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Basic Properties of Evenly Convex Sets

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A subset of a finite-dimensional real vector space is called *evenly convex* if it is the intersection of a collection of open halfspaces. The study of such sets was initiated in 1952 by Werner Fenchel, who defined a natural polarity operation and mentioned some of its properties. Over the years since then, evenly convex sets have made occasional appearances in the literature but there has been no systematic study of their basic properties. Such a study is undertaken in the present paper.

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Introduction

Throughout this paper, $\mathbb{E}$ will denote a real vector space of finite dimension $d \geq 2$, equipped with a Euclidean norm. It is precisely the closed convex subsets of $\mathbb{E}$ that can be represented as intersections of closed halfspaces. Each open convex set can be represented as an intersection of open halfspaces, but the sets so representable (here called *evenly convex* or simply *e-convex*) form a much broader class that includes also the closed convex sets and many sets that are neither open nor closed. For example, if B is the closed unit ball $\{x \in \mathbb{E} : \|x\| \leq 1\}$ of $\mathbb{E}$ and C is an arbitrary set such that $\mathrm{int}B \subset C \subset B$, then C is evenly convex.

In 1952, Fenchel [6] defined the class of evenly convex subsets of $\mathbb{E}$ and observed that this class admits a natural polarity operation whose properties are similar to those of the standard polarity that plays such an important role in the study of closed convex sets. Since the appearance of Fenchel's paper in 1952, the notion of even convexity has appeared sporadically for almost thirty years, precisely in the following connections: in 1968, the maximal separation theorems of Klee [10]; in 1970 and 1972, the linear range-domain

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implications of Schröder [17, 18]; in 1984, the partial convex polyhedra of Rockafellar (p. 510 of [16]); after 1980, evenly convex sets have been used more extensively in quasiconvex programming, permitting to define the *evenly quasiconvex functions* that play a relevant role in the theory of conjugation for quasiconvex functions. These concepts, as well as some properties of e-convex sets, have been considered both in finite-dimensional spaces, precisely for conjugation of quasiconvex functions by Martinez-Legaz in 1983 [13] and by Passy and Prisman in 1984 [14], and by Goberna, Jornet and Rodriguez in 2003 [7] for various results on sets of solutions of linear systems containing strict inequalities, and in infinite-dimensional spaces by Penot and Volle in 1990 [15] and by Daniilidis and Martinez-Legaz in 2002 [4]. However, even in finite-dimensional spaces, many of the basic properties of evenly convex sets have not appeared in the literature, and it is our intention to present those properties here.

As the term is used here, a *cone* is a subset K of $\mathbb{E}$ such that $K + K \subset K$, $]0, \infty[\, K \subset K$, and K includes a nonzero point of $\mathbb{E}$. Thus each cone K is convex, has the origin 0 as its apex, and is a union of closed or open rays issuing from 0. (Note that the origin is not required to belong to K.) The term *semispace* is used for a cone which is maximal among cones omitting their apex 0 (i.e. a cone K such that for any $x \in \mathbb{E} \setminus \{0\}$ exactly one between x and $-x$ belongs to K). Special attention is paid here to the even convexity of cones and also to the even convexity of bounded sets.

Our section headings are as follows:

1. Exposed Points and Extreme Points
2. k-Sections
3. k-Projections
4. Binary Operations and Polarities

Section 1 gives some hints on the shape of e-convex sets, presenting some properties that e-convex sets or their boundaries must enjoy. Sections 2 and 3, which contain the main results, are devoted to the study of the relationship between e-convexity of sets and e-convexity of their sections or linear projections, providing some characterizations of sets which admit only e-convex sections or projections. In the last section some natural ways of combining e-convex sets are defined via some "meet" and "join" (binary) operations, which moreover prove to be dual operations with respect to Fenchel's polarity.

1. Exposed Points and Extreme Points

As the term is used here, an *exposed face* of a closed convex subset X of $\mathbb{E}$ is a nonempty intersection of the form $X \cap H$ where H is a hyperplane in $\mathbb{E}$ that supports but does not contain X. An *exposed point* of X is a point p such that $\{p\}$ is an exposed face. Now suppose that X is a full-dimensional closed convex subset of $\mathbb{E}$, and $C = X \setminus Y$ where Y is a subset of the boundary of X. It follows immediately from the definition that C is e-convex if and only if Y is the (possibly empty) union of some collection of exposed faces of X. In particular, the even convexity of X is preserved by the deletion from X of an arbitrary subset of $\exp X$, the set of all exposed points of X.

There is a particular subset G of $\mathbb{R}^2$ which, along with its close relatives, forms the basis of several examples that appear in later sections. Hence it seems appropriate to describe the set here.

Example 1.1 (Basic Example). Let R be a closed rectangle in $\mathbb{R}^2$ and D a closed circular halfdisk that is disjoint from R's interior and whose diameter coincides with one of the sides of R, say the segment $[p, q]$. Set

$$G = (R \cup D) \setminus \{p, q\}.$$

Then G is not e-convex, because each open halfspace that contains G must also contain the two points p and q of G's complement. The points p and q are extreme points of G's closure but they are not exposed.

Note that this example, as well as the examples in the nextfollowing sections that are based on it, can be made locally polytopal except at the points p and q, i.e. the set G can be modified in such a way that each point different from p and q admits a neighborhood whose intersection with G is a polytope.

Because of the close relationship of e-convexity to exposed faces, and the relationship of exposed points to extreme points, we digress somewhat to summarize some aspects of that relationship. This summary seems worth including because the notion of exposed point, which is essential here, is a natural sharpening of the much more familiar notion of extreme point.

Let $\mathrm{ext}X$ denote the set of all extreme points of X. Then of course $\exp X \subset \mathrm{ext}X$, and when X is compact a theorem of Straszewicz [19] shows that $\exp X$ is dense in $\mathrm{ext}X$. (See [1], [5], [8], and [12] for sharpenings of this result.)

In the case of a polytope, each extreme point is exposed, and when X is a 2-dimensional convex set there are at most countable many nonexposed extreme points. For each closed convex subset X of $\mathbb{E}$ (in any dimension), the set $\mathrm{ext}X$ is a G_δ-set, hence topologically complete, and many examples suggest the conjecture that $\exp X$ must be not merely dense in $\mathrm{ext}X$ but also a residual subset of $\mathrm{ext}X$ and hence "most" of $\mathrm{ext}X$ in the sense of the Baire category theorem. However, this conjecture was disproved by an ingenious counter-example of Corson [3]. Concerning the Borel type of the set $\exp X$, a result of [2] shows that when $d = 2$ the set $\exp X$ is a G_δ-set; when $d = 3$, $\exp X$ is the union of a G_δ with a set that is the intersection of a G_δ and an F_σ; and for $d \geq 3$, $\exp X$ is the union of a G_δ, an F_σ, and $d - 2$ sets each of which is the intersection of a G_δ and an F_σ.

2. k-Sections

As the term is used here, a *k-section* of a subset X of $\mathbb{E}$ is the intersection of X with a flat (affine subspace) of dimension $k \geq 1$ in $\mathbb{E}$. We begin by noting without proof some properties of closedness and of openness of convex sets with respect to the formation of sections. These should be compared with Corollary 2.3 below.

Proof of the following basic fact is left to the reader.

Proposition 2.1. *Suppose that C is a d-dimensional convex subset of $\mathbb{E}$, that $1 \leq k \leq d$, and that $q \in \mathbb{E}$. If $C \cap M$ is open in M for every k-flat M that contains q, then C is in fact open in $\mathbb{E}$. If $k \geq 2$ or q is interior to C, then the result holds also with "open" replaced by "closed."*

Theorem 2.2. *Suppose that C is a convex subset of $\mathbb{E}$, $0 \notin C$, and $\mathbf{S}$ is a covering of C by linear subspaces of dimension ≥ 2. Suppose also that each $S \in \mathbf{S}$ includes an interior point p_S of C. Then the following two conditions are equivalent:*

(*1*) *there exists a hyperplane H that contains 0 and misses C;*
(*2*) *for each $S \in \mathbf{S}$ there exists a linear subspace L_S of S such that $L_S \cap C = \emptyset$ and $\dim L_S = (\dim S) - 1$.*

Proof. (*1*) implies (*2*). For H as in (*1*), let $L_S = S \cap H$ for each $S \in \mathbf{S}$.

(*2*) implies (*1*). Let F denote the set of all linear functionals $f : \mathbb{E} \longrightarrow \mathbb{R}$ such that $fC \subset [0, \infty[$. Since F is convex and finite-dimensional, there is a functional r that belongs to the relative interior of F. We will show that $rC \subset]0, \infty[$, whence $r^{-1}(0)$ is a hyperplane that contains 0 and misses C.

Consider an arbitrary point $x \in C$, let $S \in \mathbf{S}$ be such that $x \in S$. Let $p_S \in S \cap (\mathrm{int}C)$ and let L_S be as described in (*2*). Since $L_S \cap C = \emptyset$, a standard separation theorem yields a hyperplane H_S that contains L_S and misses $\mathrm{int}C$. Hence there exists $f_x \in F$ such that $H_S = f_x^{-1}(0)$ and $f_x(p_S) > 0$. The fact that $f_x(x) > 0$ then follows, for $f_x(x) \geq 0$, $x \in S \setminus L_S$, and (since L_S is of codimension 1 in S) p_S is a nonnegative combination of x and a point of L_S; hence $f_x(x) = 0$ implies $f_x(p_S) = 0$, a contradiction.

Now since $f_x \in F$ and $r \in \mathrm{relint}F$, there exists $g_x \in F$ and $\lambda_x \in]0, 1[$ such that $r = \lambda_x f_x + (1 - \lambda_x) g_x$. Since $f_x(x) > 0$ and $g_x(x) \geq 0$, it follows that $r(x) > 0$, and since x was an arbitrary point of C we conclude that $rC \subset]0, \infty[$. □

Corollary 2.3. *Suppose that C is a convex subset of $\mathbb{E}$, $q \in \mathbb{E}$, and $C \cap M$ is e-convex for each k-flat M that contains q. If $k \geq 3$, or $q \in \mathrm{int}C$ and $k \geq 2$, then the set C is e-convex.*

Proof. We want to show that each point y of $\mathbb{E} \setminus C$ lies in a hyperplane that misses C. Since everything is invariant under translation, we may assume that $y = 0$.

When $k \geq 3$, let $\mathbf{S}$ denote the collection of all k-flats that contain the set $\{q, y\}$ and intersect the set $\mathrm{int}C$. Note that C is covered by $\mathbf{S}$, and it follows from the theorem that the point 0 $(= y)$ belongs to a hyperplane missing C.

When $q \in \mathrm{int}C$ and $k \geq 2$, we can apply Theorem 2.2 to the collection of all 2-flats that contain $\{q, y\}$. □

Corollary 2.4. *If C is a cone with $\dim C \geq 3$ and C is not e-convex, then there is a 3-dimensional subspace M such that $C \cap M$ is not e-convex.*

Proof. Apply Corollary 2.3 with $q = 0$ and $k = 3$. □

Where "3-dimensional" appears in Corollaries 2.3 and 2.4, it cannot be replaced by "2-dimensional", as is shown, respectively, by the following two examples.

Example 2.5. Start with a copy of the set G described in Example 1.1. Suppose this copy lies in xyz-space, in the plane given by $z = 1$, and call the copy G_1. Then set

$$C_1 = \mathrm{conv}(G_1 \cup [0, p[\cup [0, q[).$$

The convex set C_1 is not e-convex, because G_1 is a 2-section of C_1. However, each 2-section of C_1 through the origin consists of a single point, a segment, a closed triangle, or a triangle with one or two of its vertices missing, and each of these sections is e-convex.

Example 2.6. Let $C_2 =]0, \infty[\, G_1$. Then C_2 is a pointed cone with its apex missing, and C_2 is not e-convex. However, each 2-section of C_2 through 0 is e-convex because each such section consists of an open ray or of an open angle with its apex missing or of the union of such an angle with one or both of its bounding open rays – and each of these sections is e-convex.

Note, however, that 3 *can* be replaced by 2 in Corollary 2.4 when the cone includes its apex. Then, in fact, having each 2-section through the apex e-convex implies that the cone is closed (see Proposition 3.3).

3. k-Projections

As the term is used here, a *k-projection* of a subset X of $\mathbb{E}$ is a translate of a linear projection of X into a k-dimensional subspace of $\mathbb{E}$. We supply examples of bounded e-convex sets that admit non e-convex projections and also examples of convex sets that are not e-convex but of which all proper projections are e-convex.

Example 3.1. Start with the basic example G defined in Example 1.1. Enlarge G by replacing its semicircular boundary arc Γ by half of a noncircular ellipse whose minor axis is $[p, q]$. Then bend the semiellipse up (rotating around $[p, q]$) until its vertical projection onto the plane of G is the semicircle Γ and call Γ_e the bent up semiellipse. Calling again R the rectangle used in the construction of G, set

$$C_3 = \operatorname{conv}(R \cup \Gamma_e) \setminus \{p, q\}.$$

Because of the bending, each of p and q now lies in a plane disjoint from C_3, so C_3 is e-convex. Of course, its projection G is not e-convex.

Example 3.2. Suppose that G lies in the xy plane of the xyz space: consider the capped cylinder generated by a complete rotation of $G \cup \{p, q\}$ about its (unique) axis of symmetry and denote by J_1 its half lying in the semispace of the negative z's. Set $J_2 = (G \times [0, 1]) \cup (\{p, q\} \times]0, 1])$. Finally, let $C_4 = J_1 \cup J_2$. Then C_4 is not e-convex, for G is a section of C_4. However, projecting C_4 onto any plane produces a set that is closed and hence e-convex.

When we pass to cones, the situation changes. It is still true that e-convex cones may admit projections that are not e-convex: consider, for instance, a closed circular cone K in $\mathbb{R}^3$, or the same cone but without its apex $(0, 0, 0)$, supported by the plane xy at the points of the positive x semiaxis. Its projection along the x axis into the plane yz is an open halfplane together with the point $(0, 0, 0)$, hence it is not e-convex. However, the property of admitting only e-convex projections has strong implications on the shape of the cone.

For cones which contain their apex the question is settled by known results, taking into account the following

Proposition 3.3. *A convex cone containing its apex is e-convex if and only if it is closed.*

Proof. Recall that, since everything is invariant under translation, we are assuming that K has apex 0 and that we have to prove only that e-convexity implies closedness. Suppose K is not closed and let $x \in \partial K \setminus K$. Since K is assumed e-convex, K is openly separated from x. Let f_x a linear functional such that $f_x(x) = \alpha > f_x K$, with $\alpha > 0$, since $0 \in K$. $x \in \partial K$ implies the existence of $p \in K$ with $f_x(p) > 0$, therefore $f_x K$ cannot be bounded from above, a contradiction. □

In fact Theorem 4.11 of [9] can be rewritten as

Proposition 3.4. *Suppose that K is a d-dimensional convex cone containing its apex: for any k, $2 \le k \le d-1$, K admits only e-convex k-projections if and only if it is a polyhedral cone.*

Therefore from now on we restrict our attention to convex cones omitting their apex. For cones which are not e-convex we have the following

Theorem 3.5. *Suppose that K is a convex cone that omits its apex 0, with $\dim K = d \ge 3$. If K is not evenly convex then it admits a (d-1)-projection that is not evenly convex. As a consequence, for any k, $2 \le k \le d-1$, it admits a k-projection that is not e-convex.*

Proof. We start remarking that, since K is a convex cone omitting its apex 0, it is contained in a semispace, and it is a semispace omitting 0 if and only if for every $q \in \mathbb{R}^d \setminus \{0\}$ exactly one between q and $-q$ belongs to K.

Since K is not e-convex, there exists p, $p \notin K$, such that K cannot be openly separated from p, i.e. for any hyperplane H, $p \in H \Rightarrow H \cap K \neq \emptyset$. ($p$ must belong to ∂K, hence H is a linear subspace of $\mathbb{R}^d$).

We split our proof in three cases.

1. Suppose $p \neq 0$ and $-p \notin K$. In this case consider the line $L =]-\infty, +\infty[p$ and the projection π_p onto $L^\perp$ along L . Since both p and $-p$ do not belong to K, $0 \notin \pi_p(K)$. Suppose $\pi_p(K)$ is e-convex: then there exists in $L^\perp$ a hyperplane H_1 (with $dim H_1 = d-2$) that openly separates $\pi_p(K)$ from 0. But then H_1+L, a hyperplane in $\mathbb{R}^d$, openly separates K from p, a contradiction.

2. If $p = 0$ or $-p \in K$ but $\exists\, q \in \mathbb{R}^d$ such that both $q \notin K$ and $-q \notin K$, set $L =]-\infty, +\infty[\, q$ and consider the projection π_q along L onto $L^\perp + p$. p cannot belong to $\pi_q(K)$. In fact, if p belonged to $\pi_q(K)$, for some $\lambda_p \in \mathbb{R}$ we should have $p + \lambda_p q \in K$ hence, immediately if $p = 0$ and recalling that $-p \in K$ if $p \neq 0$, we obtain $\lambda_p q \in K$, contradicting our assumption. Now, if $\pi_q(K)$ were e-convex, we could openly separate it from p in $L^\perp + p$ by an hyperplane H_1. Then $H_1 + L$ openly separates K from p, a contradiction.

3. If, for p and any other $q \in \mathbb{R}^d \setminus \{0\}$, exactly one between the point and its opposite belongs to K, K is a semispace without 0. We prove by induction on d that, for any $d \ge 3$, a d-dimensional semispace omitting 0 can be projected onto a $(d-1)$-dimensional semispace, containing 0, which obviously is not e-convex. Start with $d = 3$: there exist linear functionals f_1, f_2, f_3 on $\mathbb{R}^3$ such that

$$K = \{x \in \mathbb{R}^3 : \big(f_1(x) > 0\big) \vee \big(f_1(x) = 0 \wedge f_2(x) > 0\big)$$
$$\vee \big(f_1(x) = f_2(x) = 0 \wedge f_3(x) > 0\big)\}.$$

Let $q \in K$ such that $f_1(q) = f_2(q) = 0 \wedge f_3(q) > 0$. As before, call π_q the projection along $L =]-\infty, +\infty[q$ onto $L^\perp$. The set $\{x \in \mathbb{R}^3 : f_1(x) > 0\}$ is projected onto its intersection with $L^\perp$, which is an open halfplane Q; the set $\{x \in \mathbb{R}^3 : f_1(x) = 0 \wedge f_2(x) > 0\}$ is projected onto its intersection with $L^\perp$, which is an open halfline in ∂Q, and the set $\{x \in \mathbb{R}^3 : f_1(x) = 0 \wedge f_2(x) = 0 \wedge f_3 > 0\}$ is projected onto 0. Hence $\pi_q(K)$ is, in $\mathbb{R}^2$, a semispace with 0, and is not e-convex. Now assume that every d-semispace K omitting 0 has an orthogonal projection π_q such that $\pi_q(K)$ is a $(d-1)$-semispace with 0. Let K_1 be a $(d+1)$-semispace omitting 0: K_1 is the union of an halfspace Q and a d-semispace $K \subset \partial Q$ omitting 0. By the induction hypothesis, there exists $q \in \partial Q$ such that the orthogonal projection π_q onto a $(d-1)$-dimensional subspace of ∂Q sends K in a $(d-1)$-semispace with 0. π_q projects Q in a d-dimensional halfspace Q_1, whose boundary is $\pi_q(\partial Q)$, and K in $\pi_q(K)$; therefore $\pi_q(K_1) = Q_1 \cup \pi_q(K)$ is a d-semispace, with 0. □

It is known that polyhedral cones admit only closed projections, hence cones that are the intersection of a polyhedral cone with an open cone admit only e-convex projections. This condition is weakened by the following

Theorem 3.6. *Let K be a convex cone omitting its apex 0. If K is e-convex and polyhedral at each of its points, then for any k, $2 \leq k \leq d-1$, all k-projections of K are e-convex.*

Before proving Theorem 3.6, we state some new results about open separation of convex cones, that are useful in the proof.

Lemma 3.7. *Let K be a d-dimensional e-convex cone and L a linear j-dimensional subspace such that $K \cap L = \emptyset$ but $\mathrm{cl}K \cap L \neq \{0\}$. Then K is openly separated from L.*

Proof. Since K and L are disjoint sets, there exists an hyperplane H that separates K and L. Let $p \in \partial K \cap L$, $p \neq 0$: obviously $p \in H$, and since K is a convex cone with apex 0, H must be linear, hence must contain L. Now let $F = H \cap \mathrm{cl}K$, an exposed face of $\mathrm{cl}K$. Since $p \in F$ and $p \notin K$, and K is assumed to be e-convex, no point of K can belong to F, hence K is contained in one of the two open halfspaces determined by H. □

Lemma 3.8. *Let K be a d-dimensional closed convex cone in $\mathbb{E}$, such that $K \cap (-K) = \{0\}$ and L a linear subspace such that $L \cap K = \{0\}$. Then L lies in a hyperplane H through 0 such that $H \cap K = \{0\}$, and hence $K \setminus \{0\}$ is openly separated from L.*

Proof. Since K is closed and pointed, there is a compact convex set $\tilde{K}$ such that $0 \notin \tilde{K}$ and $K = [0, +\infty[\tilde{K}$. Since $\tilde{K}$ is compact and $\tilde{K} \cap L = \emptyset$, there is a ball B such that $(\tilde{K}+B) \cap L = \emptyset$. Now let $K' = [0, +\infty[(\tilde{K}+B)$, a pointed cone such that $(K \setminus \{0\}) \subset \mathrm{int}K'$ and $(K' \setminus \{0\}) \cap L = \emptyset$. The sets $K' \setminus \{0\}$ and L are weakly separated by a hyperplane H through 0: necessarily, $L \subset H$ and H misses $K \setminus \{0\}$, so $K \setminus \{0\}$ is openly separated from L. □

Lemma 3.9. *Let K be a d-dimensional closed convex cone in $\mathbb{E}$, and let $M = K \cap (-K)$, the lineality space of K. Suppose L is a linear subspace such that $L \cap K = \{0\}$. Then $K \setminus M$ is openly separated from L.*

Proof. Obviously $dimM \leq d - \dim L$, since $L \cap M = \{0\}$. Actually $dimM < d - \dim L$ for otherwise $\mathbb{E} = M \oplus L$ which is impossible because $K \cap L = \{0\}$. Hence $dim(M \oplus L) \leq d-1$ and there is a nondegenerate subspace J (i.e. $dimJ \geq 1$) such that $\mathbb{E} = M \oplus L \oplus J$. Now consider the linear transformation Π whose kernel is M and range is $L \oplus J$, projecting all of $\mathbb{E}$ onto $L \oplus J$. The space L is mapped into itself by Π, and M is mapped into the origin, so $\Pi(K)$ is a closed convex pointed cone in $L \oplus J$, with $(\Pi(K)) \cap L = \{0\}$. By Lemma 3.8, $\Pi(K) \setminus \{0\}$ lies in an open halfspace Q in $L \oplus J$ such that Q misses L. Then $Q + M (= \Pi^{-1}(Q))$ is an open halfspace in $\mathbb{E}$ such that $Q + M$ contains $K \setminus M$ and misses L and, being a subspace, L is contained in the hyperplane $\partial Q + M$ bounding $Q + M$. □

Remark 3.10. We can enumerate the different types of 2-dimensional cones that are not e-convex, namely

a) angles (or halfplanes) containing their apex and omitting at least one of their boundary rays;

b) halfplanes omitting their apex but containing one of their boundary rays.

Proof of Theorem 3.6. We proved in Theorem 3.5 that every d-dimensional convex cone that is not e-convex admits a $(d-1)$ dimensional projection that is not e-convex, hence we may assume that the cone admits a 2-projection that fails to be e-convex.

Let $\pi : \mathbb{E} \to \mathbb{R}^2$ a projection such that $\pi(K)$ is not e-convex. If $\pi(K)$ belongs to case a) of Remark 3.10, it is not polyhedral at its apex 0. Since $0 \in \pi(K)$, $K \cap \pi^{-1}(0) \neq \emptyset$ and by Klee, ([9], Theorem 4.1) , the cone K cannot be polyhedral at any point of $K \cap \pi^{-1}(0)$.

Now we want to prove that, if $\pi(K)$ is as in b) of Remark 3.10, K cannot be evenly convex.

Assume K is e-convex. Set $L = \pi^{-1}(0)$, hence $K \cap L = \emptyset$.

If $\mathrm{cl}K \cap L \neq \{0\}$ by Lemma 3.7, we may produce an open halfspace Q that contains K while its boundary ∂Q contains L.

If $\mathrm{cl}K \cap L = \{0\}$, by Lemma 3.9, we may produce an open halfspace Q such that $L \subset \partial Q$ and $\mathrm{cl}K \setminus M \subset Q$ (where M is the lineality space of $\mathrm{cl}K$). Set $F = \mathrm{cl}K \cap \partial Q$, an exposed face of $\mathrm{cl}K$. Since K is assumed e-convex, and $0 \notin K$, no point of F can belong to K, but M is contained in F, hence $K \subset \mathrm{cl}K \setminus M \subset Q$.

So in both cases we have an open halfspace Q such that $K \subset Q$ and $L \subset \partial Q$.

$\dim(\pi^{-1}(0)) = d - 2$ implies that $\pi(\partial Q)$ is a line ∂Q_1 in $\mathbb{R}^2$, and K projects in one of the open halfplanes determined in $\mathbb{R}^2$ by ∂Q_1. This contradicts the assumption that $\pi(K)$ is of the type b) of Remark 3.10, thus completing the proof of the theorem. □

The class of e-convex cones which are polyhedral at each of their points is strictly larger than the class of convex cones that are intersection of a polyhedral cone and an open cone, as shown by the following

Example 3.11. Let $\{\theta_n\}$ be an increasing sequence in $[0, 2\pi)$, such that $\theta_1 = 0$ and $\lim_{n\to+\infty} \theta_n = 2\pi$ and, in $\mathbb{R}^3$, let $\{x_n\}$ be the sequence $\{(\cos\theta_n - 1, \sin\theta_n, 1)\}$. Set $C = \mathrm{clconv}\big(\{x_n\}_{n=0}^{+\infty}\big)$ and $K =]0, +\infty[\big(C \setminus \{(0,0,1)\}\big)$. K is a convex cone omitting its apex which is e-convex and polyhedral at each of its points but it is not the intersection of a polyhedral cone with an open one.

Remark 3.12. If K is the cone of the preceding example, $\mathrm{cl}K \setminus \{0\}$ is an e-convex cone which is not polyhedral at any point of the halfline $]0, +\infty[(0,0,1)$; projecting K along that line we get an angle containing its apex but omitting one of its boundary rays.

It would be nice to have a characterization of cones that admit only e-convex k-projections, where k is a fixed integer, $2 \leq k \leq d-1$, among convex cones omitting their apex.

The characterization is obtained for $k = 2$ from Theorems 3.5 and 3.6, together with the following

Proposition 3.13. *If a convex cone K is not polyhedral at a point p, K admits a 2-projection which is not e-convex.*

Proof. It was proved in ([9], Theorem 4.1) that, for any k, $2 \leq k \leq d-1$, there exists a k-projection π into a k-flat through p such that $\pi(K)$ is not polyhedral in p. For $k = 2$, $\pi(K)$ is a 2-dimensional cone which can be non polyhedral only at its vertex, hence it cannot be e-convex. □

Hence we obtain our main result

Theorem 3.14. *A convex cone omitting its apex admits only e-convex 2-projections if and only if it is e-convex and polyhedral at each of its points.*

We close this section stating the following

Conjecture 3.15. *For any k, $3 \leq k \leq d-1$, a convex cone omitting its apex admits only e-convex k-projections if and only if it is e-convex and polyhedral at each of its points.*

4. Binary Operations and Polarities

We begin this section presenting examples that demonstrate some respects in which evenly convex sets behave differently from closed convex sets and open convex sets. These examples are cited to show that certain hypotheses cannot be abandoned. In all the examples, the involved sets are assumed to lie in a d-dimensional Euclidean space $\mathbb{E}$ with $d \geq 2$.

Vector sums. For sets X and Y, the vector sum $X+Y$ is the set $\{x+y : x \in X, y \in Y\}$. It is open if at least one of X and Y is open, and it is closed if X and Y are both closed and at least one of them is bounded.

Now let $B = \{b : \|b\| < 1\}$ and $S = \{b : \|b\| = 1\}$, the open unit ball and the unit sphere of $\mathbb{E}$, and let $X = B \cup U$ for some $U \subset S$. For each choice of U, the set X is evenly convex.

If X is not open and $u \in U$, let Y be an open segment $]0, t[$ such that the segment $u+[0,t]$ is tangent to S at u. Then Y is e-convex, but the set $X+Y$ is not e-convex because each open halfspace that contains $X+Y$ must also contain the point u of its complement.

Convex hulls. The convex hull $\mathrm{conv}(X \cup Y)$ is open if X and Y are both open, and is closed if X and Y are both closed and bounded.

With B, U, and X as in the preceding example, suppose that X is not closed and let $u \in S \setminus U$, let $Y = \{u+t\}$ where $u+[0,t]$ is tangent to S at u. Then the set $\mathrm{conv}(X \cup Y)$ is not e-convex because each open halfspace that contains $\mathrm{conv}(X \cup Y)$ must also contain the point u of its complement.

Therefore, when considering e-convex sets, it is worthwhile to define the **e-convex hull** of $X \cup Y$, denoted by e-conv$X \cup Y$, as the smallest e-convex set containing $X \cup Y$.

Obviously conv$X \cup Y \subset$ e-conv $X \cup Y \subset$ clconv$X \cup Y$.

The following part of this section describes several natural ways of combining two e-convex sets to produce a third one, and studies the interactions of these binary operations with polarities.

Proposition 4.1. *The class of all e-convex subsets of $\mathbb{E}$ is closed under each of the following "meet" operations $\wedge_i$ and "join" operations $\vee_i$:*

$$
\begin{aligned}
X \wedge_1 Y &= (\mathrm{int}X) \cap (\mathrm{int}Y)\\
X \wedge_2 Y &= X \cap Y\\
X \wedge_3 Y &= \big(X \cap (\mathrm{cl}Y)\big) \cup \big(Y \cap (\mathrm{cl}X)\big)\\
X \wedge_4 Y &= (\mathrm{cl}X) \cap (\mathrm{cl}Y)\\
X \vee_1 Y &= \mathrm{clconv}(X \cup Y)\\
X \vee_2 Y &= \text{e-conv}(X \cup Y)\\
X \vee_3 Y &= \text{e-conv}\big(X \cup (\mathrm{int}Y)\big) \cap \text{e-conv}\big(Y \cup (\mathrm{int}X)\big)\\
X \vee_4 Y &= \mathrm{conv}\big((\mathrm{int}X) \cup (\mathrm{int}Y)\big).
\end{aligned}
$$

By definition all the sets on the right side of the definitions are e-convex if X and Y are, except for the one corresponding to $X \wedge_3 Y$. Remark that convexity of X and Y does not imply convexity of $X \wedge_3 Y$. Nevertheless e-convexity of X and Y does imply e-convexity of $X \wedge_3 Y$.

Proof. E-convexity is invariant under translation, hence we assume that $0 \in X \wedge_3 Y$. To prove that $X \wedge_3 Y$ is e-convex whenever X and Y are, let $w \notin X \wedge_3 Y$; since $w \notin \mathrm{cl}X \cap Y$, and $\mathrm{cl}X \cap Y$ is e-convex, there exists a linear functional ϕ_1 such that $\phi_1(z) < 1\, \forall z \in \mathrm{cl}X \cap Y$ while $\phi_1(w) = 1$; moreover $\phi_1(z) \leq 1\, \forall z \in X \cap \mathrm{cl}Y$. Analogously there exists ϕ_2 with $\phi_2(z) < 1\, \forall z \in X \cap \mathrm{cl}Y$ while $\phi_2(w) = 1$. Set $\phi = \frac{\phi_1+\phi_2}{2}$: ϕ is a linear functional such that $\phi(z) < 1\, \forall z \in X \wedge_3 Y$ while $\phi(w) = 1$, and the proof is completed. $\square$

In the usual polarity operation for subsets of $\mathbb{E}$, each singleton $\{x\}$ has a polar $\{x\}^\circ = \{y \in \mathbb{E} : \langle x, y\rangle \leq 1\}$ (a closed halfspace when $x \neq 0$), and the polar X° of a set $X \subset \mathbb{E}$ is simply the intersection of all the sets $\{x\}^\circ$ for $x \in X$. That is,

$$X^\circ = \{y \in \mathbb{E} : \langle x, y\rangle \leq 1 \quad \text{for all } x \in X\}.$$

The set X° is always closed and convex and includes the origin, and when X itself has these properties it is true that $(X^\circ)^\circ = X$. Fenchel [6] observed that if we replace $\leq$ by $<$ (i.e. "closed halfpaces" by "open halfspaces"), and define (what we will call) the *e-polar* X^e of a set X as

$$X^e = \{y \in \mathbb{E} : \langle x, y\rangle < 1 \quad \text{for all } x \in X\},$$

then X^e is always e-convex and includes the origin, and if X itself has these properties then $(X^e)^e = X$.

There is another polarity operation, closely related to but different from Fenchel's, that is especially appropriate in studying e-convex cones. To deal efficiently with these, we use the polarity operation ^{c} defined as follows:

$$X^c = \{y \in \mathbb{E} : \langle x, y\rangle < 0 \quad \text{for all } x \in X\},$$

Then X^c is always an e-convex cone that omits the origin, and if X itself has these properties then $(X^c)^c = X$.

With respect to these polarities, the "meet" and "join" operations that we have just defined, behave as dual operations (proofs of the following Theorems 4.2 and 4.4 are trivial).

Theorem 4.2. *For $1 \le i \le 4$ the operations $\wedge_i$ and $\vee_i$ are dual in the sense that*

$$(X \wedge_i Y)^e = X^e \vee_i Y^e$$

and

$$(X \vee_i Y)^e = X^e \wedge_i Y^e$$

whenever X and Y are e-convex subsets of $\mathbb{E}$ with $0 \in X \cap Y$.

When we replace e-polarity with c-polarity, analogous results hold for convex cones that omit their apex 0. Namely

Proposition 4.3. *The class of all e-convex cones omitting their apex 0 is closed under each of the following "meet" operations $\wedge_i$ and "join" operations $\vee_i$:*

$$\begin{aligned}
X \wedge_1 Y &= (\mathrm{int}X) \cap (\mathrm{int}Y)\\
X \wedge_2 Y &= X \cap Y\\
X \wedge_3 Y &= \big(X \cap (\mathrm{cl}Y)\big) \cup \big(Y \cap (\mathrm{cl}X)\big)\\
X \wedge_4 Y &= (\mathrm{cl}X) \cap (\mathrm{cl}Y)\\
X \vee_1 Y &= \mathrm{clconv}(X \cup Y)\\
X \vee_2 Y &= \text{e-conv}(X \cup Y)\\
X \vee_3 Y &= \text{e-conv}\big(X \cup (\mathrm{int}Y)\big) \cap \text{e-conv}\big(Y \cup (\mathrm{int}X)\big)\\
X \vee_4 Y &= \mathrm{conv}\big((\mathrm{int}X) \cup (\mathrm{int}Y)\big).
\end{aligned}$$

Proof. E-convexity follows from Proposition 4.1. Moreover $X \wedge_3 Y$ and all the $X \vee_i Y$ are positively homogeneous and convex sets, hence convex cones. □

Theorem 4.4. *For $1 \le i \le 4$ the operations $\wedge_i$ and $\vee_i$ are dual in the sense that*

$$(X \wedge_i Y)^c = X^c \vee_i Y^c$$

and

$$(X \vee_i Y)^c = X^c \wedge_i Y^c$$

whenever X and Y are e-convex cones of $\mathbb{E}$ omitting their apex 0.

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