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Abstract Convexity and Connectedness

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In a former paper the concept of $n-$ary connectedness was introduced, where $1-$ary connectedness coincides with the usual notion of (abstract) connectedness. In the present paper, sets endowed with a convexity structure are studied, where the polytopes are $n-$ary connected. Interrelations between the classical Helly and Carathéodory numbers are evaluated as a pre-stage of Helly and Carathéodory type intersection theorems. Various other applications such as intersection theorems and fixed point theorems for trees and hyperconvex metric spaces are presented.

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1. Notation

Let S be a nonvoid set and 2^S the power set of S. Then every nonvoid subset $\mathcal{P} \subset 2^S$ is called a *paving* in S and $(S, \mathcal{P})$ is a *paved space*. Especially, $\mathcal{E}(S)$ denotes the paving of all nonvoid finite subsets of S, $\mathcal{E}^n(S)$ (with $\mathcal{E}^0(S) = \{\emptyset\}$) is the paving of all subsets of S with n elements, $\mathcal{P}^C$ is the paving of all complements $S \setminus P,\ P \in \mathcal{P}$, and $\mathcal{P} \cap T := \{P \cap T : P \in \mathcal{P}\}$ is the *trace* of a subset T of S.

A paving $\mathcal{P}$ is called $\cap_f-$*closed* ($\cup_f-$*closed*) iff $A, B \in \mathcal{P}$ implies $A \cap B \in \mathcal{P}$ ($A \cup B \in \mathcal{P}$), and $\cap_a-$*closed* iff $\bigcap_{R\in\mathcal{R}} R \in \mathcal{P}$ for all nonvoid $\mathcal{R} \subset \mathcal{P}$, and $\mathcal{P}$ is called *compact* iff every subpaving $\mathcal{R} \subset \mathcal{P}$ with the *finite intersection property* $\bigcap_{R\in\mathcal{F}} R \neq \emptyset \ \ \forall \mathcal{F} \in \mathcal{E}(\mathcal{R})$ has the *global intersection property* $\bigcap_{R\in\mathcal{R}} R \neq \emptyset$.

For $n \in \mathbb{N}$ a subset $T \subset S$ will be called $n-$*ary connected for* $\mathcal{P}$ [20] iff for all $\{P_0, \ldots, P_n\} \subset \mathcal{P}$ the relations $\bigcap_{j\in J} P_j \in \mathcal{P}$ for all nonvoid proper subsets J of $\{0, \ldots, n\}$, $T \subset \bigcup_{i=0}^n P_i$, and $T \cap P_{-i} \neq \emptyset,\ i \in \{0, \ldots, n\}$, with $P_{-i} := \bigcap_{j=0, j\neq i}^n P_j$, imply $T \cap \bigcap_{i=0}^n P_i \neq \emptyset$. A subset T is called *connected for* $\mathcal{P}$ iff it is $1-$ary connected for $\mathcal{P}$, and T is called *finitary connected for* $\mathcal{P}$ iff it is $n-$ary connected for $\mathcal{P}$ for every $n \in \mathbb{N}$.

If $\mathcal{K}$ is another paving in S, then we say that $\mathcal{K}$ is *($n-$ary/finitary) connected for* $\mathcal{P}$ if every $K \in \mathcal{K}$ has this property. In case $\mathcal{K} = \mathcal{P}$ we say that $\mathcal{P}$ is *($n-$ary/finitary) connected.*

If S is a topological space, then we denote by $\mathcal{F}(S)$, $\mathcal{G}(S)$, $\mathcal{K}(S)$, and $\mathcal{C}(S)$ the pavings of all closed, open, compact, and connected subsets, respectively. Here the empty set is considered to be connected. Of course, $\mathcal{F}(S)$ is compact iff S is compact, and a subset is connected iff it is (1-ary) connected for $\mathcal{F}(S)$ or for $\mathcal{G}(S)$, respectively. Observe that the paving $\mathcal{C}(S)$ is not connected in general, but $\mathcal{C}(S) \cap \mathcal{F}(S)$ and $\mathcal{C}(S) \cap \mathcal{G}(S)$ are always

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connected. Similarly, $\mathcal{K}(S)$ need not be compact, but $\mathcal{K}(S) \cap \mathcal{F}(S)$ is always compact. We write cl$T$, int$T$, and bd$T$ for the closure, interior, and boundary of a subset $T \subset S$. A topological space is called *discrete* iff $\mathcal{G}(S)$ is $\cap_a$−closed.

2. Abstract convexity

A *quasiconvex space* is a pair (X, Ψ) where X is a nonvoid set and $\Psi : \mathcal{E}(X) \cup \{\emptyset\} \to 2^X$ is a function with $\Psi(\emptyset) = \emptyset$ and $A \subset \Psi(A),\ A \in \mathcal{E}(X)$.
The function Ψ is a *quasiconvexity (for* X*)*. The subsets $C \subset X$ with $\Psi(A) \subset C$ for all $A \in \mathcal{E}(C)$ are called *convex*. In the following, $\mathcal{C}_\Psi$ denotes the paving of all convex subsets of X, and we set $\Psi(\mathcal{E}(X)) := \{\Psi(A) : A \in \mathcal{E}(X)\}$ for the paving of *polytopes*. The paving $\mathcal{C}_\Psi$ is an *alignment*, i.e., $\{\emptyset, X\} \subset \mathcal{P},\ \mathcal{P}$ is $\cap_a$−closed, and $\bigcup\{A : A \in \mathcal{A}\} \in \mathcal{P}$ for every totally ordered $\mathcal{A} \subset \mathcal{P}$.

A quasiconvexity Ψ is

- *monotone* iff $B \subset A$ implies $\Psi(B) \subset \Psi(A)$,
- a *convexity*, and (X, Ψ) is a *convex space*, provided that all polytopes are convex, i.e., $A, B \in \mathcal{E}(X),\ A \subset \Psi(B) \Longrightarrow \Psi(A) \subset \Psi(B)$.
- said to possess the *join-hull property* iff
  (1) $\Psi(A \cup \{y\}) \subset \bigcup_{x \in \Psi(A)} \Psi(\{x, y\}), A \in \mathcal{E}(X), y \in X$
  (Here a subset $T \subset X$ is convex iff $\Psi(A) \subset T$ for all $A \in \mathcal{E}^1(T) \cup \mathcal{E}^2(T)$.) and Ψ is *join-hull commutative* [27] provided relation (1) holds with equality.
- said to have the *subdivision property* [27] iff
  (2) $\Psi(A) \subset \bigcup_{x \in A} \Psi((A \setminus \{x\}) \cup \{y\})\ \forall y \in \Psi(A),\ A \in \mathcal{E}(X) \setminus \mathcal{E}^1(X)$.

Monotone quasiconvexities are also called “prehull operators” [27]. Convexities were introduced by Fuchssteiner [10] under the name ”Konvexe“. The classical example is the standard convexity on a vector space (Cf. Example 3.9 below). Other examples are metric convexity (Examples 2.7, 4.12, 6.14), order convexity (Examples 6.6, 6.8), and convexity on trees (Example 6.13). A source of further examples is van de Vel's book [27].

A subset $A \in \mathcal{E}(X)$ is called *dependent* iff there exists an $x_0 \in A$ with $x_0 \in \Psi(A \setminus \{x_0\})$. Otherwise, A is *independent*.

A subset $A \in \mathcal{E}(X)$ is called *C-dependent (Carathéodory-dependent)* provided that

(3) $\Psi(A) \subset \bigcup_{x \in A} \Psi(A \setminus \{x\})$.

A subset $A \in \mathcal{E}(X)$ is called *H-dependent (Helly-dependent)* provided that

(4) $\bigcap_{x \in A} \Psi(A \setminus \{x\}) \neq \emptyset$.

$$c = c(X, \Psi) := \sup\{\mathrm{card}A : A \in \mathcal{E}(X), A \text{ is } C-\text{independent}\}$$

is the *Carathéodory-number*, and

$$h = h(X, \Psi) := \sup\{\mathrm{card}A : A \in \mathcal{E}(X), A \text{ is } H-\text{independent}\}$$

is the *Helly-number* of Ψ or (X, Ψ), respectively.

Lemma 2.1. *In a quasiconvex space* (X, Ψ) *the following holds:*

a) *Every singleton is (C-,H-) independent.*

b) *Every dependent finite subset is H-dependent.*

c) *If Ψ is monotone, then*
   (i) *every nonvoid subset of a finite independent set is independent, and*
   (ii) *every nonvoid subset of a finite H-independent set is H-independent.*

d) *If Ψ is join-hull commutative, then every nonvoid subset of a finite C-independent set is C-independent.*

Proof. b) For $\hat{x} \in A \cap \Psi(A \setminus \{\hat{x}\})$ we have $\hat{x} \in \bigcap_{x\in A} \Psi(A \setminus \{x\})$.
c)(ii) follows from the relation $\bigcap_{x\in B} \Psi(B\setminus\{x\}) \subset \bigcap_{x\in B} \Psi(A\setminus\{x\}) \cap \bigcap_{y\in A\setminus B} \Psi(A\setminus \{y\}) = \bigcap_{x\in A} \Psi(A \setminus \{x\})$, $\emptyset \neq B \subset A \in \mathcal{E}(X)$.
d) Let $A \in \mathcal{E}(X)$ be C-independent. Suppose that $A\setminus\{\hat{x}\}$ is C-dependent for some $\hat{x} \in A$, i.e., $\Psi(A \setminus \{\hat{x}\}) \subset \bigcup_{x\in A\setminus\{\hat{x}\}} \Psi(A \setminus \{\hat{x}, x\})$. Then

$$\begin{aligned}\Psi(A) &= \bigcup_{y\in\Psi(A\setminus\{\hat{x}\})} \Psi(\{\hat{x}, y\}) \subset \bigcup_{x\in A\setminus\{\hat{x}\}} \bigcup_{t\in\Psi(A\setminus\{\hat{x},x\})} \Psi(\{\hat{x}, t\}) \\ &= \bigcup_{x\in A\setminus\{\hat{x}\}} \Psi(A \setminus \{x\}) \subset \bigcup_{x\in A} \Psi(A \setminus \{x\})\end{aligned}$$

leads to a contradiction.
The other assertions are obvious. □

Lemma 2.2 (Cf. [4], [27])**.** *In a convex space (X, Ψ) the following holds:*

a) *Ψ is monotone.*

b) *Every dependent finite subset is C-dependent.*

c) *If Ψ has the join-hull property, then Ψ is join-hull commutative.*

d) *If Ψ has the subdivision property, then relation (2) holds with equality, and the following holds:*
   (i) *Every H-dependent finite subset is C-dependent. In particular, $c(X, \Psi) \leq h(X, \Psi)$.*
   (ii) *For all $A \in \mathcal{E}^n(X)$ with $n \geq h(X, \Psi)$ we have $\Psi(A) \subset \bigcup_{x\in A} \Psi((A \setminus \{x\}) \cup \{y\})$ $\forall y \in X,\ A \in \mathcal{E}(X) \setminus \mathcal{E}^1(X)$.*

Proof. a) and c) are obvious.
b) $A = (A \setminus \{\hat{x}\}) \cup \{\hat{x}\} \subset \Psi(A \setminus \{\hat{x}\})$ implies $\Psi(A) \subset \bigcup_{x\in A} \Psi(A \setminus \{x\})$.
d)(i) Here we have $\Psi(A) = \bigcup_{x\in A} \Psi((A\setminus\{x\})\cup\{y\}) = \bigcup_{x\in A} \Psi(A\setminus\{x\})$ for $y \in \bigcap_{x\in A} \Psi(A\setminus \{x\})$.
(ii) Without loss of generality we may assume $y \notin A$. Then $A \cup \{y\}$ is H-dependent, i.e., there exists an

$$\hat{x} \in \bigcap_{x\in A\cup\{y\}} \Psi((A \cup \{y\}) \setminus \{x\}) = \Psi(A) \cap \bigcap_{x\in A} \Psi((A \cup \{y\}) \setminus \{x\})$$

and with the subdivision property we obtain

$$\Psi(A) = \bigcup_{x\in A} \Psi((A \setminus \{x\}) \cup \{\hat{x}\}) \subset \bigcup_{x\in A} \Psi((A \setminus \{x\}) \cup \{y\}).$$

□

Lemma 2.3. *Let (X, Ψ) be a quasiconvex space, and let $A \in \mathcal{E}^{n+1}(X)$ with $n \in \mathbb{N}$ be a C-dependent set. Suppose that there exists a $\cap_f-$closed paving $\mathcal{Q}$ such that*

(*i*) $\Psi(A)$ *is $n-$ary connected for $\mathcal{Q}$, and*
(*ii*) $\Psi(A \setminus \{x\}) \in \mathcal{Q}\ \forall x \in A$.

Then A is H-dependent.

Proof. For $A = \{x_0, \dots, x_n\}$ and $Q_i = \Psi(A \setminus \{x_i\})$ we have $\bigcap_{j \in J} Q_j \in \mathcal{Q}$ for $\emptyset \neq J \subset \{0, \dots, n\}$ by (ii) and $\cap_f-$closedness of $\mathcal{Q}$, $\Psi(A) \subset \bigcup_{i=0}^{n} Q_i$ since A is C-dependent, and $x_i \in \Psi(A) \cap Q_{-i}$. Together with (i) we arrive at $\Psi(A) \cap \bigcap_{i=0}^{n} Q_i \neq \emptyset$ which yields (4). □

For convexities the Lemmas 2.4 and 4.5 below are well-known ([27], Ch. II, §1, 1.7).

Lemma 2.4. *Let (X, Ψ) be a quasiconvex space with Carathéodory-number $c = c(X, \Psi)$. Then for $n \in \mathbb{N}$ the implication $(a) \Longrightarrow (b)$ holds for the following conditions:*

(*a*) $n \geq c$.
(*b*) *For each $A \in \mathcal{E}(X)$ and $y \in \Psi(A)$ there exists an $m \leq n$ and a $B \in \mathcal{E}^m(A)$ with $y \in \Psi(B)$.*

If Ψ is monotone, then both conditions are equivalent.

Proof. $(a) \Longrightarrow (b)$: Let $A \in \mathcal{E}(X)$ and $y \in \Psi(A)$. Choose $B \in \mathcal{E}(A)$ with minimal cardinality m such that $y \in \Psi(B)$. Suppose that $m > n\ (\geq c)$. Then B is C-dependent, i.e., $y \in \Psi(B) \subset \bigcup_{x \in B} \Psi(B \setminus \{x\})$, a contradiction.
$(b) \Longrightarrow (a)$: Now let Ψ be monotone. Suppose that $c > n$. Then there exists a C-independent set $A \in \mathcal{E}(X)$ with card $A > n$, and there exists a $y \in \Psi(A) \setminus \bigcup_{x \in A} \Psi(A \setminus \{x\})$ in contradiction to (b). □

Example 2.5. Every paved space $(X, \mathcal{K})$ gives rise to a convexity according to

$$\Psi_{\mathcal{K}}(A) := \bigcap \{K : K \in \mathcal{K} \cup \{\emptyset, X\}, K \supset A\}, A \in \mathcal{E}(X) \cup \{\emptyset\}.$$

Here, the following holds:

a) $\mathcal{K} \subset \mathcal{C}_{\Psi_{\mathcal{K}}}$.
b) If $\mathcal{K}$ contains $\mathcal{E}(X)$, then $\Psi_{\mathcal{K}}(A) = A,\ A \in \mathcal{E}(X) \cup \{\emptyset\}$, is the *free convexity* with $c = 1$ and $h =$ card X if X is finite and $h = \infty$ otherwise.
c) $\mathcal{K} \subset \{\emptyset, X\}$ leads to the *coarse convexity* $\Psi_{\mathcal{K}}(A) = X,\ A \in \mathcal{E}(X)$, with $c = h = 1$.
d) Suppose that $\mathcal{K}$ is $\cap_a-$closed, and for every $A \in \mathcal{E}(X)$ there exists a $K \in \mathcal{K}$ with $A \subset K$. Then $\Psi_{\mathcal{K}}(\mathcal{E}(X)) \subset \mathcal{K}$ holds. In this case, $\Psi_{\mathcal{K}}(A)$ is called the $\mathcal{K}-$*hull* of A.

Remark 2.6. For a quasiconvex space (X, Ψ) the following are equivalent:

(a) Ψ is a convexity.
(b) $\Psi = \Psi_{\mathcal{C}_\Psi}$.
(c) There exists a paving $\mathcal{K}$ in X with $\Psi = \Psi_{\mathcal{K}}$.

Example 2.7. A *segment space* is a pair $(X, \langle \cdot, \cdot \rangle)$ where X is a nonvoid set and $\langle \cdot, \cdot \rangle : X \times X \to 2^X$ is a set–valued map with $\langle x, y \rangle \supset \{x, y\}$ for all $x, y \in X$. The sets $\langle x, y \rangle$ are called *segments*, and $\langle \cdot, \cdot \rangle$ is called a *segment function* for X. A subset $T \subset X$ is

$(\langle\cdot,\cdot\rangle-)$*convex* iff $\{x,y\} \subset T$ implies $\langle x,y\rangle \subset T$. Let $\mathcal{C}$ denote the alignment of all convex subsets of X. Then $\Psi_{\mathcal{C}}$ is a convexity with $\mathcal{C}_{\Psi_{\mathcal{C}}} = \mathcal{C}$ and $\Psi_{\mathcal{C}}(\{x,y\}) \supset \langle x,y\rangle,\ x,y \in X$.

Every metric space (X,d) can be endowed with the *geodesic segment function*

$$\langle x,y\rangle_d = \{s \in X : d(x,s) + d(s,y) = d(x,y)\},\ x,y \in X.$$

The alignment $\mathcal{C}_d$ of all $\langle\cdot,\cdot\rangle_d-$convex subsets of X generates the *geodesic convexity* $\Psi_d := \Psi_{\mathcal{C}_d}$.

A metric space is called *Menger–convex* iff $\langle x,y\rangle_d \setminus \{x,y\} \neq \emptyset$ for all $x,y \in X$ with $x \neq y$. Every nonvoid convex subset of a normed linear space is Menger–convex w.r.t. the induced metric. In [11] various examples of Menger–convex metric spaces in hyperbolic geometry can be found. A classical example is the Poincaré disc. By a theorem of Menger [24, 12] in a complete Menger–convex metric space every geodesic segment is isometric to the unit interval, and therefore $\mathcal{C}_d \subset \mathcal{C}(X)$. In general, $\langle x,y\rangle_d$ is a proper subset of $\Psi_d(\{x,y\})$, i.e., a geodesic segment $\langle x,y\rangle_d$ need not be Ψ_d-convex. (Compare Example 9.3 in [1]; Ch. II.)

Many other examples of segment spaces can be found in the books of Coppel [4], van de Vel [27], and Verheul [28], and in the paper [19].

3. Quasiconvex paved spaces

A triplet $(X,\mathcal{Q},\Psi)$ is a *(quasi)convex paved space* provided that $(X,\mathcal{Q})$ is a paved space and Ψ is a (quasi)convexity for X. A (quasi)convex paved space $(X,\mathcal{Q},\Psi)$ will be called a *CP-space* (*QP-space*) provided that $\Psi(\mathcal{E}^{n+1}(X))$ is $n-$ary connected for $\mathcal{Q}$ for every $n \in \mathbb{N}$ with $n <$ card X.

Remark 3.1. Let $(X,\mathcal{Q},\Psi)$ be a QP-space with $\cap_f-$closed $\mathcal{Q}$ and T a nonvoid convex subset of S. Then the subspace $(T, \mathcal{Q} \cap T, \Psi|\mathcal{E}(T) \cup \{\emptyset\})$ is also a QP-space.

Remark 3.2.

a) In a QP-space $(X,\mathcal{Q},\Psi)$ with $\Psi(\mathcal{E}(X)) \subset \mathcal{Q}$ and $\cap_f-$closed $\mathcal{Q}$ every C-dependent finite subset is H-dependent according to Lemma 2.3. In particular, $c(X,\Psi) \geq h(X,\Psi)$ holds.

b) In a CP-space $(X,\mathcal{Q},\Psi)$ with $\Psi(\mathcal{E}(X)) \subset \mathcal{Q}$ with $\cap_f-$closed $\mathcal{Q}$ and with the subdivision property a finite subset is C-dependent iff it is H-dependent. This follows with Lemma 2.2 $d)(i)$.

Remark 3.3. Let $(X,\mathcal{Q},\Psi)$ be a quasiconvex paved space.

a) If T is a nonvoid convex subset of X, and if $\Psi(\mathcal{E}^{n+1}(T))$ is $n-$ary connected for $\mathcal{Q}$, then T is $n-$ary connected for $\mathcal{Q}$.

b) If $(X,\mathcal{Q},\Psi)$ is a QP-space such that every $Q \in \mathcal{Q}$ is convex, then $\mathcal{Q}$ is finitary connected.

Proof. a) Let $\{Q_0,\ldots,Q_n\} \subset \mathcal{Q}$ with $\bigcap_{j\in J} Q_j \in \mathcal{Q}$ for all nonvoid proper subsets J of $\{0,\ldots,n\}$ such that $T \subset \bigcup_{i=0}^n Q_i$, and let $x_i \in T \cap Q_{-i}, i \in \{0,\ldots,n\}$. We have to show that $T \cap \bigcap_{i=0}^n Q_i$ is nonvoid. Without loss of generality we may assume $A := \{x_0,\ldots,x_n\} \in \mathcal{E}^{n+1}(X)$, whence $\Psi(A)$ is $n-$ary connected for $\mathcal{Q}$. Now $\Psi(A) \subset \bigcup_{i=0}^n Q_i$

together with $x_i \in \Psi(A) \cap Q_{-i},\ i \in \{0, \dots, n\}$, implies $T \cap \bigcap_{i=0}^n Q_i \supset \Psi(A) \cap \bigcap_{i=0}^n Q_i \neq \emptyset$.
b) follows from a). $\square$

Let (X, Ψ) be a quasiconvex space. Then for $T \in 2^X \setminus \{\emptyset\}$ the set $\ker T := \{x \in X : \Psi(\{x, t\}) \subset T\ \forall t \in T\}$ is the *kernel* of T, and T is *star-shaped* iff $\ker T$ is nonvoid. Of course, $\ker T \subset T$, and $\ker T = T$ for every nonvoid convex T.

Lemma 3.4 (Compare [20]; Lemma 9)**.** *Let $(X, \mathcal{Q}, \Psi)$ be a quasiconvex paved space such that $\Psi(\mathcal{E}^1(X) \cup \mathcal{E}^2(X))$ is connected for $\mathcal{Q}$. Then the following holds:*

a) Every set $P * Q := \bigcup_{p \in P, q \in Q} \Psi(\{p, q\}),\ P, Q \in 2^X \setminus \{\emptyset\}$, is connected for $\mathcal{Q}$.
b) Every star-shaped (in particular, every convex) subset $T \subset X$ is connected for $\mathcal{Q}$.

Proof. *a*) If $\mathcal{H}$ is a paving in S with $\bigcap_{H \in \mathcal{H}} H \neq \emptyset$ such that every $H \in \mathcal{H}$ is connected for $\mathcal{Q}$, then $\bigcup_{H \in \mathcal{H}} H$ is again connected for $\mathcal{Q}$ [20]. In particular, the sets $\{p\} * Q = \bigcup_{q \in Q} \Psi(\{p, q\}), p \in P$, and $P * Q = \bigcup_{p \in P} \{p\} * Q$ are connected for $\mathcal{Q}$.
b) If T is star-shaped, then $T = \{x\} * T, x \in \ker T$, is connected for $\mathcal{Q}$. $\square$

The following two theorems were inspired by results of Levi [23] and Kołodziejczyk [21]. (Compare Examples 4.9 and 4.11 and Remark 4.10 below.)

Theorem 3.5. *Let $(X, \mathcal{Q}, \Psi)$ be a convex paved space with $\cap_f-$closed $\mathcal{Q}$ such that $2 \leq h = h(X, \Psi) < \infty$. Suppose that Ψ has the subdivision property and the join-hull property, and $\Psi(\mathcal{E}^h(X))$ is $(h-1)-$ary connected for $\mathcal{Q}$. Then for convex sets $Q_1, \dots, Q_h \in \mathcal{Q}$ the following are equivalent:*

(*a*) $\bigcap_{i=1}^h Q_i \neq \emptyset$.
(*b*) $\bigcup_{i=1}^h Q_i$ *is star-shaped, and* $Q_{-i} := \bigcap_{j \in I \setminus \{i\}} Q_j \neq \emptyset,\ i \in I := \{1, \dots, h\}$.

Proof. $(a) \implies (b)$ is obvious.
$(b) \implies (a)$: Choose $x_i \in Q_{-i},\ i \in \{1, \dots, h\}$. Without loss of generality we may assume $A := \{x_1, \dots, x_h\} \in \mathcal{E}^h(X)$. For $T := \bigcup_{i=1}^h Q_i$ let $p \in \ker T$. By Lemma 2.2 *d*)(*ii*) we have $\Psi(A) \subset \bigcup_{x \in A} \Psi((A \setminus \{x\}) \cup \{p\})$. Since Q_i is convex, we have $\Psi(A \setminus \{x_i\}) \subset Q_i$, and therefore, by the join-hull property,

$$\Psi((A \setminus \{x\}) \cup \{p\}) \subset \bigcup_{t \in \Psi(A \setminus \{x\})} \Psi(\{t, p\}) \subset T,\ x \in A$$

and we arrive at $\Psi(A) \subset T$. Now the assertion follows since $\Psi(A)$ is $(h-1)-$ary connected for $\mathcal{Q}$. $\square$

Let (X, Ψ) be a quasiconvex space. We consider an associated quasiconvexity $\Psi^\Delta : \mathcal{E}(X) \cup \{\emptyset\} \to 2^X$ defined as

$$\Psi^\Delta(A) = \begin{cases} \Psi(A) : A \in \mathcal{E}^1(X) \cup \{\emptyset\} \\ \bigcup_{x \in A} \Psi(A \setminus \{x\}) : A \in \mathcal{E}(X) \setminus \mathcal{E}^1(X) \end{cases}$$

Obviously, $\Psi^\Delta(A) \subset \Psi(A)\ \forall A \in \mathcal{E}(X)$ iff Ψ is monotone, and an $A \in \mathcal{E}(X) \setminus \mathcal{E}^1(X)$ is C-dependent iff $\Psi(A) \subset \Psi^\Delta(A)$.

Proposition 3.6. *Let X be a topological space endowed with a quasiconvexity Ψ, and let $A \in \mathcal{E}(X)$ such that*

(*i*) $\Psi(A)$ *is closed, and*
(*ii*) $\mathrm{bd}\ \Psi(A) \subset \Psi^{\Delta}(A)$.

Let T be a closed (open) subset of X such that

(*iii*) $X \setminus T$ *is connected,*
(*iv*) $T \supset \Psi^{\Delta}(A)$, *and*
(*v*) $T \subset C \neq X$ *for some convex set C.*

Then $\Psi(A) \subset T$.

Proof. We set $G_1 := X \setminus (\Psi(A) \cup T)$, $F_1 := X \setminus (\mathrm{int}\Psi(A) \cup T)$, $G_2 := \mathrm{int}\Psi(A) \setminus T$, and $F_2 := \Psi(A) \setminus T$. By (*ii*) and (*iv*) we have $G_2 = F_2$, and by (*v*), $F_1 \supset G_1 \supset X \setminus C \neq \emptyset$, since $A \subset \Psi^{\Delta}(A) \subset T \subset C$ implies $\Psi(A) \subset C$. In case $T \in \mathcal{F}(X)$ the sets G_1 and G_2 are open, and in case $T \in \mathcal{G}(X)$ the sets F_1 and F_2 are closed. Since the set $X \setminus T = G_1 \cup G_2 = F_1 \cup F_2$ is connected and $F_1 \cap F_2 = G_1 \cap G_2 = \emptyset$, we obtain $G_2 = \emptyset$ respectively $F_2 = \emptyset$, and therefore $\Psi(A) \subset T$. □

Example 3.7. Let X be a topological space endowed with a convexity Ψ. Let $A \in \mathcal{E}(X) \setminus \mathcal{E}^1(X)$ such that $\Psi(\mathcal{E}(A)) \subset \mathcal{F}(X)$ and $X \setminus \Psi^{\Delta}(A)$ is nonvoid and connected. Then the following are equivalent:

(a) A is C-dependent.
(b) $\mathrm{bd}\ \Psi(A) \subset \Psi^{\Delta}(A)$ and $\Psi^{\Delta}(A) \subset C$ for some convex set $C \neq X$.

Proof. (a) $\Longrightarrow$ (b): Take $C = \Psi(A)$.
(b) $\Longrightarrow$ (a): Apply Proposition 3.6 with $T = \Psi^{\Delta}(A)$ $(\in \mathcal{F}(X))$. □

Theorem 3.8. *Let X be a topological space endowed with a quasiconvexity Ψ, and let $\mathcal{Q}$ denote the paving of all closed (open) convex subsets of X. Let $\{Q_0, \ldots, Q_n\} \subset \mathcal{Q}$, $n \in \mathbb{N}$, such that*

(*i*) $Q_{-i} \neq \emptyset\ \forall i \in \{0, \ldots, n\}$,
(*ii*) $X \setminus \bigcup_{i=0}^n Q_i$ *is connected, and*
(*iii*) $\bigcup_{i=0}^n Q_i \subset C$ *for some convex set $C \neq X$.*

Suppose that for every H-independent subset $A \in \mathcal{E}^{n+1}(X)$

(*iv*) $\Psi(A)$ *is closed,*
(*v*) $\Psi(A)$ *is $n-$ary connected for $\mathcal{Q}$, and*
(*vi*) $\mathrm{bd}\ \Psi(A) \subset \Psi^{\Delta}(A)$.

Then $\bigcap_{i=0}^n Q_i$ is nonvoid.

Proof. We choose points $x_i \in Q_{-i}$ and set $A = \{x_0, \ldots, x_n\}$. Then $A \setminus \{x_i\} \subset Q_i \in \mathcal{Q} \subset \mathcal{C}_{\Psi}$ implies $\Psi^{\Delta}(A) \subset T := \bigcup_{i=0}^n Q_i$, and we may assume without loss of generality that $A \in \mathcal{E}^{n+1}(X)$ is H-independent. But then we have $\Psi(A) \subset T$ according to Proposition 3.6, and the assertion follows with condition (*v*). □

A vector space X can be endowed with several 'intrinsic' convexities, for example the *standard convexity*

$$\operatorname{conv}(A) := \left\{ \sum_{i=1}^{n} \lambda_i x_i : \lambda_i \geq 0,\ i \in \{1, \ldots, n\},\ \sum_{i=1}^{n} \lambda_i = 1 \right\},$$

the *affine convexity*

$$\operatorname{aff}(A) := \left\{ \sum_{i=1}^{n} \lambda_i x_i : \lambda_i \in \mathbb{R},\ i \in \{1, \ldots, n\},\ \sum_{i=1}^{n} \lambda_i = 1 \right\},$$

the *linear convexity*

$$\operatorname{lin}(A) := \left\{ \sum_{i=1}^{n} \lambda_i x_i : \lambda_i \in \mathbb{R},\ i \in \{1, \ldots, n\} \right\},$$

or the *positive convexity*

$$\operatorname{pos}(A) := \left\{ \sum_{i=1}^{n} \lambda_i x_i : \lambda_i \geq 0,\ i \in \{1, \ldots, n\} \right\},$$

for $A = \{x_1, \ldots, x_n\} \in \mathcal{E}(X)$.

The elements of $\mathcal{C}_{\text{conv}}, \mathcal{C}_{\text{aff}}, \mathcal{C}_{\text{lin}}$, and $\mathcal{C}_{\text{pos}}$ are called *convex subsets, affine subspaces* (or *flats*), *linear subspaces*, and *convex cones* (with apex 0), respectively.

In the following, if not otherwise stated, a vector space is always assumed to be endowed with the standard convexity.

Example 3.9. Let X be a vector space.

a) (Cf. [27]) The convexity conv is join-hull commutative with the subdivision property. The convexities lin and pos are join-hull commutative, but the subdivision property only holds in case $\dim X = 1$. Conversely, the convexity aff has the subdivision property, but it is join-hull commutative only in case $\dim X = 1$.

b) Let X be endowed with a topology that induces the Euclidean topology on every finite dimensional linear subspace. Then the paving $\mathcal{C} = \mathcal{C}_{\text{conv}}$ of all convex subsets of X is finitary connected for $\mathcal{C} \cap \mathcal{F}(X)$ and for $\mathcal{C} \cap \mathcal{G}(X)$. In particular, $(X, \mathcal{C} \cap \mathcal{F}(X), \operatorname{conv})$ and $(X, \mathcal{C} \cap \mathcal{G}(X), \operatorname{conv})$ are CP-spaces.
Compare the proof of [20]; Theorem 8.

c) For an $A \in \mathcal{E}(X)$ the following are equivalent.
(i) A is H-independent in (X, conv).
(ii) A is C-independent in (X, conv).
(iii) A is independent in (X, aff).
Here (i) $\iff$ (ii) follows from b) with the finite topology [14, 27] on X together with a) and Remark 3.2 b). For the proof of (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i) compare [27]; II.1.3 and 1.4.1.

4. Helly pavings

Let $n \in \mathbb{N}$. A paved space $(S, \mathcal{P})$ will be called an H_n-*space* and $\mathcal{P}$ is called an H_n-*paving* [20] iff for all $\{P_0, \ldots, P_m\} \subset \mathcal{P}$ with $m \geq n$ the relations

(5) $\bigcap_{j \in J} P_j \neq \emptyset$ for all subsets $J \in \mathcal{E}^n(\{0, \ldots, m\})$,
(6) $\bigcap_{j \in J} P_j \in \mathcal{P}$ for all nonvoid proper subsets J of $\{0, \ldots, m\}$, and
(7) $\bigcap_{i=0}^{m} P_i = \emptyset$

cannot hold simultaneously.

In a paved space $(S, \mathcal{P})$ the paving $\mathcal{P}^\top := \{T \subset S : T \cap P \in \mathcal{P}\ \forall P \in \mathcal{P}\}$ is the *transporter* of $\mathcal{P}$.

Remark 4.1.

a) ([20]) A paving $\mathcal{P}$ is an H_n-paving iff for all $\{P_0, \ldots, P_n\} \subset \mathcal{P}$ relations (5), (6), and (7) with $m = n$ cannot hold simultaneously. In this case, $\mathcal{P}$ is $l-$ary connected for $\mathcal{P}^\top$ for all $l \geq n$.

b) If $\mathcal{P}$ is a lattice (i.e., a $\cap_f-$closed and $\cup_f-$closed paving), then $\mathcal{P}$ is an H_n-paving iff it is $n-$ary connected.

c) ([20]) A $\cap_f-$closed connected H_2-paving is finitary connected.

Lemma 4.2. *Let $(X, \mathcal{Q}, \Psi)$ be a quasiconvex paved space. If every trace $\mathcal{Q} \cap \Psi(A), A \in \mathcal{E}^{n+1}(X)$, is an H_n-paving, then $\mathcal{Q}$ is also an H_n-paving.*

Proof. Let $\{Q_0, \ldots, Q_n\} \subset \mathcal{Q}$ with $\bigcap_{j \in J} Q_j \in \mathcal{Q} \setminus \{\emptyset\}$ for all nonvoid proper subsets J of $\{0, \ldots, n\}$. Choose $x_i \in Q_{-i},\ i \in \{0, \ldots, n\}$, and set $A = \{x_0, \ldots, x_n\}$. Without loss of generality we may assume $A \in \mathcal{E}^{n+1}(X)$, since otherwise $\bigcap_{i=0}^{n} Q_i \neq \emptyset$ trivially holds. Now relations (5) and (6) are satisfied for $\mathcal{P} = \mathcal{Q} \cap \Psi(A), m = n$, and $P_i = Q_i \cap \Psi(A)$. Since the trace $\mathcal{Q} \cap \Psi(A)$ is an H_n-paving, we arrive at $\Psi(A) \cap \bigcap_{i=0}^{n} Q_i \neq \emptyset$. The assertion follows with Remark 4.1 a). □

Let $(S, \mathcal{P})$ be a paved space and F a family of real-valued functions on S. Then we say that F is *point separating* iff

$$\forall \{s, t\} \in \mathcal{E}^2(S)\ \exists f \in \mathsf{F} :\ f(s) \neq f(t),$$

and F is *minimal point separating* provided that F but no proper subfamily of F is point separating.

Theorem 4.3. *Let $(S, \mathcal{P})$ be a paved space and $\mathsf{F} = \{f_1, \ldots, f_n\}$ a finite family of real-valued functions on S such that*

(8) $\{f \leq \alpha\} := \{s \in S : f(s) \leq \alpha\} \in \mathcal{P}^\top$ *for all $\alpha \in \mathbb{R}$ and all f in the linear hull of F.*

Then the following holds.

a) *If $\mathcal{P}$ is compact and connected, and if F is point separating, then $\mathcal{P}$ is an $H_{n+1}-$paving.*

b) *If F is minimal point separating with the property*
(9) $\forall s, t \in S\ \exists z \in S\ \forall i \in \{1, \ldots, n\} :\ f_i(s) = f_i(t) \Longleftrightarrow f_i(z) = 0,$
then $\mathcal{P}^\top$ is not an H_n-paving.

Proof. *a*) This is a special case of Theorem 5 in [20].
b) Since $\{f_1, \ldots, f_n\}$ is minimal point separating, there exist pairs $\{s_i, t_i\} \in \mathcal{E}^2(S),\ i \in$

$\{1,\ldots,n\}$, with $f_j(s_i) = f_j(t_i)$ for $j \neq i$ and $f_i(s_i) \neq f_i(t_i)$. By (9) there exist points $z_i \in S, i \in \{1,\ldots,n\}$, with

(10) $f_j(z_i) = 0$ for $j \neq i$ and $f_i(z_i) \neq 0$.

Now, for $i \in \{1,\ldots,n\}$ take $P_i := \{f_i = 0\}$ $(= \{f_i \leq 0\} \cap \{-f_i \leq 0\} \in \mathcal{P}^\top$ since $\mathcal{P}^\top$ is $\cap_f-$closed) and $P_0 := \{\sum_{i=1}^n f_i(z_i)^{-1} f_i = 1\} \in \mathcal{P}^\top$. Then relations (9) (with $s = t$) and (10) yield $P_{-i} \neq \emptyset, i \in \{0,\ldots,n\}$. Since $\bigcap_{i=0}^n P_i = \emptyset$, the paving $\mathcal{P}^\top$ is not an H_n-paving. □

Remark 4.4. Let S be an additive group or, more generally, an additive semigroup with the property

$$\forall s,t \in S \; \exists z \in S \quad \text{and} \quad m,n \in \mathbb{N}: \; ms = mt + nz \quad \text{or} \quad mt = ms + nz.$$

Then relation (9) is satisfied for all additive real-valued functions f_i on S.

Lemma 4.5. *Let (X,Ψ) be a quasiconvex space with finite Helly-number $h = h(X,\Psi)$. Then for $n \in \mathbb{N}$ the implication $(a) \Longrightarrow (b)$ holds for the following conditions:*

(a) $\quad n \geq h.$

(b) $\quad \mathcal{C}_\Psi$ *is an H_n-paving.*

If Ψ is a convexity, then (a), (b), and

(c) $\quad$ *There exists a $\cap_f-$closed H_n-paving $\mathcal{Q}$ in X with $\Psi(\mathcal{E}^n(X)) \subset \mathcal{Q}$.*

are equivalent.

Proof. $(a) \Longrightarrow (b)$: Let $\{Q_0,\ldots,Q_n\} \subset \mathcal{C}_\Psi$ and $x_i \in Q_{-i}, i \in \{0,\ldots,n\}$. By Remark 4.1 a) we have to show that $\bigcap_{i=0}^n Q_i \neq \emptyset$. Without loss of generality we may assume $A \in \mathcal{E}^{n+1}(X)$. By (a) the set A is H-dependent, and $\Psi(A \backslash \{x_i\}) \subset Q_i$ implies $\bigcap_{i=0}^n Q_i \neq \emptyset$.
Now let Ψ be a convexity.
$(b) \Longrightarrow (c)$: Take $\mathcal{Q} = \mathcal{C}_\Psi$.
$(c) \Longrightarrow (a)$: Let $A \in \mathcal{E}^{n+1}(X)$. Then $y \in \bigcap_{x \in A\backslash\{y\}} \Psi(A \setminus \{x\}), y \in A$, together with (c) yields $\bigcap_{x\in A} \Psi(A \setminus \{x\}) \neq \emptyset$, i.e., A is H-dependent. Together with Lemmas 2.1 $c)(ii)$ and 2.2 $a)$ we get (a). □

Corollary 4.6. *Let $(X,\mathcal{Q},\Psi)$ be a convex paved space such that $\Psi(\mathcal{E}^{n+1}(X)) \subset \mathcal{Q}^\top$ and every trace $\mathcal{Q}^\top \cap \Psi(A), A \in \mathcal{E}^{n+2}(X)$, is compact and connected. Suppose that there exists a minimal point separating family $\mathsf{F} = \{f_1,\ldots,f_n\}$ of real-valued functions on X such that*

$$\{f \leq \alpha\} \in \mathcal{Q}^\top \cap \mathcal{C}_\Psi \textit{ for all } \alpha \in \mathbb{R} \textit{ and all } f \textit{ in the linear hull of } \mathsf{F},$$

and that relation (9) with $S = X$ is satisfied. Then $h(X,\Psi) = n+1$.

Proof. Let $A \in \mathcal{E}^{n+2}(X)$ and set $\mathcal{P} = \mathcal{Q}^\top \cap \Psi(A)$. Then $\{x \in \Psi(A) : f(x) \leq \alpha\} \in \mathcal{P} \subset \mathcal{P}^\top$ for all $\alpha \in \mathbb{R}$ and all f in the linear hull of F, since $\mathcal{Q}^\top$ and therefore $\mathcal{P}$ is $\cap_f-$closed. By Theorem 4.3 $a)$ (with $S = \Psi(A)$) the trace $\mathcal{P} = \mathcal{Q}^\top \cap \Psi(A)$ is an $H_{n+1}-$paving. Hence, by Lemma 4.2, $\mathcal{Q}^\top$ is an $H_{n+1}-$paving as well, and Lemma 4.5 yields $h \leq n+1$. On the other hand, $\mathcal{C}_\Psi = \mathcal{C}_\Psi^\top$ is not an H_n-paving according to Theorem 4.3 $b)$, and with Lemma 4.5 we obtain $h \geq n+1$. □

Example 4.7. Let $(X, \mathcal{K})$ be a paved space with $\cap_a-$closed $\mathcal{K}$. Then $h(X, \Psi_{\mathcal{K}}) \leq n$ iff $\mathcal{K}$ is an H_n-paving.

Proof. Let $h(X, \Psi_{\mathcal{K}}) \leq n$. Then according to Lemma 4.5, $\mathcal{K}(\subset \mathcal{C}_{\Psi_{\mathcal{K}}})$ is an H_n-paving. Conversely, let $\mathcal{K}$ be an H_n-paving. Then $\mathcal{K} \cup \{\emptyset, X\}$ is an H_n-paving as well. Therefore, without loss of generality we may assume $\{\emptyset, X\} \subset \mathcal{K}$. Now, $h(X, \Psi_{\mathcal{K}}) \leq n$ follows from Example 2.5 d) together with Lemma 4.5 "$(c) \Longrightarrow (a)$". □

The following result is classical [3, 13].

Example 4.8.

a) $h(\mathbb{R}^d, \text{conv}) = c(\mathbb{R}^d, \text{conv}) = d + 1$.
b) For an $A \in \mathcal{E}^{d+1}(\mathbb{R}^d)$ the following are equivalent:
 (i) A is H-independent in $(\mathbb{R}^d, \text{conv})$.
 (ii) $\text{conv}(A)$ is a $d-$dimensional simplex.
 (iii) $\text{bd}\,\text{conv}(A) = \text{conv}^{\Delta}(A)$, and $\text{int}\,\text{conv}\,(A)$ is nonvoid.

Proof. a) By Corollary 4.6, applied to $(X, \mathcal{Q}, \Psi) = (\mathbb{R}^d, \mathcal{C}_{\text{conv}} \cap \mathcal{F}(\mathbb{R}^d), \text{conv})$ and to the projections $f_i(x_1, \ldots, x_i, \ldots, x_d) = x_i$, $i \in \{1, \ldots, d\}$, we obtain $h = d+1$, and by Example 3.9 c) we have $h = c$.
b) This is a simple consequence of Example 3.9 c) (Cf. [8]; Ch. 2.1). □

Example 4.9. Let $C_0, \ldots, C_d$ be closed (open) convex subsets of $\mathbb{R}^d$. Suppose that

(i) $C_{-i} \neq \emptyset$, $i \in \{0, \ldots, d\}$,
(ii) $\bigcup_{i=0}^d C_i \subset C$ for some convex set $C \neq \mathbb{R}^d$, and
(iii) $\mathbb{R}^d \setminus \bigcup_{i=0}^d C_i$ is connected.

Then $\bigcap_{i=0}^d C_i \neq \emptyset$.

Proof. Apply Theorem 3.8 together with Example 3.9 b) and Example 4.8 b). □

Remark 4.10. The "closed version" of Example 4.9 is due to Levi [23]. In [23] the assumption (ii) is missing. But without such an additional assumption the result is false. For example, take $C_0 = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : \sum_{i=1}^d x_i \overset{(>)}{\geq} 1\}$ and $C_i = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_i \overset{(<)}{\leq} 0\}$, $i \in \{1, \ldots, d\}$. On the other hand, if all sets are intersected with the halfspace $H = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 \overset{(>)}{\geq} -1\}$, then (ii) is satisfied (with C_i replaced by $C_i \cap H$), but now the conclusion of Example 4.9 fails because condition (iii) is violated.

The "closed version" of the following example is due to Kołodziejczyk [21].

Example 4.11. Let $C_0, \ldots, C_d$ be closed (open) convex subsets of $\mathbb{R}^d$ such that $C_{-i} \neq \emptyset$, $i \in \{0, \ldots, d\}$. Then $\bigcap_{i=0}^d C_i \neq \emptyset$ iff $\bigcup_{i=0}^d C_i$ is star-shaped.

Proof. Let $(X, \mathcal{Q}, \Psi)$ be $(\mathbb{R}^d, \mathcal{C}_{\text{conv}} \cap \mathcal{F}(\mathbb{R}^d), \text{conv})$ respectively $(\mathbb{R}^d, \mathcal{C}_{\text{conv}} \cap \mathcal{G}(\mathbb{R}^d), \text{conv})$. By Example 3.9 b) the paving of polytopes $\text{conv}(\mathcal{E}(X))$ is finitary connected for $\mathcal{Q}$, and the assertion follows from Theorem 3.5 together with Examples 3.9 a) and 4.8 a). □

Example 4.12. Let X be the unit circle in $\mathbb{R}^2$, and let $d(x,y)$ be the length of the minor arc joining x and y (with $d(x,-x)=\pi$). Let $\Psi_d=\Psi_{\mathcal{C}_d}$ be the corresponding geodesic convexity. Then $\Psi_d(A)$ is the least minor arc containing $A(\in\mathcal{E}(X))$ if A is contained in an arc without diametral points, and $\Psi_d(A)=X$ otherwise. Here, every polytope is a geodesic segment. Moreover [20], $\mathcal{C}_d$ is an H_n-paving $\iff$ $\mathcal{C}_d$ is $n-$ary connected $\iff$ $n\geq 3$. Hence, by Example 4.7, $h(X,\Psi_d)=3$. Since the polytope X is not $2-$ary connected for $\mathcal{Q}:=\mathcal{C}_d\cap\mathcal{F}(X)$, Lemma 2.3 cannot be applied for $n=2$. Indeed, $c(X,\Psi_d)=2<h(X,\Psi_d)$. Finally, it is easy to see that Ψ_d is join-hull commutative, but Ψ_d does not possess the subdivision property, since condition (2) is violated for $A=((1,0),(-1,0))$, say.

More on Helly and Carathéodory type theorems in $\mathbb{R}^d$ can be found in the survey papers [6, 7].

5. Set-valued functions

Let S and X be two nonvoid sets and $\Phi: X\to 2^S$ a set-valued function from X to S. The sets $\Phi(x), x\in X$, are the *values* of Φ, val$\Phi=\{\Phi(x): x\in X\}$ is the *value set* of Φ, and $\Phi(K):=\bigcup_{x\in K}\Phi(x)$ with $\Phi(\emptyset)=\emptyset$ is the *image* of $K\subset X$. A function $f: X\to S$ is a *selector* for Φ iff $f(x)\in\Phi(x), x\in X$. Clearly, a set-valued function possesses a selector iff it is a *correspondence*, i.e., iff all its values are nonvoid. If $\mathcal{P}$ is a paving in S, then Φ will be called $\mathcal{P}-$*valued* provided that val$\Phi\subset\mathcal{P}$. Finally, the *dual* of Φ is the set-valued function $\Phi^*: S\to 2^X$ with $\Phi^*(s)=\{x\in X: s\notin\Phi(x)\}, s\in S$.

We want to study the problem, whether Φ possesses a constant selector or, equivalently, whether $\bigcap_{x\in X}\Phi(x)$ is nonvoid.

Remark 5.1. Let (X,Ψ) be a quasiconvex space, S a nonvoid set and $\Phi: X\to 2^S$ a set-valued function. Then the dual Φ^* is convex-valued (i.e., every $\Phi^*(s), s\in S$, is convex) iff $\Phi(\Psi(A))=\Phi(A)$ for every $A\in\mathcal{E}(X)$.

Proposition 5.2. *Let a quasiconvex space (X,Ψ), a paved space $(S,\mathcal{P})$, a set-valued function $\Phi: X\to 2^S$, and a dependent subset $A\in\mathcal{E}^{n+2}(X)$ with $n\in\mathbb{N}\cup\{0\}$ be given. Suppose that*

(*i*) $\bigcap_{t\in B}\Phi(t)\neq\emptyset\ \forall B\in\mathcal{E}^{n+1}(A)$,
(*ii*) $\Phi(\Psi(B))=\Phi(B)\ \forall\ B\in\mathcal{E}^{n+1}(A)$,

and in case $n\neq 0$,

(*iii*) $\bigcap_{t\in C}\Phi(t)\in\mathcal{P}\ \forall C\in\mathcal{E}^m(A),\ m\in\{1,\ldots,n\}$, *and*
(*iv*) valΦ *is $n-$ary connected for $\mathcal{P}$.*

Then $\bigcap_{x\in A}\Phi(x)\neq\emptyset$.

Proof. Choose $x_0\in\Psi(A\setminus\{x_0\})\cap A$. In case $n=0$ we have $A=\{x_0,x_1\}$ and $x_0\in\Psi(\{x_1\})$, and from (*i*) and (*ii*) we infer $\emptyset\neq\Phi(x_0)=\Phi(x_0)\cap\Phi(x_1)$. In case $n\geq 1$ we have $\Phi(x_0)\subset\bigcup_{t\in A\setminus\{x_0\}}\Phi(t)$ by (*ii*), and (*i*), (*iii*) imply $\Phi(x_0)\cap\bigcap_{t\in C}\Phi(t)\neq\emptyset$ and $\bigcap_{t\in C}\Phi(t)\in\mathcal{P}$ for every proper subset C of $A\setminus\{x_0\}$. Now (*iv*) yields $\bigcap_{x\in A}\Phi(x)=\Phi(x_0)\cap\bigcap_{t\in A\setminus\{x_0\}}\Phi(t)\neq\emptyset$. $\square$

Our next two theorems were inspired by a "Dual Helly Theorem" due to Flåm and Greco (Cf. Example 5.7 below).

Theorem 5.3. *Let a quasiconvex space (X,Ψ), a paved space $(S,\mathcal{P})$ with $\cap_f-$closed $\mathcal{P}$, and a set-valued function $\Phi : X \to \mathcal{P}$ with convex-valued dual Φ^* be given. Suppose that* valΦ *is finitary connected for $\mathcal{P}$. Then every set $A \in \mathcal{E}(X)$ with $\bigcap_{x\in A}\Phi(x) = \emptyset$ contains an independent subset B with $\bigcap_{x\in B}\Phi(x) = \emptyset$.*

Proof. Since every singleton is independent, we may assume $A \in \mathcal{E}^{n+2}(X)$ for some $n \in \{0,1,2,\ldots\}$. If A is dependent, then by Remark 5.1 and Proposition 5.2 there exists an $A' \in \mathcal{E}^{n+1}(A)$ with $\bigcap_{x\in A'}\Phi(x) = \emptyset$. Proceeding inductively we obtain the desired result. $\square$

A modified version of Proposition 5.2 holds for H-independent sets:

Proposition 5.4. *Let $n \in \mathbb{N}\cup\{0\}$, let (X,Ψ) be a quasiconvex space, and let $(S,\mathcal{P})$ be a paved space. Let a set-valued function $\Phi : X \to 2^S$ be given such that*

(i) $\bigcap_{t\in B}\Phi(t) \in \mathcal{P}\setminus\{\emptyset\}\ \forall B \in \mathcal{E}^m(X),\ m \in \{1,\ldots,n+1\}$,
(ii) $\Phi(\Psi(B)) = \Phi(B)\ \forall B \in \mathcal{E}^{n+1}(X)$,

and in case $n \neq 0$,

(iii) valΦ *is $n-$ary and $(n+1)-$ary connected for $\mathcal{P}$.*

Then $\bigcap_{x\in A}\Phi(x) \neq \emptyset$ for every H-dependent set $A \in \mathcal{E}^{n+2}(X)$.

Proof. Let $A \in \mathcal{E}^{n+2}(X)$ be H-dependent, i.e.,

(11) $\exists \hat{x} \in \bigcap_{x\in A}\Psi(A\setminus\{x\})$.

In case $n = 0$ we have $\bigcap_{x\in A}\Phi(x) = \bigcap_{x\in A}\Phi(A\setminus\{x\}) = \bigcap_{x\in A}\Phi(\Psi(A\setminus\{x\})) \supset \Phi(\hat{x}) \neq \emptyset$. Now let $n > 0$. If A is dependent, then the assertion follows from Proposition 5.2. Now let A be independent. Then we have $\hat{x} \notin A$ and therefore $A_x := (A\setminus\{x\})\cup\{\hat{x}\} \in \mathcal{E}^{n+2}(X), x \in A$. By (11) every set A_x is dependent, and from Proposition 5.2 we infer

$$\emptyset \neq \bigcap_{t\in A_x}\Phi(t) = \Phi(\hat{x}) \cap \bigcap_{t\in A\setminus\{x\}}\Phi(t)\ \ \forall x \in A.$$

But (ii) and (11) imply $\Phi(\hat{x}) \subset \bigcup_{t\in A}\Phi(t)$, and together with (i) we arrive at $\Phi(\hat{x}) \cap \bigcap_{t\in A}\Phi(t) \neq \emptyset$, since val$\Phi$ is $(n+1)-$ary connected for $\mathcal{P}$. $\square$

As a consequence, we obtain as in the proof of Theorem 5.3:

Theorem 5.5. *Let a quasiconvex space (X,Ψ), a paved space $(S,\mathcal{P})$ with $\cap_f-$closed $\mathcal{P}$, and a set-valued function $\Phi : X \to \mathcal{P}$ with convex-valued dual Φ^* be given. Suppose that* valΦ *is finitary connected for $\mathcal{P}$. Then for every set $A \in \mathcal{E}(X)$ with $\bigcap_{x\in A}\Phi(x) = \emptyset$ there exists an H-independent set $B \in \mathcal{E}(X)$ with* card $B \leq$ card A *and $\bigcap_{x\in B}\Phi(x) = \emptyset$.*

Together with Remark 3.2 a) we obtain:

Corollary 5.6. *Let a QP-space $(X,\mathcal{Q},\Psi)$ with $\Psi(\mathcal{E}(X)) \subset \mathcal{Q}$ and $\cap_f-$closed $\mathcal{Q}$, a paved space $(S,\mathcal{P})$ with $\cap_f-$closed finitary connected $\mathcal{P}$, and a set-valued function $\Phi : X \to \mathcal{P}$*

with convex-valued dual Φ^* be given. Then for every set $A \in \mathcal{E}(X)$ with $\bigcap_{x\in A} \Phi(x) = \emptyset$ there exists a C-independent set $B \in \mathcal{E}(X)$ with card $B \leq$ card A *and* $\bigcap_{x\in B} \Phi(x) = \emptyset$.

The "closed version" of the following "Dual Helly Theorem" is due to Flåm and Greco [9].

Example 5.7. Let E be a vector space endowed with a topology that induces the Euclidean topology on every finite dimensional linear subspace, and let $X \subset \mathbb{R}^d$ and $S \subset E$ be nonvoid convex sets. Let $\Phi : X \to 2^S$ be a set-valued function with the following properties:

(i) every $\Phi(x), x \in X$, is closed (open) and convex,
(ii) every $\Phi^*(s), s \in S$, is convex, and
(iii) $\bigcap_{x\in B} \Phi(x) \neq \emptyset$ for all $B \in \mathcal{E}^m(X),\ m \leq d+1$.

Then $\bigcap_{x\in A} \Phi(x) \neq \emptyset$ for all $A \in \mathcal{E}(X)$.

Proof. By Example 3.9 b) the assumptions of Theorem 5.5 (or of Corollary 5.6) are satisfied, and the assertion follows together with Example 4.8. □

If $(X, \mathcal{Q})$ and $(S, \mathcal{P})$ are paved spaces, then a set-valued function $\Phi : X \to 2^S$ will be called *(n-ary) $\mathcal{Q} - \mathcal{P}-$connected* iff $\Phi(C)$ is (n-ary) connected for $\mathcal{P}$ for every subset C of X which is (n-ary) connected for $\mathcal{Q}$.

Proposition 5.8. *Let $(X, \mathcal{Q}, \Psi)$ be a quasiconvex paved space, $(S, \mathcal{P})$ a paved space, $\Phi : X \to 2^S$ a correspondence, and $A \in \mathcal{E}^{n+1}(X)$ such that the following holds:*

(i) $\bigcap_{t\in B} \Phi(t) \in \mathcal{P} \setminus \{\emptyset\}\ \forall B \in \mathcal{E}^m(A),\ m \in \{1, \ldots, n\}$,
(ii) $\Phi(\Psi(A)) = \Phi(A)$,
(iii) $\Psi(A)$ *is $n-$ary connected for $\mathcal{Q}$, and*
(iv) Φ *is $n-$ary $\mathcal{Q} - \mathcal{P}-$connected.*

Then $\bigcap_{x\in A} \Phi(x) \neq \emptyset$.

Proof. By (ii), (iii), and (iv), $T := \Phi(\Psi(A)) = \Phi(A)$ is $n-$ary connected for $\mathcal{P}$. Together with (i) the assertion follows. □

Theorem 5.9. *Let $(X, \mathcal{Q}, \Psi)$ be a QP-space, $(S, \mathcal{P})$ a paved space with $\cap_f-$closed $\mathcal{P}$, and $\Phi : X \to \mathcal{P}$ a correspondence such that Φ is $n-$ary $\mathcal{Q} - \mathcal{P}-$connected for every $n \in \mathbb{N}$,* valΦ *is compact, and Φ^* is convex-valued. Then Φ has a constant selector.*

Proof. Suppose, to the contrary, that $\bigcap_{x\in X} \Phi(x)$ is empty. Since valΦ is compact, there exists an $A \in \mathcal{E}^{n+1}(X), n \in \mathbb{N}$, with $\bigcap_{x\in A} \Phi(x) = \emptyset$. Without loss of generality we may assume that $\bigcap_{x\in B} \Phi(x) \neq \emptyset$ for all $B \in \mathcal{E}^n(A)$. Now $\Psi(A)$ is $n-$ary connected for $\mathcal{Q}$, since $(X, \mathcal{Q}, \Psi)$ is a QP-space, and Proposition 5.8 together with Remark 5.1 leads to a contradiction. □

For a set-valued function $\Phi : X \to 2^S$ we set

$$\Phi_E(x) := \bigcap_{t\in E\cup\{x\}} \Phi(t), x \in X, E \subset X.$$

Proposition 5.10. *Let a paved space $(X, \mathcal{Q})$, a nonvoid set S, a set-valued function $\Phi : X \to 2^S$, and a set $A \in \mathcal{E}^{n+1}(X), n \in \mathbb{N}$, be given such that the following holds:*

(*i*) *For every $B \in \mathcal{E}^2(A)$ there exists a subset D of X with $B \subset D$ and $\Phi(B) = \Phi(D)$ such that D is connected for $\mathcal{Q}$.*

(*ii*) *Φ_E is a $\mathcal{Q} - \mathrm{val}\Phi-$connected correspondence for every $E \in \mathcal{E}^{n-1}(A)$.*

Then $\bigcap_{x\in A} \Phi(x) \neq \emptyset$.

Proof. Take $B = \{x_1, x_2\} \in \mathcal{E}^2(A)$, choose D according to (i), and set $E = A \setminus B$. Then (i) implies $C := \Phi_E(D) = \Phi_E(B) \subset \Phi(x_1) \cup \Phi(x_2)$. Since D is connected for $\mathcal{Q}$, it follows from (ii) that C is connected for valΦ. Therefore, from $C \cap \Phi(x_i) = \Phi_E(x_i) \neq \emptyset$, $i \in \{1, 2\}$, we infer $\bigcap_{x\in A} \Phi(x) = C \cap \Phi(x_1) \cap \Phi(x_2) \neq \emptyset$. □

Lemma 5.11. *Let $(X, \mathcal{Q})$ be a paved space such that $\mathcal{Q}$ is $\cap_a-$closed and contains X, and let $\Phi : X \to 2^S$ be a correspondence with* val$\Phi^* \subset \mathcal{Q}$. *Then for $E \in \mathcal{E}(X) \cup \{\emptyset\}$ with $\Phi_E(x) \neq \emptyset \; \forall x \in X$ the implications $(a) \Longrightarrow (b) \Longrightarrow (c)$ hold for the following conditions:*

(*a*) *There exists a connected paving $\mathcal{P}$ in S with* val$\Phi \subset \mathcal{P}$ *which is $\cap_f-$closed in case $E \neq \emptyset$.*

(*b*) valΦ_E *is connected for* valΦ.

(*c*) *Φ_E is $\mathcal{Q} -$* val$\Phi-$*connected.*

Proof. $(a) \Longrightarrow (b)$ is obvious, and $(b) \Longrightarrow (c)$ follows from [18]; Lemma 3 and 4. (Compare also Remark 6.23 below.) □

Example 5.12 (Cf. [16]; Corollary 1)**.** Let $(X, \mathcal{Q})$ and $(S, \mathcal{P})$ be paved spaces such that $\mathcal{Q}$ is $\cap_a-$closed and connected with $X \in \mathcal{Q}$, and $\mathcal{P}$ is $\cap_f-$closed, compact, and connected. Then every correspondence $\Phi : X \to 2^S$ with val$\Phi \subset \mathcal{P}$ and val$\Phi^* \subset \mathcal{Q}$ has a constant selector.

Proof. Suppose, to the contrary, that $\bigcap_{x\in X} \Phi(x) = \emptyset$. Since val$\Phi \subset \mathcal{P}$ and $\mathcal{P}$ is compact, there exists an $A \in \mathcal{E}(X)$ with minimal cardinality such that $\bigcap_{x\in A} \Phi(x) = \emptyset$. Now Proposition 5.10 with $D = \Psi_{\mathcal{Q}}(B) (\in \mathcal{Q})$ together with Lemma 5.11 leads to a contradiction. □

6. Quasiconvexities with $\mathbf{c \leq 2}$ or $\mathbf{h \leq 2}$

We shall now study in some detail quasiconvex spaces (X, Ψ) with "small" Carathéodory-number or Helly-number, i.e., with $c(X, \Psi) \leq 2$ or $h(X, \Psi) \leq 2$, respectively. Convexities Ψ with $h(X, \Psi) \leq 2$ are often called *binary convexities*.

Example 6.1. In a quasiconvex paved space (X, Ψ) the following holds:

a) $c(X, \Psi) = 1 \Longrightarrow \Psi(A) \subset \bigcup_{x\in A} \Psi(\{x\}) \; \forall A \in \mathcal{E}(X) \Longrightarrow \mathcal{C}_\Psi$ is a discrete topology $\Longleftrightarrow \mathcal{C}_\Psi$ is $\cup_f-$closed. Conversely, if Ψ is a convexity and $\mathcal{C}_\Psi$ is $\cup_f-$closed, then $c(X, \Psi) = 1$.

b) If $\mathcal{C}_\Psi$ is $\cup_f-$closed, then it is finitary connected iff it is connected.

c) Ψ is *additive*, i.e., $\Psi(A) = \bigcup_{x\in A} \Psi(\{x\}), A \in \mathcal{E}(X)$, iff Ψ is monotone with $c(X, \Psi) = 1$. In this case Ψ has the join-hull property and the subdivision property.

d) Let Ψ be additive. Then (α) Ψ is a convexity. $\Longleftrightarrow$ (β) Ψ is join-hull commutative. $\Longleftrightarrow$ (γ) $\Psi(\mathcal{E}^1(X)) \subset \mathcal{C}_\Psi$.

Proof. a) The first implication follows by induction. Let $V = \bigcup_{i\in I} C_i$ with $C_i \in \mathcal{C}_\Psi,\ i \in I$. Then for $A \in \mathcal{E}(V)$ there exist i_x with $x \in C_{i_x}, x \in A$. Now $\Psi(A) \subset \bigcup_{x\in A} \Psi(\{x\})$ or $A \subset \bigcup_{x\in A} C_{i_x} \in \mathcal{C}_\Psi$ implies $\Psi(A) \subset V$, i.e., $V \in \mathcal{C}_\Psi$. But $\mathcal{C}_\Psi$ is an alignment and therefore a discrete topology. If Ψ is a convexity, then for $A \in \mathcal{E}(X) \setminus \mathcal{E}^1(X)$ the relation $A \subset \bigcup_{x\in A} \Psi(A \setminus \{x\}) \in \mathcal{C}_\Psi$ implies relation (3).
b) This follows from the fact that a lattice of sets is finitary connected iff it is connected [20].
c) The first part follows from a), and the second part is obvious.
d) $(\alpha) \Longrightarrow (\beta)$ follows from c) together with Lemma 2.2 *c*).
$(\beta) \Longrightarrow (\gamma)$: $\Psi(\{y\}) = \bigcup_{x\in\Psi(\{y\})} \Psi(\{x\}) \cup \Psi(\{y\}) \supset \Psi(\{x\}), x \in \Psi(\{y\})$, yields $\Psi(A) = \bigcup_{x\in A} \Psi(\{x\}) \subset \Psi(\{y\}),\ A \in \mathcal{E}(\Psi(\{y\}))$.
$(\gamma) \Longrightarrow (\alpha)$: For $A, B \in \mathcal{E}(X)$ with $B \subset \Psi(A)$ we have $\Psi(\{y\}) \subset \Psi(A)\ \forall y \in B$, and therefore, $\Psi(B) \subset \Psi(A)$. □

Example 6.2. In a quasiconvex paved space (X, Ψ) the following holds:

a) If $h(X, \Psi) = 1$, then $\mathcal{C}_\Psi$ is an H_1-paving. If Ψ is monotone and $\Psi(\mathcal{E}^1(X)) \subset \mathcal{C}_\Psi$, then the converse is also true.
b) For an additive convexity Ψ we have (α) $\mathcal{C}_\Psi$ is connected $\Longleftrightarrow$ (β) $\Psi(\mathcal{E}^2(X))$ is connected for $\Psi(\mathcal{E}^1(X)) \Longleftrightarrow (\gamma)$ $h(X, \Psi) = 1$.

Proof. a) The first part follows from Lemma 4.5. Let Ψ be monotone with $\Psi(\mathcal{E}^1(X)) \subset \mathcal{C}_\Psi$. If $\mathcal{C}_\Psi$ is an H_1-paving, then $\Psi(\{x\}) \cap \Psi(\{y\}) \neq \emptyset$ for $\{x, y\} \in \mathcal{E}^2(X)$, and together with Lemma 2.1 *c*)(*ii*) we obtain $h(X, \Psi) = 1$.
b) $(\alpha) \Longrightarrow (\beta)$ is obvious.
$(\beta) \Longrightarrow (\gamma)$: $\Psi(\{x, y\}) = \Psi(\{x\}) \cup \Psi(\{y\}),\ \{x, y\} \in \mathcal{E}^2(X)$, implies $\Psi(\{x\}) \cap \Psi(\{y\}) \neq \emptyset$, and therefore $h(X, \Psi) = 1$ as above.
$(\gamma) \Longrightarrow (\alpha)$: Let $C_0, C_1, C_2 \in \mathcal{C}_\Psi$ with $C_0 \subset C_1 \cup C_2$. Then for $x_i \in C_0 \cap C_i, i \in \{1, 2\}$, we have $C_0 \cap C_1 \cap C_2 \supset \Psi(\{x_1\}) \cap \Psi(\{x_2\}) \neq \emptyset$. □

Example 6.3. Let X be a topological space endowed with the convexity $\mathrm{cl} : \mathcal{E}(X) \cup \{\emptyset\} \to 2^X$. Here the following holds:

a) The convexity cl coincides with the hull convexity $\Psi_{\mathcal{F}(X)}$, and $\mathcal{F}(X) \subset \mathcal{C}_{\mathrm{cl}}$.
b) An $A \in \mathcal{E}(X)$ is dependent iff it contains an accumulation point.
c) The convexity cl is additive. In particular, by Example 6.1, cl is join-hull commutative with the subdivision property and with Carathéodory number $c(X, \mathrm{cl}) = 1$, and $\mathcal{C}_{\mathrm{cl}}$ is a discrete topology.
d) X is discrete iff $\mathcal{F}(X) = \mathcal{C}_{\mathrm{cl}}$.
(If X is discrete, then $C = \bigcup_{x\in C} \mathrm{cl}(\{x\}) \in \mathcal{F}(X)$ for every $C \in \mathcal{C}_{\mathrm{cl}}$. Together with a) we obtain $\mathcal{F}(X) = \mathcal{C}_{\mathrm{cl}}$. Conversely, by c), $\mathcal{F}(X) = \mathcal{C}_{\mathrm{cl}}$ implies that $\mathcal{F}(X)$ and $\mathcal{G}(X) = \mathcal{F}(X)^C$ are discrete topologies.)
Similarly, X is discrete iff $\mathcal{G}(X) = \mathcal{C}_{\mathrm{op}}$, where $\mathrm{op} = \Psi_{\mathcal{G}(X)}$ is the "open hull convexity".
e) The following properties are equivalent:
(i) X is a T_1-space.
(ii) The closure operation coincides with the free convexity Υ.
(iii) Every $A \in \mathcal{E}(X)$ is H-independent.
(iv) Every $A \in \mathcal{E}^2(X)$ is H-independent.

f) For $n \in \mathbb{N}$ the following properties are equivalent:
(i) $\mathcal{F}(X)$ is $n-$ary connected.
(ii) $\mathcal{F}(X)$ is $l-$ary connected for every $l \geq n$.
(iii) $\mathcal{F}(X)$ is an H_n-paving.
(iv) $h(X, \mathrm{cl}) \leq n$.
(Here, (iii) $\iff$ (iv) follows from a) together with Example 4.7, and (i) $\iff$ (ii) $\iff$ (iii) follows from Remark 4.1.)

g) The following properties are equivalent:
(i) $\mathcal{F}(X)$ is connected (i.e., X is *ultraconnected* [26]).
(ii) $\mathcal{F}(X)$ is finitary connected.
(iii) $\mathcal{F}(X)$ is an H_1-paving.
(iv) $\mathcal{F}(X) \setminus \{\emptyset\}$ is $\cap_f-$closed.
(v) $(S, \mathcal{F}(X), \mathrm{cl})$ is a CP-space.
(vi) $h(X, \mathrm{cl}) = 1$.
(Here, (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (vi) follows from f), (iii) $\iff$ (iv) and (ii) $\implies$ (v) are obvious, and (v) $\implies$ (vi) follows from c) and Remark 3.2 b).)

Example 6.4. Let X be a nonvoid set, and let $\hat{x} \in X$ be fixed. Then $\mathcal{G}(X) := \{G \subset X : \hat{x} \notin G\} \cup \{X\}$ is a topology on X, *the excluded point topology* [26]. This topology is compact, discrete, and ultraconnected. By Example 6.3 $(X, \mathcal{F}(X), \mathrm{cl})$ is a CP-space, and $h(X, \mathrm{cl}) = c(X, \mathrm{cl}) = 1$.

Example 6.5. For a quasiconvex space (X, Ψ) with monotone Ψ we have (a) $\iff$ (b) $\implies$ (c) for the following conditions:

(a) $c(X, \Psi) \leq 2$.
(b) $\Psi(A) = \bigcup_{x,y \in A} \Psi(\{x, y\})$ for all $A \in \mathcal{E}(X)$.
(c) Ψ has the join-hull property, and every $A \in \mathcal{E}^3(X)$ is C-dependent.

If Ψ is a convexity, then the three conditions are equivalent.

Proof. (a) $\iff$ (b) follows from Lemma 2.4.
(b) $\implies$ (c): For $\{x, y, z\} \in \mathcal{E}^3(A)$, $A \in \mathcal{E}(X)$, we have $\Psi(\{x, y\}) \subset \Psi(A \setminus \{z\}) \subset \Psi^\Delta(A)$, and (b) implies $\Psi(A) \subset \Psi^\Delta(A)$, i.e., every $A \in \mathcal{E}(X)$ with card $A \geq 3$ is C-dependent. For $A \in \mathcal{E}(X)$ and $y \in X$ relation (b) implies

$$\Psi(A \cup \{y\}) = \Psi(A) \cup \bigcup_{x \in A} \Psi(\{x, y\}) \subset \bigcup_{x \in \Psi(A)} \Psi(\{x, y\}).$$

The last assertion follows with Lemmas 2.1 *d*) and 2.2 *c*). □

Example 6.6. Let $\preceq \subset X \times X$ be a relation. Then

$$\Psi(A) = \{x \in X : \exists x_1, x_2 \in A \text{ with } x_1 \preceq x \text{ and } x \preceq x_2\}, A \in \mathcal{E}(X) \cup \{\emptyset\}$$

defines a quasiconvexity iff $\preceq$ is reflexive. In this case, Ψ satisfies condition (b) in Example 6.5, and therefore Ψ is monotone with $c(X, \Psi) \leq 2$. If $\preceq$ is a preorder (i.e., reflexive and transitive), then Ψ is a convexity, the *order convexity* of $(X, \preceq)$.

Example 6.7. For a quasiconvex space (X, Ψ) we have (a) $\implies$ (b) $\iff$ (c) $\implies$ (d) for the following conditions:

(a) $h(X, \Psi) \leq 2$.
(b) Ψ is *modular*, i.e., $\Psi(\{x,y\}) \cap \Psi(\{y,z\}) \cap \Psi(\{z,x\}) \neq \emptyset$ for all $\{x,y,z\} \subset X$.
(c) Every $A \in \mathcal{E}^3(X)$ is H-dependent.
(d) $\mathcal{C}_\Psi$ is an H_2-paving.

If Ψ is monotone, then (a) and (b) are equivalent, and if all segments $\Psi(A), A \in \mathcal{E}^2(X)$ are convex, then (c) and (d) are equivalent.
If Ψ is a convexity with property (a) such that every pair of points can be screened with convex sets (i.e., $\forall \{x,y\} \in \mathcal{E}^2(X)\ \exists C, D \in \mathcal{C}_\Psi : C \cup D = X, x \in C \setminus D$, and $y \in D \setminus C$), then Ψ is join-hull commutative.

Proof. (a) $\Longrightarrow$ (b) $\Longleftrightarrow$ (c) is obvious.
(b) $\Longrightarrow$ (d): Let $\{C_0, C_1, C_2\} \subset \mathcal{C}_\Psi$ and let $x_i \in C_{-i}, i \in \{0,1,2\}$. Then $C_0 \cap C_1 \cap C_2 \supset \Psi(\{x_1,x_2\}) \cap \Psi(\{x_0,x_2\}) \cap \Psi(\{x_0,x_1\}) \neq \emptyset$. Hence (d) follows with Remark 4.1 a).
If Ψ is monotone, then (c) $\Longrightarrow$ (a) follows with Lemma 2.1 $c)(ii)$.
Let $A := \{x_0, x_1, x_2\} \in \mathcal{E}^3(X)$. Suppose that each $C_i = \Psi(\{x_j, x_k\}),\ \{i,j,k\} = \{0,1,2\}$, is convex. Since $x_i \in C_{-i},\ i \in \{0,1,2\}$, condition (d) implies $C_0 \cap C_1 \cap C_2 \neq \emptyset$, i.e., A is H-dependent.
The last assertion is proved in [27]; p. 172 f. □

Example 6.8. Let $(X, \langle\cdot,\cdot\rangle)$ be a *modular segment space*, i.e., a segment space with $\langle x,y\rangle \cap \langle y,z\rangle \cap \langle z,x\rangle \neq \emptyset$ for all $\{x,y,z\} \in \mathcal{E}^3(X)$, and let $\mathcal{C}$ be the paving of all $\langle\cdot,\cdot\rangle-$convex subsets of X. Then, by Examples 2.7 and 6.7, we have $h(X, \Psi_\mathcal{C}) \leq 2$. The segment function $\langle\cdot,\cdot\rangle_d$ of a metric space (X,d) is modular iff every triple of pairwise intersecting closed balls has a common point ([27]; p. 134, [28]; p. 32). Another example is a lattice $(X, \wedge, \vee)$ [27] endowed with the segment function

$$\langle x,y\rangle = \{s \in X : x \wedge y \leq s \leq x \vee y\}, x, y \in X.$$

Here we have $(x \wedge y) \vee (x \wedge z) \vee (y \wedge z) \in \langle x,y\rangle \cap \langle y,z\rangle \cap \langle z,x\rangle$.

Theorem 6.9. *Let $\Phi : X \to 2^S$ be a correspondence. Then the following conditions are equivalent:*

(a) *Φ has a constant selector.*
(b) *There exist topologies on X and S and a quasiconvexity Ψ on X such that*
 (i) *the values of Φ are connected, closed, and compact,*
 (ii) *the values of Φ^* are convex,*
 (iii) *Φ is $\mathcal{F}(X) - \mathcal{F}(S)-$connected,*
 (iv) *every segment $\Psi(A) \in \Psi(\mathcal{E}^2(X))$ is connected, and*
 (v) *$\mathcal{K}(S) \cap \mathcal{F}(S)$ is an H_2-paving.*
(c) valΦ *is connected, and there exists a paving $\mathcal{Q}$ in X and a quasiconvexity Ψ for X such that $\Psi(\mathcal{E}^2(X))$ is connected for $\mathcal{Q}$, Φ is $\mathcal{Q} -$ val$\Phi-$connected, Φ^* is convex-valued, and there exists a compact $\cap_f-$closed H_2-paving $\mathcal{P}$ in S such that Φ is $\mathcal{P}-$valued.*

Proof. $(a) \Longrightarrow (b)$: On X take the coarse topology $\mathcal{G}(X) = \{\emptyset, X\}$ and the free convexity $\Psi(A) \equiv A$. For $\hat{s} \in \bigcap_{x \in X} \Phi(x)$ take the excluded point topology on S according to Example 6.4.
$(b) \Longrightarrow (c)$: Take $\mathcal{Q} = \mathcal{F}(X)$ and $\mathcal{P} = \mathcal{K}(S) \cap \mathcal{F}(S)$.

$(c) \Longrightarrow (a)$: From Proposition 5.8 with $n = 1$ and $\mathcal{P} = \mathrm{val}\Phi$ together with Remark 5.1 it follows that $\Phi(x_1) \cap \Phi(x_2)$ is nonvoid for all $x_1, x_2 \in X$. Since $\mathcal{P}$ is a $\cap_f$−closed H_2−paving paving we infer from $\mathrm{val}\Phi \subset \mathcal{P}$ that $\mathrm{val}\Phi$ has the finite intersection property. Since $\mathrm{val}\Phi$ is compact, we arrive at (a). □

Theorem 6.10. *Let a quasiconvex paved space $(X, \mathcal{Q}, \Psi)$ with $h(X, \Psi) \le 2$, a paved space $(S, \mathcal{P})$ with compact and $\cap_f$−closed $\mathcal{P}$, and a correspondence $\Phi : X \to \mathcal{P}$ with convex-valued dual Φ^* be given. Suppose that the paving of segments $\Psi(\mathcal{E}^2(X))$ is connected for $\mathcal{Q}$, Φ is $\mathcal{Q}-\mathrm{val}\Phi-$connected, and $\mathrm{val}\Phi$ is finitary connected for $\mathcal{P}$. Then Φ has a constant selector.*

Proof. From Proposition 5.8 with $n = 1$ together with Remark 5.1 it follows that $\Phi(x_1) \cap \Phi(x_2)$ is nonvoid for all $x_1, x_2 \in X$. Hence, val Φ has the finite intersection property according to Theorem 5.5, and the assertion follows since $\mathrm{val}\Phi$ is compact. □

Corollary 6.11. *Let $(X, \mathcal{Q})$ and $(S, \mathcal{P})$ be paved spaces such that $\mathcal{Q}$ is $\cap_a$−closed, connected and contains X, and $\mathcal{P}$ is $\cap_f$−closed and compact. Let $\Phi : X \to 2^S$ be a correspondence with $\mathrm{val}\Phi^* \subset \mathcal{Q}$ and $\mathrm{val}\Phi \subset \mathcal{P}$. Assume, moreover, that either*

(i) val Φ *is connected, and $\mathcal{P}$ is an H_2−paving, or*

(ii) val Φ *is finitary connected for $\mathcal{P}$, and $\mathcal{Q}$ is an H_2−paving.*

Then Φ has a constant selector.

Proof. By Lemma 5.11, Φ is $\mathcal{Q}$ − $\mathrm{val}\Phi$−connected. Obviously, $\Psi_{\mathcal{Q}}(\mathcal{E}^2(X))(\subset \mathcal{Q})$ is connected for $\mathcal{Q}$, and Φ^* is $\Psi_{\mathcal{Q}}$−convex-valued. In case (i) the assertion follows from Theorem 6.9 "$(c) \Longrightarrow (a)$". In case (ii) we have $h(X, \Psi_{\mathcal{Q}}) \le 2$ according to Example 4.7. Hence, Theorem 6.10 can be applied. □

Theorem 6.12. *Let two paved spaces $(X, \mathcal{Q})$ and $(S, \mathcal{P})$ be given such that $\mathcal{Q}$ is $\cap_f$−closed and compact, and $\mathcal{P}$ is $\cap_a$−closed, connected and contains S. Let $\Phi : X \to 2^S$ be a correspondence such that $\mathrm{val}\Phi^* \subset \mathcal{Q}$ and $\mathrm{val}\Phi \subset \mathcal{P}$. Assume, moreover, that either*

(i) $\mathrm{val}\Phi^*$ *is connected, and $\mathcal{Q}$ is an H_2−paving, or*

(ii) $\mathrm{val}\Phi^*$ *is finitary connected for $\mathcal{Q}$, and $\mathcal{P}$ is an H_2−paving.*

Then Φ has a constant selector.

Proof. Suppose that $\bigcap_{x \in X} \Phi(x) = \emptyset$. Then Φ^* is a correspondence, and by Corollary 6.11, applied to Φ^* instead of Φ, there exists an $\hat{x} \in \bigcap_{s \in S} \Phi^*(s)$ in contradiction to $\Phi(\hat{x}) \ne \emptyset$. □

Example 6.13. A *tree-like space* is a connected Hausdorff space X in which every two points x and y can be separated by a third point z, i.e., x and y belong to different components of $X \setminus \{z\}$. For a locally connected tree-like space X the following holds:

a) ([20, 29]) $\mathcal{C}(X)$ is an alignment, and $\Psi = \Psi_{\mathcal{C}(X)}$ is a convexity with $\Psi(\{x\}) = \{x\}, x \in X$. For $\{x, y\} \in \mathcal{E}^2(X)$ the segment

$$[x, y] := \{x, y\} \cup \{z \in X : z \text{ separates } x \text{ from } y\}$$

is compact, and it is the smallest connected set containing x and y. Hence, $[x, y] = \Psi(\{x, y\})$. By Lemma 3.4 $a)$ the polytopes $\Psi(A) = \bigcup_{x,y \in A}[x, y]$, $A \in \mathcal{E}(X)$, are connected (and compact) as well, and therefore, $\Psi = \Psi_{\mathcal{C}(X)\cap\mathcal{K}(X)}$. Moreover, the pavings of convex, star-shaped, respectively connected subsets coincide according to Lemma 3.4 *b*).

b) $\mathcal{C}_\Psi = \mathcal{C}(X)$ is an H_2-paving [20]. Therefore, $h = h(X,\Psi) \leq 2$. From $\Psi(\{x\}) \cap \Psi(\{y\}) = \{x\} \cap \{y\} = \emptyset,\ x \neq y$, we infer $h = 2$. Together with a), Example 6.5, and Lemma 2.2 we get $c = c(X,\Psi) \leq 2$, and Ψ is join-hull commutative. Hence, $\Psi(\{x,y\}) \neq \{x,y\},\ x \neq y$, implies $c = 2$. In particular, relation (3) and therefore (2) holds for every $A \in \mathcal{E}^n(X), n \geq 3$. Let $\{x,y\} \in \mathcal{E}^2(X)$ and $t \in [x,y]$. Then $D := [x,t] \cup [t,y]$ is connected, and $\{x,y\} \subset D$ implies $[x,y] \subset D$. Hence, relation (2) is also satisfied for every $A \in \mathcal{E}^2(X)$. Therefore, Ψ has the subdivision property.

c) $\mathcal{C}(X)$ is finitary connected for $\mathcal{C}(X)\cap\mathcal{F}(X)$ [20]. In particular, $(X, \mathcal{C}(X)\cap\mathcal{F}(X), \Psi)$ is a CP-space.

A metric space (X,d) is *hyperconvex* iff any paving $\{B(x_i,r_i) : i \in I\}$ of closed balls in X satisfying $d(x_i,x_j) \leq r_i + r_j$ for all $i,j \in I$, has the global intersection property $\bigcap_{i\in I} B(x_i,r_i) \neq \emptyset$. We denote by $\mathcal{A}_d$ the paving of arbitrary intersections of closed balls (the paving of 'admissible' subsets of X).

Example 6.14. Let (X,d) be a hyperconvex metric space. Then (X,d) is complete and Menger-convex [15], and $\mathcal{A}_d$ is a finitary connected and compact H_2-paving [20]. Hence, $(X, \mathcal{A}_d, \mathrm{cov})$ with the convexity $\mathrm{cov}(A) = \Psi_{\mathcal{A}_d}(A), A \in \mathcal{E}(X)$, is a CP-space with $\mathcal{A}_d \subset \mathcal{C}_{\mathrm{cov}}$. Together with $\mathrm{cov}(\{x\}) \cap \mathrm{cov}(\{y\}) = \{x\} \cap \{y\} = \emptyset,\ x \neq y$, we obtain $h(X,\mathrm{cov}) = 2$ according to Example 4.7. In particular, by Lemma 4.5, $\mathcal{C}_{\mathrm{cov}}$ is an H_2-paving. By Remark 3.3 a), $\mathcal{C}_{\mathrm{cov}}$ is finitary connected for $\mathcal{A}_d$ and, by Lemma 3.4 *b*), $\mathcal{C}_{\mathrm{cov}} \subset \mathcal{C}(X)$, since according to Examples 20 and 21 in [20] we have $\mathcal{A}_d \subset \mathcal{C}(X)$. By Remark 4.1 c) the pavings $\mathcal{F}(X) \cap \mathcal{C}_{\mathrm{cov}}$ and $\mathcal{G}(X) \cap \mathcal{C}_{\mathrm{cov}}$ are finitary connected. Hence, by Remark 3.3 a), $\mathcal{C}_{\mathrm{cov}}$ is is finitary connected for $\mathcal{F}(X) \cap \mathcal{C}_{\mathrm{cov}}$, and $(X, \mathcal{F}(X) \cap \mathcal{C}_{\mathrm{cov}}, \mathrm{cov})$ is a CP-space.

In the following, a hyperconvex metric space will always be endowed with the convexity cov.

Example 6.15. Let X and S be nonvoid sets and $\Phi : X \to 2^S$ a correspondence. Suppose that either

(i) there exists a locally connected tree-like topology on X such that the dual Φ^* has connected values, or
(ii) there exists a hyperconvex metric d on X such that the dual Φ^* has convex values.

Then the following are equivalent:

(a) Φ has a constant selector.
(b) Φ is $\mathcal{F}(X) - \mathrm{val}\Phi-$connected, and there exists a $\cap_f-$closed compact paving $\mathcal{P}$ in S such that Φ is $\mathcal{P}-$valued and $\mathrm{val}\Phi$ is finitary connected for $\mathcal{P}$.

Proof. (a) $\Longrightarrow$ (b): For $\hat{s} \in \bigcap_{x\in X} \Phi(x)$ take the excluded point topology according to Example 6.4, and take $\mathcal{P} = \mathcal{F}(S)$.
(b) $\Longrightarrow$ (a): Apply Theorem 6.10 with $\mathcal{Q} = \mathcal{F}(X)$ together with Example 6.13 in case (i) and Example 6.14 in case (ii). □

Example 6.16. Let X and S be nonvoid sets and $\Phi : X \to 2^S$ a correspondence. Suppose that one of the following three conditions is satisfied:

(i) there exists a locally connected tree-like topology on X such that the dual Φ^* has compact connected values,
(ii) there exists a hyperconvex metric d on X such that the dual Φ^* has compact convex values, or
(iii) there exists a hyperconvex metric d on X such that the dual Φ^* has admissible values.

Then the following are equivalent:

(a) Φ has a constant selector.
(b) There exists a $\cap_a-$closed connected paving $\mathcal{P}$ in S with $S \in \mathcal{P}$ such that Φ is $\mathcal{P}-$valued.

Proof. (a) $\Longrightarrow$ (b): Compare the proof of Example 6.15.
(b) $\Longrightarrow$ (a): Let $\mathcal{Q} = \mathcal{K}(X) \cap \mathcal{C}(X)$ in case (i), $\mathcal{Q} = \mathcal{K}(X) \cap \mathcal{C}_{\text{cov}}$ in case (ii), and $\mathcal{Q} = \mathcal{A}_d$ in case (iii). Then, by Examples 6.13 and 6.14, $\mathcal{Q}$ is a $\cap_a-$closed, connected, and compact H_2-paving with val$\Phi^* \subset \mathcal{Q}$. From Theorem 6.12($i$) the assertion follows.

Example 6.17. Let $(X, \mathcal{Q})$ be a paved space such that $\mathcal{Q}$ is $\cap_f-$closed and compact, and let $\Phi : X \to 2^S$ be a correspondence with val$\Phi^* \subset \mathcal{Q}$ such that valΦ^* is finitary connected for $\mathcal{Q}$. Suppose that either

(i) there exists a locally connected tree-like topology on S such that Φ has closed connected values, or
(ii) there exists a hyperconvex metric on S such that Φ has closed convex values.

Then Φ has a constant selector.

Proof. Apply Theorem 6.12(ii) with $\mathcal{P} = \mathcal{F}(S) \cap \mathcal{C}(S)$ together with Example 6.13 in case (i), and with $\mathcal{P} = \mathcal{F}(S) \cap \mathcal{C}_{\text{cov}}$ together with Example 6.14 in case (ii). □

The following generalization of Darboux's Theorem is well–known. (Compare Remark 6.23 below.)

Lemma 6.18. *Let X and S be topological spaces, and let $\Phi : X \to 2^S$ be a correspondence with a continuous selector $f : X \to S$. Then Φ is $\mathcal{F}(X) - \mathcal{F}(S)-$connected iff* val$\Phi \subset$ *$\mathcal{C}(S)$.*

Proof. Suppose that val$\Phi \subset \mathcal{C}(S)$. Let $C \in \mathcal{C}(X)$ and $F_1, F_2 \in \mathcal{F}(S)$ with $\Phi(C) \subset F_1 \cup F_2$ and $\Phi(C) \cap F_1 \cap F_2 = \emptyset$. We have to show that either $\Phi(C) \cap F_1 = \emptyset$ or $\Phi(C) \cap F_2 = \emptyset$ holds. Suppose, to the contrary, that $\Phi(x_i) \cap F_i \neq \emptyset$ for $x_i \in C, i \in \{1, 2\}$. Then we have $\Phi(x_i) \subset F_i, i \in \{1, 2\}$, since val$\Phi$ is connected for $\mathcal{F}(S)$. Now $x_i \in f^{-1}(F_i) \cap C, i \in \{1, 2\}, C \subset f^{-1}(F_1) \cup f^{-1}(F_2)$, and $f^{-1}(F_i) \in \mathcal{F}(X)$ imply $C \cap f^{-1}(F_1) \cap f^{-1}(F_2) \neq \emptyset$ in contradiction to $\Phi(C) \cap F_1 \cap F_2 = \emptyset$. The converse is obvious. □

Example 6.19. Let X and S be locally connected tree-like spaces, and let $\Phi : X \to 2^S$ be a connected-valued compact-valued correspondence with connected-valued dual Φ^*. Then the following are equivalent:

(a) Φ has a constant selector.
(b) Φ has a continuous selector.

(c) Φ is $\mathcal{F}(X) - \mathcal{F}(S)-$connected.
(d) Φ is $\mathcal{F}(X) - \mathcal{K}(S) \cap \mathcal{C}(S)-$connected.

Proof. (a) $\Longrightarrow$ (b) and (c) $\Longrightarrow$ (d) is obvious, and (b) $\Longrightarrow$ (c) follows from Lemma 6.18. Finally, (d) $\Longrightarrow$ (a) follows from Examples 6.13 and 6.15 (with $\mathcal{P} = \mathcal{C}(S) \cap \mathcal{K}(S)$). □

Example 6.20. Let (X, d_X) and (S, d_S) be two hyperconvex metric spaces, and let $\Phi : X \to 2^S$ be a correspondence such that Φ has either admissible or compact, convex values and Φ^* has convex values. Then Φ has a constant selector iff it has a continuous selector.

Proof. Apply Example 6.15 with $\mathcal{P} = \mathcal{A}_{d_S}$ or $\mathcal{P} = \mathcal{K}(S) \cap \mathcal{C}_{\text{cov}}$, respectively, together with Example 6.14 and Lemma 6.18. □

It is easy to formulate more results of the above type, for example in combination with Example 6.7 and/or Remark 6.23 below together with many examples of modular convexities which can be found in [4, 20, 27, 28].

Our theorems above can also be used to derive fixed-point theorems. Recall that a point $x \in X$ is a *fixed point* of a correspondence $\Phi : X \to 2^X$ provided that $x \in \Phi(x)$.

Example 6.21. Let X be a locally connected tree-like space, and let $\Phi : X \to 2^X$ be a connected-valued correspondence with connected-valued and compact-valued dual Φ^*. Then Φ has a fixed point.

Similar results can be found in [2, 25].

Proof. Suppose, to the contrary, that Φ has no fixed point, which means that $x \in \Phi^*(x)$ for every $x \in X$. In particular, Φ^* has a continuous selector, and by Example 6.19 – with Φ replaced by Φ^* – there exists an $\hat{x} \in \bigcap_{x \in X} \Phi^*(x)$, i.e., $\Phi(\hat{x}) = \emptyset$, a contradiction. □

Similarly together with Example 6.20 we obtain the following:

Example 6.22. Let (X, d) be a hyperconvex metric space. Then every convex-valued correspondence $\Phi : X \to 2^X$ with convex-valued and compact-valued or with admissible-valued dual Φ^* has a fixed point.

Remark 6.23. Let X and S be topological spaces. Then a correspondence $\Phi : X \to 2^S$ is *quartercontinuous* [22] (respectively semicontinuous in the sense of Correa et al. [5]) provided that for all $x \in X$ and $G \in \mathcal{G}(S)$ with $\Phi(x) \subset G$ there exists an $U \in \mathcal{G}(X)$ with $x \in U$ and $\Phi(u) \cap G \neq \emptyset$ for all $u \in U$. Obviously, every correspondence with continuous selector is quartercontinuous. Further examples of quartercontinuous correspondences can be found in [17, 18].

According to Correa et al. [5], Lemma 6.18 can be generalized as follows:

Every quartercontinuous correspondence $\Phi : X \to 2^S$ with connected values is $\mathcal{F}(X) - \mathcal{F}(S)-$connected.

An abstract version of this theorem for paved spaces was proved in [18]. For an application of Theorem 5.9 one needs practicable conditions for $n-$ary $\mathcal{Q} - \mathcal{P}-$conncetedness of a correspondence Φ for $n > 1$. It is intended to study this problem in a separate paper.

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References

[1] V. Boltyanski, H. Martini, P. S. Soltan: Excursions into Combinatorial Geometry, Springer, Berlin (1997).

[2] C. E. Capel, W. L. Strother: Multi-valued functions and partial order, Portugal. Math. 17 (1958) 41–47.

[3] C. Carathéodory: Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen, Rend. Circ. Mat. Palermo 32 (1911) 193–217.

[4] W. A. Coppel: Foundations of Convex Geometry, Cambridge University Press, Cambridge (1998).

[5] R. Correa, J. B. Hiriart-Urruty, J.-P. Penot: A note on connected set–valued mappings, Boll. Unione Mat. Ital., VI. Ser., C, Anal. Funz. Appl. 5 (1986) 357–366.

[6] L. Danzer, B. Grünbaum, V. Klee: Helly's theorem and its relatives, in: Convexity, V. Klee (ed.), Proc. Sympos. Pure Math. 7, American Mathematical Society, Providence (1963) 101–180.

[7] J. Eckhoff: Helly, Radon, and Carathéodory type theorems, in: Handbook of Convex Geometry, P. M. Gruber, J. M. Wills (eds.), Vol. A, North-Holland, Amsterdam (1993) 389–448.

[8] R. Engelking, K. Sieklucki: Topology. A Geometric Approach, Heldermannn, Berlin (1992).

[9] S. D. Flåm, G. H. Greco: Minimax and intersection theorems, in: Fixed Point Theory and Applications, M. A. Théra and J.-B. Baillon (eds.), Longman, Harlow (1991) 123–140.

[10] B. Fuchssteiner: Verallgemeinerte Konvexitätsbegriffe und der Satz von Krein-Milman, Math. Ann. 186 (1970) 149–154.

[11] K. Goebel, S. Reich: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, Marcel Dekker, New York (1984).

[12] K. Goebel, W. A. Kirk: Topics in Metric Fixed Point Theory, Cambridge University Press, Cambridge (1990).

[13] E. Helly: Über Mengen konvexer Körper mit gemeinschaftlichen Punkten, Jber. Deutsch. Math. Verein 32 (1923) 175–176.

[14] S. Kakutani, V. Klee: The finite topology of a linear space, Arch. Math. 14 (1963) 55–58.

[15] M. A. Khamsi, W. A. Kirk: An Introduction to Metric Spaces and Fixed Point Theory, Wiley, New York (2001).

[16] J. Kindler: Intersection theorems and minimax theorems based on connectedness, J. Math. Anal. Appl. 178 (1993) 529–546.

[17] J. Kindler: Intersecting sets in midset spaces I, Arch. Math. 62 (1994) 49–57; and II, Arch. Math. 62 (1994) 168–176.

[18] J. Kindler: Intersection theorems, minimax theorems and abstract connectedness, in: Minimax Theory and Applications (Erice, 1996), B. Ricceri, S. Simons (eds.), Nonconvex Optim. Appl. 26, Kluwer, Dordrecht (1998) 105–120.

[19] J. Kindler: Separation theorems for abstract convex structures, J. Convex Analysis 9 (2002) 73–96.

[20] J. Kindler: Helly and Klee type intersection theorems for finitary connected paved spaces, Topology Appl. 144 (2004) 271–291.

[21] K. Kołodziejczyk: On intersections of convex sets, Discuss. Math. 6 (1983) 185–188.

[22] H. Komiya: On minimax theorems without linear structure, Hiyoshi Rev. Nat. Sci. 8 (1990) 74–78.

[23] F. W. Levi: Eine Ergänzung zum Hellyschen Satze, Arch. Math. 4 (1953) 222–224.

[24] K. Menger: Untersuchungen über allgemeine Metrik, I, II, III, Math. Ann. 100 (1928) 75–163.

[25] R. E. Smithson: Fixed point theorems for certain classes of multifunctions, Proc. Amer. Math. Soc. 31 (1972) 595–600.

[26] L. A. Steen, J. A. Seebach: Counterexamples in Topology, Holt, Rinehart and Winston, New York (1970).

[27] M. van de Vel: Theory of Convex Structures, North-Holland, Amsterdam (1993).

[28] E. R. Verheul: Multimedians in Metric and Normed Spaces, CWI Tracts 91, CWI, Amsterdam (1993).

[29] G. T. Whyburn: Cut points in general topological spaces, Proc. Natl. Acad. Sci. USA 61 (1968) 380–387.