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Continuity Properties for the Subdifferential and ε-Subdifferential of a Convex Function and its Conjugate

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In this paper one studies several continuity properties of the subdifferential and ε-subdifferential of a convex function as well as of its conjugate. Many of these properties correspond to similar properties studied previously for the duality and preduality mappings in Banach spaces.

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1. Introduction

In the study of the geometry of Banach spaces it was observed that properties like rotundity, (very) smoothness, drop and weak drop properties are characterized by some continuity properties of the duality or pre-duality mappings (see [3, 10, 14, 17, 19, 26]). It is our aim in this paper to do a similar study for convex functions. We shall see that many results stated for the duality mapping and the slices of the closed unit ball of a normed space have their counterparts for the subdifferential and ε-subdifferential of convex functions. This was already observed by Gregory in his interesting paper [22] and also by J. Borwein as mentioned in [14, p. 108]; a few results in this direction are established by Giles and Moors [15], and Giles and Sciffer [16].

Historically, the first continuity result for the subdifferential mapping ∂f of the convex function $f : X \to \mathbb{R} \cup \{\infty\}$ which is continuous on the interior of its domain was obtained by Moreau [28] who showed that ∂f is upper semicontinous on $\operatorname{int}(\operatorname{dom} f)$. Then Asplund and Rockafellar in [1], among other important results, obtained the Hausdorff continuity of $\partial_{\cdot} f(\cdot)$ on $(0, \infty) \times \operatorname{int}(\operatorname{dom} f)$. After that, Nurminskii [29] proved that $\partial_\varepsilon f$ is locally Lipschitz on $\operatorname{int}(\operatorname{dom} f)$ when $X = \mathbb{R}^n$ and $\varepsilon > 0$ is fixed. Nurminskii's result was extended significantly by Hiriart-Urruty in [23]; in particular, for X a Banach space,

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$\partial_{\cdot} f(\cdot)$ is locally Lipschitz on $(0,\infty)\times\operatorname{int}(\operatorname{dom} f)$ (a survey of these results can be found in [24, Section 7]). Moreover, Contesse and Penot [6] showed that $\partial_\varepsilon f$ is Mosco-continuous on $\operatorname{dom} f$ when X is reflexive and the restriction of f to $\operatorname{dom} f$ is continuous. As proved by Thibault [34], the Lipschitz property of the ε–subdifferential is inherited by the V-subdifferential introduced in [35] for vector-valued convex operators. In this paper we are mainly interested by semicontinuity properties of $\partial_{\cdot} f(\cdot)$ at $(0,x)$ with $x\in\operatorname{int}(\operatorname{dom} f)$ with respect to one or both variables.

On the other hand it is common in convex analysis to consider the biconjugate f^{**} of a convex function f defined on a separated locally convex space X as being defined also on X (taking into account the pairing (X,X^*)) even when X is a normed vector space. As mentioned by H. H. Bauschke in his review of [39] in Mathematical Reviews, it would be desirable "the biconjugate f^{**} be defined and analyzed in the bidual X^{**} rather than X". This paper also tries to contribute to this desideratum.

2. Notions, notation and preliminary results

With the exception of this section where we recall (and extend slightly) the main result of Gregory in [22], presented in the framework of real separated locally convex spaces, throughout the rest of the paper X will be a real normed vector space endowed with a norm denoted by $\|\cdot\|$. The n-th dual of X will be denoted by $X^{*(n)}$ and its norm (for $n\geq 1$), denoted also by $\|\cdot\|$, is the corresponding dual norm. Of course, $X^{*(0)}$ is nothing else but X, $X^{*(1)}$ is X^* and $X^{*(2)}$ is X^{**}. The closed unit ball, the open unit ball, and the unit sphere of $X^{*(n)}$ will be denoted by $U_{X^{*(n)}}$, $B_{X^{*(n)}}$ and $S_{X^{*(n)}}$, respectively. If $u\in X^{*(n)}$ and $\varphi\in X^{*(n+1)}$, the value of φ at u is denoted by $\langle u,\varphi\rangle$. So, if $x\in X$ and $x^*\in X^*$ then $\langle x,x^*\rangle$ is $x^*(x)$, while $\langle x^*,x\rangle$ is $x(x^*)$, where x is interpreted as an element of X^{**}. The norm topology on $X^{*(n)}$ will be denoted by $\tau_{\|\cdot\|}$; this will not create confusion because the trace of the norm topology of $X^{*(n+2k)}$ on $X^{*(n)}$ is the norm topology of $X^{*(n)}$ with natural embedding of $X^{*(n)}$ into $X^{*(n+2k)}$. The weak and weak* topologies on $X^{*(n)}$ for $n\geq 1$ will be denoted by $w(n)$ and $w^*(n)$, respectively; the weak topology on X will be denoted by w and the weak* topology $w^*(1)$ on X^* will be denoted simply by w^*. The closure of $A\subset X^{*(n)}$ for a topology τ on $X^{*(n)}$ will be denoted by $\overline{A}^{\tau}$. However, the norm closure of $A\subset X^{*(n)}$ will be denoted by $\operatorname{cl} A$, even if A is considered as a subset of $X^{*(n+2k)}$ with $k\in\mathbb{N}:=\{1,2,\ldots\}$. The norm interior of $A\subset X^{*(n)}$ is denoted by $\operatorname{int} A$; of course, if X is a normed vector space which is not reflexive, even if the norm interior of the subset A of $X^{*(n)}$ is nonempty, its norm interior as a subset of $X^{*(n+2k)}$ (with $k\geq 1$) is empty. The convex hull of the subset A of a (real) linear space Y is denoted by $\operatorname{conv} A$. When Y is endowed with a linear topology τ, the closed convex hull of A is denoted by $\overline{\operatorname{conv}}^{\tau} A$; when Y is a normed space and τ is the norm topology, $\overline{\operatorname{conv}}^{\tau} A$ is denoted simply by $\overline{\operatorname{conv}} A$. Also recall that the *indicator function* of the subset A of Y is the function $\iota_A: Y\to\overline{\mathbb{R}}:=\mathbb{R}\cup\{-\infty,+\infty\}$ defined by $\iota_A(y):=0$ for $y\in A$ and $\iota_A(y):=\infty:=+\infty$ for $y\in Y\setminus A$. Before stating the result of Gregory we recall that a *bornology* on the (real) separated locally convex space (X,τ) is a family $\mathcal{B}$ of subsets of X such that any $B\in\mathcal{B}$ is τ-bounded, τ-closed and absolutely convex, $\cup\{B\mid B\in\mathcal{B}\}=X$ and $\lambda B\in\mathcal{B}$ whenever $\lambda>0$ and $B\in\mathcal{B}$. The topology $\tau_{\mathcal{B}}$ on X^* is the topology defined by the family of seminorms $\{\sigma_B\mid B\in\mathcal{B}\}$ where

$$\sigma_B: X^*\to\mathbb{R},\quad \sigma_B(x^*):=\sup\{\langle x,x^*\rangle\mid x\in B\},\tag{1}$$

is the *support function* associated to B. A base of neighborhoods for 0 in X^* is formed by the sets B^0, where B^0 is the polar set of B (that is, $B^0 := \{x^* \in X^* \mid \sigma_B(x^*) \leq 1\}$). We say that the function $f : X \to \overline{\mathbb{R}}$ is *directionally $\mathcal{B}$-differentiable at* $x \in X$ with $f(x) \in \mathbb{R}$ if the limit

$$f'(x,y) := \lim_{t \to 0_+} \frac{f(x+ty) - f(x)}{t} \tag{2}$$

exists in $\mathbb{R}$ for every $y \in X$, and the limit is uniform with respect to $y \in B$, for every $B \in \mathcal{B}$. Of course, if f is directionally $\mathcal{B}$-differentiable at x then x is in the algebraic interior of the set $\{y \in X \mid f(y) \in \mathbb{R}\}$. When $\emptyset \neq A \subset \{x \in X \mid f(x) \in \mathbb{R}\}$ and the limit in (2) exists and is uniform with respect to $x \in A$ and $y \in B$ for every $B \in \mathcal{B}$, we say that f is *uniformly directionally $\mathcal{B}$-differentiable on* A (or *for* $x \in A$). Of course, we say that f is *(uniformly) $\mathcal{B}$-differentiable at x (on A)* if f is (uniformly) directionally $\mathcal{B}$-differentiable at x (on A) and $f'(x,\cdot)$ is a continuous linear functional (for every $x \in A$); in this case $f'(x,\cdot)$ is denoted by $\nabla f(x)$.

Recall that $f : X \to \overline{\mathbb{R}}$ is *proper* if its domain $\operatorname{dom} f := \{x \in X \mid f(x) < \infty\}$ is nonempty and $f(x) \neq -\infty$ for every $x \in X$. The class of proper convex functions defined on X will be denoted by $\Lambda(X)$, while the class of lower semicontinuous (lsc for short) proper convex functions on X will be denoted by $\Gamma(X)$. The *conjugate* of $f \in \Lambda(X)$ is the function

$$f^* : X^* \to \overline{\mathbb{R}}, \quad f^*(x^*) := \sup\{\langle x, x^* \rangle - f(x) \mid x \in X\};$$

f^* is convex, w^*-lower semicontinuous, and is proper if and only if f is minorized by a continuous affine function. The *ε-subdifferential* of $f \in \Lambda(X)$ at $x \in \operatorname{dom} f$ is defined by

$$\partial_\varepsilon f(x) := \{x^* \in X^* \mid \langle y - x, x^* \rangle \leq f(y) - f(x) + \varepsilon \quad \forall y \in X\}$$

and $\partial_\varepsilon f(x) := \emptyset$ if $x \notin \operatorname{dom} f$; of course, $\partial_\varepsilon f(x) = \emptyset$ for $\varepsilon < 0$. It is obvious that $\partial_\varepsilon f(x) \subset \partial_{\varepsilon'} f(x)$ if $\varepsilon \leq \varepsilon'$. Taking $\varepsilon = 0$ we get the usual *subdifferential* $\partial f(x)$ of f at x. In general, the (ε-) subdifferential is a multifunction. Recall that the limit in (2) for $f \in \Lambda(X)$ and $x \in \operatorname{dom} f$ always exists (possibly in $\overline{\mathbb{R}}$); moreover, in such a situation, for $\varepsilon \geq 0$, we also consider the *ε-directional derivative* as being defined by

$$f'_\varepsilon(x,u) := \inf\{t^{-1}[f(x+tu) - f(x) + \varepsilon] \mid t > 0\} \in \overline{\mathbb{R}}.$$

Then $f'_\varepsilon(x,\cdot)$ is a sublinear functional and $\partial_\varepsilon f(x) = \partial f'_\varepsilon(x,\cdot)(0)$ (see f.i. [39, Th. 2.4.4]).

We are interested in the continuity properties of the multifunctions

$$\begin{aligned}
&S_f : \mathbb{R} \times X \rightrightarrows X^*, \ S_f(\varepsilon, x) := \partial_\varepsilon f(x), \\
&S_{f^*} : \mathbb{R} \times X^* \rightrightarrows X^{**}, \ S_{f^*}(\varepsilon, x^*) := \partial_\varepsilon f^*(x^*), \\
&S^X_{f^*} : \mathbb{R} \times X^* \rightrightarrows X, \ S^X_{f^*}(\varepsilon, x^*) := X \cap \partial_\varepsilon f^*(x^*) = X \cap S_{f^*}(\varepsilon, x^*).
\end{aligned}$$

Recall that the multifunction $\mathcal{R} : S \rightrightarrows T$ between the topological spaces (S,σ) and (T,τ) is *upper semicontinuous* (usc or σ-τ usc if we want to emphasize the topologies) *at* $s \in S$ if for every open set $D \subset T$ with $\mathcal{R}(s) \subset D$, there exists a neighborhood U of s in S such that $\mathcal{R}(U) := \cup_{s' \in U} \mathcal{R}(s') \subset D$. When the topology τ on T is defined by a metric d, we say that $\mathcal{R}$ is *H-usc* (or σ-τ H-usc) *at* s if for every $\varepsilon > 0$ there exists a neighborhood U of s in S such that $\mathcal{R}(U) \subset (\mathcal{R}(s))^\varepsilon$, where, for $A \subset T$, $A^\varepsilon := \{x \in T \mid d(x, A) < \varepsilon\}$; of course, for $A, B \subset (T, d)$ nonempty sets and $x \in T$, $d(x, A) := \inf\{d(x, a) \mid a \in A\}$ is the distance from x to A and $\operatorname{gap}(A, B) := \inf\{d(a, b) \mid a \in A, b \in B\}$ is the gap between A and B with $d(x, \emptyset) := \infty$ (and so $\emptyset^\varepsilon = \emptyset$) and $\operatorname{gap}(A, \emptyset) = \operatorname{gap}(\emptyset, A) := \infty$, $\operatorname{gap}(\emptyset, \emptyset) := 0$. Similarly, if (T, τ) is a real topological vector space, $\mathcal{R}$ is H-usc at s if for every neighborhood V of 0 in T, there exists a neighborhood U of s in S such that $\mathcal{R}(U) \subset \mathcal{R}(s) + V$; these two variants of Hausdorff upper semicontinuity coincide when T is a normed vector space.

The product topology on $S \times T$ of the topology σ on S and the topology τ on T will be denoted, as usual, by $\sigma \times \tau$; the usual topology on $\mathbb{R}$ is denoted by τ_0.

Gregory [22, Th. 3.1] established several characterizations of the directional $\mathcal{B}$-differentiability with an elegant proof; as he mentions in [22, p. 12], "many of the equivalences can be regarded as an extension (to the case where $\partial f(x)$ is not a singleton) of results of Asplund and Rockafellar [1] on the $\mathcal{A}$-differentiability of convex functions". An inspection of his proof shows that the following result also holds. We give its proof for reader's convenience; for the definition of σ_B see (1).

Theorem 2.1. *Let (X, τ) be a real separated locally convex space, $\mathcal{B}$ a bornology on X, $f \in \Lambda(X)$ and $A \subset X$ a nonempty set. Assume that $A + U_0 \subset \operatorname{dom} f$ and f is Lipschitz on $A + U_0$ for some τ-neighborhood U_0 of 0 (that is, there exists a continuous seminorm $p : X \to \mathbb{R}$ such that $|f(x) - f(x')| \le p(x - x')$ for all $x, x' \in A + U_0$). The following assertions are equivalent:*

(*i*) *f is uniformly directionally $\mathcal{B}$-differentiable on A;*
(*ii*) *the multifunction $S_f(\cdot, x)$ is uniformly τ_0-$\tau_{\mathcal{B}}$ H-usc at 0 for $x \in A$;*
(*iii*) *the multifunction S_f is uniformly $\tau_0 \times \tau$-$\tau_{\mathcal{B}}$ H-usc at $(0, x)$ for $x \in A$;*
(*iv*) *the multifunction $S_f(0, \cdot)$ is uniformly τ-$\tau_{\mathcal{B}}$ H-usc at x for $x \in A$;*
(*v*) $\lim_{y \to x} \inf \sigma_B(\partial f(y) - \partial f(x)) = 0$ *for every $B \in \mathcal{B}$ uniformly for $x \in A$.*

Of course, statement (*iii*) means that for every $B \in \mathcal{B}$, there exist $\eta > 0$ and a neighborhood U of 0 such that $\partial_\eta f(y) \subset \partial f(x) + B^0$ for each $x \in A$ and each $y \in x + U$; the meanings of (*ii*) and (*iv*) are similar.

Proof. (*i*) $\Rightarrow$ (*ii*) Let $B \in \mathcal{B}$ and choose $t > 0$ such that

$$\frac{f(x + ty) - f(x)}{t} - f'(x, y) < \tfrac{1}{2} \quad \forall x \in A, \quad \forall y \in B.$$

Let $\eta = t/2$. Assume that for some $x \in A$, there is $y^* \in \partial_\eta f(x)$ such that $y^* \notin \partial f(x) + B^0$, or, equivalently, $(y^* - \partial f(x)) \cap B^0 = \emptyset$. Because the first set is w^*-compact (see f.i. [39, Th. 2.4.9]), both being nonempty, w^*-closed and convex, a separation theorem yields some $y \in X$ and $\mu \in \mathbb{R}$ such that

$$\inf\{\langle y, y^* - x^*\rangle \mid x^* \in \partial f(x)\} > \mu > \sup\{\langle y, u^*\rangle \mid u^* \in B^0\}.$$

Because $0 \in B^0$, we may (and we do) assume that $\mu = 1$. So, we obtain $y \in B^{00} = B$. The left-hand side of the above inequality shows that

$$\langle y, y^*\rangle - \max\{\langle y, x^*\rangle \mid x^* \in \partial f(x)\} = \langle y, y^*\rangle - f'(x, y) > 1.$$

Taking into account that $y^* \in \partial_\eta f(x)$, we get the contradiction

$$1 > \frac{f(x + ty) - f(x)}{t} - f'(x, y) + \tfrac{1}{2} \ge \langle y, y^*\rangle - f'(x, y) > 1.$$

$(ii) \Rightarrow (iii)$ Fix some $B \in \mathcal{B}$. By hypothesis we have $\partial_\eta f(x) \subset \partial f(x) + B^0$ for some $\eta > 0$ and every $x \in A$. Consider $\gamma := \eta/4$ and a convex neighborhood U of 0 such that $2U \subset U_0$ and $p(u) \leq \gamma$ for $u \in U$. Take $x \in A$, $y \in x + U$ and $y^* \in \partial_\gamma f(y)$. Then, for every $z \in X$, we have

$$\begin{aligned} f(z) - f(x) &= f(z) - f(y) + f(y) - f(x) \\ &\geq \langle z - y, y^* \rangle - \gamma + f(y) - f(x) \\ &\geq \langle z - x, y^* \rangle - \gamma + \langle x - y, y^* \rangle + f(y) - f(x) \\ &\geq \langle z - x, y^* \rangle - 2\gamma + [f(y) - f(2y - x)] + [f(y) - f(x)] \\ &\geq \langle z - x, y^* \rangle - 2\gamma - 2p(y - x) \geq \langle z - x, y^* \rangle - 4\gamma \\ &\geq \langle z - x, y^* \rangle - \eta. \end{aligned}$$

It follows that $y^* \in \partial_\eta f(x)$, and so $y^* \in \partial f(x) + B^0$.

$(iii) \Rightarrow (iv)$ is obvious.

$(iv) \Rightarrow (v)$ Fix $B \in \mathcal{B}$ and take $\varepsilon > 0$. Then $\varepsilon^{-1}B \in \mathcal{B}$. By hypothesis, there exists a neighborhood U of 0 in X such that $\partial f(y) \subset \partial f(x) + (\varepsilon^{-1}B)^0 = \partial f(x) + \varepsilon B^0$ for every $x \in A$ and every $y \in x + U$. We may (and we do) suppose that $U \subset U_0$, and so $\partial f(y) \neq \emptyset$, f being continuous at y. Hence, taking some $y^* \in \partial f(y)$, there exists $x^* \in \partial f(x)$ such that $z^* := y^* - x^* \in (\partial f(y) - \partial f(x)) \cap \varepsilon B^0$, and so $\sigma_B(z^*) \leq \varepsilon$. Hence $\inf \sigma_B(\partial f(y) - \partial f(x)) \leq \varepsilon$ for all $x \in A$ and $y \in x + U$.

$(v) \Rightarrow (i)$ Fix some $B \in \mathcal{B}$. Then, for $\varepsilon > 0$ there exists $U_\varepsilon \subset U_0$ a convex neighborhood of 0 such that $\inf \sigma_B(\partial f(y) - \partial f(x)) \leq \varepsilon$ for all $x \in A$ and $y \in x + U_\varepsilon$. Because B is bounded, there exists some $\eta > 0$ such that $\eta B \subset U_\varepsilon$. Fix some $x \in A$ and $y \in B$. For $x^* \in \partial f(x)$ and $y^* \in \partial f(x + \eta y)$ we have

$$0 \leq \frac{f(x + \eta y) - f(x)}{\eta} - f'(x, y) \leq \langle y, y^* \rangle - \langle y, x^* \rangle \leq \sigma_B(y^* - x^*),$$

whence

$$0 \leq \frac{f(x + \eta y) - f(x)}{\eta} - f'(x, y) \leq \inf \sigma_B(\partial f(x + \eta y) - \partial f(x)) \leq \varepsilon.$$

Since the quotient is increasing with respect to (w.r.t. for short) η and the estimate is independent of $y \in B$, we obtain f is uniformly directionally $\mathcal{B}$-differentiable on A. □

The notion of bornology used above is somewhat different from the usual notion in which the sets need not be closed or absolutely convex. These supplementary conditions are not restrictive for the Fréchet bornology (the family of all bounded subsets of X), for the Hadamard bornology (the family of all compact subsets of X) when X is a Banach space (apply [25, Th. 2.11.C]), or for the weak Hadamard bornology (the family of all weakly compact subsets of X) when X is a Banach space (applying the Krein–Smulian weak compactness theorem [27, Th. 2.8.14]).

When $\mathcal{B} := \{K \subset X \mid K \text{ is compact, convex and } K = -K \neq \emptyset\}$, the function $f \in \Lambda(X)$ (which is continuous at $x \in \operatorname{dom} f$) is automatically directionally $\mathcal{B}$-differentiable at x, and so properties (ii)–(v) hold in such a case for $A = \{x\}$. (The fact that the limit in (2) is uniform on the compact set $K \subset X$ follows from Dini's theorem because the mapping $0 < t \mapsto t^{-1}[f(x+ty) - f(x)]$ is monotone and continuous w.r.t. $y \in K$ (for small t) and the limit $f'(x,\cdot)$ is continuous.)

Given the topological space (T,τ) and a net $(A_i)_{i\in I}$ of subsets of T, we say that (A_i) V$^+$-converges to $A \subset T$, and we write $A_i \xrightarrow{\mathrm{V}^+} A$, if for every open subset D of T with $A \subset D$, there exists some $i_D \in I$ such that $A_i \subset D$ for every $i \geq i_D$ (V$^+$ stands for upper Vietoris); when we want to emphasize the topology τ we write V$^+_\tau$. When (T,d) is a metric space, we say that the net $(A_i)_{i\in I}$ of subsets of T W-converges to $A \subset T$, and we write $A_i \xrightarrow{\mathrm{W}} A$, if $(d(x,A_i))_{i\in I} \to d(x,A)$ for every $x \in T$ (W stands for Wijsman). In the framework of a metric space (T,d), we say that (A_i) H$^+$-converges to $A \subset T$, and write $A_i \xrightarrow{\mathrm{H}^+} A$, if for every $\varepsilon > 0$ there exists $i_\varepsilon \in I$ such that $A_i \subset A^\varepsilon$ for $i \geq i_\varepsilon$ (H$^+$ stands for upper Hausdorff); similarly, when (T,τ) is a topological vector space, $A_i \xrightarrow{\mathrm{H}^+_\tau} A$, if for every neighborhood V of 0 in T there exists $i_V \in I$ such that $A_i \subset A + V$ for $i \geq i_V$. Note that if A_i is nonempty for every $i \in I$ and $(A_i)_{i\in I}$ converges for one of the above convergences then necessarily A is nonempty. Moreover, if T is a metric space or a topological vector space and $A_i \xrightarrow{\mathrm{V}^+} A$ then $A_i \xrightarrow{\mathrm{H}^+} A$, the converse being true if A is compact.

Given a sequence $(A_n)_{n\in\mathbb{N}}$ of subsets of a metric space (T,d) or a topological vector space (T,τ) and $A \subset T$, taking $S := \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\} \subset \mathbb{R}$ endowed with the usual topology and the multifunction $\mathcal{R} : S \rightrightarrows T$ defined by $\mathcal{R}(0) := A$, $\mathcal{R}(\frac{1}{n}) := A_n$, we have $A_n \xrightarrow{\mathrm{H}^+} A$ if and only if $\mathcal{R}$ is H-usc at 0. Similarly, $A_n \xrightarrow{\mathrm{V}^+} A$ if and only if $\mathcal{R}$ is usc at 0 (even if (T,τ) is a general topological space).

Also recall that for a sequence (A_n) of subsets of the metric space (T,d), $\limsup A_n$ is the set of those $x \in T$ such that there exist $(n_k) \subset \mathbb{N}$ an increasing sequence and a sequence $(x_k) \subset T$ convergent to x with $x_k \in A_{n_k}$ for every $k \geq 1$; so, $x \in \limsup A_n$ if and only if $\liminf d(x,A_n) = 0$. Recall that the *Hausdorff index of noncompactness* of the nonempty bounded subset A of the metric space (T,d) is the number

$$\alpha(A) := \inf\{\varepsilon > 0 \mid A \subset F^\varepsilon \text{ for some finite set } F \subset T\}.$$

We shall need the following result which seems to be new.

Proposition 2.2. *Let (T,d) be a complete metric space and let (A_n) be a sequence of nonempty closed subsets of T. The following assertions are equivalent:*

- (i) *every sequence (x_k) with $x_k \in A_{n_k}$, $(n_k) \subset \mathbb{N}$ being an increasing sequence, has a convergent subsequence;*
- (ii) *the set $A := \limsup A_n$ is nonempty and compact, and $A_n \xrightarrow{H^+} A$;*
- (iii) *there exists a nonempty compact set $A \subset T$ such that $A_n \xrightarrow{H^+} A$.*

Moreover, if (i), (ii) or (iii) holds then $\alpha(A_n) \to 0$.

Proof. $(i) \Rightarrow (ii)$ By our hypothesis the set $A := \limsup A_n$ is nonempty. Of course, A is a closed set. In order to obtain that A is compact it is sufficient to show that every sequence in A has a Cauchy subsequence. If this is not true then there exist a sequence $(x_k) \subset A$ and $\varepsilon > 0$ such that $d(x_k,x_p) \geq \varepsilon$ for all distinct k and p. Because $x_1 \in A$

we have $\liminf d(x_1, A_n) = 0$, and so there exists $n_1 \in \mathbb{N}$ such that $d(x_1, A_{n_1}) < \varepsilon/3$. Because $x_2 \in A$ we have $\liminf d(x_2, A_n) = 0$, and so there exists $n_2 \in \mathbb{N}$, $n_2 > n_1$, such that $d(x_2, A_{n_2}) < \varepsilon/3$. Continuing in this way, we find an increasing sequence $(n_k) \subset \mathbb{N}$ such that $d(x_k, A_{n_k}) < \varepsilon/3$ for every k. Consider $x'_k \in A_{n_k}$ with $d(x_k, x'_k) < \varepsilon/3$. By our hypothesis, (x'_k) has a convergent subsequence, and so we may assume that (x'_k) itself is convergent, whence (x'_k) is Cauchy. It follows that there exists $k_0 \in \mathbb{N}$ such that $d(x'_k, x'_p) < \varepsilon/3$ for $k, p \geq k_0$. Hence, for distinct $k, p \geq k_0$ we get the contradiction $\varepsilon \leq d(x_k, x_p) \leq d(x_k, x'_k) + d(x'_k, x'_p) + d(x'_p, x_p) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$. Hence A is compact.

Assume that (A_n) does not H$^+$-converge to A. Then there exist $\varepsilon > 0$ and an increasing sequence $(n_k) \subset \mathbb{N}$ such that $A_{n_k} \not\subset A^\varepsilon$. So, for every k we get $x_k \in A_{n_k}$ such that $d(x_k, A) \geq \varepsilon$. By our hypothesis (x_k) has a convergent subsequence, and so we may assume that $x_k \to x \in X$. It follows that $x \in A$. This yields the contradiction $\varepsilon \leq \lim d(x_k, A) = d(x, A) = 0$.

$(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$ First notice that A is nonempty. Let $(n_k) \subset \mathbb{N}$ be an increasing sequence and $(x_k) \subset X$ be such that $x_k \in A_{n_k}$ for every k. Consider a sequence $(\varepsilon_p)_{p\geq 1} \subset (0, \infty)$ converging to 0. Because $A_n \xrightarrow{\mathrm{H}^+} A$, there exists a sequence $(m_p) \subset \mathbb{N}$ such that $A_n \subset A^{\varepsilon_p}$ for $n \geq m_p$. As $n_k \to \infty$, there exists an increasing sequence (k_p) such that $n_{k_p} \geq m_p$ for every p, and so $d(x_{k_p}, A) < \varepsilon_p$ for every p. Hence, we get a sequence $(y_p)_{p\geq 1} \subset A$ such that $d(x_{k_p}, y_p) < \varepsilon_p$ for every p, whence $d(x_{k_p}, y_p) \to 0$. Because A is compact, (y_p) has a convergent subsequence, and so $(x_{k_p})_{p\geq 1}$ has a convergent subsequence. This shows that (x_k) has a convergent subsequence.

Assume now that (iii) holds. Let $\varepsilon > 0$. By hypothesis, there exists n_ε such that $A_n \subset A^{\varepsilon/2}$ for $n \geq n_\varepsilon$. Since A is compact, there exist $F \subset A$ finite such that $A \subset F^{\varepsilon/2}$. It follows that $A_n \subset F^\varepsilon$. Therefore $\alpha(A_n) \leq \varepsilon$, and so $\alpha(A_n) \to 0$. □

The case of a decreasing sequence (A_n) will be useful.

Corollary 2.3. *Let (T, d) be a complete metric space and (A_n) be a decreasing sequence of nonempty closed subsets of T. The following assertions are equivalent:*

(i) *every sequence (x_n) with $x_n \in A_n$ for $n \geq 1$ has a convergent subsequence;*

(ii) *$A := \cap_{n\geq 1} A_n$ is a nonempty compact set and for every $\varepsilon > 0$ there exists $n \geq 1$ such that $A_n \subset A^\varepsilon$, that is, $A_n \xrightarrow{H^+} A$;*

(iii) $\alpha(A_n) \to 0$.

Proof. Because (A_n) is decreasing we have $\limsup A_n = \cap_{n\geq 1} A_n$. It is easy to see that in our case, assertions (i) and (ii) in the preceding proposition are equivalent to assertions (i) and (ii) of the present statement, respectively. So we have $(i) \Leftrightarrow (ii) \Rightarrow (iii)$.

The implication $(iii) \Rightarrow (i)$ is an immediate consequence of the following result which we state without proof, being probably known. □

Lemma 2.4. *Let (T, d) be a metric space and (A_n) be a decreasing sequence of nonempty closed subsets of X. Then $\alpha(A_n) \to 0$ if and only if every sequence (x_n) with $x_n \in A_n$ for every n has a Cauchy subsequence.*

Note the fact that $\cap_{n\geq 1} A_n$ is a nonempty compact set provided $\alpha(A_n) \to 0$ in the preceding corollary is established in statement (f) of [2, p. 4].

3. Continuity properties for the subdifferential and ε-subdifferential of a convex function

From now on, we consider $(X, \|\cdot\|)$ to be a real normed vector space. We say, as in [5], that the function $f : X \to \overline{\mathbb{R}}$ is *directionally Fréchet differentiable at* $x \in X$ with $f(x) \in \mathbb{R}$ if f is directionally $\mathcal{B}$-differentiable when $\mathcal{B}$ is the Fréchet bornology (that is, $\mathcal{B}$ is the class of all absolutely convex closed bounded subsets of X). Of course, if f is directionally Fréchet differentiable at x then x is in the interior of the set $\{y \in X \mid f(y) \in \mathbb{R}\}$. Moreover, note that f is Fréchet differentiable at x with $f(x) \in \mathbb{R}$ iff f is Gâteaux differentiable at x (that is, $f'(x, y)$ exists in $\mathbb{R}$ for every $y \in X$ and $f'(x, \cdot)$ is a continuous linear functional) and directionally Fréchet differentiable at x. Taking $\mathcal{B}$ to be the Fréchet bornology on X in Gregory's theorem (Theorem 2.1) we obtain the following result.

Theorem 3.1. *Let $A \subset X$ be a nonempty set and $f \in \Lambda(X)$. Assume that there exists $\eta > 0$ such that $A + \eta U_X \subset \operatorname{dom} f$ and f is Lipschitz on $A + \eta U_X$. The following assertions are equivalent:*

- (i) *f is uniformly directionally Fréchet differentiable on A;*
- (ii) *the multifunction $S_f(\cdot, x)$ is uniformly τ_0-$\tau_{\|\cdot\|}$ H-usc at 0 for $x \in A$ in the sense that for every $\varepsilon > 0$, there exists $\eta > 0$ such that $\partial_\eta f(x) \subset \partial f(x) + \varepsilon U_{X^*}$ for every $x \in A$;*
- (iii) *the multifunction S_f is uniformly $\tau_0 \times \tau_{\|\cdot\|}$-$\tau_{\|\cdot\|}$ H-usc at $(0, x)$ for $x \in A$;*
- (iv) *the multifunction $S_f(0, \cdot)$ is uniformly $\tau_{\|\cdot\|}$-$\tau_{\|\cdot\|}$ H-usc at x for $x \in A$;*
- (v) $\lim_{y\to x} \operatorname{gap}(\partial f(y), \partial f(x)) = 0$ *uniformly for $x \in A$.*

The next result shows that uniformly directional Fréchet differentiability for a convex function is a strong condition.

Theorem 3.2. *Let $f \in \Lambda(X)$ be continuous on $\operatorname{int}(\operatorname{dom} f)$ and $\emptyset \neq S \subset \operatorname{dom} f$. If f is uniformly directionally Fréchet differentiable on S then f is Fréchet differentiable on $\operatorname{int} S$ and ∇f is uniformly continuous on $\operatorname{int} S$. Conversely, if f is Gâteaux differentiable on S and ∇f is uniformly continuous on S then f is uniformly directionally Fréchet differentiable on every set A with $\operatorname{gap}(A, X \setminus S) > 0$.*

Proof. Assume that f is uniformly directionally Fréchet differentiable on S. Let us show that f is Gâteaux differentiable at any $x \in \operatorname{int} S$. The conclusion then follows from the equivalence of (i) and (v) in Theorem 3.1. So, take $x \in \operatorname{int} S$ and $u \in S_X$. There exists some $r > 0$ such that $S + rU_X \subset \operatorname{dom} f$ and $x + rU_X \subset S$. Consider the function $\psi : \mathbb{R} \to \overline{\mathbb{R}}$ defined by $\psi(t) := f(x+tu)$. We have that $\psi'_+(t) := \lim_{s\downarrow 0} s^{-1}[\psi(t+s)-\psi(t)] = f'(x+tu, u)$ and $\psi'_-(t) := \lim_{s\uparrow 0} s^{-1}[\psi(t+s)-\psi(t)] = -f'(x+tu, -u)$ for every $t \in \operatorname{dom} \psi$. Since f is uniformly directionally Fréchet differentiable on S, for every $\varepsilon > 0$ there exists $\delta \in (0, r]$ such that

$$\frac{f(y + sv) - f(y)}{s} - f'(y, v) \leq \varepsilon \quad \forall y \in S,\ \forall v \in U_X,\ \forall s \in (0, \delta].$$

In particular, taking $y = x + tu$ and $v \in \{-u, u\}$, we obtain

$$\frac{\psi(t + s) - \psi(t)}{s} - \psi'_+(t) \leq \varepsilon, \quad \frac{\psi(t - s) - \psi(t)}{s} + \psi'_-(t) \leq \varepsilon$$

for all $t \in (-r, r)$ and $s \in (0, \delta]$. Fixing $t \in (-r, r)$ and replacing t by $t' \in (t, r)$ in the second inequality, and taking into account that $\lim_{t' \downarrow t} \psi'_-(t') = \psi'_+(t)$ and the continuity of ψ on $(-2r, 2r)$ we obtain

$$\frac{\psi(t-s) - \psi(t)}{s} + \psi'_+(t) \leq \varepsilon \quad \forall t \in (-r, r), \ \forall s \in (0, \delta].$$

Hence

$$\frac{\psi(t+s) - \psi(t)}{s} + \frac{\psi(t-s) - \psi(t)}{s} \leq 2\varepsilon \quad \forall t \in (-r, r), \ \forall s \in (0, \delta].$$

Taking the limit for $s \to 0$ we obtain $0 \leq \psi'_+(t) - \psi'_-(t) \leq 2\varepsilon$ for every $t \in (-r, r)$ and $\varepsilon > 0$. Hence $\psi'_+(t) = \psi'_-(t)$ for $t \in (-r, r)$. In particular, $\psi'_+(0) = \psi'_-(0)$, that is, $f'(x, -u) = -f'(x, u)$. Since $u \in U_X$ is arbitrary, we obtain $f'(x, \cdot)$ is linear (and continuous), and so f is Gâteaux differentiable at x.

Conversely, assume that f is Gâteaux differentiable on S and ∇f is uniformly continuous on S. Consider $A \subset \operatorname{dom} f$ such that $\gamma := \operatorname{gap}(A, X \setminus S) > 0$. Fix $\varepsilon > 0$; then there exists $\delta \in (0, \gamma)$ such that $\|\nabla f(x) - \nabla f(y)\| \leq \varepsilon$ for $x, y \in S$ with $\|x - y\| \leq \delta$. Taking $t \in (0, \delta)$, $u \in U_X$ and $x \in A$, we necessarily have that $x + tu \in S$. By the mean-value theorem there exists some $t' \in (0, t)$ such that $f(x + tu) - f(x) = \langle tu, \nabla f(x + t'u) \rangle$, and so

$$\frac{f(x+tu) - f(x)}{t} - f'(x, u) = \langle u, \nabla f(x + t'u) - \nabla f(x) \rangle \leq \varepsilon.$$

The conclusion follows. □

An inspection of the proof of the previous theorem shows that the next result holds, where the function $g : D \subset X \to X^*$ is said to be $\tau_{\|\cdot\|}$-$\tau_{\mathcal{B}}$ *uniformly continuous on* $A \subset D$ if

$$\forall \varepsilon > 0, \ \forall B \in \mathcal{B}, \ \exists \delta > 0, \ \forall y, y' \in A \ : \ \|y - y'\| \leq \delta \Rightarrow \sigma_B(g(y) - g(y')) \leq \varepsilon.$$

Theorem 3.3. *Let $f \in \Lambda(X)$ be continuous on* $\operatorname{int}(\operatorname{dom} f)$*, $S \subset \operatorname{dom} f$ and let $\mathcal{B}$ be a bornology on X. If f is uniformly directionally $\mathcal{B}$-differentiable on S then f is $\mathcal{B}$-differentiable on* $\operatorname{int} S$ *and ∇f is $\tau_{\|\cdot\|}$-$\tau_{\mathcal{B}}$ uniformly continuous on* $\operatorname{int} S$*. Conversely, if f is $\mathcal{B}$-differentiable on S and ∇f is $\tau_{\|\cdot\|}$-$\tau_{\mathcal{B}}$ uniformly continuous on S then f is uniformly directionally $\mathcal{B}$-differentiable on every set A with* $\operatorname{gap}(A, X \setminus S) > 0$.

Observe that the function $f := \|\cdot\|$ is uniformly directionally $\mathcal{B}$-differentiable on S_X if and only if f is uniformly directionally $\mathcal{B}$-differentiable on $[r, r^{-1}] \cdot S_X$ for every (some) $r \in (0, 1)$ (just use the positive homogeneity of f). With this remark, from Theorem 3.2 we obtain the norm is uniformly directionally Fréchet differentiable on S_X if and only if the norm is uniformly smooth, that is, [10, Prop. 4.1] (established for X complete). Moreover, the norm is uniformly directionally $\mathcal{B}$-differentiable on S_X if and only if the norm is $\mathcal{B}$-differentiable on $X \setminus \{0\}$ (that is, X is $\mathcal{B}$-smooth) and $\nabla \|\cdot\|$ is $\tau_{\|\cdot\|}$-$\tau_{\mathcal{B}}$ uniformly continuous on S_X (compare with Lemma 1.1 in [16] established for the weak Hadamard bornology).

An example of directionally Fréchet differentiable function is that of conical function. We say that $f \in \Lambda(X)$ is *conical at* $x \in \operatorname{dom} f$ if there exists some $r > 0$ and a sublinear function $p : X \to \overline{\mathbb{R}}$ such that $f(y) = f(x) + p(y - x)$ for every $y \in x + rB_X$. Of course, if this relation holds then $p = f'(x, \cdot)$, p is proper and $\partial f(y) = \partial p(y - x) \subset \partial p(0) = \partial f(x)$

for every $y \in x + rB_X$. Similar to the notion of quasi-polyhedral norm (see [18, Def. 3.1]), we say that the function $f \in \Lambda(X)$ is *quasi-polyhedral at* $x \in \operatorname{dom} f$ if there exists some $r > 0$ such that $\partial f(y) \subset \partial f(x)$ for every $y \in x + rB_X$. Hence, every function $f \in \Lambda(X)$ which is conical at $x \in \operatorname{dom} f$ is quasi-polyhedral at x. The converse is true if f is continuous at x.

Proposition 3.4. *Let $f \in \Lambda(X)$ be continuous at $x \in \operatorname{dom} f$. Then f is conical at x if and only if f is quasi-polyhedral at x.*

Proof. Assume that $\partial f(y) \subset \partial f(x)$ for every $y \in x + rB_X \subset \operatorname{dom} f$. We may suppose that $x = 0$ and $f(0) = 0$ and we have to show that $f(y) = f'(0, y)$ for $y \in rB_X$. But $f(ty) \leq tf(y)$ for every $y \in X$ and $t \in (0, 1)$, and so $f'(0, y) \leq f(y)$ for every $y \in X$. Take $y \in rB_X$. Because f is continuous on $rB_X \subset \operatorname{int}(\operatorname{dom} f)$, by the mean-value theorem in the convex case (see f.i. [39, Exer. 2.31]), there exist some $\gamma \in (0, 1)$ and $z^* \in \partial f(\gamma y)$ such that $f(y) = f(y) - f(0) = \langle y, z^*\rangle$. Because $\gamma y \in rB_X$, we have $z^* \in \partial f(0)$, and so $f(y) \leq \max\{\langle y, x^*\rangle \mid x^* \in \partial f(0)\} = f'(0, y)$. □

When X is a Banach space the continuity of f can be replaced by lower semicontinuity.

Proposition 3.5. *Let X be a Banach space, $f \in \Gamma(X)$ and $x \in \operatorname{dom} f$. Then f is conical at x if and only if f is quasi-polyhedral at x.*

Proof. We have to prove the sufficiency part. As in the proof of the preceding proposition, we take $x = 0$, $f(0) = 0$ and $\partial f(y) \subset \partial f(0)$ for every $y \in rB_X$. First, by Brøndsted–Rockafellar theorem (see f.i. [39, Th. 3.1.2]) we have $0 \in \operatorname{cl}(\operatorname{dom} \partial f)$. From our hypothesis we obtain $\partial f(0) \neq \emptyset$. We have seen that $f'(0, y) \leq f(y)$ for every $y \in X$, and so

$$s_{\partial f(0)}(y) := \sup\{\langle y, x^*\rangle \mid x^* \in \partial f(0)\} \leq f'(0, y) \leq f(y) \quad \forall y \in X. \tag{3}$$

Take $y \in rB_X$ and $s \in \mathbb{R}$, $s \leq f(y)$. Applying Zagrodny's theorem (see f.i. [39, Th. 3.2.5]), we obtain a sequence (x_n) converging to $x \in [0, y[$ with $f(x_n) \to f(x)$ and a sequence (x_n^*) such that $x_n^* \in \partial f(x_n)$ for every n and $s = s - f(0) \leq \liminf\langle y, x_n^*\rangle$. Because rB_X is open, we have $x_n \in rB_X$ for large n, and so $x_n^* \in \partial f(0)$ for such n. It follows that $s \leq \sup\{\langle y, x^*\rangle \mid x^* \in \partial f(0)\} = s_{\partial f(0)}(y)$. Letting $s \to f(y)$ we get $f(y) \leq s_{\partial f(0)}(y)$. Therefore equality holds in (3) for every $y \in rB_X$. □

Of course, if $f \in \Lambda(X)$ is continuous and conical at $x \in \operatorname{dom} f$ then f is directionally Fréchet differentiable at x; so we get again [18, Lem. 3.3] (established for the norm and X complete).

Taking into account that for metric spaces S, T, the multifunction $\mathcal{R} : S \rightrightarrows T$ is H-usc at $s \in S$ if and only if for every sequence $(s_n) \subset S$ converging to s and every sequence $(t_n) \subset T$ such that $t_n \in \mathcal{R}(s_n)$ for $n \geq 1$ we have $d(t_n, \mathcal{R}(s)) \to 0$, when A is a singleton, to the characterizations in Theorem 3.1 we can add other ones. Note that in the preceding characterization of H-upper semicontinuity we can replace sequences by nets.

Theorem 3.6. *Let $f \in \Lambda(X)$ be continuous at $x \in \operatorname{dom} f$. The following assertions are equivalent:*

(i) *f is directionally Fréchet differentiable at x;*

(ii) *the multifunction $S_f(\cdot, x)$ is τ_0-$\tau_{\|\cdot\|}$ H-usc at 0;*

(*iii*) *the multifunction* S_f *is* $\tau_0 \times \tau_{\|\cdot\|}$-$\tau_{\|\cdot\|}$ *H-usc at* $(0, x)$*;*
(*iv*) *the multifunction* $S_f(0, \cdot)$ *is* $\tau_{\|\cdot\|}$-$\tau_{\|\cdot\|}$ *H-usc at* x*;*
(*v*) $\lim_{y \to x} \operatorname{gap}(\partial f(y), \partial f(x)) = 0$*;*
(*vi*) *for every sequence* $(\eta_n) \subset [0, \infty)$ *converging to* 0 *and every sequence* (x_n^*) *with* $x_n^* \in \partial_{\eta_n} f(x)$ *for* $n \in \mathbb{N}$*, we have* $d(x_n^*, \partial f(x)) \to 0$*;*
(*vii*) *for every sequence* $(\eta_n) \subset [0, \infty)$ *converging to* 0*, for every sequence* $(x_n) \subset X$ *converging to* x *and every sequence* (x_n^*) *with* $x_n^* \in \partial_{\eta_n} f(x_n)$ *for* $n \in \mathbb{N}$*, we have* $d(x_n^*, \partial f(x)) \to 0$*;*
(*viii*) *for every sequence* $(x_n) \subset X$ *converging to* x *and every sequence* (x_n^*) *with* $x_n^* \in \partial f(x_n)$ *for* $n \in \mathbb{N}$*, we have* $d(x_n^*, \partial f(x)) \to 0$.

Note that the equivalence of conditions (*i*), (*ii*) and (*iv*) of the preceding result is proved by Robert [30]. However, the notion of directionally Fréchet differentiability was first introduced by Valadier [37, p. 61] under the name of “sous-différentiabilité au sens de Fréchet” (subdifferentiability in Fréchet’s sense); there the author provides a criterion for f to be directionally Fréchet differentiable. Also note that the equivalence of (*i*) and (*iv*) is obtained again by Giles and Moors [15, Th. 3.2]. As mentioned before Theorem 2.1, the equivalences above can be seen as an extension of some results of Asplund and Rockafellar [1] to the case where $S_f(0, x)$ is not a singleton. The case where $S_f(0, x)$ is a singleton is discussed in Theorem 3.8. On the other hand, as pointed out in Introduction, S_f has very good continuity properties on $(0, \infty) \times \operatorname{int}(\operatorname{dom} f)$ (see [23, 29, 34]).

Of course, we could state the preceding theorem with an arbitrary bornology on X. We preferred to state it only for the Fréchet bornology because the assertions (*v*)–(*viii*) have nice formulations using the distance function. For example, taking $\mathcal{B}$ to be the class of weakly compact absolutely convex subsets of X, one obtains: *f is directionally $\mathcal{B}$-differentiable at x if and only if for every $B \in \mathcal{B}$ there exists a neigborhood U of x such that $\partial f(U) \subset \partial f(x) + B^0$.* This assertion is stated by Giles and Sciffer [16, Lem. 2.1] for X complete and with $\mathcal{B}$ replaced by the class of weakly compact subsets of X, that is, the weak Hadamard bornology. Taking into account the Krein–Smulian weak compactness theorem (see [27, Th. 2.8.14]), [16, Lem. 2.1] follows from the above assertion.

Taking $f = \|\cdot\|$ and $x \in S_X$, observe that $\partial f(x) = \{x^* \in S_{X^*} \mid \langle x, x^* \rangle = 1\}$ and $\partial_\eta f(x) = \{x^* \in U_{X^*} \mid \langle x, x^* \rangle \geq 1 - \eta\}$. These sets will be denoted by $S(x, 0)$ and $S(x, \eta)$, respectively. When $f = \|\cdot\|$, $x \in S_X$ and X is complete, Franchetti and Payá [10, Th. 1.2] obtained the equivalence of conditions (*i*), (*ii*), (*iv*) and (*v*), while Hu and Lin [26, Th. 12] obtained the equivalence of conditions (*i*), (*ii*), (*iv*)–(*vi*) and (*viii*) of the preceding result.

The variant with $\partial f(x)$ compact of the preceding result is the following:

Theorem 3.7. *Let $f \in \Lambda(X)$ be continuous at $x \in \operatorname{dom} f$. The following assertions are equivalent:*

(*i*) $\partial f(x)$ *is compact and* f *is directionally Fréchet differentiable at* x*;*
(*ii*) $\partial f(x)$ *is compact and the multifunction* $S_f(\cdot, x)$ *is* τ_0-$\tau_{\|\cdot\|}$ *H-usc at* 0*;*
(*iii*) $\partial f(x)$ *is compact and the multifunction* S_f *is* $\tau_0 \times \tau_{\|\cdot\|}$-$\tau_{\|\cdot\|}$ *H-usc at* $(0, x)$*;*
(*iv*) $\partial f(x)$ *is compact and the multifunction* $S_f(0, \cdot)$ *is* $\tau_{\|\cdot\|}$-$\tau_{\|\cdot\|}$ *H-usc at* x*;*
(*v*) $\partial f(x)$ *is compact and* $\lim_{y \to x} \operatorname{gap}(\partial f(y), \partial f(x)) = 0$*;*

(*vi*) *for every sequence* $(\eta_n) \subset [0,\infty)$ *converging to* 0 *and every sequence* (x_n^*) *with* $x_n^* \in \partial_{\eta_n} f(x)$ *for* $n \in \mathbb{N}$, $(x_n^*)_{n\in\mathbb{N}}$ *has a* $\tau_{\|\cdot\|}$*-convergent subsequence;*

(*vii*) *for every sequence* $(\eta_n) \subset [0,\infty)$ *converging to* 0, *every sequence* $(x_n) \subset X$ *converging to* x *and every sequence* (x_n^*) *with* $x_n^* \in \partial_{\eta_n} f(x_n)$ *for* $n \in \mathbb{N}$, $(x_n^*)_{n\in\mathbb{N}}$ *has a* $\tau_{\|\cdot\|}$*-convergent subsequence;*

(*viii*) *for every sequence* $(x_n) \subset X$ *converging to* x *and every sequence* (x_n^*) *with* $x_n^* \in \partial f(x_n)$ *for* $n \in \mathbb{N}$, $(x_n^*)_{n\in\mathbb{N}}$ *has a* $\tau_{\|\cdot\|}$*-convergent subsequence;*

(*ix*) $\lim_{\eta\downarrow 0} \alpha\left(\partial_\eta f(x)\right) = 0$.

Proof. The equivalence of conditions (*i*)–(*v*) is immediate from Theorem 3.6. In order to have the equivalence of conditions (*i*)–(*viii*) it is sufficient to observe that every condition in the above statement is equivalent to the compactness of $\partial f(x)$ and the validity of the corresponding condition in Theorem 3.6. Assume, for example, that (*viii*) in the present statement holds. Taking $x_n = x$ for every n, we obtain immediately that $\partial f(x)$ is compact. Assume now that (*viii*) in Theorem 3.6 is not true. Then there exist a sequence (x_n) converging to x, a sequence (x_n^*) with $x_n^* \in \partial f(x_n)$ for every n and $\varepsilon > 0$ such that $d(x_n^*, \partial f(x)) \geq \varepsilon$ for every n. By hypothesis, (x_n^*) has a subsequence $(x_{n_k}^*)$ $\tau_{\|\cdot\|}$-converging to $x^* \in X^*$. Because $\operatorname{gph} \partial f$ is closed, it follows that $x^* \in \partial f(x)$, which contradicts the fact that $d(x^*, \partial f(x)) \geq \varepsilon$. The converse is (almost) immediate.

Because X^* is complete, the equivalence of conditions (*vi*) and (*ix*) follows immediately from Corollary 2.3. □

For $f = \|\cdot\|$, $x \in S_X$ and X complete, the equivalence of assertions (*i*), (*ii*), (*iv*)–(*vi*) and (*viii*) in the preceding result is stated by Hu and Lin in [26, Cor. 13], while the equivalence of (*ii*) and (*ix*) is stated by Banaś and Sadarangani in [3, Th. 7]. Note that Hu and Lin [26] say that the norm is *quasi-Fréchet differentiable* when it satisfies condition (*viii*) in Theorem 3.7 at any $x \in S_X$.

The case with $\partial f(x)$ a singleton in the preceding statement (instead of being compact) is also of interest. In such a case f is directionally Fréchet differentiable at x if and only if f is Fréchet differentiable at x. So we have the following result.

Theorem 3.8. *Let* $f \in \Lambda(X)$ *be continuous at* $x \in \operatorname{dom} f$. *The following assertions are equivalent:*

(*i*) *f is Fréchet differentiable at x;*

(*ii*) $\lim_{\eta\downarrow 0} \operatorname{diam} \partial_\eta f(x) = 0$*;*

(*iii*) $\lim_{\eta\downarrow 0, y\to x} \operatorname{diam} \partial_\eta f(y) = 0$*;*

(*iv*) $\partial f(x) = \{\overline{x}^*\}$ *for some* $\overline{x}^* \in X^*$ *and* $\lim_{y\to x} d(\overline{x}^*, \partial f(y)) = 0$*;*

(*v*) *for every sequence* $(\eta_n) \subset [0,\infty)$ *converging to* 0 *and every sequence* (x_n^*) *with* $x_n^* \in \partial_{\eta_n} f(x)$ *for* $n \in \mathbb{N}$, $(x_n^*)_{n\in\mathbb{N}}$ *is* $\tau_{\|\cdot\|}$*-convergent;*

(*vi*) *for every sequence* $(\eta_n) \subset [0,\infty)$ *converging to* 0, *every sequence* $(x_n) \subset X$ *converging to* x *and every sequence* (x_n^*) *with* $x_n^* \in \partial_{\eta_n} f(x_n)$ *for* $n \in \mathbb{N}$, $(x_n^*)_{n\in\mathbb{N}}$ *is* $\tau_{\|\cdot\|}$*-convergent;*

(*vii*) *for every sequence* $(x_n) \subset X$ *converging to* x *and every sequence* (x_n^*) *with* $x_n^* \in \partial f(x_n)$ *for* $n \in \mathbb{N}$, $(x_n^*)_{n\in\mathbb{N}}$ *is* $\tau_{\|\cdot\|}$*-convergent.*

Proof. It is obvious that any one of the assertions (*ii*)–(*vii*) implies that $\partial f(x)$ is a singleton (being nonempty). The statement follows then from Theorem 3.6. □

Note that $\lim_{y\to x} \operatorname{diam} \partial f(y) = 0$ is not equivalent to (i) in the preceding theorem. Just consider a Gâteaux differentiable convex function f which is not Fréchet differentiable; such a function is $f : L^1(0,1) \to \mathbb{R}$, $f(x) := \int_0^1 \sqrt{1 + (x(t))^2}dt$ (see [39, Exer. 2.10]).

The equivalence of (i) and (v) in the previous result for $f = \|\cdot\|$ and $x \in S_X$ is stated as Smulian Lemma in [8, p. 243].

To the characterizations above one can add those from [22, Cor. 3.3] obtained for the Fréchet bornology (see also [39, Th. 3.3.2]; in fact, taking into account [22, Cor. 3.3], the condition that X is a Banach space is superfluous for the implication (vi) ⇒ (vii) in [39, Th. 3.3.2]).

We add here a characterization for the τ_0-$w(1)$ H-upper semicontinuity at 0 of the multifunction $S_f(\cdot, x)$ which appears as a consequence of a result of Section 5 (Corollary 5.3); the notation is explained in the next section. It can be completed with the corresponding characterizations (for $\tau = w$) in Proposition 3.10 below.

Proposition 3.9. *Let $f \in \Lambda(X)$ be continuous at $x \in \operatorname{dom} f$. Then*

$$\partial f^{**}(x) = \overline{\partial f(x)}^{w^*(3)}$$

if and only if the multifunction $S_f(\cdot, x)$ is τ_0-$w(1)$ H-usc at 0.

When $f = \|\cdot\|$, $x \in S_X$ and X is complete, the preceding result is nothing else but [14, Th. 3.1].

Some of the continuity properties in the preceding results can be stated when $\tau_{\|\cdot\|}$ is replaced by another locally convex topology on X^*. These continuity results can not be obtained directly from Theorem 2.1 because such a topology is not necessarily of type $\tau_{\mathcal{B}}$.

Proposition 3.10. *Let $f \in \Lambda(X)$, $x \in \operatorname{dom} f$ and τ a locally convex topology on X^*. Consider the following assertions:*

(i) *the multifunction $S_f(\cdot, x)$ is τ_0-τ H-usc at 0;*

(ii) *the multifunction $S_f(0, \cdot)$ is $\tau_{\|\cdot\|}$-τ H-usc at x;*

(iii) *the multifunction S_f is $\tau_0 \times \tau_{\|\cdot\|}$-$\tau$ H-usc at $(0, x)$.*

If f is continuous at x then $(i) \Leftrightarrow (iii)$; if X is complete, f is lower semicontinuous and τ is weaker than the norm topology on X^*, then $(ii) \Leftrightarrow (iii)$; if X is complete, f is lsc, $x \in \operatorname{int}(\operatorname{dom} f)$ and τ is weaker than the norm topology, then $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$.

Proof. Note that $(iii) \Rightarrow (i) \wedge (ii)$ always and the last statement subsumes the first two.

$(i) \Rightarrow (iii)$ Let f be continuous at x and assume that $S_f(\cdot, x)$ is τ_0-τ H-usc at 0. Let V be a convex τ-neighborhood of 0 in X^*. By hypothesis, there exists $\eta > 0$ such that $\partial_\eta f(x) \subset \partial f(x) + V$. Because f is continuous at x, there exists $r > 0$ such that $|f(y) - f(x)| \le \eta/9$ for $\|y - x\| < 2r$. Let $\gamma := \eta/3$. Consider $(y, y^*) \in \operatorname{gph} \partial_\gamma f$ with

$\|y - x\| < r$. Then, for every $z \in X$, we have

$$\begin{aligned} f(z) - f(x) &= f(z) - f(y) + f(y) - f(x) \\ &\geq \langle z - y, y^* \rangle - \gamma + f(y) - f(x) \\ &\geq \langle z - x, y^* \rangle - \gamma + \langle x - y, y^* \rangle + f(y) - f(x) \\ &\geq \langle z - x, y^* \rangle - 2\gamma + [f(y) - f(2y - x)] + [f(y) - f(x)] \\ &\geq \langle z - x, y^* \rangle - 3\gamma = \langle z - x, y^* \rangle - \eta. \end{aligned}$$

It follows that $y^* \in \partial_\eta f(x)$, and so $y^* \in \partial f(x) + V$. The conclusion follows.

(ii) $\Rightarrow$ *(iii)* Let X be complete, f be lsc and τ be weaker than the norm topology. Assume $S_f(0, \cdot)$ is $\tau_{\|\cdot\|}$-τ H-usc at x. Because $\operatorname{dom} f \subset \operatorname{cl}(\operatorname{dom} \partial f)$, we have $\partial f(x) \neq \emptyset$ in our situation.

Let V be a convex τ-neighborhood of 0 in X^*. By hypothesis, there exists $\eta \in (0, 1)$ such that $\partial f(y) \subset \partial f(x) + \frac{1}{2}V$ for $\|y - x\| < \eta$. Let $\gamma > 0$ satisfy $2\gamma < \eta$ and $\gamma U_{X^*} \subset \frac{1}{2}V$. Consider $\|y - x\| < \gamma$ and $y^* \in \partial_{\gamma^2} f(y)$. By Brøndsted–Rockafellar theorem (see [39, Th. 2.4.2]), there exists $(z, z^*) \in \operatorname{gph} \partial f$ such that $\|z - y\| \leq \gamma$ and $\|z^* - y^*\| \leq \gamma$. It follows that $\|z - x\| \leq \|z - y\| + \|y - x\| \leq \gamma + \gamma < \eta$. Hence $z^* \in \partial f(x) + \frac{1}{2}V$, and so $y^* \in \partial f(x) + \gamma U_{X^*} + \frac{1}{2}V \subset \partial f(x) + V$. The proof is complete. □

The equivalence of (*i*) and (*ii*) in Proposition 3.10 is obtained in [14, Th. 2.1] for $f = \|\cdot\|$, $\tau \in \{w(1), w^*, \tau_{\|\cdot\|}\}$ and X complete.

Of course, other topologies on the family $\mathcal{CC}^*(X^*)$ of w^*-closed convex subsets of X^* could be considered, and not only topologies of the type H_τ^+. Such topologies are the (upper) Vietoris topologies (weak* and strong), slice*, Wijsman, etc. (see [4], [33] for some of them).

First let us note the following result.

Lemma 3.11. *Let $(A_i)_{i\in I}$ be a net of nonempty subsets of X^* and $\emptyset \neq A \subset X^*$. If $A_i \xrightarrow{V_{w^*}^+} A$ then $\liminf \operatorname{gap}(A_i, C) \geq \operatorname{gap}(A, C)$ for every w^*-closed subset C of X^*. Conversely, if A is w^*-compact and $\liminf \operatorname{gap}(A_i, C) \geq \operatorname{gap}(A, C)$ for every w^*-closed subset of X^* then $A_i \xrightarrow{V_{w^*}^+} A$.*

Proof. Assume $A_i \xrightarrow{V_{w^*}^+} A$ and take $C \subset X^*$ a nonempty w^*-closed set. Let $\liminf \operatorname{gap}(A_i, C) < \mu < \infty$. Then the set $J := \{i \in I \mid \operatorname{gap}(A_i, C) < \mu\}$ is cofinal. The set $C + \mu U_{X^*}$ is w^*-closed (as the sum of a w^*-compact set and a w^*-closed set) and $A_i \cap (C + \mu U_{X^*}) \neq \emptyset$ for every $i \in J$. It follows that $A \cap (C + \mu U_{X^*}) \neq \emptyset$, whence $\operatorname{gap}(A, C) \leq \mu$. Hence $\liminf \operatorname{gap}(A_i, C) \geq \operatorname{gap}(A, C)$.

Conversely, assume that A is w^*-compact and $\liminf \operatorname{gap}(A_i, C) \geq \operatorname{gap}(A, C)$ for every w^*-closed subset C of X^*. Let $C \subset X^*$ be a w^*-closed set such that $A \cap C = \emptyset$. Then $A - C$ is w^*-closed and $0 \notin A - C$. It follows that $\operatorname{gap}(A, C) = d(0, A - C) > 0$, and so there exists some $i_0 \in I$ such that $\operatorname{gap}(A_i, C) > 0$ for every $i \geq i_0$. Hence $A_i \cap C = \emptyset$ for $i \geq i_0$. It follows that $A_i \xrightarrow{V_{w^*}^+} A$. □

Proposition 3.12. *Let $f \in \Lambda(X)$ be continuous at $x \in \operatorname{dom} f$. Then $\partial_\eta f(y) \xrightarrow{V^+_{w^*}} \partial f(x)$ for $(\eta, y) \to (0, x)$. In particular $(\partial_\eta f(x))$ slice*-converges to $\partial f(x)$ for $\eta \downarrow 0$, and so $\partial_\eta f(x) \xrightarrow{W} \partial f(x)$ for $\eta \downarrow 0$.*

Proof. Because $\partial f(x)$ is nonempty and w^*-compact, it is sufficient to show that $\partial_\eta f(y) \xrightarrow{\mathrm{H}^+_{w^*}} \partial f(x)$. Assume that this is not true. Then there exists V a w^*-neighborhood of 0 in X^* and a net $((\eta_i, x_i))_{i \in I} \subset [0, \infty) \times X$ converging to $(0, x)$ such that $\partial_{\eta_i} f(x_i) \not\subset \partial f(x) + V$ for every $i \in I$. Take $x_i^* \in \partial_{\eta_i} f(x_i) \backslash (\partial f(x) + V)$ for every $i \in I$. Using f.i. [39, Th. 2.4.13], there exists a $r, m > 0$ such that $\partial_1 f(y) \subset mU_{X^*}$ for every $y \in x + rU_X$. There exists some $i_0 \in I$ such that $\eta_i \leq 1$ and $x_i \in x + rU_X$ for every $i \geq i_0$. Hence $(x_i^*)_{i \geq i_0}$ is bounded, and taking a subnet if necessary, we may assume that $x_i^* \to^{w^*} x^*$. Using now [39, Th. 2.4.2(ix)] we obtain $x^* \in \partial f(x)$, which contradicts our choice of the net $(x_i^*)_{i \in I}$. The rest follows using the preceding lemma and the fact that $\partial f(x) \subset \partial_\eta f(x)$ for $\eta \geq 0$. □

The preceding results can be formulated easily for the conjugate f^* of a function $f \in \Gamma(X)$. It is known that $f^* \in \Gamma^*(X^*)$, that is, $f^* \in \Gamma(X^*)$ and f^* is also w^*-lower semicontinuous. Since X^* is a Banach space, instead of asking that f^* is continuous at $x^* \in \operatorname{dom} f^*$ we have to ask that $x^* \in \operatorname{int}(\operatorname{dom} f^*)$. But when dealing with f^* it is more interesting to have continuity properties for the multifunction $S^X_{f^*}$, for example, than for the multifunction S_{f^*}. For doing such a study we need first some results concerning higher order conjugates and higher order duals.

4. Auxiliary results

Given the normed vector space $(X, \|\cdot\|)$ and a function $f \in \Gamma(X)$, we define inductively the n-th conjugate of f : $f^{*(0)} := f$, $f^{*(n+1)} := (f^{*(n)})^*$ for $n \in \mathbb{N} \cup \{0\}$; hence $f^{*(1)} = f^*$. We shall denote $f^{*(2)}$ by f^{**}. So $f^{*(n)} \in \Gamma^* \left(X^{*(n)}\right) \subset \Gamma \left(X^{*(n)}\right)$ for $n \geq 1$. We have the following result.

Lemma 4.1. *Let $f \in \Gamma(X)$. Then $f^{**}|_X = f$, $\operatorname{dom} f = X \cap \operatorname{dom} f^{**}$ and*

$$X^* \cap \partial_\eta f^{**}(x) = \partial_\eta f(x) \quad \forall x \in X,\ \forall \eta \geq 0. \tag{4}$$

Moreover,

$$f^{*(n+2k)}|_{X^{*(n)}} = f^{*(n)} \quad \forall n, k \in \mathbb{N} \tag{5}$$

and

$$X^{*(n+1)} \cap \partial_\eta f^{*(n+2)}(x) = \partial_\eta f^{*(n)}(x) \quad \forall x \in X^{*(n)},\ \forall \eta \geq 0. \tag{6}$$

Proof. The first relation is nothing else, but the biconjugate theorem (see f.i. [39, Th. 2.3.3]), while the second follows immediately from the first one. Recalling that $x^* \in \partial_\eta f(x)$ if and only if $f(x) + f^*(x^*) \leq \langle x, x^* \rangle + \eta$, one obtains immediately (4) from $f^{**}|_X = f$. By induction one gets easily (5) and (6). □

From (4) we obtain the function $f \in \Lambda(X)$ is Gâteaux differentiable at $x \in \operatorname{dom} f$ whenever f is continuous at x and f^{**} is Gâteaux differentiable at x. A more interesting result is established in Corollary 5.12.

Let now $B \subset X^*$ be a nonempty convex set and consider the function

$$s_B : X \to \overline{\mathbb{R}}, \quad s_B(x) := \sup\{\langle x, x^* \rangle \mid x^* \in B\};$$

it is known that $s_B = s_{\overline{B}^{w^*}}$ and $(s_B)^* = \iota_{\overline{B}^{w^*}}$ (see f.i. [39, Th. 2.4.14(vi) and (i)]). If $A \subset X$ is a nonempty convex set, then

$$(\iota_A)^* = \sigma_A = s_A = s_{\overline{A}^{w^*(2)}} \tag{7}$$

with A considered as a subset of X^{**}. It follows (see [39, Th. 2.4.14(i)]) that

$$(\iota_A)^{**} = \sigma_A^* = \iota_{\overline{A}^{w^*(2)}} \text{ and } \overline{A}^{w^*(2)} = \overline{\operatorname{cl} A}^{w^*(2)}.$$

The last equality is an immediate consequence of the relation $(\iota_A)^* = (\iota_{\operatorname{cl} A})^*$. On the other hand, for $C \subset X^*$ a convex set we have

$$X^* \cap \overline{C}^{w^*(3)} \subset \overline{C}^{w(1)} = \operatorname{cl} C \subset \overline{C}^{w^*},$$

but generally, $\overline{C}^{w^*} \not\subset \overline{C}^{w^*(3)}$. However, if $u, u_i \in X^{*(n)}$ for i in the upward directed set I and $A \subset X^{*(n)}$, then

$$\begin{gathered} u_i \to^{w(n)} u \Leftrightarrow u_i \to^{w^*(n+2)} u, \\ A \text{ is } w(n)\text{-compact} \Leftrightarrow A \text{ is } w^*(n+2)\text{-compact}. \end{gathered} \tag{8}$$

Proposition 4.2. *Let $f \in \Gamma(X)$, $x \in \operatorname{dom} f$ and $x^* \in \operatorname{dom} f^*$. Then for every $\varepsilon > 0$*

$$\partial_\varepsilon f^*(x^*) = \overline{X \cap \partial_\varepsilon f^*(x^*)}^{w^*(2)} \tag{9}$$

and

$$(f'_\varepsilon(x, \cdot))^{**} = (f^{**})'_\varepsilon (x, \cdot). \tag{10}$$

Proof. Fix $\varepsilon > 0$. As $f^* \in \Gamma^*(X^*)$ and $x^* \in \operatorname{dom} f^*$, using [39, Th. 2.4.11] for the duality (X^*, X), we have

$$(f^*)'_\varepsilon(x^*, u^*) = \sup\{\langle u, u^* \rangle \mid u \in X \cap \partial_\varepsilon f^*(x^*)\} = \sigma_{X \cap \partial_\varepsilon f^*(x^*)}(u^*),$$

while, applying the same result for the duality (X^*, X^{**}), we have

$$(f^*)'_\varepsilon(x^*, u^*) = \sup\{\langle u^*, u^{**} \rangle \mid u^{**} \in \partial_\varepsilon f^*(x^*)\} = s_{\partial_\varepsilon f^*(x^*)}(u^*).$$

Since $\partial_\varepsilon f^*(x^*)$ is $w^*(2)$-closed and convex, by [39, Th. 2.4.14(vi)], it follows that

$$\partial_\varepsilon f^*(x^*) = \overline{\operatorname{conv}}^{w^*(2)}(X \cap \partial_\varepsilon f^*(x^*)) = \overline{X \cap \partial_\varepsilon f^*(x^*)}^{w^*(2)}$$

($X \cap \partial_\varepsilon f^*(x^*)$ being convex).

Applying the preceding estimates for f replaced by f^* and using (4), we get

$$(f^{**})'_\varepsilon (x, \cdot) = ((f^*)^*)'_\varepsilon (x, \cdot) = s_{\partial_\varepsilon f^{**}(x)} = s_{\overline{X^* \cap \partial_\varepsilon f^{**}(x)}^{w^*(3)}} = s_{\overline{\partial_\varepsilon f(x)}^{w^*(3)}},$$

while $f'_\varepsilon(x, \cdot) = s_{\partial_\varepsilon f(x)}$, and so, using (7),

$$(f'_\varepsilon(x, \cdot))^{**} = \left(s_{\partial_\varepsilon f(x)}\right)^{**} = (\iota_{\partial_\varepsilon f(x)})^* = \sigma_{\partial_\varepsilon f(x)} = s_{\overline{\partial_\varepsilon f(x)}^{w^*(3)}}.$$

The conclusion follows. $\square$

Taking $f = \iota_{U_X}$ (hence $f^* = \|\cdot\|$) and $x^* \in S_{X^*}$, the preceding result shows that

$$S(x^*, \eta) = \overline{X \cap S(x^*, \eta)}^{w^*(2)} \quad \forall x^* \in S_{X^*},\ \forall \eta > 0. \tag{11}$$

This is [14, Lem. 2.1].

We need the following (perhaps known) result.

Lemma 4.3. *Let $f \in \Gamma(X)$ be continuous at $x \in \operatorname{dom} f$. Then f^{**} is continuous at x.*

Proof. W.l.o.g. we may assume that $x = 0$ and $f(0) = 0$. Because f is Lipschitz on a neighborhood of 0 (see f.i. [39, Cor. 2.2.13]), there exist $r, m > 0$ such that $f(y) \le m\|y\|$ for every $y \in rU_X$. This inequality can be written as $f \le m\|\cdot\| + \iota_{rU_X}$. Taking the conjugate we obtain

$$f^* \ge (m\|\cdot\| + \iota_{rU_X})^* = (m\|\cdot\|)^* \Box (\iota_{rU_X})^* = \iota_{mU_{X^*}} \Box (r\|\cdot\|)$$

(the convolution being even exact because $m\|\cdot\|$ is continuous; see [39, Th. 2.8.7]). Taking again the conjugates (for the duality (X^*, X^{**})), we obtain

$$f^{**} \le (\iota_{mU_{X^*}} \Box (r\|\cdot\|))^* = m\|\cdot\| + \iota_{rU_{X^{**}}}.$$

Hence $f^{**}(x^{**}) \le m\|x^{**}\|$ for every $x^{**} \in rU_{X^{**}}$, which implies that f^{**} is continuous at 0, too. $\square$

The proof of the preceding result shows that if $f \in \Lambda(X)$ is bounded on bounded sets then f^{**} is also bounded on bounded sets (see [32, p. 134]); as observed in [32], if f is continuous on $X = \operatorname{dom} f$, it does not follow that f^{**} is continuous (and so $\operatorname{dom} f^{**}$ could be distinct from X^{**}).

Consider $f \in \Gamma(X)$ and $x \in \operatorname{dom} f$. Because for every $\alpha > 0$ we have $f'(x, u) \le \alpha^{-1}(f(x + \alpha u) - f(x))$ for every $u \in X$, proceeding as above (taking conjugates with respect to the dualities (X, X^*) and (X^*, X^{**})), we obtain

$$(f'(x, \cdot))^{**}(u^{**}) \le \alpha^{-1}(f^{**}(x + \alpha u^{**}) - f(x)) \quad \forall u^{**} \in X^{**},$$

which shows that

$$(f'(x, \cdot))^{**} \le (f^{**})'(x, \cdot). \tag{12}$$

A natural question is whether

$$(f^{**})'(x, \cdot) = (f'(x, \cdot))^{**}, \tag{13}$$

at least when $f \in \Gamma(X)$ is continuous at $x \in \operatorname{dom} f$. But in this case ($f$ continuous at $x \in \operatorname{dom} f$), we have (see [39, Th. 2.4.9]) that $f'(x, \cdot) = s_{\partial f(x)}$, and so $(f'(x, \cdot))^{**} = (s_{\partial f(x)})^{**} = s_{\overline{\partial f(x)}^{w^*(3)}}$. On the other hand $(f^{**})'(x, \cdot) = s_{\partial f^{**}(x)}$. Hence (13) holds if and only if

$$\partial f^{**}(x) = \overline{\partial f(x)}^{w^*(3)}. \tag{14}$$

The next result gives a condition under which (13) holds. In fact it gives more information.

Proposition 4.4. *Let $f \in \Gamma(X)$ be continuous at $x \in \operatorname{dom} f$. Then f is directionally Fréchet differentiable at x if and only if f^{**} is directionally Fréchet differentiable at x; moreover, if f is directionally Fréchet differentiable at x then* (13) *and* (14) *hold.*

Proof. First note from Lemma 4.1 that $f(x+tu) = f^{**}(x+tu)$ for all $t \in \mathbb{R}$ and $u \in X$. It follows that

$$(f^{**})'(x,u) = f'(x,u) \quad \forall u \in X. \tag{15}$$

Assume that f is directionally Fréchet differentiable at x and take $\varepsilon > 0$. There exists $\alpha > 0$ such that

$$\frac{f(x+\alpha u) - f(x)}{\alpha} \leq f'(x,u) + \varepsilon \quad \forall u \in U_X,$$

or, equivalently, $f(x+\alpha\cdot) \leq f(x) + \alpha\varepsilon + \alpha f'(x,\cdot) + \iota_{U_X}$. Taking the biconjugates we obtain

$$\frac{f^{**}(x+\alpha u^{**}) - f^{**}(x)}{\alpha} = \frac{f^{**}(x+\alpha u^{**}) - f(x)}{\alpha} \leq (f'(x,\cdot))^{**}(u^{**}) + \varepsilon \tag{16}$$

for all $u^{**} \in U_{X^{**}}$. Using (12), it follows that

$$\frac{f^{**}(x+\alpha u^{**}) - f^{**}(x)}{\alpha} \leq (f^{**})'(x,u^{**}) + \varepsilon \quad \forall u^{**} \in U_{X^{**}}.$$

Since f^{**} is continuous at x (see Lemma 4.3), f^{**} is directionally Fréchet differentiable at x. Moreover, from (16) we also obtain that $(f^{**})'(x,u^{**}) \leq (f'(x,\cdot))^{**}(u^{**}) + \varepsilon$ for all $u^{**} \in U_{X^{**}}$ and $\varepsilon > 0$. Letting $\varepsilon \to 0$ we obtain $(f^{**})'(x,\cdot) \leq (f'(x,\cdot))^{**}$ on $U_{X^{**}}$, and by positive homogeneity of both functions, we get the inequality on X^{**}. As the converse is always true (see (12)), we have (13) holds.

Assume now that f^{**} is directionally Fréchet differentiable at x. Taking (15) into account and the fact that $U_X \subset U_{X^{**}}$, we obtain immediately that f is directionally Fréchet differentiable at x. $\square$

Taking into account the equivalence of (i) and (iv) in Theorem 3.6, from the preceding result we get [14, Cor. 2.1] which is stated for X complete, $f = \|\cdot\|$ and $x \in S_X$.

Before studying continuity properties of the multifunction $S^X_{f^*}$, we establish other useful auxiliary results.

Lemma 4.5. *Let $y \in X$, $f \in \Gamma(X)$, $x^* \in \operatorname{dom} f^*$, $\gamma \geq 0$ and $x^{**} \in \partial_\gamma f^*(x^*)$ be such that $\|y - x^{**}\| < \alpha$. Then for every $\delta > 0$ and every $w^*(2)$-neighborhood V of x^{**} in X^{**}, there exists $x_\delta \in X \cap V \cap \partial_{\gamma+\delta} f^*(x^*)$ such that $\|y - x_\delta\| < \alpha$.*

Proof. Let us fix some $\delta > 0$ and V_0 a convex $w^*(2)$-neighborhood of x^{**} in X^{**}. For every $\eta \in (0,\delta]$ we have $x^{**} \in \partial_\gamma f^*(x^*) \subset \partial_{\gamma+\eta} f^*(x^*) = \overline{X \cap \partial_{\gamma+\eta} f^*(x^*)}^{w^*(2)}$ (the last equality being provided by (9)). Hence, for every $i = (\eta, V) \in I := (0,\delta] \times \mathcal{N}_{w^*(2)}(x^{**})$, there exists $x_i \in X \cap V \cap \partial_{\gamma+\eta} f^*(x^*)$. Endowing I with the order defined by $(\eta, V) \geq (\eta', V')$ if $\eta \leq \eta'$ and $V \subset V'$, we have $(x_i)_{i\in I} \to^{w^*(2)} x^{**}$. It follows that there exists some $i_0 \in I$ such that $x_i \in V_0$ for $i \geq i_0$. Let $C := \operatorname{conv}\{x_i \mid i \in I,\, i \geq i_0\} \subset X \cap V_0 \cap \partial_{\gamma+\delta} f^*(x^*)$. We have that

$d(y,C) < \alpha$. In the contrary case $(y+\alpha B_X) \cap C = \emptyset$, and so, by a separation theorem, there exists $u^* \in S_{X^*}$ such that

$$\langle y+\alpha u, u^*\rangle \geq \langle z, u^*\rangle \quad \forall u \in U_X,\ \forall z \in C.$$

It follows that $\langle y, u^*\rangle - \alpha \geq \langle x_i, u^*\rangle$ for every $i \in I$ with $i \geq i_0$. Taking the limit, because $(x_i) \to^{w^*(2)} x^{**}$, we obtain $\langle y, u^*\rangle - \alpha \geq \langle u^*, x^{**}\rangle$, and so we get the contradiction $\|y - x^{**}\| \geq \langle u^*, y - x^{**}\rangle \geq \alpha$. The conclusion follows. □

When $\gamma = 0$ the conclusion of the preceding result can be reformulated (partially) as follows: If $f \in \Gamma(X)$, $x^{**} \in \partial f^*(x^*)$, $y \in X$ and $\|y - x^{**}\| < \alpha$ then

$$f^*(x^*) = \sup\{\langle x, x^*\rangle - f(x) \mid x \in y + \alpha B_X\}. \tag{17}$$

The preceding result yields the next one.

Corollary 4.6. *Let $y \in X$, $f \in \Gamma(X)$ and $x^{**} \in \operatorname{dom} f^{**}$ be such that $\|y - x^{**}\| < \alpha$. Then for every $\mu > f^{**}(x^{**})$ and every $w^*(2)$-neighborhood V of x^{**} in X^{**}, there exists $x_\mu \in X \cap V$ such that $f(x_\mu) \leq \mu$ and $\|y - x_\mu\| < \alpha$.*

Proof. Fix some $x^* \in \operatorname{dom} f^*$ and take $\gamma := f^*(x^*) + f^{**}(x^{**}) - \langle x^*, x^{**}\rangle \geq 0$; then $x^{**} \in \partial_\gamma f^*(x^*)$. Let $\mu > f^{**}(x^{**})$ and V a $w^*(2)$-neighborhood of x^{**} in X^{**}. Consider $\delta := \frac{1}{2}[\mu - f^{**}(x^{**})] > 0$ and $V' := V \cap \{y^{**} \in X^{**} \mid |\langle x^*, y^{**} - x^{**}\rangle| < \delta\} \in \mathcal{N}_{w^*(2)}(x^{**})$. Applying the preceding result for δ and V' we get some $x_\delta \in X \cap V' \cap \partial_{\gamma+\delta} f^*(x^*)$ such that $\|y - x_\delta\| < \alpha$. Then

$$\begin{aligned} f(x_\delta) &\leq \langle x_\delta, x^*\rangle + \gamma + \delta - f^*(x^*) = \langle x^*, x_\delta - x^{**}\rangle + f^{**}(x^{**}) + \delta \\ &= \langle x^*, x_\delta - x^{**}\rangle + \mu - 2\delta + \delta \leq \mu. \end{aligned}$$

Setting $x_\mu := x_\delta$, the conclusion follows. □

From the preceding result we get the following description of f^{**} stated by Gossez [21, Lem. 3.1] for X complete (see also the proof of Prop. 1 in Rockafellar [31] and Voisei [38]).

Corollary 4.7. *Let $f \in \Gamma(X)$ and $x^{**} \in X^{**}$. Then there exists a net $(x_i)_{i \in I} \subset X$ such that*

$$x_i \to^{w^*(2)} x^{**},\ \|x_i\| \to \|x^{**}\| \ \text{and}\ f(x_i) \to f^{**}(x^{**}).$$

Proof. Assume first that $x^{**} \in \operatorname{dom} f^{**}$ and take $\mu_n := f^{**}(x^{**}) + n^{-1}$ and $\alpha_n := \|x^{**}\| + n^{-1}$ for $n \in \mathbb{N}$. Applying the preceding result with $y = 0$, for every $i = (n, V) \in I := \mathbb{N} \times \mathcal{N}_{w^*(2)}(x^{**})$, there exists $x_i \in X \cap V$ such that $f(x_i) \leq \mu_n$ and $\|x_i\| < \alpha_n$. Endowing I with the order given by $(n, V) \geq (n', V')$ if $n \geq n'$ and $V \subset V'$, we have $x_i \to^{w^*(2)} x^{**}$, $\limsup \|x_i\| \leq \|x^{**}\|$ and $\limsup f(x_i) \leq f^{**}(x^{**})$. Since f^{**} is $w^*(2)$-lsc and $f^{**}|_X = f$, we obtain $f^{**}(x^{**}) \leq \liminf f(x_i)$, and so $f(x_i) \to f^{**}(x^{**})$. For $x^* \in S_{X^*}$ we have $\|x_i\| \geq \langle x_i, x^*\rangle \to \langle x^*, x^{**}\rangle$, and so $\liminf \|x_i\| \geq \langle x^*, x^{**}\rangle$. Taking the supremum w.r.t. $x^* \in S_{X^*}$, we obtain $\liminf \|x_i\| \geq \|x^{**}\|$, and so $\|x_i\| \to \|x^{**}\|$. If $x^{**} \notin \operatorname{dom} f^{**}$ take an arbitrary net $(x_i)_{i \in I} \subset \|x^{**}\| \cdot U_X$ $w^*(2)$-converging to x^{**}. As above we get the conclusion. The proof is complete. □

Using Lemma 4.5 we obtain once again the next result stated by Rockafellar [31, Prop. 1].

Corollary 4.8. *Let X be complete, $x^* \in X^*$, $x^{**} \in X^{**}$ and $f \in \Gamma(X)$. Then $x^{**} \in \partial f^*(x^*)$ if and only if there exists a bounded net $(x_i)_{i\in I} \subset X$ $w^*(2)$-converging to x^{**} and a net $(x_i^*)_{i\in I} \subset X^*$ norm-converging to x^* such that $x_i^* \in \partial f(x_i)$ for every $i \in I$.*

Proof. The sufficiency is immediate (even without X complete); just note that $f^{**}(x_i) + f^*(x_i^*) = f(x_i) + f^*(x_i^*) = \langle x_i, x_i^*\rangle$ and f^* and f^{**} are norm-lsc and $w^*(2)$-lsc, respectively.

Assume that $x^{**} \in \partial f^*(x^*)$. By Lemma 4.5, for every $i = (n, V) \in I := \mathbb{N} \times \mathcal{N}_{w^*(2)}(x^{**})$, there exists $x_i' \in X \cap V \cap \partial_{n^{-1}} f^*(x^*)$ such that $\|x_i'\| < \|x^{**}\| + n^{-1}$. By the Brøndsted–Rockafellar theorem (see f.i. [39, Th. 3.1.2]), there exists $x_i \in X$ and $x_i^* \in X^*$ such that $\|x_i - x_i'\| \le n^{-1/2}$, $\|x_i^* - x^*\| \le n^{-1/2}$ and $x_i^* \in \partial f(x_i)$. It is obvious that the nets $(x_i)_{i\in I}$ and $(x_i^*)_{i\in I}$ satisfy the desired conditions. □

Corollary 4.9. *Let $A \subset X$ be a nonempty convex set, $y \in X$ and $x^{**} \in \overline{A}^{w^*(2)}$ with $\|y - x^{**}\| < \alpha$. Then for every $\delta > 0$ and every $w^*(2)$-neighborhood V of x^{**} in X^{**} there exists $x_\delta \in A \cap V$ such that $\|y - x_\delta\| < \alpha$.*

Proof. Consider $f := \iota_{\operatorname{cl} A} \in \Gamma(X)$. Since $x^{**} \in \overline{A}^{w^*(2)} = \partial f^*(0)$, applying Lemma 4.5 for $x^* = 0$ and $\gamma = 0$, for every $\delta > 0$ and every $w^*(2)$-open neighborhood V of x^{**} in X^{**} there exists $x_\delta' \in X \cap V \cap \partial_\delta f^*(0) = V \cap \operatorname{cl} A$ such that $\|y - x_\delta'\| < \alpha$. Taking into account the continuity of the norm and the ($w^*(2)$-) openness of V, we get some $x_\delta \in A \cap V$ with the desired properties. □

As an immediate consequence of the preceding result we get the following one which is related to (9).

Corollary 4.10. *Let $A \subset (X, \|\cdot\|)$ be a nonempty convex set. Then*

$$d(x, A) = d(x, \overline{A}^{w^*(2)}) \quad \forall x \in X.$$

Proof. Fix some $x \in X$. Of course, we have $d(x, A) \ge d(x, \overline{A}^{w^*(2)})$. Let $d(x, \overline{A}^{w^*(2)}) < \gamma < \infty$. Then $\|x - x^{**}\| < \gamma$ for some $x^{**} \in \overline{A}^{w^*(2)}$. From Corollary 4.9 we get some $x' \in A$ such that $\|x - x'\| < \gamma$, and so $d(x, A) < \gamma$. It follows that $d(x, A) \le d(x, \overline{A}^{w^*(2)})$. The conclusion follows. □

Related to this result we mention that for every nonempty bounded set $A \subset X$ we have $\operatorname{diam} A = \operatorname{diam} \overline{A}^{w^*(2)}$; this formula was established in [14, Cor. 3.1] in a special case, but the proof is the same.

Another immediate consequence of Corollary 4.9 is the following.

Corollary 4.11. *Let $x \in X$ and $x^{**} \in X^{**}$ be such that $\|x - x^{**}\| < \alpha$. Then for every $\delta > 0$ and every $w^*(2)$-neighborhood V of x^{**} in X^{**}, there exists $x_\delta \in V \cap X$ such that $\|x_\delta\| \le \|x^{**}\|$ and $\|x - x_\delta\| < \alpha$.*

Proof. If $x^{**} \in X$ we just take $x_\delta := x^{**}$. In the other case, take $\gamma := \|x^{**}\| > 0$ and $A := \gamma U_X$. It is clear that $x^{**} \in \overline{A}^{w^*(2)}$. From Corollary 4.9 we get the desired conclusion. □

We are now in a position to establish continuity results for the multifunction $S^X_{f^*}$.

5. Continuity properties for the ε-subdifferential of the conjugate of a convex function

Because we are interested in the continuity properties of the multifunction $S^X_{f^*}$, we shall assume directly that $f \in \Gamma(X)$; this can always be done if we want f^* to be proper (because in such a case $f^* = \overline{f}^*$, where $\overline{f}$ is the lsc envelope of f). Also note that f^* is automatically continuous at $x^* \in \operatorname{int}(\operatorname{dom} f^*)$ because f^* is lsc and X^* is complete.

In this section the continuity results will be presented in an (almost) reversed order than that in Section 3. We begin with a result which corresponds to Proposition 3.10 for the conjugate function f^*.

Proposition 5.1. *Let $f \in \Gamma(X)$, $x^* \in \operatorname{dom} f^*$ and τ a locally convex topology on X. Consider the following assertions:*

(i) *the multifunction $S^X_{f^*}(\cdot, x^*)$ is τ_0-τ H-usc at 0;*

(ii) *the multifunction $S^X_{f^*}(0, \cdot)$ is $\tau_{\|\cdot\|}$-τ H-usc at x^*;*

(iii) *the multifunction $S^X_{f^*}$ is $\tau_0 \times \tau_{\|\cdot\|}$-$\tau$ H-usc at $(0, x^*)$.*

If $x^* \in \operatorname{int}(\operatorname{dom} f^*)$ then (i) $\Leftrightarrow$ (iii); if X is complete and τ is weaker than the norm topology, then (ii) $\Leftrightarrow$ (iii); if X is complete, $x^* \in \operatorname{int}(\operatorname{dom} f^*)$ and τ is weaker than the norm topology, then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

Proof. Of course, (iii) $\Rightarrow$ (i) $\wedge$ (ii) always.

(i) $\Rightarrow$ (iii) The proof being very close to that of the proof of the implication (i) $\Rightarrow$ (iii) of Proposition 3.10, we omit it.

(ii) $\Rightarrow$ (iii) Assume that X is complete, τ is weaker than the norm topology and $S^X_{f^*}(0, \cdot)$ is $\tau_{\|\cdot\|}$-τ H-usc at x^*. Because $\operatorname{dom} f^* \subset \operatorname{cl}(\operatorname{Im} \partial f)$ (see f.i. [39, Th. 3.1.2]), we have $X \cap \partial f^*(x^*) \neq \emptyset$ in our situation.

Let V be a convex τ-neighborhood of 0 in X. By hypothesis, there exists $\eta \in (0, 1)$ such that $X \cap \partial f^*(y^*) \subset (X \cap \partial f^*(x^*)) + \frac{1}{2}V$ for $\|y^* - x^*\| < \eta$. Let $\gamma > 0$ satisfy $2\gamma < \eta$ and $\gamma U_X \subset \frac{1}{2}V$. Consider $\|y^* - x^*\| < \gamma$ and $y \in X \cap \partial_{\gamma^2} f^*(y^*)$, that is, $y^* \in \partial_{\gamma^2} f(y)$. By Brøndsted–Rockafellar theorem (see f.i. [39, Th. 3.1.2]), there exists $(z, z^*) \in \operatorname{gph} \partial f$ such that $\|z - y\| \le \gamma$ and $\|z^* - y^*\| \le \gamma$. It follows that $\|z^* - x^*\| \le \|z^* - y^*\| + \|y^* - x^*\| \le \gamma + \gamma < \eta$. Hence $z \in (X \cap \partial f^*(x^*)) + \frac{1}{2}V$, and so $y \in (X \cap \partial f^*(x^*)) + \gamma U_X + \frac{1}{2}V \subset (X \cap \partial f^*(x^*)) + V$. □

The equivalence of (i) and (ii) in Proposition 5.1 is obtained in [19, Lem. 2.1] for $f = \iota_{U_X}$ (that is, $f^* = \|\cdot\|$), $\tau = w$ and X complete.

We have seen that (9) holds for every $\varepsilon > 0$ provided $f \in \Gamma(X)$ and $x^* \in \operatorname{dom} f^*$. The next result provides a useful characterization of (9) for $\varepsilon = 0$.

Proposition 5.2. *Let $f \in \Gamma(X)$ and $x^* \in \operatorname{int}(\operatorname{dom} f^*)$. Then*

$$\partial f^*(x^*) = \overline{X \cap \partial f^*(x^*)}^{w^*(2)} \tag{18}$$

if and only if the multifunction $S^X_{f^*}(\cdot, x^*)$ is τ_0-w H-usc at 0.

Proof. Assume (18) holds but the multifunction $S^X_{f^*}(\cdot,x^*)$ is not τ_0-w usc at 0; of course, $X\cap\partial f^*(x^*)\neq\emptyset$ in our conditions. It follows that there exists a $w^*(2)$-open, convex and symmetric neighborhood V of 0 in X^{**} such that for every $\eta>0$ there exists

$$x_\eta\in(X\cap\partial_\eta f^*(x^*))\setminus((X\cap\partial f^*(x^*))+V_X),$$

where $V_X:=V\cap X$. Since $(x_\eta)_{\eta\in(0,1]}\subset X\cap\partial_1 f^*(x^*)\subset\partial_1 f^*(x^*)$ and f^* is continuous at x^*, we have $(x_\eta)_{\eta\in(0,1]}$ is bounded (see f.i. [39, Th. 2.4.9]). It follows that the net $(x_\eta)_{\eta\in(0,1]}$ (for $\eta\to 0$) has a subnet $(x_i)_{i\in I}$ $w^*(2)$-converging to x^{**} in X^{**}. Of course,

$$f^{**}(x_i)+f^*(x^*)=f(x_i)+f^*(x^*)\le\langle x_i,x^*\rangle+\eta_i\quad\forall i\in I\text{ (with }\eta_i\to 0),$$

and so, taking the liminf, we get $f^{**}(x^{**})+f^*(x^*)\le\langle x^*,x^{**}\rangle$. Hence $x^{**}\in\partial f^*(x^*)$. Because $x_i\notin(X\cap\partial f^*(x^*))+V$ and this set is $w^*(2)$-open in X^{**}, we obtain $x^{**}\notin(X\cap\partial f^*(x^*))+V$. This is a contradiction because $\overline{X\cap\partial f^*(x^*)}^{w^*(2)}\subset(X\cap\partial f^*(x^*))+V$.

Assume now that $S^X_{f^*}(\cdot,x^*)$ is τ_0-w usc at 0; of course, $X\cap\partial f^*(x^*)$ is nonempty. Take $x^{**}\notin\overline{X\cap\partial f^*(x^*)}^{w^*(2)}$. By a separation theorem, there exist $\overline{x}^*\in X^*$ and $\alpha'\in\mathbb{R}$ such that

$$\langle\overline{x}^*,x^{**}\rangle>\alpha'>\alpha:=\sup\{\langle x,\overline{x}^*\rangle\mid x\in X\cap\partial f^*(x^*)\}.$$

Because $V_X:=\{u\in X\mid\langle u,\overline{x}^*\rangle\le\alpha'-\alpha\}$ is a weak neighborhood of 0 in X, it follows that there exists $\eta>0$ such that $X\cap\partial_\eta f^*(x^*)\subset(X\cap\partial f^*(x^*))+V_X$. It follows (using also Proposition 4.2) that

$$\partial f^*(x^*)\subset\partial_\eta f^*(x^*)=\overline{X\cap\partial_\eta f^*(x^*)}^{w^*(2)}\subset\{u^{**}\in X^{**}\mid\langle\overline{x}^*,u^{**}\rangle\le\alpha'\}.$$

Hence $x^{**}\notin\partial f^*(x^*)$. The conclusion follows. □

When X is complete, $f=\iota_{U_X}$ (hence $f^*=\|\cdot\|$) and $x^*\in S_{X^*}$, the preceding result is equivalent to [19, Lem. 2.2] (taking into account Proposition 5.1).

Corollary 5.3. *Let $f\in\Gamma(X)$ be continuous at $x\in\operatorname{dom}f$. Then*

$$\partial f^{**}(x)=\overline{\partial f(x)}^{w^*(3)}\tag{19}$$

if and only if the multifunction $S_f(\cdot,x)$ is τ_0-$w(1)$ H-usc at 0.

Proof. Because f is continuous at x, by Lemma 4.3, f^{**} is also continuous at x. Taking into account Lemma 4.1, (19) can be written as $\partial f^{**}(x)=\overline{X^*\cap\partial f^{**}(x)}^{w^*(3)}$. Applying Proposition 5.2 for f^* and $x\in\operatorname{int}(\operatorname{dom}(f^*)^*)$, the last equality is equivalent to the fact that the mapping $\eta\to X^*\cap\partial_\eta f^{**}(x)=\partial_\eta f(x)$ (by Lemma 4.1) is τ_0-$w(1)$ H-usc at 0. □

When X is complete, $f=\|\cdot\|$ and $x\in S_X$, Corollary 5.3 is nothing else but [14, Th. 3.1].

Corollary 5.4. *Let X be a Banach space, $f\in\Gamma(X)$ and $x\in\operatorname{int}(\operatorname{dom}f)$. The following assertions are equivalent:*

(*i*) $\partial f^{**}(x)=\partial f(x)$;

(*ii*) $\partial f(x)$ *is $w(1)$-compact and $S_f(\cdot,x)$ is τ_0-$w(1)$ H-usc at 0;*

(*iii*) $\partial f(x)$ *is $w(1)$-compact and $S_f(0,\cdot)$ is $\tau_{\|\cdot\|}$-$w(1)$ H-usc at x;*

(iv) $\partial f(x)$ *is* $w(1)$*-compact and* S_f *is* $\tau_0 \times \tau_{\|\cdot\|}$-$w(1)$ *H-usc at* $(0, x)$.

Proof. In view of Proposition 3.10, it is enough to prove that $(i) \Leftrightarrow (ii)$.

$(i) \Rightarrow (ii)$ By Lemma 4.3, f^{**} is continuous at x, and so, by [39, Th. 2.4.9], $\partial f^{**}(x)$ is $w^*(3)$-compact in $X^{*(3)}$. Hence, by (i) and (8), $\partial f(x)$ is $w(1)$-compact in X^*. The rest follows from Corollary 5.3.

$(ii) \Rightarrow (i)$ By Corollary 5.3, we have $\partial f^{**}(x) = \overline{\partial f(x)}^{w^*(3)}$. Since $\partial f(x)$ is $w(1)$-compact, using (8) we get $\partial f^{**}(x) = \partial f(x)$. □

Corollary 5.5. *Let* X *be a Banach space,* $f \in \Gamma(X)$ *and* $x \in \operatorname{int}(\operatorname{dom} f)$*. The following assertions are equivalent:*

(i) f *is Gâteaux differentiable at* x *and* $S_f(0, \cdot)$ *is* $\tau_{\|\cdot\|}$-$w(1)$ *H-usc at* x*;*
(ii) f^{**} *is Gâteaux differentiable at* x.

Proof. $(i) \Rightarrow (ii)$ This follows from the implication $(iii) \Rightarrow (i)$ of Corollary 5.4 because $\partial f(x)$ is a singleton.

$(ii) \Rightarrow (i)$ Since $\partial f^{**}(x)$ is a singleton, $\partial f(x) \neq \emptyset$ (because f is continuous), and (by (4)) $\partial f(x) = \partial f^{**}(x) \cap X^*$, it follows that $\partial f(x)$ is a singleton; hence f is Gâteaux differentiable at x; it follows that assertion (i) in Corollary 5.4 holds. The second part of (i) also follows from Corollary 5.4. □

From Corollary 5.5 one obtains the following result in Giles [12, p. 72]: *For a Banach space* X *and* $x \in S_X$, X^{**} *is smooth at* $\hat{x} \in S_{X^{**}}$ *if and only if every support mapping of* X *into* X^* *is* $\tau_{\|\cdot\|}$-$w(1)$ *continuous at* x.

Corollary 5.6. *Let* X *be a Banach space,* $f \in \Gamma(X)$ *and* $x \in \operatorname{int}(\operatorname{dom} f)$*. If* f *is Fréchet differentiable at* x*, then* f^{**} *is Gâteaux differentiable at* x *and* $\nabla f^{**}(x) = \nabla f(x) \in X^*$.

Proof. The assertion follows from the previous corollary and Theorem 3.8. □

We continue this section with the version for convex functions of [19, Th. 2.3] where the implications $(i) \Leftrightarrow (iii) \Rightarrow (iv) \Leftrightarrow (vi)$ below were stated for $f = \iota_{U_X}$ (that is, $f^* = \|\cdot\|$) and $x^* \in S_{X^*}$.

Theorem 5.7. *Let* X *be a Banach space,* $f \in \Gamma(X)$ *and* $x^* \in \operatorname{int}(\operatorname{dom} f^*)$*. Consider the following assertions:*

(i) *the multifunction* $S_{f^*}(0, \cdot)$ *is* $\tau_{\|\cdot\|}$-$w(2)$ *H-usc at* x^**;*
(ii) *the multifunction* $S_{f^*}(\cdot, x^*)$ *is* τ_0-$w(2)$ *H-usc at* 0*;*
(iii) $\partial f^{*(3)}(x^*) = \overline{\partial f^*(x^*)}^{w^*(4)}$*;*
(iv) *the multifunction* $S^X_{f^*}(0, \cdot)$ *is* $\tau_{\|\cdot\|}$-w *H-usc at* x^**;*
(v) *the multifunction* $S^X_{f^*}(\cdot, x^*)$ *is* τ_0-w *H-usc at* 0*;*
(vi) $\partial f^*(x^*) = \overline{X \cap \partial f^*(x^*)}^{w^*(2)}$.

Then $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv) \Leftrightarrow (v) \Leftrightarrow (vi)$.

Proof. The equivalence of (i) and (ii) follows from Proposition 3.10, the equivalence of (ii) and (iii) follows from Corollary 5.3, while the equivalence of (iv) and (v) follows from

Proposition 5.1, the equivalence of (v) and (vi) follows from Proposition 5.2 (applied all for f^*). In order to obtain the implication $(iii) \Rightarrow (vi)$ we shall follow several steps, the final one, Fact 5.11, providing our implication; we follow the lines of the proof of [19, Th. 2.3].

Fact 5.8. *Let $f \in \Gamma(X)$ and $x^* \in \operatorname{dom} f^*$. Assume that $C \subset X$ is a convex bounded set such that $f^*(x^*) = \sup\{\langle x, x^*\rangle - f(x) \mid x \in C\}$. Then $\overline{C}^{w^*(2k+2)} \cap \partial f^{*(2k+1)}(x^*) \neq \emptyset$ for every $k \geq 0$. Moreover, if $\partial f^{*(3)}(x^*) = \overline{\partial f^*(x^*)}^{w^*(4)}$, then $\operatorname{gap}(C, \partial f^*(x^*)) = 0$.*

Proof. Of course, taking into account (5), we have (also) that $f^{*(2k+1)}(x^*) = \sup\{\langle x, x^*\rangle - f^{*(2k)}(x) \mid x \in C\}$. There exists a net $(x_i)_{i\in I} \subset C$ such that $\gamma_i := \langle x_i, x^*\rangle - f^{*(2k)}(x_i) \to f^{*(2k+1)}(x^*)$. Because (x_i) is bounded, taking possibly a subnet, we have $x_i \to^{w^*(2k+2)} u \in \overline{C}^{w^*(2k+2)}$. As $f^{*(2k+2)}(x_i) = f^{*(2k)}(x_i) = \langle x_i, x^*\rangle - \gamma_i$, taking the lim inf, we get $f^{*(2k+2)}(u) \leq \langle x^*, u\rangle - f^{*(2k+1)}(x^*)$, whence we get $u \in \partial f^{*(2k+1)}(x^*)$.

Assume now that $\partial f^{*(3)}(x^*) = \overline{\partial f^*(x^*)}^{w^*(4)}$. Then $\overline{C}^{w^*(4)} \cap \overline{\partial f^*(x^*)}^{w^*(4)} \neq \emptyset$, whence $0 \in \overline{C}^{w^*(4)} - \overline{\partial f^*(x^*)}^{w^*(4)} \subset \overline{C - \partial f^*(x^*)}^{w^*(4)}$, and so

$$0 \in X^{*(2)} \cap \overline{C - \partial f^*(x^*)}^{w^*(4)} \subset \overline{C - \partial f^*(x^*)}^{w(2)} = \overline{C - \partial f^*(x^*)}^{\|\cdot\|}.$$

Hence $\operatorname{gap}(C, \partial f^*(x^*)) = 0$.

Fact 5.9. *Let X be a Banach space, $f \in \Gamma(X)$ and $x^* \in \operatorname{dom} f^*$. Assume that $\partial f^{*(3)}(x^*) = \overline{\partial f^*(x^*)}^{w^*(4)}$. Then $d(y, \partial f^*(x^*)) = d(y, X \cap \partial f^*(x^*))$ for every $y \in X$. In particular $X \cap \partial f^*(x^*) \neq \emptyset$ if $\partial f^*(x^*) \neq \emptyset$ (for example if $x^* \in \operatorname{int}(\operatorname{dom} f^*)$).*

Proof. If $\partial f^*(x^*) = \emptyset$ there is nothing to prove. So, let $\partial f^*(x^*)$ be nonempty, fix some $y \in X$ and take $d(y, \partial f^*(x^*)) < \beta < \infty$. Take some $\alpha \in \mathbb{R}$ such that $d(y, \partial f^*(x^*)) < \alpha < \beta$ and $\varepsilon := \beta - \alpha > 0$. There exists $x^{**} \in \partial f^*(x^*)$ such that $\|y - x^{**}\| < \alpha$. By Lemma 4.5 relation (17) holds. Using Fact 5.8 with $C := y + \alpha B_X$, we have $\operatorname{gap}(y + \alpha B_X, \partial f^*(x^*)) = 0$. Then, there exists $x_1 \in y + \alpha B_X$ such that $d(x_1, \partial f^*(x^*)) < \varepsilon/2$. Applying Lemma 4.5 for y and α replaced by x_1 and $\varepsilon/2$ we get the relation (17) for these elements, and then applying Fact 5.8 we obtain $\operatorname{gap}(x_1 + (\varepsilon/2)B_X, \partial f^*(x^*)) = 0$. So, there exists $x_2 \in x_1 + (\varepsilon/2)B_X$ such that $d(x_2, \partial f^*(x^*)) < \varepsilon/2^2$. Continuing in this way, we get a sequence $(x_n)_{n\geq 1}$ such that $x_{n+1} \in x_n + (\varepsilon/2^n)B_X$ and $d(x_n, \partial f^*(x^*)) < \varepsilon/2^n$ for every $n \geq 1$. It is clear that $(x_n)_{n\geq 1} \subset X$ is a Cauchy sequence, and so it converges to some $x \in X$. Moreover, from the continuity of $d(\cdot, \partial f^*(x^*))$, we obtain $d(x, \partial f^*(x^*)) = 0$, and so $x \in X \cap \partial f^*(x^*)$. But

$$\|y - x_{n+1}\| \leq \|y - x_1\| + \|x_1 - x_2\| + \cdots + \|x_n - x_{n+1}\| < \alpha + \frac{\varepsilon}{2} + \cdots + \frac{\varepsilon}{2^n} < \beta.$$

Taking the limit, we obtain $\|y - x\| \leq \beta$, and so $d(y, X \cap \partial f^*(x^*)) \leq \beta$. Hence $d(y, X \cap \partial f^*(x^*)) \leq d(y, \partial f^*(x^*))$. As the reverse inequality is obvious, the conclusion follows.

An immediate consequence of the preceding result is the following reinforcement of Fact 5.8.

Fact 5.10. *Let X be a Banach space, $f \in \Gamma(X)$ and $x^* \in \operatorname{dom} f^*$. Assume that $C \subset X$ is a convex bounded set such that $f^*(x^*) = \sup\{\langle x, x^*\rangle - f(x) \mid x \in C\}$. If $\partial f^{*(3)}(x^*) = \overline{\partial f^*(x^*)}^{w^*(4)}$, then $\operatorname{gap}(C, X \cap \partial f^*(x^*)) = 0$.*

Proof. By Fact 5.8 we have $\operatorname{gap}(C, \partial f^*(x^*)) = 0$ (in particular $\partial f^*(x^*)$ is nonempty). Using Fact 5.9 we obtain

$$\begin{aligned} 0 &= \operatorname{gap}(C, \partial f^*(x^*)) = \inf_{y \in C} d(y, \partial f^*(x^*)) = \inf_{y \in C} d(y, X \cap \partial f^*(x^*)) \\ &= \operatorname{gap}(C, X \cap \partial f^*(x^*)). \end{aligned}$$

Fact 5.11. *Let X be a Banach space, $f \in \Gamma(X)$ and $x^* \in \operatorname{int}(\operatorname{dom} f^*)$. Assume that $\partial f^{*(3)}(x^*) = \overline{\partial f^*(x^*)}^{w^*(4)}$; then $\partial f^*(x^*) = \overline{X \cap \partial f^*(x^*)}^{w^*(2)}$.*

Proof. Let $x^{**} \in \partial f^*(x^*)$. Because $\partial f^*(x^*) \subset \partial_\eta f^*(x^*) = \overline{X \cap \partial_\eta f^*(x^*)}^{w^*(2)}$ for $\eta > 0$, for every $w^*(2)$-neighborhood V of 0 in X^{**} and every $\eta \in (0,1]$, there exists $x_{\eta,V} \in X \cap (x^{**} + V) \cap \partial_\eta f^*(x^*)$. Taking $I := (0,1] \times \mathcal{N}_{w^*(2)}(0)$ with the order $(\eta, V) \geq (\eta', V')$ if $\eta \leq \eta'$ and $V \subset V'$, the net $(x_{\eta,V})_{(\eta,V)\in I}$ $w^*(2)$-converges to x^{**}. Let us fix some V_0 a convex $w^*(2)$-neighborhood of 0 in X^{**} and take $A := \operatorname{conv}\{x_{\eta,V} \mid \eta \in (0,1], V \subset \frac{1}{2}V_0\}$. Then $A \subset \partial_1 f^*(x^*)$, and so A is bounded (because f^* is continuous at x^*); moreover, $A \subset x^{**} + \frac{1}{2}V_0$. Because $f^*(x^*) - \eta \leq \langle x_{\eta,V}, x^*\rangle - f(x_{\eta,V})$, we have also that $\sup\{\langle x, x^*\rangle - f(x) \mid x \in A\} = f^*(x^*)$. From Fact 5.10 we have $\operatorname{gap}(A, X \cap \partial f^*(x^*)) = 0$. Taking $\varepsilon > 0$ such that $\varepsilon U_{X^{**}} \subset \frac{1}{2}V_0$, we find $x \in A$, and $x' \in X \cap \partial f^*(x^*)$ such that $x' - x \in \frac{1}{2}V_0$. It follows that $x' \in x + \frac{1}{2}V_0 \subset x^{**} + \frac{1}{2}V_0 + \frac{1}{2}V_0 = x^{**} + V_0$. Hence $x^{**} \in \overline{X \cap \partial f^*(x^*)}^{w^*(2)}$. □

Corollary 5.12. *Let X be a Banach space, $f \in \Gamma(X)$ and $x^* \in \operatorname{int}(\operatorname{dom} f^*)$. If $f^{*(3)}$ is Gâteaux differentiable at x^*, then f^* is also Gâteaux differentiable at x^* and $\nabla f^{*(3)}(x^*) = \nabla f^*(x^*) \in X$.*

Proof. Suppose $f^{*(3)}$ is Gâteaux differentiable at x^*. Since $\partial f^*(x^*)$ is nonempty and, by (2), $\partial f^*(x^*) = \partial f^{*(3)}(x^*) \cap X^{**}$, we have f^* is Gâteaux differentiable at x^* and $\nabla f^*(x^*) = \nabla f^{*(3)}(x^*)$. By the implication $(iii) \Rightarrow (vi)$ of Theorem 5.7 we have $\nabla f^*(x^*) \in X$. □

By taking $f = \iota_{U_X}$ (that is, $f^* = \|\cdot\|$) and $x^* \in S_{X^*}$ in Corollary 5.12 we obtain the main result of [11] that a non reflexive Banach space has non smooth third dual (see also [13, Th. 3.4.8]).

Corollaries 5.6 and 5.12 yield immediately the next result.

Corollary 5.13. *Let X be a Banach space, $f \in \Gamma(X)$ and $x^* \in \operatorname{int}(\operatorname{dom} f^*)$. If f^* is Fréchet differentiable at x^*, then $f^{*(3)}$ is Gâteaux differentiable at x^* and $\nabla f^{*(3)}(x^*) = \nabla f^*(x^*) \in X$.*

Corollary 5.14. *Let X be a Banach space, $f \in \Gamma(X)$ and $x^* \in \operatorname{int}(\operatorname{dom} f^*)$. Then the following assertions are equivalent:*

(i) $\partial f^*(x^*) \subset X$;
(ii) $X \cap \partial f^*(x^*)$ *is w-compact and* $S^X_{f^*}(0, \cdot)$ *is* $\tau_{\|\cdot\|}$-w *H-usc at* x^*.

Proof. $(i) \Rightarrow (ii)$ Since $\partial f^*(x^*)$ is $w^*(2)$-compact in X^{**}, by (i) and (8), $X \cap \partial f^*(x^*) = \partial f^*(x^*)$ is w-compact. The rest of the conclusion follows from the implication $(vi) \Rightarrow (iv)$ of Theorem 5.7.

(ii) $\Rightarrow$ *(i)* By the implication *(iv)* $\Rightarrow$ *(vi)* of Theorem 5.7, we have $\partial f^*(x^*) = \overline{\partial f^*(x^*) \cap X}^{w^*(2)}$. Since $\partial f^*(x^*) \cap X$ is w-compact, by (8) we have $\overline{\partial f^*(x^*) \cap X}^{w^*(2)} = \partial f^*(x^*) \cap X = \partial f^*(x^*)$. □

Corollary 5.15. *Let X be a Banach space, $f \in \Gamma(X)$ and $x \in \operatorname{int}(\operatorname{dom} f)$. Consider the following assertions:*

(i) $S_f(0,\cdot)$ *is* $\tau_{\|\cdot\|}$-$w(1)$ *H-usc at* x*;*
(ii) $S_{f^{**}}^{X^*}(0,\cdot)$ *is* $\tau_{\|\cdot\|}$-$w(1)$ *H-usc at* x*;*
(iii) $S_{f^{**}}(0,\cdot)$ *is* $\tau_{\|\cdot\|}$-$w(3)$ *H-usc at* x*.*

Then (iii) $\Rightarrow$ *(ii)* $\Leftrightarrow$ *(i).*

Proof. *(i)* $\Leftrightarrow$ *(ii)* By Proposition 3.10 and Corollary 5.3, the statement *(i)* is equivalent to the fact $\partial f^{**}(x) = \overline{\partial f(x)}^{w^*(3)}$, which in turn is equivalent to $\partial f^{**}(x) = \overline{\partial f^{**}(x) \cap X^*}^{w^*(3)}$ by (2). By Lemma 4.3 f^{**} is continuous at x, and so, by Proposition 5.2, the last equality is equivalent to *(ii)*.

(iii) $\Rightarrow$ *(ii)* This is just the implication *(i)* $\Rightarrow$ *(iv)* of Theorem 5.7 for f^*. □

By taking $f = \|\cdot\|$ and $x \in S_X$ in Corollary 5.15 we obtain [19, Th. 4.2].

In the sequel we are interested in the continuity of the multifunction $S_{f^*}^X$ when we consider the norm topology on X. We begin with the following interesting result; its proof is inspired by that of [10, Th. 3.3], given for $f = \iota_{U_X}$ (that is, $f^* = \|\cdot\|$) and $x^* \in S_{X^*}$.

Lemma 5.16. *Let X be a Banach space and $f \in \Gamma(X)$. If f^* is directionally Fréchet differentiable at $x^* \in \operatorname{int}(\operatorname{dom} f^*)$ then $X \cap \partial f^*(x^*) \neq \emptyset$ and (18) holds.*

Proof. Because f^* is continuous at x^*, by [39, Th. 2.4.9], we have

$$f^{*\prime}(x^*, u^*) = \max\{\langle u^*, u^{**}\rangle \mid u^{**} \in \partial f^*(x^*)\} \quad \forall u^* \in X^*.$$

Let us prove that $f^{*\prime}(x^*, \cdot)$ is w^*-lsc (when f^* is directionally Fréchet differentiable at x^*). If this is done, then, by [39, Th. 2.4.14(ii) and Th. 2.4.4(i)], we obtain $X \cap \partial f^*(x^*) = X \cap \partial f^{*\prime}(x^*, \cdot)(0) \neq \emptyset$ and

$$f^{*\prime}(x^*, u^*) = \sup\{\langle u, u^*\rangle \mid u \in X \cap \partial f^*(x^*)\} \quad \forall u^* \in X^*,$$

and so, by [39, Th. 2.4.14(vi)], our conclusion follows (like in Proposition 4.2).

So consider first (for $r > 0$) the function $\psi_r : (0, \infty) \to \overline{\mathbb{R}}$ defined by

$$\psi_r(t) := \sup\left\{ \frac{f^*(x^* + tu^*) - f^*(x^*)}{t} - f^{*\prime}(x^*, u^*) \;\middle|\; u^* \in rU_{X^*} \right\}.$$

Then $\lim_{t\to 0_+} \psi_r(t) = 0$. Let L_γ be the sublevel set $\{u^* \in X^* \mid f^{*\prime}(x^*, u^*) \le \gamma\}$ at height $\gamma \in \mathbb{R}$. Because X is complete, in order to show that L_γ is w^*-closed, by Krein–Smulian Theorem (see [7, Th. V.5.7]), it is sufficient to show that $L_\gamma \cap rU_{X^*}$ is w^*-closed for every $r > 0$. Indeed, take the net $(u_i^*)_{i\in I} \subset L_\gamma \cap rU_{X^*}$ w^*-converging to u^*. Of course, $u^* \in rU_{X^*}$. Fix $t > 0$. Then

$$\frac{f^*(x^* + tu_i^*) - f^*(x^*)}{t} \le f^{*\prime}(x^*, u_i^*) + \psi_r(t) \le \gamma + \psi_r(t).$$

Since f^* is w^*-lsc, we obtain $[f^*(x^*+tu^*)-f^*(x^*)]/t \le \gamma + \psi_r(t)$ for every $t>0$, and so $f^{*\prime}(x^*,u^*) \le \gamma$. Hence $u^* \in L_\gamma \cap rU_{X^*}$. □

The preceding result extends [20, Cor. 2.4], where f is the indicator function of a bounded closed convex set.

The next result corresponds to Theorem 3.6; we do not write those characterizations of (i) which are obtained by just replacing f by f^* in Theorem 3.6. The equivalence of (i) and (ii) below extends [20, Prop. 2.2].

Theorem 5.17. *Let X be a Banach space, $f \in \Gamma(X)$ and $x^* \in \operatorname{int}(\operatorname{dom} f^*)$. The following assertions are equivalent:*

- (i) *f^* is directionally Fréchet differentiable at x^*;*
- (ii) *the multifunction $S^X_{f^*}(\cdot, x^*)$ is τ_0-$\tau_{\|\cdot\|}$ H-usc at 0;*
- (iii) *the multifunction $S^X_{f^*}$ is $\tau_0 \times \tau_{\|\cdot\|}$-$\tau_{\|\cdot\|}$ H-usc at $(0,x^*)$;*
- (iv) *the multifunction $S^X_{f^*}(0,\cdot)$ is $\tau_{\|\cdot\|}$-$\tau_{\|\cdot\|}$ H-usc at x^*;*
- (v) $\lim_{y^* \in \operatorname{Im}\partial f,\ y^* \to x^*} \operatorname{gap}(X \cap \partial f^*(y^*), X \cap \partial f^*(x^*)) = 0$;
- (vi) *for every sequence $(\eta_n) \subset [0,\infty)$ converging to 0 and every sequence (x_n) with $x_n \in X \cap \partial_{\eta_n} f^*(x^*)$ ($\Leftrightarrow x^* \in \partial_{\eta_n} f(x_n)$) for $n \in \mathbb{N}$, we have $d(x_n, X \cap \partial f^*(x^*)) \to 0$;*
- (vii) *for every sequence $(\eta_n) \subset [0,\infty)$ converging to 0, for every sequence $(x_n^*) \subset X^*$ converging to x^* and every sequence (x_n) with $x_n \in X \cap \partial_{\eta_n} f^*(x_n^*)$ for $n \in \mathbb{N}$, we have $d(x_n, X \cap \partial f^*(x^*)) \to 0$;*
- $(viii)$ *for every sequence $(x_n^*) \subset X$ converging to x^* and every sequence (x_n) with $x_n \in X \cap \partial f^*(x_n^*)$ for $n \in \mathbb{N}$, we have $d(x_n, X \cap \partial f^*(x^*)) \to 0$;*
- (ix) *for every $\varepsilon > 0$ there exists $\eta > 0$ such that $X \cap \partial_\eta f^*(x^*) \subset \partial f^*(x^*) + \varepsilon U_{X^{**}}$.*

Proof. $(i) \Leftrightarrow (ii)$ Taking into account Theorem 3.6 (for f^*), we have (i) is equivalent to

$$\forall \varepsilon > 0,\ \exists \eta > 0\ :\ \partial_\eta f^*(x^*) \subset \partial f^*(x^*) + \varepsilon U_{X^{**}}. \tag{20}$$

So we have to show that (20) is equivalent to

$$\forall \varepsilon > 0,\ \exists \eta > 0\ :\ X \cap \partial_\eta f^*(x^*) \subset X \cap \partial f^*(x^*) + \varepsilon U_X. \tag{21}$$

(Of course, if (21) holds then necessarily $X \cap \partial f^*(x^*)$ is nonempty.)

If (21) holds, using Proposition 4.2 and taking the $w^*(2)$-closures, we obtain

$$\begin{aligned} \partial_\eta f^*(x^*) = \overline{X \cap \partial_\eta f^*(x^*)}^{w^*(2)} &\subset \operatorname{cl}_{w^*(2)}\big(\overline{X \cap \partial f^*(x^*)}^{w^*(2)} + \overline{\varepsilon U_X}^{w^*(2)}\big) \\ &\subset \partial f^*(x^*) + \varepsilon U_{X^{**}}, \end{aligned} \tag{22}$$

the last set being even $w^*(2)$-compact. Hence (20) holds.

Assume now that (20) holds. Again by Theorem 3.6 we have f^* is directionally Fréchet differentiable at x^*, and so, by Lemma 5.16, (18) holds. Fix $\varepsilon > 0$ and let $\eta > 0$ correspond to $\varepsilon' := \varepsilon/2$ given by (20). Let $u \in X \cap \partial_\eta f^*(x^*)$. By (20) we have $d(u, \partial f^*(x^*)) \le \varepsilon'$. From Corollary 4.10 we have $d(u, \partial f^*(x^*)) = d(u, X \cap \partial f^*(x^*))$, and so $u \in (X \cap \partial f^*(x^*)) + \varepsilon U_X$.

The equivalence of (ii), (iii) and (iv) follows from Proposition 5.1 taking $\tau = \tau_{\|\cdot\|}$; (vi), (vii) and $(viii)$ are the sequential characterizations of (ii), (iii) and (iv), respectively.

$(iv) \Rightarrow (v)$ Let $\varepsilon > 0$; by hypothesis, there exists $r > 0$ such that $X \cap \partial f^*(y^*) \subset (X \cap \partial f^*(x^*)) + \varepsilon U_X$ for all $y^* \in x^* + rU_{X^*}$. So, taking $y^* \in \operatorname{Im} \partial f \cap (x^* + rU_{X^*})$, we find $y \in X \cap \partial f^*(y^*) = (\partial f)^{-1}(y^*)$. Then there exists $x \in X \cap \partial f^*(x^*)$ and $u \in U_X$ such that $y = x + \varepsilon u$. It follows that $\operatorname{gap}(X \cap \partial f^*(y^*), X \cap \partial f^*(x^*)) \le \|y - x\| \le \varepsilon$. Hence (v) holds.

$(v) \Rightarrow (i)$ Let $\varepsilon > 0$. Then there exists $\eta > 0$ such that $\operatorname{gap}(X \cap \partial f^*(y^*), X \cap \partial f^*(x^*)) \le \varepsilon$ for every $y^* \in \operatorname{Im} \partial f \cap (x^* + \eta U_{X^*})$; we may assume that $x^* + \eta U_{X^*} \subset \operatorname{int}(\operatorname{dom} f^*)$, and so f^* is continuous at any point of $x^* + \eta U_{X^*}$. Let $u^* \in U_{X^*}$ be such that $y^* := x^* + \eta u^* \in \operatorname{Im} \partial f$ and take $y \in X \cap \partial f^*(y^*) \subset \partial f^*(y^*)$; take also $x \in X \cap \partial f^*(x^*) \subset \partial f^*(x^*)$ (such an element exists!). Then

$$\frac{f^*(x^* + \eta u^*) - f^*(x^*)}{\eta} - f^{*\prime}(x^*, u^*) \le \langle u^*, y\rangle - \langle u^*, x\rangle = \langle y - x, u^*\rangle \le \|y - x\|.$$

It follows that

$$\frac{f^*(x^* + \eta u^*) - f^*(x^*)}{\eta} - f^{*\prime}(x^*, u^*) \le \operatorname{gap}(X \cap \partial f^*(x^* + \eta u^*), X \cap \partial f^*(x^*)) \le \varepsilon. \quad (23)$$

Consider now $v^* \in B_{X^*}$; then $z^* := x^* + \eta v^* \in x^* + \eta U_{X^*}$. By Brøndsted–Rockafellar theorem (see f.i. [39, Th. 3.1.2]), there exists a sequence $(z_n^*) \subset \operatorname{Im} \partial f$ converging to z^*; we may assume that $z_n^* \in x^* + \eta B_{X^*}$ for every n. Taking $v_n^* := \eta^{-1}(z_n^* - x^*) \in B_{X^*}$, from (23) we obtain

$$\frac{f^*(x^* + \eta v_n^*) - f^*(x^*)}{\eta} - f^{*\prime}(x^*, v_n^*) \le \varepsilon.$$

Taking into account that f^* and $f^{*\prime}(x^*, \cdot)$ are continuous on $\operatorname{int}(\operatorname{dom} f^*) \supset x^* + \eta U_{X^*}$ and X^*, respectively, passing to the limit for $n \to \infty$ in the above inequality we obtain

$$\frac{f^*(x^* + \eta v^*) - f^*(x^*)}{\eta} - f^{*\prime}(x^*, v^*) \le \varepsilon \quad \forall v^* \in B_{X^*}.$$

This proves that f^* is directionally Fréchet differentiable at x^*.

$(i) \Leftrightarrow (ix)$ We have seen above that (i) is equivalent to (20) which obviously implies (ix). For the converse, observe that the set $\partial f^*(x^*) + \varepsilon U_{X^{**}}$ is $w^*(2)$-closed (even $w^*(2)$-compact), and so, using Proposition 4.2, we obtain (20) holds when (ix) is satisfied. □

The equivalence of assertions (iv) and (ix) for $f = \iota_{U_X}$ and $x^* \in S_{X^*}$ in the preceding theorem is [14, Th. 2.2].

Similar to Corollary 5.15, we have the following result for the norm topology.

Corollary 5.18. *Let X be a Banach space, $f \in \Gamma(X)$ and $x \in \operatorname{int}(\operatorname{dom} f)$. The following assertions are equivalent:*

(i) $S_f(0, \cdot)$ *is* $\tau_{\|\cdot\|}$-$\tau_{\|\cdot\|}$ *H-usc at* x;

(ii) $S_{f^{**}}^{X^*}(0, \cdot)$ *is* $\tau_{\|\cdot\|}$-$\tau_{\|\cdot\|}$ *H-usc at* x;

(iii) $S_{f^{**}}(0, \cdot)$ *is* $\tau_{\|\cdot\|}$-$\tau_{\|\cdot\|}$ *H-usc at* x.

Proof. $(i) \Leftrightarrow (iii)$ follows from Theorem 3.6 and Proposition 4.4.

$(ii) \Leftrightarrow (iii)$ Taking into account Theorem 3.6 (for f^*), the proof follows from the equivalence of (i) and (iv) of Theorem 5.17. □

The result which corresponds to Theorem 3.7 is the following.

Theorem 5.19. *Let X be a Banach space, $f \in \Gamma(X)$ and $x^* \in \operatorname{int}(\operatorname{dom} f^*)$. The following assertions are equivalent:*

(*i*) $X \cap \partial f^*(x^*)$ *is compact and f^* is directionally Fréchet differentiable at x^*;*
(*ii*) $X \cap \partial f^*(x^*)$ *is compact and the multifunction $S_{f^*}^X(\cdot, x^*)$ is τ_0-$\tau_{\|\cdot\|}$ H-usc at 0;*
(*iii*) $X \cap \partial f^*(x^*)$ *is compact and the multifunction $S_{f^*}^X$ is $\tau_0 \times \tau_{\|\cdot\|}$-$\tau_{\|\cdot\|}$ H-usc at $(0, x^*)$;*
(*iv*) $X \cap \partial f^*(x^*)$ *is compact and the multifunction $S_{f^*}^X(0, \cdot)$ is $\tau_{\|\cdot\|}$-$\tau_{\|\cdot\|}$ H-usc at x^*;*
(*v*) $X \cap \partial f^*(x^*)$ *is compact and $\lim_{y^* \in \operatorname{Im} \partial f,\ y^* \to x^*} \operatorname{gap}(X \cap \partial f^*(y^*), X \cap \partial f^*(x^*)) = 0$;*
(*vi*) *for every sequence $(\eta_n) \subset [0, \infty)$ converging to 0 and every sequence (x_n) with $x_n \in X \cap \partial_{\eta_n} f^*(x^*)$ for $n \in \mathbb{N}$, (x_n) has a convergent subsequence;*
(*vii*) *for every sequence $(\eta_n) \subset [0, \infty)$ converging to 0, for every sequence $(x_n^*) \subset X^*$ converging to x^* and every sequence (x_n) with $x_n \in X \cap \partial_{\eta_n} f^*(x_n^*)$ for $n \in \mathbb{N}$, (x_n) has a convergent subsequence;*
(*viii*) *for every sequence $(x_n^*) \subset X$ converging to x^* and every sequence (x_n) with $x_n \in X \cap \partial f^*(x_n^*)$ for $n \in \mathbb{N}$, (x_n) has a convergent subsequence;*
(*ix*) $\lim_{\eta \downarrow 0} \alpha\left(X \cap \partial_\eta f^*(x^*)\right) = 0$.
(*x*) $\lim_{\eta \downarrow 0} \alpha\left(\partial_\eta f^*(x^*)\right) = 0$.

Proof. The equivalence of (*i*)–(*viii*) follows immediately from Theorem 5.17, while the equivalence of (*vi*) and (*ix*) follows from Corollary 2.3.

(*ii*) $\Rightarrow$ (*x*) Because $X \cap \partial f^*(x^*)$ is $\tau_{\|\cdot\|}$-compact in X, it is also $\tau_{\|\cdot\|}$-compact in X^{**}. It follows that $X \cap \partial f^*(x^*)$ is $w^*(2)$-compact in X^{**}. As in the proof of (*i*) $\Leftrightarrow$ (*ii*) of Theorem 5.17, for $\varepsilon > 0$ there exists $\eta > 0$ such that (22) holds. Because $X \cap \partial f^*(x^*)$ is $w^*(2)$-compact in X^{**}, we obtain $\partial_\eta f^*(x^*) \subset (X \cap \partial_\eta f^*(x^*)) + \varepsilon U_{X^{**}}$. Hence

$$\begin{aligned}\alpha\left(\partial_\eta f^*(x^*)\right) &\leq \alpha\left((X \cap \partial_\eta f^*(x^*)) + \varepsilon U_{X^{**}}\right) \leq \alpha(X \cap \partial_\eta f^*(x^*)) + \alpha\left(\varepsilon U_{X^{**}}\right) \\ &= 0 + \varepsilon = \varepsilon.\end{aligned}$$

It follows that $\lim_{\eta \downarrow 0} \alpha\left(\partial_\eta f^*(x^*)\right) = 0$, that is, (*x*) holds.

(*x*) $\Rightarrow$ (*i*) By Theorem 3.7, we have $\partial f^*(x^*)$ is compact and f^* is directionally Fréchet differentiable at x^*. Since X is closed in X^{**}, it follows that $X \cap \partial f^*(x^*)$ is compact. The proof is complete. □

The following result corresponds to Theorem 3.8.

Theorem 5.20. *Let X be a Banach space, $f \in \Gamma(X)$ and $x^* \in \operatorname{int}(\operatorname{dom} f^*)$. The following assertions are equivalent:*

(*i*) *f^* is Fréchet differentiable at x^*;*
(*ii*) $\lim_{\eta \downarrow 0} \operatorname{diam}\left(X \cap \partial_\eta f^*(x^*)\right) = 0$;
(*iii*) $\lim_{\eta \downarrow 0, y^* \to x^*} \operatorname{diam}\left(X \cap \partial_\eta f^*(y^*)\right) = 0$;
(*iv*) $\partial f^*(x^*) = \{\overline{x}^{**}\}$ *and* $\lim_{y^* \in \operatorname{Im} \partial f, y^* \to x^*} d(\overline{x}^{**}, X \cap \partial f^*(y^*)) = 0$;
(*v*) *for every sequence $(\eta_n) \subset [0, \infty)$ converging to 0 and any sequence (x_n) with $x_n \in X \cap \partial_{\eta_n} f^*(x^*)$ for $n \in \mathbb{N}$, (x_n) is $\tau_{\|\cdot\|}$-convergent;*
(*vi*) *for every sequence $(\eta_n) \subset [0, \infty)$ converging to 0, for every sequence $(x_n^*) \subset X^*$ converging to x^* and every sequence (x_n) with $x_n \in X \cap \partial_{\eta_n} f^*(x_n^*)$ for $n \in \mathbb{N}$, (x_n) is $\tau_{\|\cdot\|}$-convergent;*

(*vii*) *for every sequence* $(x_n^*) \subset X$ *converging to* x^* *and every sequence* (x_n) *with* $x_n \in X \cap \partial f^*(x_n^*)$ *for* $n \in \mathbb{N}$, (x_n) *is* $\tau_{\|\cdot\|}$*-convergent.*

Note that from the preceding result we obtain $\nabla f^*(x^*) \in X$ when f^* is Fréchet differentiable at x^*; see also Corollary 5.13 (and [39, Cor. 3.3.4]). When $f = \iota_{U_X}$ and $x^* \in S_{X^*}$, the equivalence of (*i*) and (*v*) in the previous result is stated by Smulian [36, p. 645].

We end this section with a result related to continuity with respect to other convergences.

Theorem 5.21. *Let* $f \in \Gamma(X)$ *and* $x^* \in \operatorname{int}(\operatorname{dom} f^*)$. *Consider the following assertions:*

(*i*) $X \cap \partial_\eta f^*(x^*) \xrightarrow{V_w^+} X \cap \partial f^*(x^*)$ *for* $\eta \downarrow 0$;

(*ii*) $X \cap \partial_\eta f^*(x^*) \xrightarrow{H_w^+} X \cap \partial f^*(x^*)$ *for* $\eta \downarrow 0$;

(*iii*) $\partial f^*(x^*) = \overline{X \cap \partial f^*(x^*)}^{w^*(2)}$;

(*iv*) $d(x, X \cap \partial f^*(x^*)) = d(x, \partial f^*(x^*))$ *for every* $x \in X$;

(*v*) $X \cap \partial_\eta f^*(x^*) \xrightarrow{W} X \cap \partial f^*(x^*)$ *for* $\eta \downarrow 0$.

Then (*i*) $\Rightarrow$ (*ii*) $\Leftrightarrow$ (*iii*) $\Rightarrow$ (*iv*) $\Leftrightarrow$ (*v*).

Proof. (*i*) $\Rightarrow$ (*ii*) is obvious, (*ii*) $\Leftrightarrow$ (*iii*) is established in Proposition 5.2, while (*iii*) $\Rightarrow$ (*iv*) follows directly from Corollary 4.10.

(*iv*) $\Rightarrow$ (*v*) Fix $x \in X$ and take $\liminf d(x, X \cap \partial_\eta f^*(x^*)) = \sup_{\eta > 0} d(x, X \cap \partial_\eta f^*(x^*)) < \gamma < \infty$. Then for every $n \geq 1$ there exists $x_n \in X \cap \partial_{1/n} f^*(x^*)$ such that $\|x - x_n\| < \gamma$. The sequence $(x_n)_{n\geq 1} \subset \partial_1 f^*(x^*)$ being bounded, it has a subnet $(x_i)_{i\in I}$ $w^*(2)$-convergent to $x^{**} \in X^{**}$. Because

$$f^{**}(x_i) + f^*(x^*) = f(x_i) + f^*(x^*) \leq \langle x_i, x^* \rangle + \eta_i = \langle x^*, x_i \rangle + \eta_i,$$

where $(\eta_i)_{i\in I}$ is the subnet of $(1/n)$ corresponding to $(x_i)_{i\in I}$, and taking into account that f^{**} is $w^*(2)$-lsc, we obtain $f^{**}(x^{**}) + f^*(x^*) \leq \langle x^*, x^{**} \rangle$, that is, $x^{**} \in \partial f^*(x^*)$. We get

$$d(x, X \cap \partial f^*(x^*)) = d(x, \partial f^*(x^*)) \leq \|x - x^{**}\| \leq \liminf \|x - x_i\| \leq \gamma.$$

Hence $d(x, X \cap \partial f^*(x^*)) \leq \liminf_{\eta\downarrow 0} d(x, X \cap \partial_\eta f^*(x^*)) = \lim_{\eta\downarrow 0} d(x, X \cap \partial_\eta f^*(x^*))$. The reverse inequality being always true, we get the conclusion.

(*v*) $\Rightarrow$ (*iv*) Fix $x \in X$ and take $\alpha > d(x, \partial f^*(x^*))$. Then there exists $x^{**} \in \partial f^*(x^*)$ such that $\|x - x^{**}\| < \alpha$. Take now $\delta > 0$. By Lemma 4.5, there exists $x_\delta \in X \cap \partial_\delta f^*(x^*)$ such that $\|x_\delta - x\| < \alpha$. It follows that $d(x, X \cap \partial_\delta f^*(x^*)) < \alpha$. Because $X \cap \partial_\eta f^*(x^*) \xrightarrow{W} X \cap \partial f^*(x^*)$ for $\eta \downarrow 0$, we obtain $d(x, X \cap \partial f^*(x^*)) \leq \alpha$. Hence $d(x, X \cap \partial f^*(x^*)) \leq d(x, \partial f^*(x^*))$. The converse inequality being always true, the conclusion follows. □

6. Convergence of slices of the unit ball of X

In this section we shall point out some new results concerning the convergence of slices of the closed unit ball of X for several convergences (upper Vietoris, Wijsman, ...) either as applications of the preceding results or with direct proofs.

Recall that for $x \in S_{X^{*(n)}}$ and $\delta \geq 0$, $S(x, \delta)$ is the nonempty $w^*(n+1)$-compact convex set $\{x^* \in U_{X^{*(n+1)}} \mid \langle x, x^* \rangle \geq 1 - \delta\} \subset X^{*(n+1)}$; of course, $S(x, \delta) \subset S(x, \delta')$ if $0 \leq \delta \leq \delta'$

and $S(x,0) \subset S_{X^{*(n+1)}}$. Because $S(x,\delta) = \partial_\delta \|\cdot\|(x)$ for $x \in S_{X^{*(n)}}$ and $\delta \geq 0$ (see f.i. [39, Th. 2.4.14(iii) and Cor. 2.4.16]), the results established in the previous section can be applied. If $x^* \in S_{X^*}$, we always have that $X \cap S(x^*,\delta)$ is nonempty for $\delta > 0$, but $X \cap S(x^*,0)$ might be empty.

It is worth observing that if for some $x^* \in S_{X^*}$ and some $(\delta_n) \subset (0,\infty)$ converging to 0 there exists a sequence $(x_n) \subset X$ converging weakly to $x \in X$ with $x_n \in S(x^*,\delta_n)$ for every $n \geq 1$, then $x \in S(x^*,0)$. Indeed, $\langle x, x^*\rangle = \lim \langle x_n, x^*\rangle \geq 1$; as U_X is w-closed, we have also $x \in U_X$, and so $\langle x, x^*\rangle \leq 1$, whence $x \in S(x^*,0)$. In the next statements $(\delta_n)_{n\geq 1} \subset (0,1]$ is an arbitrary decreasing sequence with limit 0; we can take $\delta_n = 1/n$ for $n \geq 1$.

Theorem 6.1. *Let $x^* \in S_{X^*}$. Consider the following assertions:*

(*i*) $X \cap S(x^*,\delta_n) \xrightarrow{V_w^+} X \cap S(x^*,0)$;

(*ii*) $X \cap S(x^*,0)$ *is w-totally bounded and* $X \cap S(x^*,\delta_n) \xrightarrow{H_w^+} X \cap S(x^*,0)$;

(*iii*) $X \cap S(x^*,\delta_n) \xrightarrow{H_w^+} X \cap S(x^*,0)$;

(*iv*) $\forall u^* \in S_{X^*}$, $\forall \alpha > \sup\{\langle x, u^*\rangle \mid x \in X \cap S(x^*,0)\}$, $\exists n_0$, $\forall n \geq n_0$, $X \cap S(x^*,\delta_n) \subset \{x \in X \mid \langle x, u^*\rangle < \alpha\}$;

(*v*) $S(x^*,0) = \overline{X \cap S(x^*,0)}^{w^*(2)}$;

(*vi*) $d(x, X \cap S(x^*,0)) = d(x, S(x^*,0))$ *for every* $x \in X$;

(*vii*) $X \cap S(x^*,\delta_n) \xrightarrow{W} X \cap S(x^*,0)$.

Then (*i*) $\Rightarrow$ (*ii*) $\Rightarrow$ (*iii*) $\Leftrightarrow$ (*iv*) $\Leftrightarrow$ (*v*) $\Rightarrow$ (*vi*) $\Leftrightarrow$ (*vii*). *Moreover, each of these conditions implies that* $X \cap S(x^*,0) \neq \emptyset$.

Proof. The fact that $X \cap S(x^*,0) \neq \emptyset$ whenever one of the conditions (*i*)–(*vii*) holds is evident. The implications (*ii*) $\Rightarrow$ (*iii*) $\Rightarrow$ (*iv*) are obvious. Applying Theorem 5.21 to the function $f = \iota_{U_X}$, we have (*iii*) $\Leftrightarrow$ (*v*) $\Rightarrow$ (*vi*) $\Leftrightarrow$ (*vii*). So, we have to show that (*i*) $\Rightarrow$ (*ii*) and (*iv*) $\Rightarrow$ (*v*).

(*i*) $\Rightarrow$ (*ii*) It is sufficient to show that $X \cap S(x^*,0)$ is w-totally bounded. In the contrary case there exists a symmetric convex neighborhood V of 0 for the weak topology such that $X \cap S(x^*,0)$ is not included in any set $\cup_{x\in F}(x+V)$ with $F \subset X \cap S(x^*,0)$ finite. Hence, there exists a sequence $(x_n)_{n\geq 1} \subset X \cap S(x^*,0)$ such that $x_{n+1} \notin \{x_1,\ldots,x_n\} + V$ for every $n \geq 1$. Let $y_n := (1-\delta_n)x_n$ and $E := \{y_n \mid n \geq 1\}$. The set E is w-closed; for if not there exists some $y \in \overline{E}^w \setminus E$, and so $y + \frac{1}{4}V$ contains an infinity of elements of E, say $\{y_{n_k} \mid k \geq 1\}$, where $(n_k) \subset \mathbb{N}$ is an increasing sequence. Since (x_n) is (norm) bounded, we have $\delta_n x_n \to^w 0$, and so there exists $n_0 \geq 1$ such that $\delta_n x_n - \delta_m x_m \in \frac{1}{2}V$ for $n, m \geq n_0$. Taking $k \geq n_0$, we have $y_{n_{k+1}} - y_{n_k} \in \frac{1}{2}V$, whence

$$x_{n_{k+1}} - x_{n_k} = y_{n_{k+1}} - y_{n_k} + \delta_{n_{k+1}}x_{n_{k+1}} - \delta_{n_k}x_{n_k} \in \tfrac{1}{2}V + \tfrac{1}{2}V = V,$$

a contradiction.

(*iv*) $\Rightarrow$ (*v*) First observe that $X \cap S(x^*,0)$ is nonempty in our conditions. Assume that (*v*) does not hold. Then there exists some $x^{**} \in S(x^*,0) \setminus \overline{X \cap S(x^*,0)}^{w^*(2)}$. By a separation theorem there exist $u^* \in S_{X^*}$ and $\alpha \in \mathbb{R}$ such that

$$\langle u^*, x^{**}\rangle > \alpha > \sup\{\langle x, u^*\rangle \mid x \in X \cap S(x^*,0)\}.$$

From our hypothesis, there exists n_0 such that

$$X \cap S(x^*, \delta_{n_0}) \subset \{u \in X \mid \langle u, u^* \rangle < \alpha\} \subset H_{**} := \{u^{**} \in X^{**} \mid \langle u^*, u^{**} \rangle \leq \alpha\}.$$

Taking into account (11) we obtain

$$S(x^*, 0) \subset S(x^*, \delta_{n_0}) = \overline{X \cap S(x^*, \delta_{n_0})}^{w^*(2)} \subset H_{**}.$$

This yields the contradiction $\langle u^*, x^{**} \rangle \leq \alpha$. □

If $X \cap S(x^*, 0)$ is weakly compact then $(i) \Leftrightarrow (iii)$ in Theorem 6.1. Note that the equivalence of (iii) and (v) in Theorem 6.1 is stated in [19, Lem. 2.2] for X complete.

Corollary 6.2. *Let X be complete. The following assertions are equivalent:*

(i) $X \cap S(x^*, \delta_n) \xrightarrow{V_w^+} X \cap S(x^*, 0)$ *for every* $x^* \in S_{X^*}$*;*

(ii) $X \cap S(x^*, \delta_n) \xrightarrow{H_w^+} X \cap S(x^*, 0)$ *for every* $x^* \in S_{X^*}$*;*

(iii) $X \cap S(x^*, \delta_n) \xrightarrow{W} X \cap S(x^*, 0)$ *for every* $x^* \in S_{X^*}$*;*

(iv) X *is reflexive.*

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are immediate consequences of the preceding theorem.

$(iii) \Rightarrow (iv)$ Because $d(0, X \cap S(x^*, \delta_n)) \leq 1$, we obtain $d(0, X \cap S(x^*, 0)) < \infty$, and so $X \cap S(x^*, 0) \neq \emptyset$. Hence every $x^* \in S_{X^*}$ attains its supremum on U_X. By James' theorem we obtain X is reflexive.

$(iv) \Rightarrow (i)$ Taking f to be the norm on X^* and $x^* \in S_{X^*}$, from Proposition 3.12 we obtain $S(x^*, \delta_n) \xrightarrow{V_{w^*}^+} S(x^*, 0)$. Because X is reflexive this means that $X \cap S(x^*, \delta_n) \xrightarrow{V_w^+} X \cap S(x^*, 0)$. □

Applying Theorem 5.17 and the preceding result for $f = i_{U_X}$ and $x^* \in S_X$, we obtain [10, Th. 3.3] that X is reflexive if the norm of X^* is directionally Fréchet differentiable on X^*. When the weak topology is replaced by the strong topology we have the following result.

Proposition 6.3. *Let $x^* \in S_{X^*}$. Consider the following assertions:*

(i) $S(x^*, \delta_n) \xrightarrow{V^+} S(x^*, 0)$*;*

(ii) $S(x^*, 0)$ *is compact and* $S(x^*, \delta_n) \xrightarrow{H^+} S(x^*, 0)$*;*

(iii) $X \cap S(x^*, \delta_n) \xrightarrow{V^+} X \cap S(x^*, 0)$*;*

(iv) $X \cap S(x^*, 0)$ *is compact and* $X \cap S(x^*, \delta_n) \xrightarrow{H^+} X \cap S(x^*, 0)$*;*

(v) *for every sequence* $(x_n) \subset U_X$ *with* $\langle x_n, x^* \rangle \to 1$*,* (x_n) *has a convergent subsequence;*

(vi) $X \cap S(x^*, \delta_n) \xrightarrow{H^+} X \cap S(x^*, 0)$*;*

(vii) *for every sequence* $(x_n) \subset U_X$ *with* $\langle x_n, x^* \rangle \to 1$ *we have* $d(x_n, X \cap S(x^*, 0)) \to 0$*;*

$(viii)$ $S(x^*, \delta_n) \xrightarrow{H^+} S(x^*, 0)$.

Then $(i) \Leftrightarrow (ii)$ *and* $(iii) \Leftrightarrow (iv) \Leftrightarrow (v) \Rightarrow (vi) \Leftrightarrow (vii)$.

Moreover, if X is complete then $(ii) \Leftrightarrow (iv)$, $(vi) \Leftrightarrow (viii)$, *and each of the above assertions implies that* $X \cap S(x^*, 0)$ *is nonempty.*

Proof. $(i) \Leftrightarrow (ii)$ It is sufficient to show that $S(x^*, \delta_n) \xrightarrow{\mathrm{V}^+} S(x^*, 0)$ implies that $S(x^*, 0)$ is compact. Indeed, assume that $S(x^*, \delta_n) \xrightarrow{\mathrm{V}^+} S(x^*, 0)$ but $S(x^*, 0)$ is not compact. Then there exists a sequence $(x_n^{**}) \subset S(x^*, 0)$ which has not convergent subsequences. Consider $y_n^{**} := (1 - \delta_n)x_n^{**}$ for $n \geq 1$. It follows that also (y_n^{**}) has not convergent subsequences, and so the set $F := \{y_n^{**} \mid n \geq 1\}$ is closed and disjoint of $S(x^*, 0)$. But $y_n^{**} \in S(x^*, \delta_n) \cap F$, and so we must have that $S(x^*, 0) \cap F$ is nonempty, a contradiction.

$(iii) \Leftrightarrow (iv)$ The proof follows the same lines as that of $(i) \Leftrightarrow (ii)$.

$(vi) \Rightarrow (vii)$ Take $(x_n) \subset U_X$ with $\langle x_n, x^* \rangle \to 1$. Consider $\delta_n' := \max\{\delta_n, 1 - \langle x_n, x^* \rangle\} \in (0, \infty)$. Then $\delta_n' \to 0$ and $x_n \in X \cap S(x^*, \delta_n')$ for every n. We get that $d(x_n, X \cap S(x^*, 0)) \to 0$ by the sequential characterization of the H^+-convergence.

$(vii) \Rightarrow (vi)$ Take the sequence (x_n) such that $x_n \in X \cap S(x^*, \delta_n)$ $(\subset U_X)$ for every n. Since $1 \geq \langle x_n, x^* \rangle \geq 1 - \delta_n$, we obtain $\langle x_n, x^* \rangle \to 1$. By hypothesis, we get $d(x_n, X \cap S(x^*, 0)) \to 0$. The conclusion follows by the sequential characterization of the H^+-convergence.

$(iv) \Rightarrow (v)$ Take $(x_n) \subset U_X$ with $\langle x_n, x^* \rangle \to 1$. Because $(vi) \Rightarrow (vii)$, we have $d(x_n, X \cap S(x^*, 0)) \to 0$. Hence there exists $(x_n') \subset X \cap S(x^*, 0)$ with $\| x_n - x_n' \| \to 0$. Since $X \cap S(x^*, 0)$ is compact, (x_n') has a convergent subsequence (x_{n_k}'), and so (x_{n_k}) is convergent.

$(v) \Rightarrow (iv)$ Consider first $(x_n) \subset X \cap S(x^*, 0)$. Because $\langle x_n, x^* \rangle = 1 \to 1$, (x_n) has a convergent subsequence. As $X \cap S(x^*, 0)$ is closed, $X \cap S(x^*, 0)$ is compact. Consider now the sequence (x_n) such that $x_n \in X \cap S(x^*, \delta_n)$ $(\subset U_X)$ for every n and take a subsequence (x_{n_k}) of (x_n) such that $\gamma := \limsup d(x_n, X \cap S(x^*, 0)) = \lim d(x_{n_k}, X \cap S(x^*, 0))$. Since $1 \geq \langle x_{n_k}, x^* \rangle \geq 1 - \delta_{n_k}$, we obtain $\langle x_{n_k}, x^* \rangle \to 1$. By (v), possibly taking a subsequence, we may suppose (and we do) that (x_{n_k}) converges to $x \in X \cap S(x^*, 0)$. This implies that $\gamma = \lim d(x_{n_k}, X \cap S(x^*, 0)) = 0$.

$(v) \Rightarrow (vii)$ Take $(x_n) \subset U_X$ with $\langle x_n, x^* \rangle \to 1$. As in the proof of the implication $(v) \Rightarrow (iv)$ we obtain $\gamma := \limsup d(x_n, X \cap S(x^*, 0)) = 0$.

Assume now that X is complete. Then the equivalence of (vi) and $(viii)$ is nothing else but the equivalence of (20) and (21) for $f := \iota_{U_X}$ stated in the proof of Theorem 5.17. The equivalence of (ii) and (iv) follows from the equivalence of (ix) and (x) in Theorem 5.19 and Corollary 2.3. $\square$

Because assertion (iv) of the preceding result is nothing else but assertion (ii) of Theorem 5.19 for the function $f := \iota_{U_X}$ (that is, $f^* = \|\cdot\|$) and $x^* \in S_{X^*}$, when X is complete, to the equivalent conditions (i)–(v) above we can add those obtained from Theorem 5.19; similarly, because assertion (vi) of the preceding result is nothing else but assertion (ii) of Theorem 5.17, to the equivalent conditions (vi)–$(viii)$ above we can add those obtained from Theorem 5.17 (for example that the norm of X^* is directionally Fréchet differentiable at x^*).

The V^+-convergence of $(X \cap S(x^*, \delta_n))$ for every $x^* \in S_{X^*}$ is related to important properties of the space X, as the next result shows.

Corollary 6.4. *Let X be complete. The following assertions are equivalent:*

(i) *X is reflexive and has the Kadec–Klee property;*

(ii) $X \cap S(x^*, \delta_n) \xrightarrow{\mathrm{V}^+} X \cap S(x^*, 0)$ *for every* $x^* \in S_{X^*}$;

(*iii*) $S(x^*, \delta_n) \xrightarrow{V^+} S(x^*, 0)$ *for every* $x^* \in S_{X^*}$.

Recall that a normed vector space X has the Kadec–Klee property if $\|x_n - x\| \to 0$ whenever $x_n \to^w x$ and $\|x_n\| \to \|x\|$ (or equivalently, $\|x_n - x\| \to 0$ whenever $x, x_n \in S_X$ and $x_n \to^w x$).

Proof. The equivalence of (*ii*) and (*iii*) follows immediately from Proposition 6.3 [(*i*) ⇔ (*iii*)].

(*ii*) ⇒ (*i*) It follows from Corollary 6.2 that X is reflexive. Let $x, x_n \in S_X$ with $x_n \to^w x$. Assuming that for some $\varepsilon > 0$ and an infinite set $P \subset \mathbb{N}$ we have $\|x_n - x\| \geq \varepsilon$ for every $n \in P$, from the equivalence of (*iii*) and (*v*) in Proposition 6.3, a subsequence $(x_n)_{n \in Q}$ converges strongly to $x' \in S_X$ (with $Q \subset P$ an infinite set). By the uniqueness of the weak limit, it follows that $x' = x$, contradicting the choice of $(x_n)_{n \in P}$.

(*i*) ⇒ (*ii*) Fix $x^* \in S_{X^*}$. It is sufficient to show that assertion (*v*) in Proposition 6.3 holds. For this take $(x_n) \subset U_X$ with $\langle x_n, x^* \rangle \to 1$. It follows that $\|x_n\| \to 1$. Because X is reflexive there exists an infinite set $P \subset \mathbb{N}$ such that $(x_n)_{n \in P}$ converges weakly to $x \in U_X$. It follows that $\langle x, x^* \rangle = 1$, and so $\|x\| = 1 = \lim_{n \in P} \|x_n\|$. By our hypothesis we obtain $\lim_{n \in P} \|x_n - x\| = 0$, and so our aim is satisfied. □

Of course, we could write the result obtained from Theorem 5.20 for $f = \iota_{U_X}$ and $x^* \in S_{X^*}$. Assertion (*ii*) of Theorem 5.20 is related to an important class of normed vector spaces, namely that of strongly convex spaces.

The normed vector space $(X, \|\cdot\|)$ is strongly convex if for every nonempty convex set $C \subset X$ one has $\lim_{t \downarrow d(0,C)} \operatorname{diam}(C \cap tU_X) = 0$. The next result can be found in [9] (see f.i. [27]).

Proposition 6.5. *Let X be a normed vector space. The following assertions are equivalent:*

(*i*) $(X, \|\cdot\|)$ *is strongly convex;*
(*ii*) $\lim_{\delta \downarrow 0} \operatorname{diam}(X \cap S(x^*, \delta)) = 0$ *for every* $x^* \in S_{X^*}$.

Moreover, if X is complete, the assertions above are equivalent to

(*iii*) *X is reflexive, strictly convex and has the Kadec–Klee property.*

Asking the limit to be uniform with respect to $x^* \in S_{X^*}$ in Proposition 6.5 is a very strong condition, as the next result due to Smulian [36] shows.

Proposition 6.6. *Let X be a Banach space. The following assertions are equivalent:*

(*i*) *X is uniformly convex;*
(*ii*) $\lim_{\delta \downarrow 0} \operatorname{diam}(X \cap S(x^*, \delta)) = 0$ *uniformly with respect to* $x^* \in S_{X^*}$;
(*iii*) $\lim_{\delta \downarrow 0} \operatorname{diam}(S(x^*, \delta)) = 0$ *uniformly with respect to* $x^* \in S_{X^*}$;
(*iv*) $\|\cdot\|$ *is uniformly Fréchet differentiable on* S_{X^*}.

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