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Generalized Convexity and Separation Theorems

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Kakutani's classical theorem (stating that two disjoint convex sets can be separated by complementary convex sets) is extended to the setting when convexity is meant in the sense of Beckenbach. Then a characterization of pairs of functions that can be separated by a generalized convex function or by a function belonging to a two-parameter family (in the sense of Beckenbach) is presented. As consequences, stability results of the Hyers–Ulam-type are also obtained.

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1. Introduction

Geometrically, the convexity of a function $f : I \to \mathbb{R}$ means that, for any two distinct points on the graph of f, the segment joining these points lies above the corresponding part of the graph. In 1937 Beckenbach [2] generalized this concept by replacing the segments by graphs of continuous functions belonging to a certain two-parameter family $\mathcal{F}$ of functions (for the precise definition, see Section 3). The so obtained *generalized convex* functions have many properties known for classical convexity (cf., e.g., [2], [3], [16], [5], [4], [11], [17]).

Using a similar idea, Krzyszkowski [10] introduced the notion of generalized convex sets and proved, among others, a Caratheodory-type result for them.

The aim of this paper is to present further results on generalized convex sets and generalized convex functions. In Section 2, we prove a Kakutani-type separation theorem for generalized convex sets. A characterization of pairs of functions that can be separated by a generalized convex functions is given in Section 3. The problem of separation by functions belonging to $\mathcal{F}$ is discussed in Section 4. As corollaries, some Hyers–Ulam-type stability results related to generalized convexity are obtained.

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2. Separation of generalized convex sets

Let $\mathcal{F}$ be a family of continuous real functions defined on an interval $I \subseteq \mathbb{R}$. We say that $\mathcal{F}$ is a *two-parameter family* if for any two points $(x_1, y_1), (x_2, y_2) \in I \times \mathbb{R}$ with $x_1 \neq x_2$ there exists exactly one $\varphi \in \mathcal{F}$ such that

$$\varphi(x_i) = y_i \quad \text{for } i = 1, 2.$$

The unique function $\varphi \in \mathcal{F}$ determined by the points $a = (x_1, y_1)$, $b = (x_2, y_2)$ will be denoted by $\varphi_{(x_1,y_1)(x_2,y_2)}$ or φ_{ab}. Let $a = (x_1, y_1)$, $b = (x_2, y_2) \in I \times \mathbb{R}$. Define the generalized segment $[a, b] \subset I \times \mathbb{R}$ joining a, b as follows:

$$[a, b] := \big\{\big(x, \varphi_{(x_1,y_1)(x_2,y_2)}(x)\big) : \min\{x_1, x_2\} \leqslant x \leqslant \max\{x_1, x_2\}\big\}, \quad \text{if } x_1 \neq x_2$$

and

$$[a, b] := \{(x_1, y) : \min\{y_1, y_2\} \leqslant y \leqslant \max\{y_1, y_2\}\}, \quad \text{if } x_1 = x_2.$$

A set $A \subset I \times \mathbb{R}$ is called *convex with respect to* $\mathcal{F}$ (in short $\mathcal{F}$*-convex*) if for any $a, b \in A$ we have $[a, b] \subset A$ (cf. [10]).

The celebrated theorem of Kakutani [9] (see also [18], [19, p. 19], [14], [15]) says that each two disjoint convex sets in a real vector space can be separated by complementary convex sets. The main result of this section is an analogue of Kakutani's theorem for generalized convex sets.

Theorem 2.1. *Let $A, B \subset I \times \mathbb{R}$ be disjoint $\mathcal{F}$-convex sets. Then there exist disjoint $\mathcal{F}$-convex sets C, D such that $A \subset C$, $B \subset D$ and $C \cup D = I \times \mathbb{R}$.*

We start with three lemmas needed in the proof of this theorem. The first one is known and expresses a basic property of two-parameter families.

Lemma 2.2 ([2]). *Let φ_1, φ_2 be distinct elements of $\mathcal{F}$ and $\varphi_1(x_0) = \varphi_2(x_0)$ for some $x_0 \in \operatorname{int} I$. Then $\varphi_1(x) > \varphi_2(x)$ for all $x \in I$ on one side of x_0 and $\varphi_1(x) < \varphi_2(x)$ for all $x \in I$ on the other side of x_0.*

Remark 2.3. It follows from this lemma that if for $\varphi_1, \varphi_2 \in \mathcal{F}$ $\varphi_1(x_0) = \varphi_2(x_0)$ and $\varphi_1(x_1) < \varphi_2(x_1)$ for some $x_0, x_1 \in I$, $x_0 \neq x_1$, then $\varphi_1 < \varphi_2$ on the half-line with the origin x_0 containing x_1.

Remark 2.4. As a consequence of Lemma 2.2 we obtain also that if $\varphi_1, \varphi_2 \in \mathcal{F}$ satisfy the inequalities $\varphi_1(x_1) < \varphi_2(x_1)$ and $\varphi_1(x_2) < \varphi_2(x_2)$ for some $x_1 < x_2$, then $\varphi_1(x) < \varphi_2(x)$ for all $x \in [x_1, x_2]$.

The next lemma extends an easily visualizable property of line segments.

Lemma 2.5. *Let a, b, c be distinct points in $I \times \mathbb{R}$. If $a_1 \in [a, c]$, $b_1 \in [b, c]$, then $[a, b_1] \cap [a_1, b] \neq \varnothing$.*

Proof. We denote the coordinates of a given point $p \in I \times \mathbb{R}$ by (x_p, y_p). Without loss of generality we may assume that $x_a \leqslant x_b$. We may also assume that $a_1 \in [a, c] \setminus \{a, c\}$ and $b_1 \in [b, c] \setminus \{b, c\}$ (otherwise the proof is trivial). We can distinguish the following four cases (*i*)-(*iv*) as it is detailed below.

Case (i): $x_a < x_b$ and $c \notin \operatorname{graph}\varphi_{ab}$. Assume, for instance, that $y_c > \varphi_{ab}(x_c)$. In this case a few subcases are possible. Assume first that $x_a < x_c < x_b$. Then b_1 lies above the graph of $\varphi_{a_1 b}$ and a_1 lies above the graph of φ_{ab_1}. Indeed, by Lemma 2.2 (cf. also Remark 2.3), $\varphi_{ab}(x) < \varphi_{ac}(x)$ for all $x > x_a$. In particular, $\varphi_{ab}(x_{a_1}) < y_{a_1}$, whence $\varphi_{ba}(x) < \varphi_{ba_1}(x)$, $x < x_b$, and, consequently, $y_a < \varphi_{ba_1}(x_a)$. Hence, using Lemma 2.2 once more and the fact that $\varphi_{ac}(x_{a_1}) = \varphi_{ba_1}(x_{a_1})$, we obtain $y_c > \varphi_{ba_1}(x_c)$. Therefore $\varphi_{bc}(x) > \varphi_{ba_1}(x)$ for all $x < x_b$, and, consequently,

$$y_{b_1} > \varphi_{ba_1}(x_{b_1}). \tag{1}$$

Similarly we can also show that

$$y_{a_1} > \varphi_{ab_1}(x_{a_1}). \tag{2}$$

Now, define $\psi = \varphi_{ba_1} - \varphi_{ab_1}$. Then, by (1) and (2), $\psi(x_{b_1}) < 0$ and $\psi(x_{a_1}) > 0$. Since ψ is continuous, there exists an $x_0 \in [x_{a_1}, x_{b_1}]$ such that $\psi(x_0) = 0$, i.e., $\varphi_{ab_1}(x_0) = \varphi_{a_1 b}(x_0) =: y_0$. Hence (x_0, y_0) is the desired point of the intersection $[a, b_1] \cap [a_1, b]$.

The further subcases of this case are $x_c = x_a$, $x_c < x_a$ and symmetrically $x_c = x_b$, $x_c > x_b$. Their proofs are similar, so they are omitted.

Case (ii): $x_a < x_b$ and $c \in \operatorname{graph}\varphi_{ab}$. Then $\varphi_{ab_1} = \varphi_{a_1 b}$ and obviously $[a, b_1] \cap [a_1, b] \neq \varnothing$.

Case (iii): $x_a = x_b$ and $x_c \neq x_a$. Assume, for instance, that $y_a < y_b$, $x_a < x_c$ and $x_{a_1} \leqslant x_{b_1}$. Then a_1 lies below the graph of φ_{ab_1}. Indeed, $\varphi_{ca}(x) < \varphi_{cb}(x)$ for all $x < x_c$. In particular, $\varphi_{ca}(x_{b_1}) < y_{b_1}$. Hence $\varphi_{ac}(x) < \varphi_{ab_1}(x)$, $x > x_a$, and, consequently, $y_{a_1} < \varphi_{ab_1}(x_{a_1})$. Now, for $\psi = \varphi_{ba_1} - \varphi_{ab_1}$ we have $\psi(x_a) = y_b - y_a > 0$ and $\psi(x_{a_1}) = y_{a_1} - \varphi_{ab_1}(x_{a_1}) < 0$. Hence there exists an $x_0 \in [x_a, x_{a_1}]$ such that $\psi(x_0) = 0$, which implies $[a, b_1] \cap [a_1, b] \neq \varnothing$.

Case (iv): $x_a = x_b = x_c$. The statement is obvious in this case. □

Given a set $A \subset I \times \mathbb{R}$ we denote by $\operatorname{conv}_{\mathcal{F}} A$ the *$\mathcal{F}$-convex hull* of A, i.e., the intersection of all $\mathcal{F}$-convex subsets of $I \times \mathbb{R}$ containing A (cf. [10]).

Lemma 2.6. *Let $A \subset I \times \mathbb{R}$ be $\mathcal{F}$-convex and $p \in I \times \mathbb{R}$. Then* $\operatorname{conv}_{\mathcal{F}}(A \cup \{p\}) = \bigcup_{a \in A} [a, p]$.

Proof. Of course $\bigcup_{a \in A}[a, p] \subset \operatorname{conv}_{\mathcal{F}}(A \cup \{p\})$. To prove the reverse inclusion it suffices to show that the set $\bigcup_{a \in A}[a, p]$ is $\mathcal{F}$-convex.

Take arbitrary $c_1, c_2 \in \bigcup_{a \in A}[a, p]$, $c_1 \in [a_1, p]$, $c_2 \in [a_2, p]$. If either $x_{a_1} \neq x_{a_2}$ and $p \in \operatorname{graph}\varphi_{a_1 a_2}$ or $x_{a_1} = x_{a_2}$ and $p \in L := \{(x_{a_1}, y) : y \in \mathbb{R}\}$ then, clearly, $[c_1, c_2] \subset [a_1, p] \cup [a_2, p]$. If $x_{a_1} \neq x_{a_2}$ and $p \notin \operatorname{graph}\varphi_{a_1 a_2}$ or $x_{a_1} = x_{a_2}$ and $p \notin L$, then for every $c \in [c_1, c_2]$ the graph of φ_{pc} (or the line $\{(x_p, y) : y \in \mathbb{R}\}$ if $x_p = x_c$) crosses $[a_1, a_2]$. Thus there exists an $a \in [a_1, a_2] \subset A$ such that $c \in [a, p]$, which completes the proof. □

Proof of Theorem 2.1. Consider the class $\mathcal{P}$ of all pairs (A_t, B_t) of $\mathcal{F}$-convex sets such that $A_t \cap B_t = \varnothing$ and $A \subset A_t$, $B \subset B_t$. This class is not empty ($(A, B) \in \mathcal{P}$) and is partially ordered by the relation: $(A_s, B_s) \prec (A_t, B_t)$ if $A_s \subset A_t$ and $B_s \subset B_t$. Moreover, every chain $\mathcal{L}$ of elements of $\mathcal{P}$ has an upper bound in $\mathcal{P}$ (the pair $(\bigcup_{(A_t,B_t)\in\mathcal{L}} A_t, \bigcup_{(A_t,B_t)\in\mathcal{L}} B_t)$). Therefore, by the Kuratowski-Zorn lemma, there exists a maximal element (C, D) in $\mathcal{P}$. Now, it remains to prove that $C \cup D = I \times \mathbb{R}$. On the contrary, suppose that there exists a point $p \in I\times\mathbb{R}\setminus(C\cup D)$. Consider the sets $C_1 = \mathrm{conv}_{\mathcal{F}}(C\cup\{p\})$ and $D_1 = \mathrm{conv}_{\mathcal{F}}(D\cup\{p\})$. By the maximality of (C, D) we infer that $C_1 \cap D \neq \varnothing$ and $C \cap D_1 \neq \varnothing$. Let $c_1 \in C_1 \cap D$ and $d_1 \in C \cap D_1$. By Lemma 2.6 there exist $c \in C$, $d \in D$ such that $c_1 \in [c, p]$ and $d_1 \in [d, p]$. Hence, using Lemma 2.5, we obtain $[c, d_1] \cap [c_1, d] \neq \varnothing$. Since the sets C, D are $\mathcal{F}$-convex, we have $[c, d_1] \subset C$ and $[c_1, d] \subset D$. Consequently, $C \cap D \neq \varnothing$, a contradiction. □

Remark 2.7. The above theorem can be also deduced from an abstract version of Kakutani's theorem obtained by Ellis [7] in terms of convexity defined axiomatically (cf. also Chepoi [6] and Kubiś [12]).

The property expressed in our Lemma 2.5 is a version of a Pasch axiom (cf. [6], [12]) for generalized convexity. As we have seen, Lemma 2.5 played a crucial role in the proof of Theorem 2.1. But it is worth noting that, using Theorem 2.1, we can also obtain a short proof of Lemma 2.5. Indeed, let $a_1 \in [a, c]$, $b_1 \in [b, c]$ and suppose that $[a, b_1]\cap[a_1, b] = \varnothing$. Then, by Theorem 2.1, there exist complementary $\mathcal{F}$-convex sets C, D such that $[a, b_1] \subset C$ and $[a_1, b] \subset D$. Since $C \cup D = I \times \mathbb{R}$, the point c belongs to C or D. In the first case, $[a, c] \subset C$, whence $a_1 \in C$, which is impossible because $a_1 \in D$. Similarly, the second case is impossible. Thus $[a, b_1] \cap [a_1, b] \neq \varnothing$.

3. Separation by $\mathcal{F}$-convex functions

In this section we characterize functions that can be separated by an $\mathcal{F}$-convex function. The presented result generalizes the sandwich theorem obtained by Baron, Matkowski and Nikodem [1] stating that two functions $f, g : I \to \mathbb{R}$ satisfy

$$f(tx + (1 - t)y) \leqslant tg(x) + (1 - t)g(y), \quad x, y \in I, \quad t \in [0, 1],$$

if and only if there exists a convex function $h : I \to \mathbb{R}$ such that $f \leqslant h \leqslant g$.

Let $\mathcal{F}$ be a two-parameter family. A function $f : I \to \mathbb{R}$ is said to be $\mathcal{F}$-*convex* [2] if for any $x_1, x_2 \in I$, $x_1 < x_2$

$$f(x) \leqslant \varphi_{(x_1,f(x_1))(x_2,f(x_2))}(x), \quad x_1 \leqslant x \leqslant x_2.$$

A function f is called $\mathcal{F}$-*concave* if $-f$ is $\mathcal{F}$-convex. Of course the only functions which are simultaneously $\mathcal{F}$-convex and $\mathcal{F}$-concave are the functions belonging to $\mathcal{F}$. Therefore they will be also called $\mathcal{F}$-*affine*. Note that in the case when $\mathcal{F} = \{\varphi(x) = mx + b : m, b \in \mathbb{R}\}$, we obtain the notion of standard convex, concave and affine functions. Given a function $f : I \to \mathbb{R}$ we denote by $\mathrm{epi}\, f$ the *epigraph* of f, i.e. the set $\{(x, y) : x \in I, y \geqslant f(x)\}$. The set $\{(x, y) : x \in I, y > f(x)\}$ will be called the *strict epigraph* of f. Before formulating the main result of this section we present two lemmas needed in its proof. The first one gives a characterization of the $\mathcal{F}$-convex hulls of epigraphs. By the Caratheodory-type theorem obtained by Krzyszkowski [11] it is known that every point of $\mathrm{conv}_{\mathcal{F}} A$, where $A \subset I \times \mathbb{R}$ is a "combination" of at most three points from A (more precisely $\mathrm{conv}_{\mathcal{F}} A = \bigcup\{[a, b] : a \in A, b \in [a_1, a_2] \text{ with } a_1, a_2 \in A\}$). It appears that if A is an epigraph, then its $\mathcal{F}$-convex hull consists of "combinations" of two points from A. We prove this fact directly, i.e. without using a Caratheodory-type result.

Lemma 3.1. *Let $f : I \to \mathbb{R}$ be a given function and let E denote its epigraph or strict epigraph. Then*

$$\mathrm{conv}_{\mathcal{F}} E = \bigcup\{[a, b] : a, b \in E,\ x_a \neq x_b\}.$$

Proof. Assume that $E = \operatorname{epi} f$ (the proof is analogous in the case when E is the strict epigraph of f). Denote

$$E_1 = \operatorname{conv}_{\mathcal{F}} E \quad \text{and} \quad E_2 = \bigcup\{[a,b] : a,b \in E,\ x_a \neq x_b\}.$$

Of course $E_2 \subset E_1$. To prove the reversed inclusion it suffices to show that E_2 is $\mathcal{F}$-convex.

Let $p,q \in E_2$. Then there exist $a,b,c,d \in E$ such that $x_a \neq x_b$, $x_c \neq x_d$ and $p \in [a,b]$, $q \in [c,d]$. Assume first that $x_p \neq x_q$. Since $\varphi_{ab}(x_p) = \varphi_{pq}(x_p)$, it follows from Lemma 2.2 that at least at one of the points x_a, x_b, φ_{ab} has non greater value than φ_{pq}. Assume, for instance, that

$$\varphi_{ab}(x_a) \leqslant \varphi_{pq}(x_a).$$

Similarly, since $\varphi_{cd}(x_q) = \varphi_{pq}(x_q)$, at least at one of the points x_c, x_d the value of φ_{cd} is non greater then the value of φ_{pq}. Assume, that

$$\varphi_{cd}(x_c) \leqslant \varphi_{pq}(x_c).$$

Now take an arbitrary $s \in [p,q]$. There are three possible cases.

Case (i): $\min\{x_a, x_c\} \leqslant x_s \leqslant \max\{x_a, x_c\}$. Define $a_1 = (x_a, \varphi_{pq}(x_a))$, $c_1 = (x_c, \varphi_{pq}(x_c))$. Since $a \in E$ and $y_a = \varphi_{ab}(x_a) \leqslant \varphi_{pq}(x_a) = y_{a_1}$, it follows that $a_1 \in E$. Similarly $c_1 \in E$. Note also that $\varphi_{a_1 c_1} = \varphi_{pq}$ because these functions have the same values at x_a and x_c. In particular

$$y_s = \varphi_{pq}(x_s) = \varphi_{a_1 c_1}(x_s).$$

Hence $s = (x_s, y_s) = (x_s, \varphi_{a_1 c_1}(x_s)) \in [a_1, c_1]$ and, consequently, $s \in E_2$.

Case (ii): $x_s > \max\{x_a, x_c\}$ (then, necessarily, $x_d \geqslant x_s$). Since $\varphi_{cd}(x_q) = \varphi_{pq}(x_q)$ and $\varphi_{cd}(x_c) \leqslant \varphi_{pq}(x_c)$, we have also $\varphi_{cd}(x_s) \leqslant \varphi_{pq}(x_s) = y_s$. Hence, by Lemma 2.2,

$$\varphi_{cd}(x) \leqslant \varphi_{cs}(x) \quad \text{for all } x \geqslant x_c.$$

In particular $y_d = \varphi_{cd}(x_d) \leqslant \varphi_{cs}(x_d)$. Define $d_1 = (x_d, \varphi_{cs}(x_d))$. Then $d_1 \in E$ and $s = (x_s, y_s) = (x_s, \varphi_{cs}(x_s)) \in [c, d_1]$. Consequently, $s \in E_2$.

Case (iii): $x_s < \min\{x_a, x_c\}$. The proof in this case is analogous.

Finally, it remains to consider the case $x_p = x_q$. Assume, for instance, that $y_p \leqslant y_q$ and $x_a < x_b$. Since $\varphi_{as}(x_a) = \varphi_{ab}(x_a)$ and $\varphi_{as}(x_s) = y_s \geqslant y_p = \varphi_{ab}(x_p) = \varphi_{ab}(x_s)$ we have, by Lemma 2.2, that $\varphi_{as}(x) \geqslant \varphi_{ab}(x)$ for all $x \geqslant x_a$. In particular $\varphi_{as}(x_b) \geqslant \varphi_{ab}(x_b) = y_b$. Hence $b_1 := (x_b, \varphi_{as}(x_b)) \in E$ and $s \in [a, b_1]$. Consequently $s \in E_2$. This completes the proof. □

Lemma 3.2. *Let E be an $\mathcal{F}$-convex subset of $I \times \mathbb{R}$ such that for every $x \in I$ the section $\{y : (x,y) \in E\}$ is nonempty and bounded from below. Then the function $h : I \to \mathbb{R}$ defined by $h(x) = \inf\{y : (x,y) \in E\}$ is $\mathcal{F}$-convex.*

Proof. Fix $x_1, x_2 \in I$, $x_1 < x_2$ and denote $\varphi = \varphi_{(x_1,h(x_1))(x_2,h(x_2))}$. By the definition of h there exist points $(x_1, y_n), (x_2, z_n) \in E$, $n \in \mathbb{N}$ such that $y_n \searrow h(x_1)$, $z_n \searrow h(x_2)$. Let $\varphi_n = \varphi_{(x_1,y_n)(x_2,z_n)}$, $n \in \mathbb{N}$. It is known (cf. [2]) that $\varphi_n(x) \to \varphi(x)$ for every $x \in I$. By the $\mathcal{F}$-convexity of E, $(x, \varphi_n(x)) \in E$ for every $x \in [x_1, x_2]$ and $n \in \mathbb{N}$. Hence $h(x) \leqslant \varphi_n(x)$, $n \in \mathbb{N}$ and passing to the limit, $h(x) \leqslant \varphi(x)$, $x \in [x_1, x_2]$, which yields that h is $\mathcal{F}$-convex. □

Remark 3.3. If $E = \operatorname{epi} h$ then the above lemma follows from the fact that h is $\mathcal{F}$-convex if and only if $\operatorname{epi} h$ is $\mathcal{F}$-convex (cf. [10]).

Now we can prove the main result of this section generalizing the sandwich theorem from [1].

Theorem 3.4. *Let $f, g : I \to \mathbb{R}$. There exists an $\mathcal{F}$-convex function $h : I \to \mathbb{R}$ such that $f \leqslant h \leqslant g$ if and only if*

$$f(x) \leqslant \varphi_{(x_1,g(x_1))(x_2,g(x_2))}(x) \tag{3}$$

for all $x_1, x_2 \in I$ such that $x_1 < x_2$, and all $x \in [x_1, x_2]$.

Proof. Assume that $f \leqslant h \leqslant g$ with an $\mathcal{F}$-convex h and fix $x_1, x_2 \in I$, $x_1 < x_2$. Then, using Lemma 2.2 (cf. also Remark 2.4), we get for every $x \in I$

$$f(x) \leqslant h(x) \leqslant \varphi_{(x_1,h(x_1))(x_2,h(x_2))}(x) \leqslant \varphi_{(x_1,g(x_1))(x_2,g(x_2))}(x).$$

Now, assume that (3) holds. Put $E = \operatorname{conv}_{\mathcal{F}} \operatorname{epi} g$. Fix an arbitrary $x \in I$ and take any $y \in \mathbb{R}$ such that $(x, y) \in E$. By Lemma 3.1 there exist $(x_1, y_1), (x_2, y_2) \in \operatorname{epi} g$, such that $x_1 < x_2$, $x \in [x_1, x_2]$ and $y = \varphi_{(x_1,y_1)(x_2,y_2)}(x)$. Hence, by Lemma 2.2 (cf. also Remark 2.4) and (3)

$$y \geqslant \varphi_{(x_1,g(x_1))(x_2,g(x_2))}(x) \geqslant f(x).$$

This allows us to define a function $h : I \to \mathbb{R}$ by the formula

$$h(x) = \inf\{y : (x, y) \in E\}$$

and gives $f \leqslant h$. Moreover, since $(x, g(x)) \in E$ for every $x \in I$, we also have $h \leqslant g$. Finally, by Lemma 3.2, we obtain that h is $\mathcal{F}$-convex. □

As an immediate consequence of the above theorem we obtain the Hyers-Ulam stability result for generalized convex functions proved by Krzyszkowski [11] (cf. also [8]).
Let $\varepsilon > 0$. We say that a function $g : I \to \mathbb{R}$ is *$\varepsilon - \mathcal{F}$-convex* if

$$g(x) \leqslant \varphi_{(x_1,g(x_1))(x_2,g(x_2))}(x) + \varepsilon$$

for all $x_1, x_2 \in I$, $x_1 < x_2$ and $x \in [x_1, x_2]$.

Corollary 3.5. *Let $\varepsilon > 0$. If a function $g : I \to \mathbb{R}$ is $\varepsilon - \mathcal{F}$-convex then there exists an $\mathcal{F}$-convex function $h : I \to \mathbb{R}$ such that*

$$g - \varepsilon \leqslant h \leqslant g.$$

Proof. Put $f = g - \varepsilon$ in Theorem 3.4. □

4. Separation by $\mathcal{F}$-affine functions

In this section we present some necessary and sufficient conditions under which two functions $f, g : I \to \mathbb{R}$ can be separated by an $\mathcal{F}$-affine function. This result is a generalization of the theorem proved by Nikodem and Wąsowicz [13] stating that for given $f, g : I \to \mathbb{R}$

there exists an affine function $h : I \to \mathbb{R}$ such that $f \leqslant h \leqslant g$ if and only if the following inequalities hold

$$\begin{aligned} f(tx + (1-t)y) &\leqslant tg(x) + (1-t)g(y) \\ g(tx + (1-t)y) &\geqslant tf(x) + (1-t)f(y) \end{aligned}$$

for all $x, y \in I$, $t \in [0,1]$. In the proof of that result the classical Helly's theorem was applied. The method used by us is completely different and is based on the Kakutani-type theorem from Section 2.

Theorem 4.1. *Let $f, g : I \to \mathbb{R}$. The following conditions are equivalent:*

(*i*) *there exists a function $h \in \mathcal{F}$ such that $f \leqslant h \leqslant g$;*

(*ii*) *there exists an $\mathcal{F}$-convex function $h_1 : I \to \mathbb{R}$ and an $\mathcal{F}$-concave function $h_2 : I \to \mathbb{R}$ such that $f \leqslant h_1 \leqslant g$ and $f \leqslant h_2 \leqslant g$;*

(*iii*) *for all $x_1, x_2 \in I$ such that $x_1 < x_2$ and all $x \in [x_1, x_2]$*

$$f(x) \leqslant \varphi_{(x_1,g(x_1))(x_2,g(x_2))}(x) \tag{4}$$

$$g(x) \geqslant \varphi_{(x_1,f(x_1))(x_2,f(x_2))}(x); \tag{5}$$

(*iv*) *for all $x_1, x_2, x_3, x_4 \in I$ such that $x_1 < x_2$, $x_3 < x_4$ and all $x \in [x_1, x_2] \cap [x_3, x_4]$*

$$\varphi_{(x_1,f(x_1))(x_2,f(x_2))}(x) \leqslant \varphi_{(x_3,g(x_3))(x_4,g(x_4))}(x).$$

Proof. Implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are obvious. To prove $(iii) \Rightarrow (iv)$ suppose, contrary to our claim, that there are elements $x_1, x_2, x_3, x_4 \in I$, with $x_1 < x_2$, $x_3 < x_4$, and $x \in [x_1, x_2] \cap [x_3, x_4]$ such that

$$\varphi_{(x_1,f(x_1))(x_2,f(x_2))}(x) > \varphi_{(x_3,g(x_3))(x_4,g(x_4))}(x). \tag{6}$$

Denote $\overline{\varphi} = \varphi_{(x_1,f(x_1))(x_2,f(x_2))}$, $\underline{\varphi} = \varphi_{(x_3,g(x_3))(x_4,g(x_4))}$. By (6) and Lemma 2.2, it follows that $\underline{\varphi} < \overline{\varphi}$ at least on one of the intervals $[\min\{x_1, x_3\}, x]$, $[x, \max\{x_2, x_4\}]$. Assume, for instance, that $\underline{\varphi} < \overline{\varphi}$ on the first interval (the proof in the second instance is analogous). There are two possible cases: $x_1 \leqslant x_3 \leqslant x$ or $x_3 \leqslant x_1 \leqslant x$. In the first one we have

$$g(x_3) = \underline{\varphi}(x_3) < \overline{\varphi}(x_3) = \varphi_{(x_1,f(x_1))(x_2,f(x_2))}(x_3),$$

which contradicts (5). In the second case we get

$$f(x_1) = \overline{\varphi}(x_1) > \underline{\varphi}(x_1) = \varphi_{(x_3,g(x_3))(x_4,g(x_4))}(x_1),$$

which contradicts (4).

$(iv) \Rightarrow (i)$. It follows from (iv) that $f \leqslant g$ on I. Define

$$\begin{aligned} A &= \operatorname{conv}_{\mathcal{F}}\{(x,y) : x \in I, \quad y < f(x)\} \\ B &= \operatorname{conv}_{\mathcal{F}}\{(x,y) : x \in I, \quad y > g(x)\}. \end{aligned}$$

Clearly, these sets are $\mathcal{F}$-convex. We will show that they are also disjoint. On the contrary, suppose that there exists an $(x, y) \in A \cap B$. By Lemma 3.1, there are points (x_1, y_1),

(x_2, y_2) with $x_1 < x_2$ and $y_i < f(x_i)$, $i = 1, 2$, and points (x_3, y_3), (x_4, y_4) with $x_3 < x_4$ and $y_i > g(x_i)$, $i = 3, 4$, such that

$$(x, y) \in [(x_1, y_1), (x_2, y_2)] \cap [(x_3, y_3), (x_4, y_4)].$$

Hence

$$y = \varphi_{(x_1,y_1)(x_2,y_2)}(x) = \varphi_{(x_3,y_3)(x_4,y_4)}(x).$$

From this, using the fact that $y_i < f(x_i)$, $i = 1, 2$ and $y_i > g(x_i)$, $i = 3, 4$, we obtain

$$\varphi_{(x_1,f(x_1))(x_2,f(x_2))}(x) > y > \varphi_{(x_3,g(x_3))(x_4,g(x_4))}(x).$$

This contradicts (iv) and proves that $A \cap B = \varnothing$. Now, by Theorem 2.1, there exist disjoint $\mathcal{F}$-convex sets C, D such that $A \subset C$, $B \subset D$ and $C \cup D = I \times \mathbb{R}$. Define $h(x) = \inf\{y : (x, y) \in D\}$. Since the sets C, D are complementary, we have also $h(x) = \sup\{y : (x, y) \in C\}$. In view of Lemma 3.2, we obtain that h is $\mathcal{F}$-convex and $\mathcal{F}$-concave. Therefore $h \in \mathcal{F}$. It is also clear that $f \leqslant h \leqslant g$. This completes the proof. □

As an immediate consequence of Theorem 2.1 we obtain the following sandwich result for $\mathcal{F}$-convex and $\mathcal{F}$-concave functions.

Corollary 4.2. *Let $f, g : I \to \mathbb{R}$ and $f \leqslant g$. If either f is $\mathcal{F}$-convex and g is $\mathcal{F}$-concave or f is $\mathcal{F}$-concave and g is $\mathcal{F}$-convex then there exists an $h \in \mathcal{F}$ such that $f \leqslant h \leqslant g$.*

Proof. Note, that f, g satisfy condition (ii) from Theorem 2.1. □

As another corollary to Theorem 4.1 we get a stability result for $\mathcal{F}$-affine functions.

Corollary 4.3. *Let $\varepsilon > 0$. If $f : I \to \mathbb{R}$ satisfies the inequalities*

$$\begin{aligned} f(x) &\leqslant \varphi_{(x_1,f(x_1)+\varepsilon)(x_2,f(x_2)+\varepsilon)}(x) \\ f(x) &\geqslant \varphi_{(x_1,f(x_1))(x_2,f(x_2))}(x) - \varepsilon \end{aligned}$$

for all $x_1, x_2 \in I$, $x_1 < x_2$ and all $x \in [x_1, x_2]$, then there exists a function $h \in \mathcal{F}$ such that

$$f \leqslant h \leqslant f + \varepsilon.$$

Proof. Apply Theorem 4.1 for f and $g = f + \varepsilon$. □

If the family $\mathcal{F}$ is invariant under translations (i.e., for every $\varphi \in \mathcal{F}$ and $c \in \mathbb{R}$, we have $\varphi + c \in \mathcal{F}$) then the above result can be formulated in a simpler form. It generalizes the stability result for affine functions proved in [13].

Corollary 4.4. *Let $\mathcal{F}$ be a two-parameter family invariant under translations and $\varepsilon > 0$. If $f : I \to \mathbb{R}$ satisfies the inequality*

$$|f(x) - \varphi_{(x_1,f(x_1))(x_2,f(x_2))}(x)| \leqslant \varepsilon \tag{7}$$

for all $x_1, x_2 \in I$, $x_1 < x_2$ and all $x \in [x_1, x_2]$ then there exists an $h \in \mathcal{F}$ such that

$$|f - h| \leqslant \frac{\varepsilon}{2}.$$

Proof. Condition (7) can be rewritten in the form of two inequalities

$$f(x) \leqslant \varphi_{(x_1,f(x_1))(x_2,f(x_2))}(x) + \varepsilon = \varphi_{(x_1,f(x_1)+\varepsilon)(x_2,f(x_2)+\varepsilon)}(x)$$

and

$$f(x) \geqslant \varphi_{(x_1,f(x_1))(x_2,f(x_2))}(x) - \varepsilon.$$

By Corollary 4.3 there exists an $h_1 \in \mathcal{F}$ such that

$$f \leqslant h_1 \leqslant f + \varepsilon.$$

Define $h = h_1 - \frac{\varepsilon}{2}$. Then $h \in \mathcal{F}$ and $|f - h| \leqslant \frac{\varepsilon}{2}$. □

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