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On the Closedness of the Sum of Closed Convex Cones in Reflexive Banach Spaces

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In this paper, we establish a sufficient condition for the sum of two closed convex cones K and L of a reflexive Banach space to be closed.

1. Introduction

Consider two closed convex cones K and L of a reflexive Banach space X. In general, the sum of two closed convex cones K and L of a reflexive Banach needs not be closed. However, it is a well known fact that the direct sum of two closed subspaces U and V of a Banach space Y is closed if and only if there exists a constant $d > 0$ such that

$$||u - v|| \geq d, \quad \forall u \in U, \forall v \in V, \;\; ||u|| = ||v|| = 1. \tag{1}$$

For example, a proof can be found in [3].

It is natural to ask if one can replace the subspaces U and V by closed convex cones K and L. In Section 2, we provide an easy example of two closed convex cones in $l_2 \times l_2$ which shows that condition (1) does not guarantee the closedness of the direct sum of two closed convex cones. The sum $K + L$ of two closed convex cones is called a direct sum if $K \cap L = \{0\}$. In this case we write $K + L$ as $K \oplus L$. As a main result we establish in Section 3 a sufficient condition for the sum (not necessarily the direct sum) of two closed convex cones K and L of a reflexive Banach X space to be closed. Finally, we provide an easy example which shows that our condition is not necessary.

2. Counterexample

First, we give an example which shows that (1) does not guarantee the closedness of the direct sum of two closed convex cones.

Example 2.1. Consider the Hilbert space l_2 consisting of all sequences $\xi = (\xi_1, \xi_2, ...)$ such that

$$||\xi||_2^2 = \sum_{i=1}^{\infty} (\xi_i)^2 < \infty.$$

Set $K = \{\xi \in l_2 | \xi_n \geq 0, \; \forall n \in \mathbb{N}\}$. Let $A : l_2 \to l_2$ be defined by

$$A(\xi_1, \xi_2, ...) = (\frac{\xi_1}{1}, \frac{\xi_2}{2}, \frac{\xi_3}{3}, ...).$$

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Consider now

$$G_K := \{(x, y) \in l_2 \times l_2 : \ x \in K, y = A(x)\}.$$

G_K is the graph of K under the linear mapping A. Since K is a closed convex cone and A is a continuous mapping, G_K is a closed convex cone of $l_2 \times l_2$.

Set $L = \{(k, 0) | k \in -K\} \subset l_2 \times l_2$. Then L is a closed convex cone of $l_2 \times l_2$. As A is injective, we have $G_K \cap L = \{0\}$. According to our definition on the previous page we write the sum of G_K and L as $G_K \oplus L$.

The sequence $z_n := (x_n, y_n)$ where $x_n = (0, 0, ...)$, $\forall n \in \mathbb{N}$, and $y_n = (1, \frac{1}{2}, ..., \frac{1}{n}, 0, ...)$, $\forall n \in \mathbb{N}$, is contained in $G_K \oplus L$. This can be seen by choosing

$$\begin{aligned} &\big((1, 1, ..., 1, 0, 0..), (1, \frac{1}{2}, ..., \frac{1}{n}, 0, ...)\big) \in G_K \\ \text{and} \quad &\big((-1, -1, ..., -1, 0, 0...), (0, 0, ...)\big) \in L. \end{aligned}$$

The limit of this sequence is $\big((0, 0, ...), (1, \frac{1}{2}, \frac{1}{3}, ...)\big)$. As $G_K \cap L = \{0\}$ and

$$\sum_{i=1}^{\infty} (-1)^2 = \infty$$

the limit is not contained in $G_K \oplus L$.

Next, we show that there exists a constant $d > 0$ such that

$$||x - y|| \geq d, \quad \forall x \in G_K, \forall y \in L, \quad ||x|| = ||y|| = 1.$$

Let $x = \big((x_1, x_2, ...), (x_1, \frac{x_2}{2}, ...)\big) \in G_K$ and $y = \big((y_1, y_2, ...), (0, 0, ...)\big) \in L$. Then we have

$$\begin{aligned} ||x - y|| &= ||\big((x_1 - y_1, x_2 - y_2, ...), (x_1, \frac{x_2}{2}, ...)\big)|| \\ &= \big(\sum_{i=1}^{\infty} |x_i - y_i|^2 + \sum_{i=1}^{\infty} |x_i/i|^2\big)^{\frac{1}{2}} \\ &\geq \big(\sum_{i=1}^{\infty} |x_i|^2 + \sum_{i=1}^{\infty} |x_i/i|^2\big)^{\frac{1}{2}} \\ &= ||x||. \end{aligned}$$

The inequality follows from the fact that $x_i \geq 0$ and $y_i \leq 0$, $\forall i \in \mathbb{N}$. Hence

$$||x - y|| \geq 1, \quad \forall x \in G_K, \forall y \in L, \ ||x|| = ||y|| = 1.$$

3. Main result

We continue by recalling a known result.

Lemma 3.1. *Let M be a closed convex subset of a Banach space X. Then M is weakly closed.*

For a proof see ([7], page 122).

Theorem 3.2. *Let X be a reflexive Banach space. Let K and L be closed convex cones of X. Suppose there exists a constant $d > 0$ such that*

$$||x+y|| \geq d, \quad \forall x \in K, \forall y \in L, \; ||x|| = ||y|| = 1. \tag{2}$$

Then $K + L$ is closed.

Proof. Let $(x_n + y_n)$ be a convergent sequence in $K + L$ where $x_n \in K$, $\forall n \in \mathbb{N}$, and $y_n \in L$, $\forall n \in \mathbb{N}$. Let z be the limit of this sequence. There are two possible cases:

1. The sequences of norms $(||x_n||)$ and $(||y_n||)$ are both bounded.
2. One of the sequences of norms $(||x_n||)$ and $(||y_n||)$, say $(||x_n||)$ is unbounded.

Case 1: Since $(||x_n||)$ and $(||y_n||)$ are both bounded and X is a reflexive Banach space there exists a subsequence denoted by $\tilde{n}$ such that the subsequences $(x_{\tilde{n}})$ and $(y_{\tilde{n}})$ are both weak convergent. Denote the weak limit of $(x_{\tilde{n}})$ by z_1 and the weak limit of $(y_{\tilde{n}})$ by z_2. According to our assumption K and L are both closed convex cones. From Lemma 3.1 we conclude that they are also weakly closed. Therefore, $z_1 \in K$ and $z_2 \in L$. Hence, the sequence $(x_{\tilde{n}} + y_{\tilde{n}})$ converges weakly to $z_1 + z_2 \in K + L$. Since the strong and the weak limit coincide, we get $z = z_1 + z_2$ and therefore $z \in K + L$.

Case 2: Since $(||x_n||)$ is unbounded we can pass to a subsequence denoted by $\tilde{n}$ such that $||x_{\tilde{n}}|| \to \infty$ and $||x_{\tilde{n}}|| > 0$. From our assumption $(x_n + y_n) \to z$ we get $||x_n + y_n|| \to ||z||$. Therefore, $||x_{\tilde{n}} + y_{\tilde{n}}|| \to ||z||$. Since $||x_{\tilde{n}}|| > 0$ and $||x_{\tilde{n}}|| \to \infty$, we get

$$||\frac{x_{\tilde{n}} + y_{\tilde{n}}}{||x_{\tilde{n}}||}|| \to 0.$$

As K and L are cones, we have $\hat{x}_{\tilde{n}} = \frac{x_{\tilde{n}}}{||x_{\tilde{n}}||} \in K$ and $\hat{y}_{\tilde{n}} = \frac{y_{\tilde{n}}}{||x_{\tilde{n}}||} \in L$. Moreover, $||\hat{x}_{\tilde{n}}|| = 1$. This implies

$$\begin{aligned} |\, 1 - ||\hat{y}_{\tilde{n}}||\, | &= |\, ||\hat{x}_{\tilde{n}}|| - ||\hat{y}_{\tilde{n}}||\, | \\ &\leq ||\hat{x}_{\tilde{n}} + \hat{y}_{\tilde{n}}|| \to 0. \end{aligned}$$

Hence there exists a subsubsequence denoted by $\bar{n}$ such that $||\hat{y}_{\bar{n}}|| \neq 0 \; \forall \bar{n}$. Note that $(\hat{x}_{\bar{n}})$ is a sequence of K such that $||\hat{x}_{\bar{n}}|| = 1$. As we can divide by $||\hat{y}_{\bar{n}}||$ for all $\bar{n}$ we get from

$$\begin{aligned} ||\hat{x}_{\bar{n}} + \frac{\hat{y}_{\bar{n}}}{||\hat{y}_{\bar{n}}||}|| &= ||\hat{x}_{\bar{n}} + \hat{y}_{\bar{n}} + \frac{\hat{y}_{\bar{n}}}{||\hat{y}_{\bar{n}}||} - \hat{y}_{\bar{n}}|| \\ &\leq ||\hat{x}_{\bar{n}} + \hat{y}_{\bar{n}}|| + ||\frac{\hat{y}_{\bar{n}}}{||\hat{y}_{\bar{n}}||}(1 - ||\hat{y}_{\bar{n}}||)|| \\ &= ||\hat{x}_{\bar{n}} + \hat{y}_{\bar{n}}|| + |\, 1 - ||\hat{y}_{\bar{n}}||\, | \to 0 \end{aligned}$$

a contradiction to our assumption (2). □

The following example shows that condition (2) is not necessary to ensure the closedness of the sum of two closed convex cones K and L even if we have $K \cap L = \{0\}$.

Example 3.3. Consider the following subsets of $\mathbb{R}^2$

$$\begin{aligned} K &= \{(x,y) \in \mathbb{R}^2 | x \geq 0, y \geq 0\}, \\ L &= \{(x,y) \in \mathbb{R}^2 | x \leq 0, y = x\}. \end{aligned}$$

K and L are closed convex cones. Moreover $K \cap L = \{0\}$.

Since the vectors $(1,0), (-1,0), (0,1)$, and $(0,-1)$ are all elements of $K \oplus L$ we have $K \oplus L = \mathbb{R}^2$.

Choose $x = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \in K$ and $y = (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}) \in L$. As $||x|| = ||y|| = 1$ and $||x + y|| = 0$ condition (2) can not be necessary to ensure the closedness of two closed convex cones.

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