\right)$, we say that u is a function with Bounded Variation in Ω, we write $u \in BV\left(\Omega;\mathbb{R}^d\right)$, if the distributional derivative Du of u is representable by a $d \times N$ matrix valued measure on Ω with finite total variation $|Du|(\Omega)$ whose entries are denoted by $D_j u_i$, i.e., if $\varphi \in C_c^1(\Omega)$ then

$$\int_\Omega u_i \partial_j \varphi \, dx = -\int_\Omega \varphi dD_j u_i.$$

Define the *approximate upper and lower limit* of each component u_i, $i = 1, \ldots, d$, by

$$u_i^+(x) := \inf\left\{t \in \mathbb{R} : \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \mathcal{L}^N(\{u_i > t\} \cap B(x,\varepsilon)) = 0\right\},$$

$$u_i^-(x) := \sup\left\{t \in \mathbb{R} : \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \mathcal{L}^N(\{u_i < t\} \cap B(x,\varepsilon)) = 0\right\}.$$

Then the *jump set* of u , denoted by $S(u)$, is defined by

$$S(u) := \cup_{i=1}^d \{x \in \Omega : u_i^- < u_i^+\}.$$

If $u \in BV\left(\Omega;\mathbb{R}^d\right)$, then $S(u)$ turns out to be countably $\left(\mathcal{H}^{N-1}, N-1\right)$ rectifiable, i.e.,

$$S(u) = N \cup \bigcup_{i \geq 1} K_i,$$

where $\mathcal{H}^{N-1}(N) = 0$ and each K_i is a compact subset of a C^1 manifold. If $x \in \Omega \setminus S(u)$, then $u(x)$ is understood as the common value of $(u_1^+, \ldots, u_d^+)$ and $(u_1^-, \ldots, u_d^-)$ with $u_i^\pm(x) \in [-\infty, +\infty]$ for $i = 1, \ldots, d$. It can be shown that $u(x) \in \mathbb{R}^d$ for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega \setminus S(u)$.

Theorem 2.4. *If $u \in BV(\Omega;\mathbb{R}^d)$, then*

(*i*) *for $\mathcal{L}^N$-a.e. $x \in \Omega$*

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{B(x,\varepsilon)} |u(y) - u(x) - \langle \nabla u(x), x - y\rangle| \, dy = 0,$$

where ∇u is the density of the absolutely continuous part of Du with respect to the Lebesgue measure $\mathcal{L}^N$;

(*ii*) *for $\mathcal{H}^{N-1}$ a.e. $x \in S(u)$ there exists a unit vector $\nu(x) \in S^{N-1}$ and there exist $u^-(x), u^+(x) \in \mathbb{R}^d$ such that*

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{\{y \in B(x,\varepsilon) : \langle y - x, \nu(x)\rangle > 0\}} |u(y) - u^+(x)| \, dx = 0,$$

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{\{y \in B(x,\varepsilon) : \langle y - x, \nu(x)\rangle < 0\}} |u(y) - u^-(x)| \, dx = 0;$$

(iii) *for $\mathcal{H}^{N-1}$ a.e. $x \in \Omega \setminus S(u)$,*

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{B(x,\varepsilon)} |u(y) - u(x)| \, dx = 0.$$

In what follows u^+ and u^- will denote the vectors introduced in *(ii)* above.

The next result will be used in Section 5.2.

Lemma 2.5. *For $\mathcal{H}^{N-1}$ a.e. $x_0 \in S(u)$,*

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^{N-1}} \int_{S(u) \cap (x_0 + \varepsilon Q_{\nu(x_0)})} |u^+(x) - u^-(x)| \, d\mathcal{H}^{N-1}(x) = |u^+(x_0) - u^-(x_0)|.$$

If $u \in BV(\Omega; \mathbb{R}^d)$, then Du can be decomposed into three orthogonal measure as

$$Du = \nabla u \, dx + (u^+ - u^-) \otimes \nu_u d\mathcal{H}^{N-1} \llcorner S(u) + C(u).$$

Here $C(u)$ is the so called Cantor part of Du and satisfies the property that $|C(u)|(B) = 0$ for any $B \in \mathcal{B}(\Omega)$ such that $\mathcal{H}^{N-1}(B) < +\infty$. We recall that, by a result of Alberti in [1], the density of the Cantor part $C(u)$ defined by

$$A(x) := \lim_{\varepsilon \to B(x,\varepsilon)} \frac{C(u)(B(x,\varepsilon))}{|C(u)|(B(x,\varepsilon))},$$

is a rank-one matrix for $|C(u)|$ a.e. $x \in \Omega$.

We denote by $BV(\Omega; S^{d-1})$ the space of functions $u \in BV(\Omega; \mathbb{R}^d)$ such that $u(x) \in S^{d-1}$ for a.e. $x \in \Omega$.

Remark 2.6. It is easy to prove that, if $u \in BV(\Omega; S^{d-1})$, then, for a.e $x \in \Omega$, $\nabla u(x) \in [T_{u(x)}(S^{d-1})]^N$ and, for $|C(u)|$ a.e. $x \in \Omega$, $A(x) \in [T_{u(x)}(S^{d-1})]^N$, where, given $y \in S^{d-1}$, $T_y(S^{d-1})$ denotes the tangent space to S^{d-1} at y.

The next lemma (see [4], Lemma 4.5) will be used in Section 6.

Lemma 2.7. *Let $u \in BV(\Omega : \mathbb{R}^d)$, let ρ be a convolution kernel, and let*

$$u_n(x) := (u * \rho_n)(x),$$

where $\rho_n(x) := n^N \rho(nx)$. Then

$$\int_{B(x_0,\varepsilon)} h(x) |\nabla u_n(x)| \, dx \leq \int_{B(x_0, \varepsilon + \frac{1}{n})} (h * \rho_n)(x) |Du(x)|,$$

whenever $\operatorname{dist}(x_0, \partial\Omega) > \varepsilon + \frac{1}{n}$ *and h is a nonnegative Borel function;*

$$\lim_{n \to +\infty} \int_{B(x_0,\varepsilon)} \theta(\nabla u_n(x)) \, dx = \int_{B(x_0,\varepsilon)} \theta(Du(x)),$$

for every function θ positively homogeneous of degree one and for every $\varepsilon \in (0, \operatorname{dist}(x_0, \partial\Omega))$ such that $|Du|(\partial B(x_0, \varepsilon)) = 0$; if, in addition, $u \in L^\infty(\Omega; \mathbb{R}^d)$, then for every $x_0 \in \Omega \backslash S(u)$,

$$\lim_{n \to +\infty} u_n(x_0) = u(x_0), \qquad \lim_{n \to +\infty} (|u_n - u| * \rho_n)(x_0) = 0.$$

2.3. Quasiconvexity and relaxation results

A function $f : \mathbb{R}^{d\times N} \mapsto \mathbb{R}$ is said to be quasiconvex if

$$f(\xi) \leq \int_{(0,1)^N} f(\xi + \nabla\varphi)\, dx$$

for all $\xi \in \mathbb{R}^{d\times N}$ and for all $\varphi \in W_0^{1,\infty}((0,1)^N;\mathbb{R}^d)$. We recall that if f is quasiconvex and

$$|f(\xi)| \leq C(1+|\xi|), \tag{8}$$

then f is Lipschitz continuous (see [8]). We define the recession function of f by

$$f^\infty(\xi) := \limsup_{t\to+\infty} \frac{f(t\xi)}{t}. \tag{9}$$

Note that f^∞ is positively homogeneous of degree one and it can be proved that if f is quasiconvex and satisfies (8), then also f^∞ is quasiconvex (see [12]).

Let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \mapsto [0,+\infty)$ and $F : L^1(\Omega;\mathbb{R}^d) \mapsto [0,+\infty]$ defined by

$$F(u) := \begin{cases} \displaystyle\int_\Omega f(x,u,\nabla u)\, dx & \text{if } u \in W^{1,1}(\Omega;\mathbb{R}^d) \\ +\infty & \text{otherwise.} \end{cases} \tag{10}$$

Let $\overline{F} : L^1(\Omega;\mathbb{R}^d) \to [0,+\infty]$ the relaxation of F with respect to the L^1 topology, namely,

$$\overline{F}(u) = \inf\{\liminf_n F(u_n) : \ u_n \to u \text{ in } L^1(\Omega)\}. \tag{11}$$

If $(a,b,\nu) \in \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1}$, let $\{\nu_1,\dots \nu_{N-1},\nu\}$ form an orthonormal basis of $\mathbb{R}^N$ and define

$$\begin{aligned} \mathcal{A}(a,b,\nu) := \Big\{\varphi \in W^{1,1}(Q_\nu;\mathbb{R}^d) :\ & \varphi(x) = a \text{ if } x\cdot\nu = -\frac{1}{2},\ \varphi(x) = b \text{ if } x\cdot\nu = \frac{1}{2}, \\ & \varphi \text{ is periodic with period 1 in the } \nu_1,\dots,\nu_{N-1} \text{ directions}\Big\}. \end{aligned} \tag{12}$$

In [12] the following theorem was proved.

Theorem 2.8. *Let f satisfy the following hypotheses:*

(F1) *f is continuous;*

(F2) *$f(x,u,\cdot)$ is quasiconvex;*

(F3) *There exist two positive constant c, C such that*

$$c|\xi| \leq f(x,u,\xi) \leq C(1+|\xi|)$$

for all $(x,u,\xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N}$;

(F4) *For every compact $J \subset\subset \Omega \times \mathbb{R}^d$ there exist a continuous function ω with $\omega(0) = 0$ such that*

$$|f(x,u,\xi) - f(x',u',\xi)| \leq \omega(|x-x'| + |u-u'|)(1+|\xi|)$$

for all (x,u,ξ), $(x',u',\xi) \in J \times \mathbb{R}^{d\times N}$. *In addition, for every* $x_0 \in \Omega$ *and for all* $\delta > 0$ *there exists* $\varepsilon > 0$ *such that if* $|x - x_0| \le \varepsilon$, *then*

$$f(x,u,\xi) - f(x_0,u,\xi) \ge -\delta(1+|\xi|)$$

for every $(u,\xi) \in \mathbb{R}^d \times \mathbb{R}^{d\times N}$;

(F5) *there exist* $C' > 0$, $0 < m < 1$ *such that*

$$|f^\infty(x,u,\xi) - f(x,u,\xi)| \le C(1+|\xi|^{1-m})$$

for every $(x,u,\xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N}$. *Then*

$$\overline{F}(u) = \begin{cases} \displaystyle\int_\Omega f(x,u,\nabla u)\,dx + \int_{S(u)} H(x,u^-,u^+,\nu_u)\,d\mathcal{H}^{N-1} & \\ \qquad\displaystyle + \int_\Omega f^\infty(x,u,dC(u)) & \text{if } u \in BV(\Omega;\mathbb{R}^d) \\ +\infty & \text{otherwise,} \end{cases} \tag{13}$$

where $H : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1} \to \mathbb{R}$ *is defined by*

$$H(x,a,b,\nu) := \inf\left\{ \int_{Q_\nu} f^\infty(x,\varphi,\nabla\varphi)\,dx : \ \varphi \in \mathcal{A}(a,b,\nu) \right\}. \tag{14}$$

Let, now, $\mathcal{M} \subseteq \mathbb{R}^d$ be a C^1 manifold and, given $y \in \mathcal{M}$, denote by $T_y(\mathcal{M})$ the tangent space to $\mathcal{M}$ at y.

The following definition has been introduced in [9].

Definition 2.9. Given a function $f : \mathbb{R}^{d\times N} \mapsto \mathbb{R}$, the tangential quasiconvexification of f is defined by

$$Q_T f(y,\xi) := \inf\{ \int_{(0,1)^N} f(y,\xi+\nabla\varphi)\,dx : \ \varphi \in W_0^{1,\infty}((0,1)^N; T_y(\mathcal{M}))\}$$

for all $y \in \mathcal{M}$ and $\xi \in [T_y(\mathcal{M})]^N$.

In [9] the following relaxation result was proved.

Theorem 2.10. *If* $f : \mathbb{R}^{d\times N} \to [0,+\infty)$ *is a continuous function satisfying*

$$0 \le f(\xi) \le C(1+|\xi|^p)$$

for some $p \ge 1$, $C > 0$, *and all* $\xi \in \mathbb{R}^{d\times N}$, *then*

$$\mathcal{F}(u) = \int_\Omega Q_T f(u,\nabla u)\,dx, \quad u \in W^{1,p}(\Omega;\mathcal{M}),$$

where

$$\mathcal{F}(u) := \inf\{\liminf_{n\to+\infty} \int_\Omega f(\nabla u_n)\,dx : \ u_n \in W^{1,p}(\Omega;\mathcal{M}), \ u_n \rightharpoonup u \ \textit{in } W^{1,p}\}.$$

Definition 2.11. A function $f : \mathcal{M} \times \mathbb{R}^{d\times N} \mapsto \mathbb{R}$, is said to be tangential quasiconvex if

$$f(y,\xi) := \int_{(0,1)^N} f(y, \xi + \nabla\varphi)\, dx$$

for all $y \in \mathcal{M}$ and $\xi \in [T_y(\mathcal{M})]^N$ and $\varphi \in W_0^{1,\infty}((0,1)^N; T_y(\mathcal{M})$.

Remark 2.12. In [2], it was observed that the result given by Theorem 2.10 still holds true if f depends continuously also on u. In particular, we infer that if f is tangential quasiconvex, then the functional

$$\mathcal{F}(u) := \int_\Omega f(u, \nabla u)\, dx, \quad u \in W^{1,p}(\Omega; \mathcal{M})$$

is $W^{1,p}$-sequentially weakly lower semicontinuous.

Let P_y the orthogonal projection of $\mathbb{R}^d$ onto the tangent space $T_y(\mathcal{M})$ and consider the function $\overline{f} : \mathcal{M} \times \mathbb{R}^{d\times N} \mapsto \mathbb{R}$ defined by

$$\overline{f}(y,\xi) := f(y, P_y\xi),$$

with $P_y\xi := (P_y\xi^1, \dots, P_y\xi^N)$ and ξ^i the i^{th} columns of $\xi \in \mathbb{R}^{d\times N}$. In [9] it was proved that for any $y \in \mathcal{M}$ and $\xi \in [T_y(\mathcal{M})]^N$

$$Q_T f(y,\xi) = Q\overline{f}(y,\xi),$$

where $Q\overline{f}$ is the quasiconvex envelope of $\overline{f}$ defined by

$$Q\overline{f}(y,\xi) := \inf\{\int_{(0,1)^N} f(y, \xi + \nabla\varphi)\, dx : \varphi \in W_0^{1,\infty}((0,1)^N; \mathbb{R}^d)\}.$$

In particular, if f is tangential quasiconvex, then $\overline{f}$ is quasiconvex.

In the rest of the paper a tangential quasiconvex function f will be identified by the function $\overline{f}$ defined above. So we may think a tangential quasiconvex function as the restriction of a quasiconvex function on the set $T(\mathcal{M}) \subseteq \mathcal{M} \times \mathbb{R}^{d\times N}$ defined by

$$T(\mathcal{M}) := \{(y,\xi) : y \in \mathcal{M},\ \xi \in [T_y(\mathcal{M})]^N\}. \tag{15}$$

3. Statement of the main result

Let $N, d \in \mathbb{N}$, with $d \geq 2$. Given a bounded open subset Ω of $\mathbb{R}^N$ and a function $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to [0, +\infty)$, define the functional $F : L^1(\Omega; \mathbb{R}^d) \to \bar{[}0, +\infty]$ as

$$F(u) := \begin{cases} \displaystyle\int_\Omega f(x, u, \nabla u)\, dx & \text{if } u \in W^{1,1}(\Omega; S^{d-1}) \\ +\infty & \text{otherwise.} \end{cases} \tag{16}$$

Note that, in order that the definition of F is well posed, it suffices that the integrand f is defined only on $\Omega \times T(S^{d-1})$, where $T(S^{d-1})$ is given by (15) with $\mathcal{M} = S^{d-1}$.

On f we will consider the following set of hypotheses:

(H1) f is continuous;

(H2) $f(x,\cdot,\cdot)$ is a tangential quasiconvex function according to Definition 2.11 with $\mathcal{M} = S^{d-1}$;

(H3) there exist two positive constants $c_1,\ c_2$ such that

$$c_1|\xi| \leq f(x,y,\xi) \leq c_2(|\xi|+1)$$

for every $x \in \Omega,\ y \in S^{d-1},\ \xi \in [T_y(S^{d-1})]^N$;

(H4) For every compact $J \subset \Omega$, there exist a continuous function ω with $\omega(0) = 0$ such that

$$|f(x,y,\xi) - f(x',y',\xi)| \leq \omega(|x-x'| + |y-y'|)(1+|\xi|)$$

for every $(x,y,\xi),\ (x',y',\xi) \in J \times S^{d-1} \times \mathbb{R}^{d\times N}$;

(H5) there exist $C > 0$, $0 \leq m < 1$ such that

$$|f^\infty(x,y,\xi) - f(x,y,\xi)| \leq C(1+|\xi|^{1-m})$$

for every $x \in \Omega,\ y \in S^{d-1},\ \xi \in [T_y(S^{d-1})]^N$.

The main result of the paper is the integral representation for the relaxation $\overline{F} : L^1(\Omega;\mathbb{R}^d) \to \overline{[}0,+\infty)$ of F with respect to the L^1 topology (see (11)).

Define

$$\mathcal{P}(a,b,\nu) := \Big\{\varphi \in W^{1,1}(Q_\nu;S^{d-1}) : \varphi(x) = a \text{ if } x\cdot\nu = -\frac{1}{2},\ \varphi(x) = b \text{ if } x\cdot\nu = \frac{1}{2},$$
$$\varphi \text{ is periodic with period 1 in the } \nu_1,\nu_2,\ldots,\nu_{N-1} \text{ directions}\Big\}.$$

Theorem 3.1. *If* (H1)–(H5) *hold, then*

$$\overline{F}(u) = \begin{cases} \displaystyle\int_\Omega f(x,u,\nabla u)\,dx + \int_{S(u)} K(x,u^-,u^+,\nu_u)\,d\mathcal{H}^{N-1} & \\ \qquad \displaystyle + \int_\Omega f^\infty(x,u,dC(u)) & \textit{if } u \in BV(\Omega;S^{d-1}) \\ +\infty & \textit{otherwise,} \end{cases} \tag{17}$$

where $K : \Omega \times S^{d-1} \times S^{d-1} \times S^{N-1} \to \mathbb{R}$ *is defined by*

$$K(x,a,b,\nu) := \inf\left\{\int_{Q_\nu} f^\infty(x,\varphi,\nabla\varphi)\,dx :\ \varphi \in \mathcal{P}(a,b,\nu)\right\} \tag{18}$$

Remark 3.2. If f satisfies (H2)–(H4), then, using the definition of recession function, one can easily prove that:

(H2') $f^\infty(x,\cdot,\cdot)$ is tangentially quasiconvex;

(H3') $c_1|\xi| \leq f^\infty(x,y,\xi) \leq c_2|\xi|$ for every $x \in \Omega,\ y \in S^{d-1},\ \xi \in [T_y(S^{d-1})]^N$;

(H4') For every compact $J \subset \Omega$, there exist a continuous function ω with $\omega(0) = 0$ such that

$$|f^\infty(x,y,\xi) - f^\infty(x',y',\xi)| \leq \omega(|x-x'| + |y-y'|)|\xi|$$

for every $(x,y,\xi),\ (x',y',\xi) \in J \times S^{d-1} \times \mathbb{R}^{d\times N}$.

Remark 3.3. By Theorem 2.2 and by (H1) and (H3), we can restrict to sequences of smooth approximating functions in the definition of $\overline{F}$, that is

$$\overline{F}(u) = \inf\{\liminf_n \int_\Omega f(x, u_n, \nabla u_n)\, dx : \ u_n \to u \text{ in } L^1(\Omega; \mathbb{R}^d), \ u_n \in \mathcal{D}(\Omega; S^{d-1})\},$$

where $\mathcal{D}(\Omega; S^{d-1})$ is defined in (6) and (7).

4. Properties of the surface density function

Before proving Theorem 3.1 we state some properties of the surface energy density K, we will need in the sequel, and we show a more explicit characterization of it under isotropy assumption on f^∞.

The following lemma is the analogue of [12] Lemma 2.15 .

Lemma 4.1. *Let* (H1)–(H4) *hold and let* $K : \Omega \times S^{d-1} \times S^{d-1} \times S^{N-1} \to [0, +\infty)$ *be defined by* (18). *Then*

(a) $|K(x, a, b, \nu) - K(x, a', b', \nu)| \le c(|a - a'| + |b - b'|)$ *for every* (x, a, b, ν), $(x, a', b', \nu) \in \Omega \times S^{d-1} \times S^{d-1} \times S^{N-1}$*;*

(b) $(x, \nu) \mapsto K(x, a, b, \nu)$ *is upper semicontinuous for every* $(a, b) \in S^{d-1} \times S^{d-1}$*;*

(c) K *is upper semicontinuous in* $\Omega \times S^{d-1} \times S^{d-1} \times S^{N-1}$*;*

(d) $K(x, a, b, \nu) \le C|b - a|$ *for every* $(x, a, b, \nu) \in \Omega \times S^{d-1} \times S^{d-1} \times S^{N-1}$*;*

(e) *for all* $x \in \Omega$, $\varepsilon > 0$, *there exists* $\delta > 0$ *such that if* $|x' - x| < \delta$, *then* $|K(x', a, b, \nu) - K(x, a, b, \nu)| \le \varepsilon C|b - a|$.

Proof. (a) Let $\varphi \in \mathcal{P}(a, b, \nu)$ and let $\gamma_1, \gamma_2 : [1/4, 1/2] \to S^{d-1}$ be smooth and such that

$$\gamma_1(1/4) = b, \ \gamma_1(1/2) = b', \quad \gamma_2(1/4) = a, \ \gamma_2(1/2) = a',$$

$$\int_{1/4}^{1/2} |\gamma_1'(t)|\, dt \le C|b - b'|, \quad \int_{1/4}^{1/2} |\gamma_2'(t)|\, dt \le C|a - a'|. \tag{19}$$

Then define $\varphi^* \in \mathcal{P}(a', b', \nu)$ by

$$\varphi^*(y) = \begin{cases} \varphi(2y) & \text{if } |\langle y, \nu\rangle| < 1/4 \\ \gamma_1(\langle y, \nu\rangle) & \text{if } 1/4 < \langle y, \nu\rangle < 1/2 \\ \gamma_2(-\langle y, \nu\rangle) & \text{if } -1/2 < \langle y, \nu\rangle < -1/4. \end{cases}$$

So far, arguing as in the proof of [12] Lemma 2.15 (a), by using the periodicity of φ, the growth condition (H3') on f^∞ and (19), we get

$$\begin{aligned} K(x, a', b', \nu) &\le \int_{Q_\nu} f^\infty(x, \varphi^*(y), \nabla\varphi^*(y)\, dy \\ &\le \int_{Q_\nu} f^\infty(x, \varphi(y), \nabla\varphi(y)\, dy + C(|a - a'| + |b - b'|). \end{aligned}$$

Then, by the arbitrariness of $\varphi \in \mathcal{P}(a, b, \nu)$, we conclude that

$$K(x, a', b', \nu) \le K(x, a, b, \nu) + C(|a - a'| + |b - b'|).$$

The proof of (b) is exactly analogous to that of [12] Lemma 2.15 (b) and hypotheses (H3') and (H4') on f^∞ are needed. Note that (c) is an immediate consequence of (a) and (b).

(d) Use the growth condition (H3') and consider the characterization of K given by Lemma 4.2 and Remark 4.3 below when $f^\infty(x,u,\xi) = |\xi|$.

Finally the proof of (e) can be carried out exactly as in Proposition 2.4 (ii) of [13]. $\square$

The following lemma is the analogue of [13] Proposition 2.6 (iii) and show that, if f^∞ satisfies an isotropy assumption, then the surface density K can be calculated by restricting the infimum to functions with one-dimensional profile. We omit the proof since it is exactly the same of that of [13] Proposition 2.6 (iii).

Lemma 4.2. *Let $K : \Omega \times S^{d-1} \times S^{d-1} \times S^{N-1} \to [0,+\infty)$ be defined by* (18) *and let f^∞ isotropic, i.e., for every $(x,u,z) \in \Omega \times S^{d-1} \times \mathbb{R}^{d\times N}$ and $\nu \in S^{n-1}$ there holds*

$$f^\infty(x,u,z\nu\otimes\nu) \le f^\infty(x,u,z)\,. \tag{20}$$

Then

$$\begin{aligned} & K(x,a,b,\nu) \\ = \;& \inf\left\{\int_0^1 f^\infty(x,\gamma(t),\gamma'(t)\otimes\nu)\,dt : \gamma \in W^{1,1}((0,1);S^{d-1}),\ \gamma(0)=a,\ \gamma(1)=b\right\}. \end{aligned}$$

Remark 4.3. In the particular case $f(x,u,\xi) = h(|\xi|)$ with $\lim_{t\to+\infty} \frac{h(t)}{t} = 1$, then $f^\infty(x,u,\xi) := |\xi|$ and so condition (20) is satisfied. Thus, by Lemma 4.2, we get

$$\begin{aligned} K(x,a,b,\nu) &= \inf\left\{\int_0^1 |\gamma'(t)|\,dt : \gamma \in W^{1,1}((0,1);S^{d-1}),\ \gamma(0)=a,\ \gamma(1)=b\right\} \\ &=: d_g(a,b), \end{aligned}$$

where d_g denotes the geodesic distance on S^{d-1}.

5. Estimate from below

Set, for $u \in BV(\Omega;S^{d-1})$, $B \in \mathcal{B}(\Omega)$

$$G(u,B) := \int_B f(x,u,\nabla u)\,dx + \int_{S(u)\cap B} K(x,u^-,u^+,\nu_u)\,d\mathcal{H}^{N-1} + \int_B f^\infty(x,u,dC(u)).$$

By simplicity, we denote $G(u) = G(u,\Omega)$.

Proposition 5.1. *Let* (H1)–(H5) *hold and let $u_n \in W^{1,1}(\Omega;S^{d-1})$ such that $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$ and $\liminf_n F(u_n) < +\infty$. Then $u \in BV(\Omega;S^{d-1})$ and*

$$\liminf_n F(u_n) \ge G(u) \tag{21}$$

Proof. We may assume without loss of generality that

$$\liminf_n F(u_n) = \lim_n F(u_n) < +\infty.$$

Then by the growth hypothesis (H3), we get

$$\sup_n \|u_n\|_{W^{1,1}(\Omega;S^{d-1})} < +\infty,$$

from which we immediately derive that $u \in BV(\Omega; S^{d-1})$.

Since $f \geq 0$, up to passing to a subsequence, we may assume that there exists a non-negative finite Radon measure μ on Ω such that

$$f(\cdot, (u_n(\cdot), \nabla u_n(\cdot))\mathcal{L}^N \llcorner \Omega \to \mu$$

weakly* in the sense of measures. Using the Radon-Nykodim's Theorem we decompose μ in the sum of four mutually orthogonal measures

$$\mu = \mu_a \mathcal{L}^N + \mu_c |D^c u| + \mu_S |u^+ - u^-| \mathcal{H}^{N-1} \llcorner S(u) + \mu_o,$$

we claim that

$$\mu_a(x_0) \geq f\left(x_0, u(x_0), \nabla u(x_0)\right) \tag{22}$$

for a.e. $x_0 \in \Omega$;

$$\mu_c(x_0) \geq f^\infty\left(x_0, \tilde{u}(x_0), \frac{dC(u)}{d\|C(u)\|}(x_0)\right) \tag{23}$$

for $\|C(u)\|$ a.e. $x_0 \in \Omega$;

$$\mu_S(x_0) \geq \frac{1}{|u^+(x_0) - u^-(x_0)|} K\left(x_0, u^+(x_0), u^-(x_0), \nu_u(x_0)\right) \tag{24}$$

for $|u^+ - u^-|\mathcal{H}^{N-1} \llcorner S(u)$ a.e. $x_0 \in \Omega$.

Assuming the previous inequalities shown, to conclude consider an increasing sequence of smooth cut-off functions $(\varphi_i) \subset C_0^\infty(\Omega)$ such that $0 \leq \varphi_i \leq 1$ and $\sup_i \varphi_i(x) = 1$ on Ω, then for every $i \in \mathbb{N}$ we have

$$\begin{aligned}
\lim_n F(u_n) \;&\geq\; \liminf_n \int_\Omega f\left(x, u_n, \nabla u_n\right) \varphi_i \, dx = \int_\Omega \varphi_i d\mu \\
&\geq\; \int_\Omega f\left(x, u, \nabla u\right) \varphi_i \, dx + \int_\Omega f^\infty\left(x, \tilde{u}, \frac{dC(u)}{d\|C(u)\|}\right) \varphi_i d\|C(u)\| \\
&\quad + \int_{S(u)} K\left(x, u^+, u^-, \nu_u\right) \varphi_i \, d\mathcal{H}^{N-1}.
\end{aligned}$$

Eventually, by letting $i \to +\infty$ and applying the Monotone Convergence Theorem, we get (21). $\square$

In the following subsections we prove (22), (23) and (24).

5.1. The density of the diffuse part

Let $\varphi : [0, +\infty) \to [0, 1]$ a Lipschitz function such that $\varphi \equiv 0$ on $[0, 1/2]$ and $\varphi \equiv 1$ on $[1, +\infty)$, and consider the function $\tilde{f} : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d \times N} \to [0, +\infty)$ defined by

$$\tilde{f}(x, u, \xi) := \varphi(|u|) f(x, \frac{u}{|u|}, P_u \xi).$$

Then $\tilde{f}$ is an extension of f and, for any $\varepsilon > 0$, it can be easily verified that hypotheses (H1)–(H5) on f imply that the function

$$f_\varepsilon(x, u, \xi) := \tilde{f}(x, u, \xi) + \varepsilon|\xi|$$

satisfies hypotheses (F1)–(F5) of Theorem 2.8. Hence, given u_n as in Proposition 5.1, for every $A \in \mathcal{A}(\Omega)$ there holds, by Theorem 2.8,

$$\begin{aligned} \liminf_n \int_A f(x, u_n, \nabla u_n)\, dx &\geq \int_A f_\varepsilon(x, u_n, \nabla u_n)\, dx - \varepsilon \sup_n \|\nabla u_n\|_{L^1(\Omega;\mathbb{R}^{d\times N})} \\ \geq \int_A f_\varepsilon(x, u, \nabla u)\, dx &+ \int_A f_\varepsilon^\infty(x, u, dC(u)) - \varepsilon \sup_n \|\nabla u_n\|_{L^1(\Omega;\mathbb{R}^{d\times N})} \end{aligned}$$

Then, letting ε tend to 0, we get

$$\liminf_n \int_A f(x, u_n, \nabla u_n)\, dx \geq \int_A f(x, u, \nabla u)\, dx + \int_A f^\infty(x, u, dC(u))$$

for every $A \in \mathcal{A}(\Omega)$. From this, it is easy to infer (22) and (23).

5.2. The density of the jump part

To prove (24) we apply the same blow-up argument of [12] Section 3. Recall that Lemma 2.5, Theorem 3.77 [3] and Radon-Nykodym's Theorem yield for $\mathcal{H}^{N-1}$ a.e. $x_0 \in J_u$

$$\lim_{t\to 0^+} \frac{1}{t^{n-1}} \int_{S(u)\cap\left(x_0 + tQ_{\nu(x_0)}\right)} \left|u^+(x) - u^-(x)\right|\, d\mathcal{H}^{N-1} = \left|u^+(x_0) - u^-(x_0)\right|, \tag{25}$$

$$\lim_{t\to 0^+} \frac{1}{t^n} \int_{x_0 + tQ^\pm_{\nu(x_0)}} \left|u(x) - u^\pm(x_0)\right|\, dx = 0, \tag{26}$$

$$\mu_S(x_0) = \lim_{t\to 0^+} \frac{\mu\left(x_0 + tQ_{\nu(x_0)}\right)}{|u^+ - u^-|\,\mathcal{H}^{N-1}\left(S(u) \cap \left(x_0 + tQ_{\nu(x_0)}\right)\right)}, \tag{27}$$

exists and is finite.

By (25) and (27), and since the function $\mathcal{X}_{x_0 + tQ_{\nu(x_0)}}$ is upper semicontinuous and with compact support in Ω if t is sufficiently small, we get

$$\begin{aligned} |u^+(x_0) - u^-(x_0)|\mu_S(x_0) &= \lim_{t\to 0^+} \frac{1}{t^{n-1}} \int_{x_0 + tQ_{\nu(x_0)}} d\mu(x) \\ \geq\ & \limsup_{t\to 0^+} \limsup_n \frac{1}{t^{n-1}} \int_{x_0 + tQ_{\nu(x_0)}} f(x, u_n, \nabla u_n)\, dx \\ =\ & \limsup_{t\to 0^+} \limsup_n \int_{Q_{\nu(x_0)}} tf(x_0 + ty, u_n(x_0 + ty), \nabla u_n(x_0 + ty)\, dy \\ =\ & \limsup_{t\to 0^+} \limsup_n \int_{Q_{\nu(x_0)}} tf(x_0 + ty, u_{n,t}(y), \frac{1}{t}\nabla u_{n,t}(y)\, dy, \end{aligned} \tag{28}$$

where

$$u_{n,t}(y) := u_n(x_0 + ty).$$

Note that, by (26), we get that, set

$$u_0(y) := \begin{cases} u^+(x_0) & \text{if } \langle y, \nu_u(x_0)\rangle \geq 0 \\ u^-(x_0) & \text{if } \langle y, \nu_u(x_0)\rangle < 0, \end{cases}$$

then

$$\lim_{t\to 0^+} \lim_n \int_{Q_{\nu(x_0)}} |u_{n,t}(y) - u_0(y)|\, dx = 0. \tag{29}$$

So far, from (28), by using hypotheses (H3)–(H5) and following the same steps of the proof in [12] Section 3, we get

$$|u^+(x_0) - u^-(x_0)|\mu_S(x_0) \geq \limsup_{t\to 0^+} \limsup_n \int_{Q_{\nu(x_0)}} f^\infty(x_0, u_{n,t}(y), \nabla u_{n,t}(y))\, dy. \tag{30}$$

Then, by (29) and (30), using a standard diagonalization procedure we construct a sequence (v_k) such that $v_k \to u_0$ in $L^1(Q_{\nu(x_0)}; \mathbb{R}^d)$ and

$$|u^+(x_0) - u^-(x_0)|\mu_S(x_0) \geq \lim_{k\to+\infty} \int_{Q_{\nu(x_0)}} f^\infty(x_0, v_k(y), \nabla v_k(y))\, dy.$$

In order to establish (24), by the definition of K, it suffices to replace (v_k) by a sequence in $\mathcal{P}(u^-(x_0), u^+(x_0), \nu_u(x_0))$ whithout increasing the energy in the limit. Assuming Lemma 5.2 below proved, we are done.

Lemma 5.2. *Let $f : \Omega \times S^{d-1}\mathbb{R}^{d\times N}$ be a Carathéodory function such that*

$$0 \leq f(x, u, \xi) \leq c(1 + |\xi|)$$

for some $C > 0$ and for all $(x, u) \in \Omega \times S^{d-1}$ and $\xi \in [T_u(S^{d-1})]^N$. Let $a, b \in S^{d-1}$ and let $v_n \in W^{1,1}(Q_\nu; S^{d-1})$ converge in $L^1(Q_\nu; \mathbb{R}^d)$ to the function u_0 defined by

$$u_0(x) := \begin{cases} b & \text{if } \langle x, \nu\rangle \geq 0 \\ a & \text{if } \langle x, \nu\rangle < 0. \end{cases}$$

Then there exists a sequence $w_n \in \mathcal{P}(a, b, \nu)$ such that $w_n \to u_0$ in $L^1(Q_\nu; \mathbb{R}^d)$ and

$$\liminf_{n\to+\infty} \int_{Q_\nu} f(x, v_n, \nabla v_n)\, dx \geq \limsup_{n\to+\infty} \int_{Q_\nu} f(x, w_n, \nabla w_n)\, dx.$$

Proof. For simplicity of notations, assume $\nu = e_N$ and set $Q := Q_{e_N}$. Without loss of generality, we may suppose that

$$\liminf_{n\to+\infty} \int_Q f(x, v_n, \nabla v_n)\, dx = \lim_{n\to+\infty} \int_Q f(x, v_n, \nabla v_n)\, dx < +\infty.$$

Moreover, by Theorem 2.2, we may assume $v_n \in \mathcal{D}(\Omega; S^{d-1})$, defined by (6) and (7). Let ρ be a mollifier and set $\rho_n := n^N \rho(nx)$. Then define

$$\tilde{\psi}_n(x) := (\rho_n * u_0)(x) = \int_{B(x,1/n)} \rho_n(x-y) u_0(y)\, dy.$$

Note that for all $x \in \mathbb{R}^N$, $\tilde{\psi}_n(x) \in \overline{ab} := \{ta + (1-t)b : \ t \in [0,1]\}$. Let $\pi : \overline{ab} \to S^{d-1}$ a C^1 function such that $\pi(a) = a$ and $\pi(b) = b$ and set

$$\psi_n := \pi \circ \tilde{\psi}_n.$$

It can be easily seen that $\psi_n \in \mathcal{P}(a, b, e_N)$. Moreover

$$\psi_n(x) = \begin{cases} b & \text{if } x_N > 1/n \\ a & \text{if } x_N < 1/n, \end{cases} \qquad \|\nabla \psi_n\|_\infty = O(n).$$

So far, we argue as in the proof of [12] Lemma 3.1. Let

$$\alpha_n := \sqrt{\|v_n - \psi_n\|_{L^1(Q)}}, \quad k_n := n[1 + \|v_n\|_{1,1} + \|\psi_n\|_{1,1}], \quad s_n := \frac{\alpha_n}{k_n}$$

where $[k]$ denotes the largest integer less than or equal to k. Since $\alpha_n \to 0^+$ we may assume $0 \le \alpha_n < 1$ and we set

$$Q_0 := (1-\alpha_n)Q, \quad Q_i := (1 - \alpha_n + is_n), \quad i = 1, \ldots, k_n.$$

Then, let φ_i be a cut-off function between Q_{i-1} and Q_i, with $\|\nabla \varphi_i\|_\infty = O(\frac{1}{s_n})$ for $i = 1, \ldots, k_n$, and define

$$w_n^i := \varphi_i v_n + (1 - \varphi_i)\psi_n.$$

Note that $w_n^i \in W^{1,1}(Q; B^d(0,1))$, and $w_n^i \equiv v_n$ on Q_{i-1}, $w_n^i \equiv \psi_n$ on $Q \setminus Q_i$. Moreover, since

$$\nabla w_n^i = \varphi_i \nabla v_n + (1-\varphi_i)\nabla \psi_n + (v_n - \psi_n) \otimes \nabla \varphi_i,$$

we get

$$\int_{Q_i \setminus Q_{i-1}} |\nabla w_n^i| \, dx \le C \int_{Q_i \setminus Q_{i-1}} (|\nabla v_n| + |\nabla \psi_n| + \frac{1}{s_n}|v_n - \psi_n|) \, dx. \tag{31}$$

We now need to suitably project the functions w_n^i on S^{d-1}. First of all observe that if $N = 1$, the sets $w_n^i(Q)$ are embedded curves so that you can find a sequence of points in the ball $B^d(0,1)$ from which the projection of w_n^i into the sphere S^{d-1} is in $W^{1,1}(Q, S^{d-1})$ and its $W^{1,1}$-norm is uniformly controlled by $\|w_n^i\|_{W^{1,1}}$.

Let us deal now with $N > 1$. To this purpose, given $y \in B^d(0, 1/2)$, let $\pi_y : B^d(0,1) \setminus \{y\} \to S^{d-1}$ the function projecting $x \in B^d(0,1)$ on S^{d-1} along the direction $x - y$. An easy computation shows that π_y is given by

$$\pi_y(x) = y + \frac{-\langle y, x-y\rangle + \sqrt{(\langle y, x-y\rangle)^2 + |x-y|^2(1-|y|^2)}}{|x-y|^2}(x-y). \tag{32}$$

Note that

$$\pi_{y_{|S^{d-1}}} = Id_{S^{d-1}}. \tag{33}$$

Moreover, it is easy to show that

$$|\nabla \pi_y(x)| \le \frac{C}{|x-y|}, \ \forall x \in B^d(0,1), \tag{34}$$

with C independent on $y \in B^d(0,1/2)$. Let G_n^i the set of critical values in $B^d(0,1/2)$ of w_n^i, that is

$$G_n^i := \{y \in B^d(0,1/2) : \exists x \in Q \text{ with } w_n^i(x) = y \text{ and } \operatorname{rank}(\nabla w_n^i(x)) < N \wedge d\},$$

and set

$$G := \cup_{n,i} G_n^i.$$

By Sard's Lemma, $\mathcal{H}^d(G) = 0$. Then, for $y \in B^d(0,1/2) \setminus G$, the function $\pi_y \circ w_{n,i}$ is smooth except on a submanifold of $\mathbb{R}^N$ of codimension greater than 2. Moreover, by Fubini's Theorem and by (34), we get

$$\begin{aligned}
&\int_{B^d(0,1/2)} \int_{Q_i \setminus Q_{i-1}} |\nabla \pi_y \circ w_n^i| \, dx \, dy \\
\leq\ & C \int_{Q_i \setminus Q_{i-1}} |\nabla w_n^i(x)| \Big(\int_{B^d(0,1/2)} |w_n^i(x) - y|^{-1} \, dy \Big) \, dx \\
\leq\ & C \Big(\int_{B^d(0,3/2)} |z|^{-1} \, dz \Big) \int_{Q_i \setminus Q_{i-1}} |\nabla w_n^i(x)| \, dx
\end{aligned} \tag{35}$$

Then, we may find $y_n^i \in B^d(0,1/2) \setminus G$ such that

$$\int_{Q_i \setminus Q_{i-1}} |\nabla \pi_{y_n^i} \circ w_{n,i}| \, dx \leq C \int_{Q_i \setminus Q_{i-1}} |\nabla w_{n,i}| \, dx. \tag{36}$$

Set, then,

$$\tilde{w}_n^i := \pi_{y_n^i} \circ w_n^i.$$

Observe that, by (33), $\tilde{w}_n^i \to u_0$ in $L^1(Q, \mathbb{R}^d)$ and

$$\begin{aligned}
\tilde{w}_n^i &\equiv v_n \text{ on } Q_{i-1}, \\
\tilde{w}_n^i &\equiv \psi_n \text{ on } Q \setminus Q_i.
\end{aligned}$$

Moreover, by (36) $\tilde{w}_n^i \in W^{1,1}(Q; S^{d-1})$ and so $\tilde{w}_n^i \in \mathcal{P}(a, b, e_N)$. Hence, by the growth condition on f, (31) and (36), we get

$$\begin{aligned}
&\int_Q f(x, \tilde{w}_n^i, \nabla \tilde{w}_n^i) \, dx \\
\leq\ & \int_Q f(x, v_n, \nabla v_n) \, dx + C \int_{Q_i \setminus Q_{i-1}} (1 + |\nabla v_n| + |\nabla \psi_n| + \frac{1}{s_n} |v_n - \psi_n|) \, dx \\
&\quad + C \int_{Q \setminus Q_i} (1 + |\nabla \psi_n|) \, dx.
\end{aligned}$$

So far, proceeding exactly as in the proof of Lemma 3.1 [12], one proves that for any $n \in \mathbb{N}$ there exists an index $i(n) \in \{1, \ldots, k_n\}$ such that

$$\int_Q f(x, \tilde{w}_n^{i(n)}, \nabla \tilde{w}_n^{i(n)}) \, dx \leq \int_Q f(x, v_n, \nabla v_n) \, dx + O(1).$$

To conclude, it suffices to set

$$w_n := \tilde{w}_n^{i(n)}.$$

□

6. Estimate from above

In this section we conclude the proof of Theorem 3.1, by showing that

$$\overline{F}(u) \le G(u). \tag{37}$$

To do this, we follow the same argument of [12] Section 5.

As a first step, we localize the functional $\overline{F}$ by setting, for any $u \in BV(\Omega; S^{d-1})$ and $A \in \mathcal{A}(\Omega)$,

$$\overline{F}(u, A) := \inf\{\liminf_n \int_A f(x, u_n, \nabla u_n)\, dx : u_n \in W^{1,1}(A; S^{d-1}),\ u_n \to u \text{ in } L^1(A; \mathbb{R}^d)\}.$$

We claim that

$$\overline{F}(u, A) \le C(|A| + |Du|(A)) \quad \forall (u, A) \in BV(\Omega; S^{d-1}) \times \mathcal{A}(\Omega), \tag{38}$$

and that $\overline{F}(u, A)$ is a *variational functional with respect to the L^1 topology*, that is

(i) $\overline{F}(\cdot, A)$ is local, *i.e.*,

$$\overline{F}(u, A) = \overline{F}(v, A) \quad \text{if } u = v \text{ a.e. in } A;$$

(ii) for every $A \in \mathcal{A}(\Omega)$, $\overline{F}(\cdot, A)$ is lower semicontinuous with respect to the $L^1(A; \mathbb{R}^d)$ topology;

(iii) for every $u \in BV(\Omega; S^{d-1})$, the set function $F(u, \cdot)$ is the restriction of a Borel measure to $\mathcal{A}(\Omega)$.

Inequality (38) follows by the growth hypothesis (H3) and by the following lemma.

Lemma 6.1. *For every $u \in BV(\Omega; S^{d-1})$ and $A \in \mathcal{A}(\Omega)$, there exists a sequence $(u_n) \subset W^{1,1}(A; S^{d-1})$ such that $u_n \to u$ in $L^1(A; \mathbb{R}^d)$ and*

$$\limsup_n |Du_n|(A) \le C|Du|(A),$$

where $C > 0$ is a constant independent on u and A.

Proof. By a standard density result in BV theory, there exists $v_n \in C^\infty(A; B^d(0,1))$ such that $v_n \to u$ in $L^1(A; \mathbb{R}^d)$ and

$$\lim_n |Dv_n|(A) = |Du|(A).$$

For $y \in B^d(0, 1/2)$, let $\pi_y : B^d(0,1) \setminus \{y\} \to S^{d-1}$ defined by (32). Then, as in the proof of Lemma 5.2, we may find $y_n \in B^d(0, 1/2)$ such that $\pi_{y_n} \circ v_n \in W^{1,1}(A; S^{d-1})$ and

$$\int_A |\nabla \pi_{y_n} \circ v_n|\, dx \le C \int_A |\nabla v_n|\, dx.$$

Then, it suffices to set

$$u_n := \pi_{y_n} \circ v_n.$$

□

Properties (i) and (ii) are direct consequence of the definition of $\overline{F}(u,A)$. To prove (iii), thanks to De Giorgi-Letta criterion (see [10]), we have to show that the set function $F(u,\cdot)$ is superadditive, subadditive and inner regular. The proof of the superadditivity property is straightforward. By (38) and using a standard argument (see for example the proof of Theorem 4.3 in [4]), the proof of the last two properties follows by the following Proposition, in which we establish the so called weak subbaditivity for $\overline{F}(u,\cdot)$.

Proposition 6.2. *Let $u \in BV(\Omega; S^{d-1})$. Then, for any $A', A, B \in \mathcal{A}(\Omega)$ such that $A' \subset\subset A$, there holds*

$$\overline{F}(u, A' \cup B) \leq \overline{F}(u, A) + \overline{F}(u, B). \tag{39}$$

Proof. Let $u_n \in W^{1,1}(A; S^{d-1})$, $v_n \in W^{1,1}(B; S^{d-1})$ be such that $u_n \to u$ in $L^1(A; \mathbb{R}^d)$, $v_n \to u$ in $L^1(B; \mathbb{R}^d)$ and

$$\lim_n \int_A f(x, u_n, \nabla u_n)\, dx = \overline{F}(u, A),$$

$$\lim_n \int_B f(x, v_n, \nabla u_n)\, dx = \overline{F}(u, B).$$

By Theorem 2.2, we may suppose $u_n \in C^\infty(A; S^{d-1})$, $v_n \in C^\infty(B; S^{d-1})$. Set

$$d := \text{dist}\,(A', A^c)$$

and, given $M \in \mathbb{N}$, for any $i \in \{1, \ldots, M\}$ define

$$A_i := \{x \in A : \text{dist}\,(x, A') < i\frac{d}{M}\},$$

$$C_i = (A_{i+1} \setminus \overline{A}_i) \cap B.$$

Let φ_i be a cut-off function between A_i and A_{i+1}, with $\|\nabla\varphi_i\|_\infty \leq 2\frac{M}{d}$, and set

$$w_n^i := \varphi_i u_n + (1 - \varphi_i) v_n.$$

Then, for any $i \in \{1, \ldots, M\}$, $w_n^i \in W^{1,1}(A' \cup B; B^d(0,1)) \cap C^\infty(A' \cup B; B^d(0,1))$ and $w_n^i \to u$ in $L^1(A' \cup B; \mathbb{R}^d)$. Moreover, since

$$\nabla w_n^i(x) = \varphi_i(x)\nabla u_n(x) + (1 - \varphi_i(x))\nabla v_n(x) + \nabla\varphi_i(x) \otimes (u_n(x) - v_n(x)),$$

we get

$$\int_{C_i} |\nabla w_n^i|\, dx \leq \int_{C_i} (|\nabla u_n| + |\nabla v_n| + 2\frac{M}{d}|u_n - v_n|)\, dx. \tag{40}$$

In order to find a good recovery sequence for $\overline{F}(u, A' \cup B)$, we now argue as in the proof of Lemma 5.2. For $y \in B^d(0,1/2)$, let $\pi_y : B^d(0,1) \setminus \{y\} \to S^{d-1}$ defined by (32). Then, as in the proof of Lemma 5.2, we may find $y_n^i \in B^d(0,1/2)$ such that $\pi_{y_n^i} \circ w_n^i \in W^{1,1}(A' \cup B; S^{d-1})$ and

$$\int_{C_i} |\nabla\pi_{y_n^i} \circ w_{n,i}|\, dx \leq C \int_{C_i} |\nabla w_{n,i}|\, dx. \tag{41}$$

Set, then,

$$\tilde{w}_n^i := \pi_{a_n^i} \circ w_n^i.$$

Observe that $w_n^i \to u$ in $L^1(A' \cup B; \mathbb{R}^d)$ and

$$\begin{aligned} \tilde{w}_n^i &\equiv u_n \text{ on } A^i, \\ \tilde{w}_n^i &\equiv v_n \text{ on } B \setminus A^{i+1}. \end{aligned}$$

Hence, by (H3), (40) and (41), we get

$$\begin{aligned} \int_{A' \cup B} f(x, \tilde{w}_n^i, \nabla \tilde{w}_n^i)\, dx \;\; &\le \;\; \int_A f(x, u_n, \nabla u_n)\, dx + \int_B f(x, v_n, \nabla v_n)\, dx \\ &\quad + C \int_{C_i} (1 + |\nabla u_n| + |\nabla v_n| + 2\frac{M}{d}|u_n - v_n|)\, dx. \end{aligned}$$

Thus, summing over $i \in \{1, \dots M\}$ and averaging, we get

$$\begin{aligned} \frac{1}{M}\sum_{i=1}^{M} \int_{A' \cup B} f(x, \tilde{w}_n^i, \nabla \tilde{w}_n^i)\, dx \;\; &\le \;\; \int_A f(x, u_n, \nabla u_n)\, dx + \int_B f(x, v_n, \nabla v_n)\, dx \\ &\quad + \frac{C}{M} \int_{A \cap B} (1 + |\nabla u_n| + |\nabla v_n| + 2\frac{M}{d}|u_n - v_n|)\, dx. \end{aligned} \tag{42}$$

For any $n \in \mathbb{N}$ there exists $i(n) \in \{1, \dots M\}$ such that

$$\int_{A' \cup B} f(x, \tilde{w}_n^{i(n)}, \nabla \tilde{w}_n^{i(n)})\, dx \le \frac{1}{M}\sum_{i=1}^{M} \int_{A' \cup B} f(x, \tilde{w}_n^i, \nabla \tilde{w}_n^i)\, dx. \tag{43}$$

Then, since $\tilde{w}_n^{i(n)}$ still converges to u in $L^1(A' \cup B; \mathbb{R}^d)$ and, by (H3),

$$\sup_n \int_{A \cup B} |\nabla u_n| + |\nabla v_n|\, dx < +\infty,$$

from (42) and (43) we deduce that

$$\overline{F}(u, A' \cup B) \le \liminf_n \int_{A' \cup B} f(x, \tilde{w}_n^{i(n)}, \nabla \tilde{w}_n^{i(n)})\, dx \le \overline{F}(u, A) + \overline{F}(u, B) + \frac{C}{M}.$$

Eventually, letting $M \to +\infty$, we obtain the thesis. □

Remark 6.3. $\overline{F}$ enjoys the following locality property on $\mathcal{B}(\Omega)$:
let $u, v \in BV(\Omega; S^{d-1})$ and let $B \in \mathcal{B}(\Omega)$ be such that

$$B \subseteq S(u) \cap S(v), \qquad (u^-(x), u^+(x), \nu_u(x)) = (v^-(x), v^+(x), \nu_v(x)) \;\forall x \in B,$$

then

$$\overline{F}(u, B) = \overline{F}(v, B).$$

The proof of this property can be carried out as in Step 1 of the proof of Proposition 4.4 in [4], where the same property is stated in the non constrained case.

So far, we can obtain inequality (37) by showing that

$$\overline{F}(u, \Omega \setminus S(u)) \leq \int_\Omega f(x, u, \nabla u)\, dx + \int_\Omega f^\infty(x, u, dC(u)), \tag{44}$$

and

$$\overline{F}(u, S(u)) \leq \int_{S(u)} K(x, u^-, u^+, \nu_u)\, d\mathcal{H}^{N-1}. \tag{45}$$

Inequality (44) will follow by Lemma 6.4 below, while (45) will be proved in Lemma 6.5.

Lemma 6.4. *If $u \in BV(\Omega, S^{d-1})$, then for $\mathcal{L}^N$ a.e. $x_0 \in \Omega$,*

$$\frac{d\overline{F}(u,\cdot)}{d\mathcal{L}^N}(x_0) \leq f(x_0, u(x_0), \nabla u(x_0)), \tag{46}$$

and for $|C(u)|$ a.e $x_0 \in \Omega$,

$$\frac{d\overline{F}(u,\cdot)}{d|C(u)|}(x_0) \leq f^\infty(x_0, u(x_0), A(x_0)). \tag{47}$$

Proof. The proof follows the lines of *Step 2* of Theorem 2.16 in [12]. We will enter into details only when the changes are significant, otherwise reminding to [12].

Let us prove first (46). By Theorem 2.4, and by Theorems 2.7-2.8 in [12], for $\mathcal{L}^N$ a.e. $x_0 \in \Omega$ we have

$$\lim_{\varepsilon\to 0} \frac{1}{|B(x_0,\varepsilon)|} \int_{B(x_0,\varepsilon)} |u(x) - u(x_0)|(1 + |\nabla u(x)|)\, dx = 0, \tag{48}$$

$$\lim_{\varepsilon\to 0} \frac{1}{|B(x_0,\varepsilon)|} \int_{B(x_0,\varepsilon)} |\nabla u(x) - \nabla u(x_0)|\, dx = 0, \tag{49}$$

$$\lim_{\varepsilon\to 0} \frac{|D_s u|(B(x_0,\varepsilon))}{|B(x_0,\varepsilon)|} = 0, \quad \lim_{\varepsilon\to 0} \frac{|Du|(B(x_0,\varepsilon))}{|B(x_0,\varepsilon)|} \text{ exists and is finite,} \tag{50}$$

$$\frac{d\overline{F}(u,\cdot)}{d\mathcal{L}^N}(x_0) \text{ exists and is finite.} \tag{51}$$

Let $\{u_n\}$ be the sequence defined in Lemma 2.7. Fix a sequence of numbers $\varepsilon \in (0, \frac{\operatorname{dist}(x_0,\partial\Omega)}{2})$ such that $|Du|(\partial B(x_0,\varepsilon)) = 0$, and a subsequence of u_n, not relabeled, such that $w_n := \pi_{a_n^\varepsilon} \circ u_n$, defined as in the proof of Lemma 5.2 and Proposition 6.2, is such that $w_n \in W^{1,1}(\Omega, S^{d-1})$ and, for some $\delta \in (\frac{3}{4}, 1)$,

$$\int_{\{x \in B(x_0,\varepsilon): |u_n(x)| \leq \delta\}} |\nabla w_n|\, dx \leq C \int_{\{x \in B(x_0,\varepsilon): |u_n(x)| \leq \delta\}} |\nabla u_n|\, dx, \tag{52}$$

where $a_n^\varepsilon \in B^d(0, \frac{1}{2})$ satisfies

$$\lim_{\varepsilon\to 0} \lim_{n\to+\infty} a_n^\varepsilon = \lim_{\varepsilon\to 0} a^\varepsilon = a_0.$$

Thanks to this choice of δ we have also that

$$|\nabla w_n|\chi_{\{|u_n|>\delta\}} \leq C|\nabla u_n|\chi_{\{|u_n|>\delta\}}, \tag{53}$$

so

$$\int_{B(x_0,\varepsilon)} |\nabla w_n|\, dx \leq C \int_{B(x_0,\varepsilon)} |\nabla u_n|\, dx. \tag{54}$$

Then

$$\begin{aligned} \frac{d\overline{F}(u,\cdot)}{d\mathcal{L}^N}(x_0) &= \lim_{\varepsilon\to 0} \frac{\overline{F}(u, B(x_0,\varepsilon))}{|B(x_0,\varepsilon)|} \\ &\leq \liminf_{\varepsilon\to 0} \liminf_{n\to+\infty} \frac{1}{|B(x_0,\varepsilon)|} \int_{B(x_0,\varepsilon)} f(x, w_n(x), \nabla w_n(x))\, dx. \end{aligned} \tag{55}$$

Introducing, as in [12], the Yosida transforms of f, given by

$$f_\lambda(x,u,\xi) := \sup\{f(x',u',\xi) - \lambda[|x-x'| + |u-u'|](1+|\xi|) : (x',u') \in \Omega\times\mathbb{R}^d\},$$

we have

$$\begin{aligned} f(x, w_n(x), \nabla w_n(x)) \;\leq\; & f(x_0, u(x_0), \nabla w_n(x)) + \eta(1+|\nabla w_n(x)|) \\ & + \lambda[|x-x_0| + C|w_n(x) - u(x_0)|](1+|\nabla w_n(x)|), \end{aligned}$$

for $x \in \overline{B}\left(x_0, \frac{\operatorname{dist}(x_0,\partial\Omega)}{2}\right)$ and $\eta > 0$. Thus, by (55) and (54),

$$\begin{aligned} \frac{d\overline{F}(u,\cdot)}{d\mathcal{L}^N}(x_0) \;\leq\; & \liminf_{\varepsilon\to 0} \liminf_{n\to+\infty} \frac{1}{|B(x_0,\varepsilon)|} \Big[\int_{B(x_0,\varepsilon)} f(x_0, u(x_0), \nabla w_n(x))\, dx \\ & + C(\eta + \lambda\varepsilon) \int_{B(x_0,\varepsilon)} |\nabla u_n|\, dx + (\lambda\varepsilon + \eta)|B(x_0,\varepsilon)| \\ & + C\lambda \int_{B(x_0,\varepsilon)} |w_n(x) - u(x_0)|(1+|\nabla w_n|)\, dx \Big]. \end{aligned}$$

Since, by (53) and (54),

$$\begin{aligned} & \int_{B(x_0,\varepsilon)} |w_n(x) - u(x_0)||\nabla w_n|\, dx \\ \leq\; & C \int_{B(x_0,\varepsilon)} |u_n(x) - u(x_0)||\nabla w_n|\, dx \\ =\; & C \int_{\{x\in B(x_0,\varepsilon): |u_n(x)|\leq\delta\}} |u_n(x) - u(x_0)||\nabla w_n|\, dx \\ & + C \int_{\{x\in B(x_0,\varepsilon): |u_n(x)|>\delta\}} |u_n(x) - u(x_0)||\nabla w_n|\, dx \\ \leq\; & (1+\delta) C \int_{\{x\in B(x_0,\varepsilon): |u_n(x)|\leq\delta\}} |\nabla u_n|\, dx \\ & + C \int_{\{x\in B(x_0,\varepsilon): |u_n(x)|>\delta\}} |u_n(x) - u(x_0)||\nabla u_n|\, dx \end{aligned}$$

$$\leq \; C \int_{B(x_0,\varepsilon)} |u_n(x) - u(x_0)||\nabla u_n| \, dx,$$

we deduce

$$\begin{aligned} \frac{d\overline{F}(u,\cdot)}{d\mathcal{L}^N}(x_0) \;\; \leq \;\; & \liminf_{\varepsilon\to 0} \liminf_{n\to+\infty} \frac{1}{|B(x_0,\varepsilon)|} \Big[\int_{B(x_0,\varepsilon)} f(x_0, u(x_0), \nabla w_n(x)) \, dx \\ & + C(\eta + \lambda\varepsilon) \int_{B(x_0,\varepsilon)} |\nabla u_n| \, dx + (\lambda\varepsilon + \eta)|B(x_0,\varepsilon)| \\ & + C\lambda \int_{B(x_0,\varepsilon)} |u_n(x) - u(x_0)|(1 + |\nabla u_n|) \, dx \Big]. \end{aligned}$$

Taking into account that $f(x_0, u(x_0), \cdot)$ is a Lipschitz function we get

$$\begin{aligned} \frac{d\overline{F}(u,\cdot)}{d\mathcal{L}^N}(x_0) \;\; \leq \;\; & \liminf_{\varepsilon\to 0} \liminf_{n\to+\infty} \frac{1}{|B(x_0,\varepsilon)|} \Big[\int_{B(x_0,\varepsilon)} f(x_0, u(x_0), \nabla u_n(x)) \, dx \\ & + C \int_{B(x_0,\varepsilon)} |\nabla u_n - \nabla w_n| \, dx + (C\eta + \lambda\varepsilon) \int_{B(x_0,\varepsilon)} |\nabla u_n| \, dx \qquad (56) \\ & + (\lambda\varepsilon + \eta)|B(x_0,\varepsilon)| + C\lambda \int_{B(x_0,\varepsilon)} |u_n(x) - u(x_0)|(1 + |\nabla u_n|) \, dx \Big]. \end{aligned}$$

Now, the first and the third term of (56) can be treated as in *Step 2* of Theorem 2.16 in [12], getting

$$\begin{aligned} \frac{d\overline{F}(u,\cdot)}{d\mathcal{L}^N}(x_0) \;\; \leq \;\; & f(x_0, u(x_0), \nabla u(x_0)) + C\eta \\ & + C \liminf_{\varepsilon\to 0} \limsup_{n\to+\infty} \frac{1}{|B(x_0,\varepsilon)|} \Big[\int_{B(x_0,\varepsilon)} |\nabla u_n - \nabla w_n| \, dx \qquad (57) \\ & + \lambda \int_{B(x_0,\varepsilon)} |u_n(x) - u(x_0)|(1 + |\nabla u_n|) \, dx \Big]. \end{aligned}$$

Splitting $B(x_0,\varepsilon)$ into the sets where $|u_n| \leq \delta$ and where $|u_n| > \delta$, the term $I_\varepsilon^n := \int_{B(x_0,\varepsilon)} |\nabla u_n - \nabla w_n| \, dx$ can be estimated in the following way

$$\begin{aligned} I_\varepsilon^n \leq C & \int_{\{x\in B(x_0,\varepsilon): |u_n(x)|\leq\delta\}} |u_n(x) - u(x_0)||\nabla u_n| \, dx \\ & + \int_{\{x\in B(x_0,\varepsilon): |u_n(x)|>\delta\}} |\nabla u_n - \nabla w_n| \, dx, \end{aligned} \qquad (58)$$

the firts term being of the same type of the last term of (57). Let us deal with the second term. It yields

$$\begin{aligned} & \int_{\{x\in B(x_0,\varepsilon): |u_n(x)|>\delta\}} |\nabla u_n - \nabla w_n| \, dx \\ \leq \; & \int_{\{x\in B(x_0,\varepsilon): |u_n(x)|>\delta\}} |\nabla \pi_{a_n^\varepsilon}(u_n) - \nabla \pi_{a_n^\varepsilon}(u(x_0))||\nabla u_n| \, dx \end{aligned}$$

$$
\begin{aligned}
&+\int_{\{x\in B(x_0,\varepsilon):|u_n(x)|>\delta\}} |(I-\nabla\pi_{a_n^\varepsilon}(u(x_0)))\nabla u_n|\,dx \qquad (59)\\
\leq\ & C\int_{B(x_0,\varepsilon)} |u_n(x)-u(x_0)||\nabla u_n|\,dx + \int_{\{x\in B(x_0,\varepsilon):|u_n(x)|>\delta\}} |L_n^\varepsilon \nabla u_n|\,dx,
\end{aligned}
$$

where $L_n^\varepsilon : \mathbb{R}^{d\times d}\to\mathbb{R}^{d\times d}$ defined by $L_n^\varepsilon = I-\nabla\pi_{a_n^\varepsilon}(u(x_0))$ is such that

$$\lim_{\varepsilon\to 0}\lim_{n\to+\infty} L_n^\varepsilon = \lim_{\varepsilon\to 0} L^\varepsilon = L_0,$$

with $L^\varepsilon = I-\nabla\pi_{a^\varepsilon}(u(x_0))$ and $L_0 = I-\nabla\pi_{a_0}(u(x_0))$. Let us note that

$$
\begin{aligned}
L_n^\varepsilon\nabla u_n &= L_n^\varepsilon(\rho_n * Du) = L_n^\varepsilon\left(\int_{B(x,\frac{1}{n})}\rho_n(x-y)Du(y)\right)\\
&= \int_{B(x,\frac{1}{n})}\rho_n(x-y)L_n^\varepsilon Du(y) = \rho_n * (L_n^\varepsilon Du),
\end{aligned}
$$

so that, by Lemma 2.7,

$$
\begin{aligned}
&\int_{\{x\in B(x_0,\varepsilon):|u_n(x)|>\delta\}} |L_n^\varepsilon\nabla u_n|\,dx\\
\leq\ & \int_{B(x_0,\varepsilon)} |\rho_n * (L_n^\varepsilon Du)|\,dx \leq |L_n^\varepsilon Du|(B(x_0,\varepsilon+\frac{1}{n})).
\end{aligned}
$$

Taking into account that $|Du|(\partial B(x_0,\varepsilon))=0$, passing to the limit as n goes to infinity in the previous inequality, we get

$$\limsup_{n\to+\infty}\int_{\{x\in B(x_0,\varepsilon):|u_n(x)|>\delta\}} |L_n^\varepsilon\nabla u_n|\,dx \leq |L^\varepsilon Du|(B(x_0,\varepsilon)). \qquad (60)$$

Let us divide by $|B(x_0,\varepsilon)|$ and denote by μ^ε the measure $\mu^\varepsilon = L^\varepsilon((u^+-u^-)\otimes\nu_u)\mathcal{H}^{N-1}\lfloor S(u)$ $+L^\varepsilon(A(x))|C(u)|$, obtaining

$$
\begin{aligned}
\frac{|L^\varepsilon Du|(B(x_0,\varepsilon))}{|B(x_0,\varepsilon)|} &\leq \frac{|\mu^\varepsilon|(B(x_0,\varepsilon))}{|B(x_0,\varepsilon)|} + \fint_{B(x_0,\varepsilon)}|L^\varepsilon\nabla u|\,dx\\
&\leq \frac{|\mu^\varepsilon|(B(x_0,\varepsilon))}{|B(x_0,\varepsilon)|} + \fint_{B(x_0,\varepsilon)}|L^\varepsilon\nabla u - L_0\nabla u|\,dx + \fint_{B(x_0,\varepsilon)}|L_0\nabla u|\,dx\\
&\leq \frac{|\mu^\varepsilon|(B(x_0,\varepsilon))}{|B(x_0,\varepsilon)|} + |L^\varepsilon - L_0|\fint_{B(x_0,\varepsilon)}|\nabla u|\,dx \qquad (61)\\
&\qquad +|L_0|\fint_{B(x_0,\varepsilon)}|\nabla u(x)-\nabla u(x_0)|\,dx,
\end{aligned}
$$

where we have used Remark 2.6 for the last term. Putting together (60) and (61) and using (49) and $(50)_1$, we get

$$\lim_{\varepsilon\to 0}\limsup_{n\to+\infty}\frac{1}{|B(x_0,\varepsilon)|}\int_{\{x\in B(x_0,\varepsilon):|u_n(x)|>\delta\}} |L_n^\varepsilon\nabla u_n|\,dx = 0,$$

so that

$$\lim_{\varepsilon\to 0}\limsup_{n\to+\infty}\frac{1}{|B(x_0,\varepsilon)|}\Big[\int_{\{x\in B(x_0,\varepsilon):|u_n(x)|>\delta\}}|\nabla u_n-\nabla w_n|\,dx$$
$$-\int_{B(x_0,\varepsilon)}|u_n(x)-u(x_0)||\nabla u_n|\,dx\Big]=0.$$

At this point to prove (46) it remains to show that

$$\liminf_{\varepsilon\to 0}\limsup_{n\to+\infty}\frac{1}{|B(x_0,\varepsilon)|}\int_{B(x_0,\varepsilon)}|u_n(x)-u(x_0)|(1+|\nabla u_n|)\,dx=0,$$

and this can be done exactly as in *Step 2* of Theorem 2.16 in [12], so the proof of (46) is complete.

Next we prove (47). Denoting by ν the measure $\nu=|Du|-|C(u)|$, by Theorems 2.7, 2.8 and 2.11 in [12], for $|C(u)|$ a.e. $x_0\in\Omega$ we have

$$\lim_{\varepsilon\to 0}\frac{\nu(B(x_0,\varepsilon))}{|C(u)|(B(x_0,\varepsilon))}=0,\qquad \lim_{\varepsilon\to 0}\frac{|Du|(B(x_0,\varepsilon))}{|C(u)|(B(x_0,\varepsilon))}\text{ exists and is finite,}\tag{62}$$

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^N}{|C(u)|(B(x_0,\varepsilon))}=0,\tag{63}$$

$$\lim_{\varepsilon\to 0}\frac{1}{|C(u)|(B(x_0,\varepsilon))}\int_{B(x_0,\varepsilon)}|u(x)-u(x_0)|\,d|C(u)|=0,\tag{64}$$

$$\lim_{\varepsilon\to 0}\frac{1}{|C(u)|(B(x_0,\varepsilon))}\int_{B(x_0,\varepsilon)}|A(x)-A(x_0)|\,d|C(u)|=0,\tag{65}$$

$$\begin{aligned}&\liminf_{\varepsilon\to 0}\frac{1}{|C(u)|(B(x_0,\varepsilon))}\int_{B(x_0,\varepsilon)}f^\infty(x_0,u(x_0),A(x))\,d|C(u)|\\ =\;&f^\infty(x_0,u(x_0),A(x_0)),\end{aligned}\tag{66}$$

$$\frac{d\overline{F}(u,\cdot)}{d|C(u)|}(x_0)\ \text{ exists and is finite.}\tag{67}$$

As for (56), we get

$$\begin{aligned}\frac{d\overline{F}(u,\cdot)}{d|C(u)|}(x_0)\ \le\ &\liminf_{\varepsilon\to 0}\liminf_{n\to+\infty}\frac{1}{|C(u)|(B(x_0,\varepsilon))}\Big[\int_{B(x_0,\varepsilon)}f(x_0,u(x_0),\nabla u_n(x))\,dx\\ &+C\int_{B(x_0,\varepsilon)}|\nabla u_n-\nabla w_n|\,dx+(C\eta+\lambda\varepsilon)\int_{B(x_0,\varepsilon)}|\nabla u_n|\,dx\\ &+(\lambda\varepsilon+\eta)|B(x_0,\varepsilon)|+C\lambda\int_{B(x_0,\varepsilon)}|u_n(x)-u(x_0)|(1+|\nabla u_n|)\,dx\Big].\end{aligned}\tag{68}$$

Using (58), (59), (60), and the fact that

$$\begin{aligned}&\limsup_{n\to+\infty}\int_{B(x_0,\varepsilon)}|u_n(x)-u(x_0)||\nabla u_n|\,dx\\ \le\ &\int_{\overline{B}(x_0,\varepsilon)\setminus S(u)}|u(x)-u(x_0)||Du|(x)+4|Du|(\overline{B}(x_0,\varepsilon)\cap S(u)),\end{aligned}$$

which is proved in [12], since $C(B(x_0,\varepsilon)\cap S(u))=0$, we conclude that

$$\begin{aligned}\frac{d\overline{F}(u,\cdot)}{d|C(u)|}(x_0) \leq & \liminf_{\varepsilon\to 0}\liminf_{n\to+\infty}\frac{1}{|C(u)|(B(x_0,\varepsilon))}\int_{B(x_0,\varepsilon)} f(x_0,u(x_0),\nabla u_n(x))\,dx\\ & +C\limsup_{\varepsilon\to 0}\frac{1}{|C(u)|(B(x_0,\varepsilon))}|L^\varepsilon Du|(B(x_0,\varepsilon))\\ & +\limsup_{\varepsilon\to 0}\frac{1}{|C(u)|(B(x_0,\varepsilon))}(C\eta+\lambda\varepsilon)(|Du|(B(x_0,\varepsilon))+|B(x_0,\varepsilon)|)\\ & +C\lambda\limsup_{\varepsilon\to 0}\frac{1}{|C(u)|(B(x_0,\varepsilon))}\Big[\int_{B(x_0,\varepsilon)}|u(x)-u(x_0)|\,d|C(u)|\\ & +2|B(x_0,\varepsilon)|+6|\nu|(B(x_0,\varepsilon))\Big]\\ \leq & \liminf_{\varepsilon\to 0}\liminf_{n\to+\infty}\frac{1}{|C(u)|(B(x_0,\varepsilon))}\int_{B(x_0,\varepsilon)} f(x_0,u(x_0),\nabla u_n(x))\,dx\\ & +C\limsup_{\varepsilon\to 0}\frac{1}{|C(u)|(B(x_0,\varepsilon))}|L^\varepsilon Du|(B(x_0,\varepsilon))\\ & +\limsup_{\varepsilon\to 0}\frac{1}{|C(u)|(B(x_0,\varepsilon))}(C\eta+\lambda\varepsilon)(|Du|(B(x_0,\varepsilon))+|B(x_0,\varepsilon)|)\\ & +C\lambda\limsup_{\varepsilon\to 0}\frac{1}{|C(u)|(B(x_0,\varepsilon))}\Big[\int_{B(x_0,\varepsilon)}|u(x)-u(x_0)|\,d|C(u)|\\ & +6\nu(B(x_0,\varepsilon))+2|B(x_0,\varepsilon)|\Big].\end{aligned} \tag{69}$$

Now, denoting by $\tilde{\mu}^\varepsilon$ the measure $\tilde{\mu}^\varepsilon = L^\varepsilon(\nabla u)\mathcal{L}^N + L^\varepsilon((u^+-u^-)\otimes\nu_u)\mathcal{H}^{N-1}\lfloor S(u)$, as for (61), one sees that

$$\begin{aligned}\frac{|L^\varepsilon Du|(B(x_0,\varepsilon))}{|C(u)|(B(x_0,\varepsilon))} &\leq \frac{|\tilde{\mu}^\varepsilon|(B(x_0,\varepsilon))}{|C(u)|(B(x_0,\varepsilon))}+\frac{1}{|C(u)|(B(x_0,\varepsilon))}\int_{B(x_0,\varepsilon)}|L^\varepsilon A(x)||C(u)|\\ &\leq \frac{|\tilde{\mu}^\varepsilon|(B(x_0,\varepsilon))}{|C(u)|(B(x_0,\varepsilon))}+\frac{1}{|C(u)|(B(x_0,\varepsilon))}\int_{B(x_0,\varepsilon)}|L^\varepsilon A(x)-L_0A(x)|C(u)|\\ &\quad +\frac{1}{|C(u)|(B(x_0,\varepsilon))}\int_{B(x_0,\varepsilon)}|L_0A(x)||C(u)|\\ &\leq \frac{|\tilde{\mu}^\varepsilon|(B(x_0,\varepsilon))}{|C(u)|(B(x_0,\varepsilon))}+\frac{|L^\varepsilon-L_0|}{|C(u)|(B(x_0,\varepsilon))}\int_{B(x_0,\varepsilon)}|A(x)||C(u)|\\ &\quad +\frac{|L_0|}{|C(u)|(B(x_0,\varepsilon))}\int_{B(x_0,\varepsilon)}|A(x)-A(x_0)||C(u)|,\end{aligned}$$

where we have used Remark 2.6 for the last term. Therefore, by $(62)_1$ and (65), we obtain

$$\limsup_{\varepsilon\to 0}\frac{1}{|C(u)|(B(x_0,\varepsilon))}|L^\varepsilon Du|(B(x_0,\varepsilon))=0. \tag{70}$$

Now, applying (62)-(64) to (69), and using (70) we deduce

$$\frac{d\overline{F}(u,\cdot)}{d|C(u)|}(x_0) \leq \liminf_{\varepsilon\to 0}\liminf_{n\to+\infty}\frac{1}{|C(u)|(B(x_0,\varepsilon))}\int_{B(x_0,\varepsilon)} f(x_0,u(x_0),\nabla u_n(x))\,dx + C\eta, \tag{71}$$

and the same arguments of *Step 2* of Theorem 2.16 in [12] lead to (47). □

Eventually the following Lemma provides us the inequality of the surface term that concludes the proof of Theorem 3.1.

Lemma 6.5. *For any $u \in BV(\Omega; S^{d-1})$ there holds*

$$\overline{F}(u, S(u)) \leq \int_{S(u)} K(x, u^-, u^+, \nu_u)\, d\mathcal{H}^{N-1}. \tag{72}$$

Proof. The proof closely follows that of Step 3 in Section 5 of [12]. Here we outline the main steps and enter into details of the proof only when significant changes occur.

Step 1. If $u(x) = a\chi_E(x) + b(1 - \chi_E(x))$ with $Per_\Omega(E) < +\infty$, then (72) holds. As in [12], it is enough to prove that for every $A \in \mathcal{A}(\Omega)$,

$$\overline{F}(u, A) \leq \int_A f(x, u, 0)\, dx + \int_{S(u)\cap A} K(x, u^-, u^+, \nu_u)\, d\mathcal{H}^{N-1}. \tag{73}$$

(i) Suppose first $E = \{x \in \mathbb{R}^N : \langle x, \nu\rangle = 0\}$ for some $\nu \in S^{N-1}$. Without loss of generality we set $\nu := e_N$. In case f does not depend on x, by the definition of K let $\varphi \in \mathcal{P}(a, b, \nu)$ be such that

$$K(a, b, e_N) + \eta \geq \int_Q f^\infty(\varphi, \nabla\varphi)\, dx, \tag{74}$$

for some $\eta > 0$. Then, for $n \in \mathbb{N}$, define $u_n \in W^{1,1}(A; S^{d-1})$ by

$$u_n(x) := \begin{cases} b & \text{if } x_N > 1/2n \\ \varphi(nx) & \text{if } |x_N| \leq 1/2n \\ a & \text{if } x_N < -1/2n. \end{cases}$$

Then it is easy to see that $u_n \to u$ in $L^1(A; \mathbb{R}^d)$. Moreover, set $A_n := \{x \in A : |x_N| \leq 1/2n\}$, $A'_n := \pi(A_n)$, π denoting the orthogonal projection onto E, by Fubini Theorem and by a change of variables, we get

$$\begin{aligned}
\int_A f(u_n, \nabla u_n)\, dx &= |A \cap \{x_N \geq 1/2n\}| f(b, 0) + |A \cap \{x_N \leq -1/2n\}| f(a, 0) \\
&\quad + \int_{A_n} f(\varphi(nx), n\nabla\varphi(nx))\, dx \\
&\leq |A \cap \{x_N \geq 1/2n\}| f(b, 0) + |A \cap \{x_N \leq -1/2n\}| f(a, 0) \\
&\quad + \int_{A'_n} dx' \int_{-1/2n}^{1/2n} f(\varphi(nx', nt), n\nabla\varphi(nx', nt))\, dt \\
&= |A \cap \{x_N \geq 1/2n\}| f(b, 0) + |A \cap \{x_N \leq -1/2n\}| f(a, 0) \\
&\quad + \int_{A'_n} dx' \int_{-1/2}^{1/2} \frac{1}{n} f(\varphi(nx', s), n\nabla\varphi(nx', s))\, ds \\
&=: I_n^1 + I_n^2 + I_n^3.
\end{aligned}$$

Thus, one easily gets that

$$\lim_n (I_n^1 + I_n^2) = \int_A f(u, 0)\, dx,$$

while, by (H5) and Riemann-Lebesgue Theorem, there holds

$$\lim_n I_n^3 = \mathcal{H}^{N-1}(S(u) \cap A) \int_Q f^\infty(\varphi, \nabla\varphi)\, dx.$$

The conclusion follows by (74) and the arbitrariness of η.

In the general case, when f depends also on x, we can argue as in the proof of Proposition 4.1 in [13], by using assumption (H4), property (a) of Lemma 4.1 and Lemma 5.2.

(ii) Suppose E is a polyhedral set, that is E is a bounded strongly Lipschitz domain and $\partial E = H_1 \cup \cdots \cup H_m$, H_i being closed subsets of hyperplanes. Then the proof of (73) can be obtained as in Step 3 (c) of Section 5 in [12], by using the same argument of the proof of Lemma 5.2.

(iii) Finally, if E is an arbitrary set of finite perimeter in Ω, the proof of (73) is exactly the same of Step 3 (f) of Section 5 in [12], where property (a) and (b) of Lemma 4.1 are needed.

Step 2. Inequality (72) holds if

$$u(x) = \sum_{i=1}^{h} a_i \chi_{E_i}(x), \tag{75}$$

with $h \in \mathbb{N}$, $a_1, \ldots, a_h \in S^{d-1}$, $E_1, \ldots, E_h$ mutually disjoint sets of finite perimeter in Ω covering Ω. The proof can be done exactly as in Step 1 of the proof of Proposition 4.8 in [4], where the property of $\overline{F}$ stated in Remark 6.3 is needed.

Step 3. Inequality (72) holds if $u \in BV(\Omega; S^{d-1})$. First of all, note that the function K can be extended to a function $\tilde{K} : \Omega \times \mathbb{R}^d \setminus \{0\} \times \mathbb{R}^d \setminus \{0\} \times S^{N-1} \to [0, +\infty)$ as

$$\tilde{K}(x, a, b, \nu) := K(x, \frac{a}{|a|}, \frac{b}{|b|}, \nu),$$

so that $\tilde{K}$ inherites the properties of K stated in Lemma 4.1 on $\Omega \times J \times J \times S^{N-1}$, for every compact set J in $\mathbb{R}^d$. Given $A \in \mathcal{A}(\Omega)$, as in Step 2 of the proof of Proposition 4.8 in [4], by using the upper semicontinuity of $\tilde{K}$, one constructs a sequence $(u_n) \subset BV(\Omega; \mathbb{R}^d)$ of the type (75), such that

$$\lim_n \|u_n - u\|_\infty = 0 \tag{76}$$

and

$$\begin{aligned} &\liminf_n \int_{S(u_n)\cap A} \tilde{K}(x, u_n^-, u_n^+, \nu_{u_n})\, d\mathcal{H}^{N-1} \\ \leq\ & C|Du|(A \setminus S(u) + \int_{S(u)\cap A} K(x, u^-, u^+, \nu_u)\, d\mathcal{H}^{N-1} \end{aligned}$$

Thus, by (76), for n large enough, the sequence

$$v_n := \frac{u_n}{|u_n|}$$

is in $BV(\Omega; S^{d-1})$ of the type (75) and, thanks to Lemma 4.1 (a),

$$\begin{aligned} &\liminf_n \int_{S(v_n)\cap A} K(x, v_n^-, v_n^+, \nu_{v_n})\, d\mathcal{H}^{N-1} \\ \leq\ & \liminf_n \int_{S(u_n)\cap A} \tilde{K}(x, u_n^-, u_n^+, \nu_{u_n})\, d\mathcal{H}^{N-1} + o(1). \end{aligned}$$

Then, by the previous Step and the lower semicontinuity of $\overline{F}$, there holds

$$\begin{aligned}\overline{F}(u,A) &\leq \liminf_n \overline{F}(v_n,A) \leq \liminf_n \int_{S(v_n)\cap A} K(x,v_n^-,v_n^+,\nu_{v_n})\,d\mathcal{H}^{N-1} \\ &\leq C|Du|(A\setminus S(u) + \int_{S(u)\cap A} K(x,u^-,u^+,\nu_u)\,d\mathcal{H}^{N-1}.\end{aligned}$$

From this and since $\overline{F}(u,\cdot)$ is a bounded measure, we easily infer (72). □

7. Relaxation of energies in micromagnetics

In this section we study the relaxation of constrained functionals $\mathcal{G} : L^1(\Omega;\mathbb{R}^N) \to [0,+\infty]$ of the type

$$\mathcal{G}(u) = \begin{cases} \int_\Omega f(x,u,\nabla u) + \int_{\mathbb{R}^N} |h_u|^2\,dx - \int_\Omega \langle h_{ext}, u\rangle\,dx & \text{if } u \in W^{1,1}(\Omega;S^{N-1}) \\ \infty & \text{otherwise,} \end{cases} \tag{77}$$

where $h_{ext} \in L^1(\Omega;\mathbb{R}^N)$ and $h_u \in L^2(\mathbb{R}^N;\mathbb{R}^N)$ is defined by

$$\begin{cases} \text{curl } h_u = 0 \\ \text{div } (h_u + u\chi_\Omega) = 0. \end{cases} \tag{78}$$

Functionals of this kind generalize those involved in variational models for micromagnetics, where u represents the magnetization of a ferromagnetic material subject to an external magnetic field h_{ext} and h_u is the induced magnetic field related to u through the Maxwell's equations (78) (see [7], [17] for a detailed explanation of the model).

System (78) is understood in the following way. Using the first equation of (78) it can be introduced the potential v with $h_u = -\nabla v$. Thus the second equation can be written as

$$\text{div}(-\nabla v + u) = 0 \text{ in } \mathbb{R}^N, \tag{79}$$

where we have extended $u = 0$ outside Ω. The equation (79) means that

$$\int_{\mathbb{R}^N} (-\nabla v + u)\nabla w\,dx = 0 \quad \forall w \in V, \tag{80}$$

where $V = \{w \in H^1(B) : \nabla w \in L^2(\mathbb{R}^N) \text{ and } \int_B w\,dx = 0\}$ is a Hilbert space with inner product $(v,w) = \int_{\mathbb{R}^N} \nabla v \nabla w\,dx + \int_B vw\,dx$, $B \subset \mathbb{R}^N$ a fixed ball with $\overline{\Omega} \subset B$.

In [17] (see Lemma 3.1) the following lemma is proved.

Lemma 7.1. *Let $u \in L^2(\Omega,\mathbb{R}^N)$. The equation (80) admits a unique solution $v \in V$. The mapping $T : L^2(\Omega,\mathbb{R}^N) \to V$, defined by $T(u) = v$ is linear and continuous.*

We are now in position to prove the following integral representation result for the relaxation $\overline{\mathcal{G}} : L^1(\Omega;\mathbb{R}^N) \to \overline{[}0,+\infty)$ of the functional $\mathcal{G}$, given by (77), with respect to the L^1 topology.

Theorem 7.2. *Let f satisfy* $(H1)$*–*$(H5)$*. Then*

$$\overline{\mathcal{G}}(u) = \overline{F}(u) + \int_{\mathbb{R}^N} |h_u|^2 \, dx - \int_{\Omega} \langle h_{ext}, u \rangle \, dx,$$

where $\overline{F}(u)$ is given by (17)*.*

Proof. Observe that a sequence $u_n \in W^{1,1}(\Omega, S^{N-1})$ converging with respect to the L^1 norm is also compact in the strong topology of L^2. So, thanks to Lemma 7.1, the result follows by the continuity of the last two terms of the functional $\mathcal{G}$ and by Theorem 3.1. □

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