Journal of Convex Analysis
Volume 14 (2007), No. 1, 149–167

# On the Directions of Segments and $r$-Dimensional Balls on a Convex Surface

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Received: April 7, 2005
Revised manuscript received: September 8, 2005

We prove that the set of directions of $(n-2)$-dimensional balls which are contained in the boundary $\partial K$ of a convex body $K \subset \mathbb{R}^n$ but in no $(n-1)$-dimensional convex subset of $\partial K$ is $\sigma$-1-rectifiable. We also show that there exists a close connection between smallness of the set of directions of line segments on $\partial K$ and smallness of the set of tangent hyperplanes to the graph of a d.c. (delta-convex) function on $R^{n-2}$. Using this connection, we construct $K \subset \mathbb{R}^3$ such that the set of directions of segments on $\partial K$ cannot be covered by countably many simple Jordan arcs having half-tangents at all points. Also new results on directions of $r$-dimensional balls in $\partial K$ parallel to a fixed linear subspace are proved.

*Keywords:* Convex body, boundary structure, Hausdorff measure, tangent hyperplane, d.c. function

*2000 Mathematics Subject Classification:* Primary 52A20; Secondary 26B25

## 1. Introduction

Put $S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$ and denote by $G(n,r)$ the Grassmann manifold of all $r$-dimensional linear subspaces of $\mathbb{R}^n$. (We recall that $G(n,r)$ is a compact differentiable manifold of dimension $r(n-r)$.) If $K \subset \mathbb{R}^n$ is a convex closed set with non-empty interior, we denote by $D(K)$ the set of all $u \in S^{n-1}$ that are parallel to a segment in $\partial K$. For $1 \le r \le n-2$, we denote by $F_r(K)$ ($F_r^*(K)$) the set of all linear spaces $V \in G(n,r)$ which are parallel to an $r$-dimensional ball in $\partial K$ (to an $r$-dimensional ball in $\partial K$ which is contained in no $(r+1)$-dimensional convex subset of $\partial K$, respectively).

In the following, if $K$ is not specified, we suppose that it is a convex body in $\mathbb{R}^n$. Klee ([7]) in 1957 (see also [8]) asked whether (for $n \ge 3$) the set $D(K)$ is always of $(n-1)$-dimensional measure zero. Mc Minn [12] (see [2] for a more transparent proof) showed that for $n = 3$ the set $D(K)$ is even $\sigma$-1-rectifiable. Ewald, Larman and Rogers [5] (see also [17]) proved the following deep results:

(ELR 1) *The set $D(K)$ has always $\sigma$-finite $(n-2)$-dimensional Hausdorff measure.*
(ELR 2) *The set $F_r(K)$ has always $\sigma$-finite $[r(n-r-1)]$-dimensional Hausdorff measure.*
(ELR 3) *If $F_{n-1}(K) = \emptyset$ and $r \ge 2$, then $F_r(K)$ has zero $[r(n-r-1)]$-dimensional Hausdorff measure.*

ISSN 0944-6532 / $ 2.50 

The proofs of these results from [5] use rather sophisticated geometrical methods and it seems that no simple proof of $D(K) \neq S^{n-1}$ for $n > 3$ is known (cf. [18, p. 277]). Also the following natural question seems to be open.

**Question 1.1.** Is it true that (for $n \geq 4$) the set $D(K)$ is always

a) $\sigma$-$(n-2)$-rectifiable?

b) $\sigma$-$H^{n-2}$-rectifiable?

We recall that a subset $M$ of a metric space $X$ is called $\sigma$-$n$-rectifiable ($\sigma$-$H^n$-rectifiable) if there exist Lipschitz mappings $\varphi_k$ $(k = 1, 2, \dots)$ from $[0,1]^n$ to $X$ whose images cover $M$ (whose images cover $M$ except a set of zero $n$-dimensional Hausdorff measure, respectively).

McMinn's result shows that Question 1.1 has positive answer for $n = 3$. In this case the following natural question is open.

**Question 1.2.** Does there exist a simple characterization of the smallest $\sigma$-ideal $\mathcal{I}$ containing all sets of the form $D(K)$, where $K \subset \mathbb{R}^3$ is a convex body? In particular, is $\mathcal{I}$ equal to the system of all $\sigma$-1-rectifiable sets?

(Recall that a system $\mathcal{I}$ of sets is called a $\sigma$-ideal if it is stable with respect to countable unions and $\mathcal{I}$ contains all subsets of any set $A \in \mathcal{I}$.)

In Section 3 we prove Theorem 3.2 which says that the set $F^*_{n-2}(K)$ is $\sigma$-1-rectifiable. It improves the theorem (ELR 3) in the case $r = n-2$ and can be considered as a generalization of McMinn's [12] result (see Remark 3.4). Our proof is a modification of Besicovitch's proof [2] of McMinn's result.

Theorem 3.2 suggests that (ELR 3) can perhaps be improved also for some $1 < r < n-2$. Also the following question naturally arises.

**Question 1.3.** Is it true that $F^*_r$ has $\sigma$-finite $(n-r-1)$-dimensional Hausdorff measure for each convex body $K \subset \mathbb{R}^n$ and $1 < r < n-2$ ?

In Section 4 we show how Besicovitch's method [2] and the Fenchel duality imply a dual formulation (Proposition 4.2) of results and problems concerning smallness of the set $D(K)$. As an immediate consequence, we show that (ELR 1) implies that the set $T(f)$ of all tangent hyperplanes to the graph of any d.c. function (i.e. a function which is the difference of two convex functions) $f : \mathbb{R}^k \to \mathbb{R}$ has $\sigma$-finite $k$-dimensional Hausdorff measure. Note that there exists a $C^1$-function $f : \mathbb{R}^k \to \mathbb{R}$ such that $T(f)$ has positive $(k+1)$-dimensional Hausdorff measure (see [3] for a stronger example).

Further, we show that the Morse-Sard theorem for d.c. functions $f : \mathbb{R}^2 \to \mathbb{R}$ of Landis [9] easily implies that $T(f)$ has zero 3-dimensional Hausdorff measure for such $f$ (see Remark 4.6) and that this result on $T(f)$ implies (using Besicovitch's method) that $D(K)$ has zero 3-dimensional Hausdorff measure for $K \subset \mathbb{R}^4$. Thus the answer to Klee's question in the case $n = 4$ was easily available after Besicovitch's note [2]. Underline, however, that the proof in [9] is not easy and it is sketched only. A complete proof, using the main idea of [9] and results of [1], is given in [15].

Using the duality result of Proposition 4.2 and a theorem of Larman and Rogers [10], we also obtain that the set $T(f)$ of all tangent hyperplanes to the graph of a d.c. function $f : \mathbb{R}^k \to \mathbb{R}$ which contain a fixed point (or are parallel to a fixed line) has zero $k$-dimensional Hausdorff measure.

In Section 5 we show, using the duality result of Proposition 4.2, that McMinn's result is in a sense close to the best one. In particular there exists a convex body $K \subset \mathbb{R}^3$ such that $D(K)$ cannot be covered by countably many simple Jordan arcs having half-tangents at all points. (For more information see Remark 5.5.) Remember that the set $N(K)$ of all non-smooth points of the boundary of such $K$ admits a covering of this type ([19, Theorem 3]).

We believe that Proposition 4.2 can be also used (in a way similar to that used in Section 5) to give a negative answer to Question 1.1 a).

To describe the results of Section 6 we introduce the following notation. For $S \in G(n,s)$ denote by $F_r^S(K)$ the set of all $V \in G(n,r)$ such that $V \subset S$ and $V$ is parallel to an $r$-dimensional ball $B \subset \partial K$ for which $(B+S) \cap \operatorname{int} K \neq \emptyset$. Using (ELR 1), we can reformulate a result of [10] (see Theorem 4.7 and Remark 4.8 below) as follows.

(LR) *The set $F_1^S(K)$ has zero $(n-2)$-dimensional Hausdorff measure whenever $n \geq 3$ and $S \in G(n,n-1)$.*

In Section 6 we prove two analogous results. Theorem 6.2 asserts that the set $F_{n-2}^S(K)$ has zero 1/2-dimensional Hausdorff measure whenever $n \geq 3$ and $S \in G(n,n-1)$. Thus, in the case $n = 3$, we obtain an improvement of (LR).

Theorem 6.5 asserts that the set $F_{n-3}^S(K)$ has zero 1-dimensional Hausdorff measure whenever $n \geq 4$ and $S \in G(n,n-2)$.

The proofs are based on Besicovitch's method and Morse-Sard theorems for d.c. mappings of [15] (see Theorem 2.6 and Theorem 2.7 below).

## 2. Preliminaries

If $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is a convex function, we denote by $\operatorname{epi} f$ its closed epigraph $\{(x,y) \in \mathbb{R}^n : f(x) \leq y\}$. The following lemma is an easy consequence of the Hahn-Banach theorem.

**Lemma 2.1.** *Let $f$ be a convex function on $\mathbb{R}^{n-1}$ and let $C$ be a convex subset of* $\operatorname{graph} f$. *Then there exists an affine function $\varphi$ on $\mathbb{R}^{n-1}$ such that $C \subset \operatorname{graph} \varphi$ and $\varphi(x) \leq f(x)$ for each $x \in \mathbb{R}^{n-1}$.*

The relative interior of a convex set $C \subset \mathbb{R}^n$, which we denote following Rockafellar [16] by $\operatorname{ri} C$, is defined as the interior of $C$ relative to the affine hull of $C$.

We shall need also the following easy well known fact.

**Lemma 2.2.** *Let $f : \mathbb{R}^k \to \mathbb{R}$ be a convex function and let $I, J \subset \operatorname{graph} f$ be convex sets such that $I \cap \operatorname{ri} J \neq \emptyset$. Then $\operatorname{conv}(I \cup J) \subset \operatorname{graph} f$.*

**Proof.** By Lemma 2.1 there exists an affine function $\varphi$ on $\mathbb{R}^n$ such that $I \subset \operatorname{graph} \varphi$ and $\varphi \leq f$. From $\operatorname{graph} \varphi \cap \operatorname{ri} J \neq \emptyset$ and $J \subset \operatorname{epi} \varphi$ we easily get that $J \subset \operatorname{graph} \varphi$. Therefore $\operatorname{conv}(I \cup J) \subset \operatorname{graph} f \cap \operatorname{graph} \varphi$, since $\operatorname{graph} f \cap \operatorname{graph} \varphi$ is a convex set. □

Whenever $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function, we define the dual function

$$f^*(x^*) = \sup_{x \in \mathbb{R}^n} \left( \langle x, x^* \rangle - f(x) \right), \quad x^* \in (\mathbb{R}^n)^*.$$

As usual, we identify the dual space $(\mathbb{R}^n)^*$ with $\mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ denotes both the duality and the scalar product. We shall use frequently (without a reference) the following basic facts on Fenchel duality (see [16, Theorem 12.2 and Theorem 23.5]).

**Lemma 2.3.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that $f^*$ is finite everywhere. Then*

(*1*) $(f^*)^* = f$,
(*2*) $x^* \in \partial f(x) \Leftrightarrow x \in \partial f^*(x^*)$ *and*
(*3*) *if* $x^* \in \partial f(x)$ *then* $f^*(x^*) = \langle x, x^* \rangle - f(x)$.

We will need also the following easy fact.

**Lemma 2.4.** *Let $s$ be a finite convex function on $\mathbb{R}^k$ ($k > 1$) and $m \in \mathbb{N}$. Then there exists a finite convex function $\tilde{s}$ on $\mathbb{R}^k$ such that $s(x) = \tilde{s}(x)$ for $x \in B(0, m)$ and the dual convex functions $(\tilde{s})^*$ and $(\tilde{s}(0, \cdot))^*$, $(\tilde{s}(1, \cdot))^*$ are finite on $\mathbb{R}^k$ and $\mathbb{R}^{k-1}$, respectively.*

**Proof.** Obviously, we can put $\tilde{s}(x) := \max\{s(x), \|x\|^2 - C\}$, where $C \in \mathbb{R}$ is sufficiently large. □

**Remark 2.5.** All above results and questions on $D(K)$ ($F_r(K)$, $F_r^*(K)$) concern the problem whether this set belongs to a diffeomorphism invariant $\sigma$-ideal. So we can write equivalently "$K \subset \mathbb{R}^n$ is a convex set with a non-empty interior" or "$K$ is the closed epigraph of a convex function $f : \mathbb{R}^{n-1} \to \mathbb{R}$" instead of "$K \subset \mathbb{R}^n$ is a convex body" in all these results and questions. We omit the obvious proof, whose only non-trivial ingredients are the following well-known facts:

1. The boundary of a convex body $K \subset \mathbb{R}^n$ is locally described by an equation $y_n = g(y_1, \ldots, y_{n-1})$ where $(y_1, \ldots, y_n)$ is a suitable system of cartesian coordinates and $g$ is a convex function on an open convex subset of $\mathbb{R}^{n-1}$.
2. If $f$ is a convex function on an open convex set $C \subset \mathbb{R}^{n-1}$ and $x \in C$ then there exists a convex function $\tilde{f}$ on $\mathbb{R}^{n-1}$ such that $f$ and $\tilde{f}$ coincide on a neighbourhood of $x$.

A function on $\mathbb{R}^n$ is said to be d.c. (delta-convex) if it is the difference of two convex functions. A mapping $f : \mathbb{R}^n \to \mathbb{R}^k$ is said to be a d.c. mapping if its components are d.c. functions.

We will use the following (relatively deep) Morse-Sard theorem for d.c. mappings of [15] which is contained in [9] (with a sketched proof) in the case $k = 1$.

**Theorem 2.6.** *Let $f : \mathbb{R}^2 \to \mathbb{R}^k$ be a d.c. mapping. Let $C := \{u \in \mathbb{R}^2 : \ f'(u) = 0\}$. Then $\mathcal{H}^1(f(C)) = 0$.*

We also use (relatively easy) Morse-Sard theorem, whose proof can be found in [13].

**Theorem 2.7.** *Let $f : \mathbb{R} \to \mathbb{R}^k$ be a d.c. mapping. Let $C := \{u \in \mathbb{R} : \ f'(u) = 0\}$. Then $\mathcal{H}^{1/2}(f(C)) = 0$.*

The basic properties of the Grassmann manifold $G(n,k)$ are described in [5, p. 2] (where it is denoted by $I_k^n$). We will use only the following easy fact.

**Lemma 2.8 (see [5, p. 15]).** *Let $1 < k < n$. Then the set $\mathcal{N}$ of all linearly independent $k$-tuples $(v_1,\dots,v_k) \in (\mathbb{R}^n)^k$ is open and the mapping $G : \mathcal{N} \to G(n,k)$; $G(v_1,\dots,v_k) = \mathrm{Lin}\{v_1,\dots,v_k\}$ is locally Lipschitz.*

Finally, we introduce the following notation.

**Definition 2.9.** Let $K \subset R^n$ be a convex closed set and $I \subset \partial K$ be a convex set. Then we define $m(I,K) \in \mathbb{N}$ as the maximum number such that $I$ is contained in an $m(I,K)$-dimensional convex subset of $\partial K$.

Further, for $j = 1,\dots,n-1$, denote by $U_j(K)$ the set of all segments $I \subset \partial K$ such that $m(I,K) = j$ and denote by $D_j(K)$ the set of all $u \in S^{n-1}$ that are parallel to some $I \in U_j(K)$.

## 3. Directions of maximal $(n-2)$-dimensional convex subsets of $\partial K$

**Lemma 3.1.** *Let $f : \mathbb{R}^{n-1} \to \mathbb{R}$ be a convex function and let $u_i \in \mathbb{R}^{n-2}$, $i = 1,\dots n-1$, be affinely independent vectors. Denote by $\mathcal{F}^*$ the set of all $(n-2)$-dimensional convex subsets of $\partial K$, where $K := \mathrm{epi}\, f$, which are contained in no $(n-1)$-dimensional convex subset of $\partial K$ and intersect each hyperplane $\{(u_i, v, y) : v, y \in \mathbb{R}\}$. Then the set $\mathcal{V}^*$ of all $V \in G(n,n-2)$, which are parallel to some $F \in \mathcal{F}^*$, is $\sigma$-1-rectifiable.*

**Proof.** Let $i \in \{1,\dots,n-1\}$. Then put $f_i(v) = f(u_i, v)$ and for each $t \in \mathbb{R}$ choose a number $s_i(t)$ for which $t \in \partial f_i(s_i(t))$, whenever such a number exists. Obviously, each $s_i$ is an increasing function defined on an interval $I_i$.

Now consider $F \in \mathcal{F}^*$ and an affine function $\varphi$ on $\mathbb{R}^{n-1}$ such that $F \subset \mathrm{graph}\,\varphi$ and $\varphi(x) \le f(x)$ for each $x \in \mathbb{R}^{n-1}$ (see Lemma 2.1). By the definition of $\mathcal{F}^*$, we can choose numbers $v_1,\dots,v_{n-1}$ such that $(u_i, v_i, f_i(v_i)) \in F$, $i = 1,\dots,n-1$. There exists $t \in \mathbb{R}$ ($t = \varphi'_{n-1}(x)$ for each $x \in \mathbb{R}^{n-1}$) such that $t \in \partial f_i(v_i)$, $i = 1,\dots,n-1$.

We have $v_i = s_i(t)$ for each $i$. Indeed, otherwise there exists $1 \le j \le n-1$ and $v_j^* \ne v_j$ such that $t \in \partial f_j(v_j^*)$. Consequently $(u_j, v_j^*, f_j(v_j^*)) \in \mathrm{graph}\,\varphi$. Therefore $n$ affinely independent points

$$(u_1, v_1, f_1(v_1)),\dots,(u_{n-1}, v_{n-1}, f_{n-1}(v_{n-1})),\ (u_j, v_j^*, f_j(v_j^*))$$

are contained in the convex set $C := \mathrm{epi}\, f \ \cap\ \{(x,y) : \ y \le \varphi(x)\}$. This contradicts the definition of $\mathcal{F}^*$, since clearly $F \subset C \subset \mathrm{graph}\, f = \partial K$.

Consequently $\mathcal{V}^*$ is contained in the image of the mapping

$$\Phi(t) := G(w_1(t),\dots,w_{j-1}(t), w_{j+1}(t),\dots,w_{n-1}(t)), \qquad t \in I := \bigcap_{i=1}^{n-1} I_i,$$

(see the definition of $G$ in Section 2), where

$$w_i(t) = (u_i - u_j, s_i(t) - s_j(t), f_i(s_i(t)) - f_j(s_j(t))), \quad 1 \le i \le n-1,\ i \ne j.$$

To prove that the image of $\Phi$ is $\sigma$-1-rectifiable, we can clearly suppose that $I$ is a compact non-degenerate interval. Since the functions $f_i$ are locally Lipschitz, the functions $f_i \circ s_i$ have bounded variation on $I$ and therefore, by Lemma 2.8, $\Phi$ has bounded variation. Using [6, 2.5.16], we obtain that $\Phi(I)$, and therefore also $\mathcal{V}^*$, is $\sigma$-1-rectifiable. □

**Theorem 3.2.** *For each $n$-dimensional convex closed set $K \subset \mathbb{R}^n$, the set $F^*_{n-2}(K)$ is $\sigma$-1-rectifiable.*

**Proof.** By Remark 2.5, it suffices to prove the theorem for $K = \operatorname{epi} f$, where $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is a convex function.

Let $F$ be an $(n-2)$-dimensional convex subset of $\partial K$ that is contained in no $(n-1)$-dimensional convex subset of $\partial K$. Choose $n-1$ affinely independent points $(e_1^{(r)}, \dots, e_n^{(r)})$, $r = 0, \dots, n-2$, in $F$. Clearly $(e_1^{(r)}, \dots, e_{n-1}^{(r)})$, $r = 0, \dots, n-2$, are also affinely independent. Therefore the matrix $(e_j^{(r)} - e_j^{(0)})$, $r = 1, \dots, n-2$, $j = 1, \dots, n-1$, has rank $n-2$.

Thus there exists a sequence $1 \le l_1 < l_2 < \cdots < l_{n-2} \le n-1$ such that $(e_{l_1}^{(r)}, \dots, e_{l_{n-2}}^{(r)})$, $r = 0, \dots, n-2$, are affinely independent. Therefore there exist $(n-1)$ affinely independent points $(s_1^{(i)}, s_2^{(i)}, \dots, s_{n-2}^{(i)}) \in \mathbb{Q}^{n-2}$, $i = 1, \dots, n-1$, in

$$\operatorname{conv}\{(e_{l_1}^{(r)}, \dots, e_{l_{n-2}}^{(r)}) :\ r = 0, \dots, n-2\}.$$

So $F$ intersects each hyperplane

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n :\ x_{l_q} = s_q^{(i)},\ q = 1, \dots, n-2\},\ i = 1, \dots, n-1.$$

Since there are finitely many sequences $l_1, \dots, l_{n-2}$ and countably many sequences $(s_1^{(i)}, \dots, s_{n-2}^{(i)}) \in \mathbb{Q}^{n-2}$, $i = 1, \dots, n-1$, Theorem 3.2 follows from Lemma 3.1 (using suitable permutations of coordinates). □

**Remark 3.3.** It follows from the proof of Lemma 3.1 that, for an $n$-dimensional convex closed set $K \subset \mathbb{R}^n$, $D_{n-2}(K)$ is $\sigma$-$(n-2)$-rectifiable. Indeed, there exist Lipschitz functions $z_{i,k}$, $i = 1, \dots, n-2$, $k \in \mathbb{N}$, on $\mathbb{R}$ such that $\mathcal{V}^*$ is contained in the union of sets

$$Z_k := \{\operatorname{Lin}\{z_{1,k}(t), \dots, z_{n-2,k}(t)\},\ t \in \mathbb{R}\},\ k \in \mathbb{N}.$$

Further, for every $k \in \mathbb{N}$ and every line segment $J$ in

$$\operatorname{Lin}\{z_{1,k}(t), \dots, z_{n-2,k}(t)\}, \quad t \in \mathbb{R},$$

$J$ is parallel to

$$\psi\Big(\sum_{i=1}^{j-1} t_i \cdot z_{i,k}(t) + z_{j,k}(t) + \sum_{i=j+1}^{n-2} t_{i-1} \cdot z_{i,k}(t)\Big),$$

for some $(t_1, \dots, t_{n-3}) \in \mathbb{R}^{n-3}$ and $j = 1, \dots, n-2$, where $\psi(x) = \frac{x}{\|x\|}$.

Since $\psi$ is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$, the set of all $u \in S^{n-1}$ parallel to a segment in $Z_k$ is contained in a union of $n-2$ locally Lipschitz images of $\mathbb{R} \times \mathbb{R}^{n-3}$.

**Remark 3.4.** For $n = 3$, Theorem 3.2 is clearly equivalent to the statement that $D_1(K)$ (see Definition 2.9) is $\sigma$-1-rectifiable, and so generalizes Mc Minn's result. Indeed, $D_2(K)$ consists of countably many circles, since $F_2(K)$ is obviously countable.

## 4. A dual formulation of questions on $D(K)$; results on tangent hyperplanes of d.c. functions

As usual, a hyperplane $T \subset \mathbb{R}^{k+1}$ of the form $T = \{(x,y) \in \mathbb{R}^k \times \mathbb{R} : \ y = \langle a, x\rangle + c\}$ will be identified with $(a,c) \in \mathbb{R}^{k+1}$.

If $f : \mathbb{R}^k \to \mathbb{R}$ is a function, then the set of all tangent hyperplanes to the graph of $f$ will be denoted by $T(f)$. So $T(f) \subset \mathbb{R}^{k+1}$ consists of all $(c,d)$, where $c = f'(a)$ and $d = f(a) - \langle a, f'(a)\rangle$.

We will need the following easy lemma.

**Lemma 4.1.** *Let $g$, $h$ be convex functions on $\mathbb{R}^k$. Set*

$$E(g,h) := \{(x^* - y^*, g(a) - h(a) - \langle a, x^* - y^*\rangle) : \\ a \in \mathbb{R}^{n-2},\ x^* \in \partial g(a),\ y^* \in \partial h(a)\}.$$

*Then $T(g-h) \subset E(g,h)$.*

**Proof.** Let $c \in \mathbb{R}^k$, $d \in \mathbb{R}$ and $(c,d) \in T(g-h)$. Choose $a \in \mathbb{R}^k$ such that $c = f'(a)$, $d = f(a) - \langle a, f'(a)\rangle$. Further, choose an arbitrary $y^* \in \partial h(a)$. Then it is easy to see that $x^* := y^* + f'(a) \in \partial g(a)$ and therefore $(c,d) = (f'(a), f(a) - \langle a, f'(a)\rangle)$ belongs to $E(g,h)$. □

For a short and sufficiently general formulation of our duality results, suppose that $\mathcal{I}_1$ is a $\sigma$-ideal of subsets $S^{n-1}$ and $\mathcal{I}_2$ is a $\sigma$-ideal of subsets of $\mathbb{R}^{n-1}$ with the following property: If $G \subset \mathbb{R}^{n-1}$, $H \subset S^{n-1}$ are open sets and $\varphi : G \to H$ is a diffeomorphism (where we consider $H$ as an $(n-1)$-dimensional $C^1$ manifold), then $A \subset G$ belongs to $\mathcal{I}_2$ if and only if $\varphi(A)$ belongs to $\mathcal{I}_1$.

**Proposition 4.2.** *Let $n \geq 3$ and $\mathcal{I}_1$, $\mathcal{I}_2$ be as above. Then the following two assertions hold.*

$(i)$ *The following statements are equivalent.*
  - $(a)$ *If $K \subset \mathbb{R}^n$ is a convex body, then $D(K) \in \mathcal{I}_1$.*
  - $(b)$ *For each pair of convex functions $g : \mathbb{R}^{n-2} \to \mathbb{R}$, $h : \mathbb{R}^{n-2} \to \mathbb{R}$, the set*

$$E(g,h) := \{(x^* - y^*, g(a) - h(a) - \langle a, x^* - y^*\rangle) : \\ a \in \mathbb{R}^{n-2},\ x^* \in \partial g(a),\ y^* \in \partial h(a)\}$$

  *belongs to $\mathcal{I}_2$.*

$(ii)$ *Let $T(w) \in \mathcal{I}_2$ for each d.c. function $w : \mathbb{R}^{n-2} \to \mathbb{R}$. Then $D_1(K) \in \mathcal{I}_1$ for each convex body $K \subset \mathbb{R}^n$.*

(Of course, $E(g,h) \subset \mathbb{R}^{n-1}$ after the usual identifications $(\mathbb{R}^{n-2})^* = \mathbb{R}^{n-2}$ and $\mathbb{R}^{n-2} \times \mathbb{R} = \mathbb{R}^{n-1}$.)

**Proof.** Denote $P := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \ x_1 > 0\}$ and define the function $\Phi_1 : P \to \mathbb{R}^{n-1}$ by

$$\Phi_1(x_1, x_2, \ldots, x_n) = \left(-\frac{x_2}{x_1}, -\frac{x_3}{x_1}, \ldots, -\frac{x_{n-1}}{x_1}, \frac{x_n}{x_1}\right).$$

The restriction of $\Phi_1$ to $P \cap S^{n-1}$ is clearly a diffeomorphism of $P \cap S^{n-1}$ onto $\mathbb{R}^{n-1}$.

First suppose that $(a)$ holds. It is clearly sufficient to prove that for all $m \in \mathbb{N}$ the set

$$\begin{aligned} E_m(g,h) := \{(x^* - y^*, g(a) - h(a) - \langle a, x^* - y^* \rangle) : \\ a \in B_{\mathbb{R}^{n-2}}(0,m), x^* \in \partial g(a), y^* \in \partial h(a)\} \end{aligned}$$

belongs to $\mathcal{I}_2$. Fix $m \in \mathbb{N}$. By Lemma 2.4 we can suppose that $g^*$ and $h^*$ are finite. Set

$$B_0 := \{0\} \times \text{epi } g^*, \quad B_1 := \{1\} \times \text{epi } h^*, B := \overline{\text{conv}(B_0 \cup B_1)}.$$

It is clear that the closed convex set $B$ has a non-empty interior. By $(a)$ and Remark 2.5 we obtain $D(B) \in \mathcal{I}_1$ and so $\Phi_1(D(B) \cap P) \in \mathcal{I}_2$. Thus it is sufficient to prove that $\Phi_1(D(B) \cap P) \supset E_m(g,h)$.

Let $a \in B_{\mathbb{R}^{n-2}}(0,m)$, $x^* \in \partial g(a)$, $y^* \in \partial h(a)$. Then $a \in \partial g^*(x^*) \cap \partial h^*(x^*)$. Let us define the affine function $q$ on $\mathbb{R}^{n-1} = \mathbb{R} \times \mathbb{R}^{n-2}$ by

$$\begin{aligned} q(t,v) := g^*(x^*) + \langle a, v \rangle - \langle a, x^* \rangle \\ +t \cdot (h^*(y^*) - g^*(x^*) + \langle a, x^* - y^* \rangle), \ t \in \mathbb{R}, \ v \in \mathbb{R}^{n-2}. \end{aligned}$$

Obviously,

$$\begin{aligned} q(0,v) = g^*(x^*) + \langle a, v - x^* \rangle \le g^*(v), \\ q(1,v) = h^*(y^*) + \langle a, v - y^* \rangle \le h^*(v). \end{aligned}$$

Thus, for $(e,f) \in B_0 \cup B_1$ with $e \in \mathbb{R}^{n-1}$, $f \in \mathbb{R}$, we have $q(e) \le f$. Hence $q(e) \le f$ also for each $(e,f) \in B$ with $e \in \mathbb{R}^{n-1}$, $f \in \mathbb{R}$. Clearly $q(0,x^*) = g^*(x^*)$ and $q(1,y^*) = h^*(y^*)$. Therefore the line segment with endpoints $(0, x^*, g^*(x^*)), (1, y^*, h^*(y^*))$ lies in $\partial B$. Letting $w := (1, y^*, h^*(y^*)) - (0, x^*, g^*(x^*))$, we obtain $w/\|w\| \in D(B) \cap P$. Since

$$\begin{aligned} \Phi_1(w/\|w\|) &= \Phi_1(w) = (x^* - y^*, h^*(y^*) - g^*(x^*)) \\ &= (x^* - y^*, -h(a) + \langle a, y^* \rangle + g(a) - \langle a, x^* \rangle) \\ &= ((x^* - y^*), g(a) - h(a) - \langle a, x^* - y^* \rangle), \end{aligned}$$

we have proved $\Phi_1(D(B) \cap P) \supset E_m(g,h)$.

Now we will prove simultaneously the opposite implication of $(i)$ and the statement $(ii)$. Consider an arbitrary convex function $f : \mathbb{R}^{n-1} \to \mathbb{R}$. By Remark 2.5 and a similar observation concerning $D_1(K)$, it is sufficient to prove that

$(\alpha)$ $D(\text{epi}\, f) \in \mathcal{I}_1$, if $(b)$ holds,
$(\beta)$ $D_1(\text{epi}\, f) \in \mathcal{I}_1$, if $T(w) \in \mathcal{I}_2$ for each d.c. function $w : \mathbb{R}^{n-2} \to \mathbb{R}$.

Denote $\pi_{r,i} := \{(x_1, \dots, x_n) \in \mathbb{R}^n : \ x_i = r\}$ for $r \in \mathbb{Q}$, $i = 1, \dots, n-1$. For every pair of parallel hyperplanes $\pi_{r_0,i}, \pi_{r_1,i}$ with $r_0, r_1 \in \mathbb{Q}, r_0 \neq r_1$, $i = 1, \dots, n-1$, and for every $m \in \mathbb{N}$, let us denote by $F(f, r_0, r_1, i, m)$ the set of all $u \in S^{n-1}$ parallel to a segment $J$ on the graph of $f$ such that the endpoints of $J$ belong to $B(0,m)$ and $\text{ri}\, J$ intersects both $\pi_{r_0,i}$ and $\pi_{r_1,i}$. It is easy to see that

$$D(\text{epi}\, f) = \bigcup F(f, r_0, r_1, i, m), \quad D_1(\text{epi}\, f) = \bigcup (F(f, r_0, r_1, i, m) \cap D_1), \tag{1}$$

where the union is taken over all $r_0$, $r_1$, $i$, $m$ considered above. Now consider $m \in \mathbb{N}$ and an arbitrary $s \in F_m := P \cap F(f, 0, 1, 1, m)$. Choose a segment $J$ parallel to $s$ on the graph of $f$ such that the endpoints of $J$ belong to $B(0, m)$ and $\operatorname{ri} J$ contains points $(a_0, f(a_0))$, $(a_1, f(a_1))$, where $a_0 = (0, a_0')$ and $a_1 = (1, a_1')$. Obviously,

$$\Phi_1(s) = \Phi_1((a_1, f(a_1)) - (a_0, f(a_0))).$$

Denote $f_0 := f(0, \cdot)$, $f_1 := f(1, \cdot)$, and $g := (f_0)^*$, $h := (f_1)^*$. Lemma 2.4 easily implies that we can suppose that the dual functions $g$, $h$ are finite.

By Lemma 2.1 there exists an affine function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\begin{aligned} \varphi(z) &\le f(z) \qquad \text{for} \quad z \in \mathbb{R}^{n-1}, \\ \varphi(\lambda a_0 + (1-\lambda)a_1) &= f(\lambda a_0 + (1-\lambda)a_1) \qquad \text{for} \ \ \lambda \in [0,1]. \end{aligned}$$

Let $u_1, v \in \mathbb{R}$, $u_2 \in \mathbb{R}^{n-2}$ be such that

$$\varphi(z) = v + \langle (u_1, u_2), z \rangle, \qquad z \in \mathbb{R}^{n-1}.$$

Since, for each $x \in \mathbb{R}^{n-2}$,

$$f_0(x) \ge \varphi(0, x) = v + \langle u_2, a_0' \rangle + \langle u_2, x - a_0' \rangle = f_0(a_0') + \langle u_2, x - a_0' \rangle,$$

we get $u_2 \in \partial f_0(a_0')$. Similarly we get $f_1(x) \ge f_1(a_1') + \langle u_2, x - a_1' \rangle$ and $u_2 \in \partial f_1(a_1')$. Hence $a_0' \in \partial g(u_2)$, $a_1' \in \partial h(u_2)$.

Further,

$$\begin{aligned} \Phi_1(s) &= \Phi_1((a_1, f(a_1)) - (a_0, f(a_0))) = (a_0' - a_1', f_1(a_1') - f_0(a_0')) \\ &= (a_0' - a_1', h^*(a_1') - g^*(a_0')) \\ &= (a_0' - a_1', -h(u_2) + \langle u_2, a_1' \rangle + g(u_2) - \langle u_2, a_0' \rangle) \\ &= (a_0' - a_1', g(u_2) - h(u_2) - \langle u_2, a_0' - a_1' \rangle) \in E(g, h). \end{aligned}$$

We have proved $\Phi_1(F_m) \subset E(g, h)$. Therefore $\Phi_1(F_m) \in \mathcal{I}_2$, and so $F_m \in \mathcal{I}_1$. Obviously, also $F(f, 0, 1, 1, m) \in \mathcal{I}_1$. By a similar argument we can prove that each $F(f, r_0, r_1, i, m)$ belongs to $\mathcal{I}_1$. So (1) implies ($\alpha$).

To prove ($\beta$), it is clearly sufficient to prove that $\Phi_1(s) \in T(g - f)$, if $J$ is as above and moreover $J \in U_1(\operatorname{epi} f)$ (see Definition 2.9).

Proceeding as above, we then obtain that $g$ is differentiable at $u_2$. Otherwise there exists $a_0'' \ne a_0'$, $a_0'' \in \partial g(u_2)$. Then $u_2 \in \partial f_0(a_0') \cap \partial f_0(a_0'')$. Hence the function $f$ is affine on $\operatorname{conv}\{(0, a_0'), (0, a_0'')\}$ and therefore

$$I' := \operatorname{conv}\{(0, a_0', f(a_0)), (0, a_0'', f(0, a_0''))\} \subset \operatorname{graph} f.$$

By Lemma 2.2, $\operatorname{conv}(J \cup I') \subset \operatorname{graph} f$ and $\dim \operatorname{conv}(J \cup I') = 2$. But this is impossible because $J \in U_1(\operatorname{epi} f)$.

Similarly we see that $h$ is differentiable at $u_2$. Thus, $a_0' - a_1' = (g - h)'(u_2)$. So (2) implies

$$\Phi_1(s) = ((g-h)'(u_2), (g-h)(u_2) - \langle u_2, (g-h)'(u_2) \rangle) \in T(g - h).$$

□

Combining Lemma 4.1, Proposition 4.2 and (ELR 1), we obtain the following result on tangent hyperplanes of d.c. functions.

**Theorem 4.3.** *Let $f : \mathbb{R}^k \to \mathbb{R}$ be a d.c. function (i.e. a difference of two convex functions). Let $T(f)$ denote the set of all tangent hyperplanes to the graph $G_f$ of $f$. Then $T(f)$ has $\sigma$-finite $k$-dimensional Hausdorff measure.*

We note that there exists a $C^1$-function $f : \mathbb{R}^k \to \mathbb{R}$ such that $T(f)$ has positive $(k+1)$-dimensional Hausdorff measure (see [3]).

**Remark 4.4.** Theorem 4.3 can be slightly generalized. Indeed, the proof of Lemma 4.1 shows that the set $T(f)$ can be replaced, in Lemma 4.1 and hence also in Theorem 4.3, by the larger set $T^s(f)$ of all "almost supporting hyperplanes". We say that a hyperplane $H \subset \mathbb{R}^{k+1}$ belongs to $T^s(f)$, if it is the graph of an affine function $g$ such that there exists $a \in \mathbb{R}^k$ for which $f(a) = g(a)$ and $\limsup_{x \to a}(g(x) - f(x))/\|x - a\| \le 0$.

Consequently, if $f$ is convex, then the set $T^s(f)$ (which is equal to the set of all supporting hyperplanes to epi $f$ in this case) has $\sigma$-finite $k$-dimensional Hausdorff measure. However, it is an obvious fact, since $T^s(f) = \{(x, -f^*(x)) :\ f^*(x) < \infty\}$ is even $\sigma$-$k$-rectifiable.

Using Proposition 4.2 and Remark 3.3, we easily show (without using any deeper result from literature) that the duality between $D(K)$ and $T(f)$ is very sharp for $n = 3$ and $n = 4$.

**Proposition 4.5.** *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be $\sigma$-ideals having the property formulated before Proposition 4.2. Suppose that $n = 4$ and $\mathcal{I}_1$ contains all $\sigma$-2-rectifiable sets or $n = 3$ and $\mathcal{I}_1$ contains all intersections $S^2 \cap V$, where $V \in G(3, 2)$.*

*Then the following assertions are equivalent.*

(*i*) *For each convex body $K \subset \mathbb{R}^n$, the set $D(K)$ belongs to $\mathcal{I}_1$.*
(*ii*) *For each d.c. function $f : \mathbb{R}^{n-2} \to \mathbb{R}$, the set $T(f)$ belongs to $\mathcal{I}_2$.*

**Proof.** Combining Lemma 4.1 and Proposition 4.2, we get the implication $(i) \Rightarrow (ii)$ in both cases.

Now suppose $(ii)$ holds. Then Proposition 4.2 implies $D_1(K) \in \mathcal{I}_1$ in both cases. Further, in both cases, $F_{n-1}$ is countable and consequently $D_{n-1}$ is covered by countably many sets of the form $S^{n-1} \cap V$, where $V \in G(n, n-1)$. So, in the case $n = 3$, $D_2(K) \in \mathcal{I}_1$ and therefore $(i)$ holds.

Now, let $n = 4$. Since, for each $V \in G(4, 3)$, $S^3 \cap V$ is $\sigma$-2-rectifiable, we obtain $D_3(K) \in \mathcal{I}_1$. Since $D_2(K)$ is $\sigma$-2-rectifiable by Remark 3.3, we obtain $(i)$. □

Applying Proposition 4.5 in the case when $n = 4$ and $\mathcal{I}_1$ and $\mathcal{I}_2$ are systems of all $\sigma$-2-rectifiable sets (or $\sigma$-$H^2$-rectifiable sets), we obtain a reformulation of Question 1.1 (for $n = 4$) in the language of tangent hyperplanes of d.c. functions $f : \mathbb{R}^2 \to \mathbb{R}$.

Using Proposition 4.5 in the case $n = 3$, we obtain a reformulation of Question 1.2 in the language of tangent hyperplanes of d.c. functions $f : \mathbb{R} \to \mathbb{R}$.

Another application of Proposition 4.5 is contained in the following remark.

**Remark 4.6.** Proposition 4.5 and the Morse-Sard theorem of Landis [9] (Theorem 2.6

above, case $k = 1$) easily imply that the set $D(K)$ has zero 3-dimensional Hausdorff measure for each convex body $K \subset \mathbb{R}^4$.

Indeed, for each d.c. function $f$ on $\mathbb{R}^2$ and for each $a \in \mathbb{R}^2$, Theorem 2.6 (applied to the function $\tilde{f} = f - \langle a, \cdot \rangle$) implies that the set

$$\{f(x) - \langle a, x\rangle : \ x \in \mathbb{R}^2, \ f'(x) - a = 0\}$$

is Lebesgue null.

In other words, the set $\{c \in \mathbb{R} : \ (a, c) \in T(f)\}$ is Lebesgue null for each $a \in \mathbb{R}^2$. Consequently Fubini's theorem implies that $T(f)$ is Lebesgue null. Applying Proposition 4.5 in the case when $n = 4$ and $\mathcal{I}_1$ and $\mathcal{I}_2$ are the systems of all sets of zero 3-dimensional Hausdorff measure, we obtain the desired result.

Thus, for $n = 4$, we get the answer to the question of Klee ([7]) using a completely different proof than that in [5].

Larman and Rogers proved the following theorem [10, Theorem 2].

**Theorem 4.7.** *Let $B \subset \mathbb{R}^n$ be an $n$-dimensional convex closed set, $n \geq 3$, and $S$ be a hyperplane in $\mathbb{R}^n$. Let $D_S(B)$ be the set of all $u \in S^{n-1}$ parallel to $S$ and parallel to some line segment in $\partial B$ which do not meet the intersection of $B$ with supporting hyperplanes, that are parallel to $S$, at points in the interior of these intersections relative to the supporting hyperplanes. Then $D_S(B)$ has zero $(n-2)$-dimensional Hausdorff measure.*

**Remark 4.8.** 1. In fact, this theorem is proved in [10] in the case when $B$ is a convex body. But this special case obviously implies the general case (cf. Remark 2.5).

2. Using (ELR 1), one easily sees that (LR) in Section 1 is a reformulation of Theorem 4.7. Indeed, the vectors in $D_S(B) \setminus F_1^S(B)$ are parallel to a segment in the relative boundary of the intersection of $B$ with a supporting hyperplane parallel to $S$ (relative to this supporting hyperplane). Therefore (ELR 1) easily implies that $D_S(B) \setminus F_1^S(B)$ has $\sigma$-finite $(n-3)$-dimensional Hausdorff measure.

From Theorem 4.7 we now deduce two theorems on special tangent hyperplanes to the graph of a d.c. function. In fact, we repeat the proof of Proposition 4.2, the implication $(a) \Rightarrow (b)$, with some additional arguments.

**Proposition 4.9.** *Let $n \geq 3$, $z \in \mathbb{R}^{n-1}$ and $f : \mathbb{R}^{n-2} \to \mathbb{R}$ be a d.c. function. Denote by $T_z(f)$ the set of all tangent hyperplanes to the graph of $f$ containing $z$. Then $T_z(f)$ has zero $(n-2)$-dimensional Hausdorff measure.*

**Proof.** We can assume that $z = 0$. Indeed, $(a, c) \in \mathbb{R}^{n-2} \times \mathbb{R}$ belongs to $T_z(f)$, if and only if $(a, c)$ belongs to $T_0(g)$, where $g(u) = f(u + z_1) - z_2$, $u \in \mathbb{R}^{n-2}$ and $z = (z_1, z_2) \in \mathbb{R}^{n-2} \times \mathbb{R}$.

We can also suppose that $f = g - h$, where $g, h$ are convex and $f^*, g^*$ are finite (otherwise we take $g + \|\cdot\|^2, h + \|\cdot\|^2$ instead of $g, h$).

Let $P$, $\Phi_1$ and $B$ have the same meaning as in the proof of Proposition 4.2, implication $(a) \Rightarrow (b)$. Set

$$S := \{(u_1, \ldots, u_n) \in \mathbb{R}^n : \ u_n = 0\}.$$

Then the set $D_S(B)$ in Theorem 4.7 has zero $(n-2)$-dimensional Hausdorff measure. We shall prove that

$$\tilde{T}_0(f) := \{(f'(a),0) \in \mathbb{R}^{n-1} : \ 0 \neq a \in \mathbb{R}^{n-2}, \\ f'(a) \text{ exists}, f(a) = \langle a, f'(a)\rangle\} \subset \Phi_1(D_S(B) \cap P). \tag{2}$$

Let $0 \neq a \in \mathbb{R}^{n-2}$, $f(a) = \langle a, f'(a)\rangle$. Then there exist $x^* \in \partial g(a), y^* \in \partial h(a)$, such that $x^* - y^* = f'(a)$ (see Lemma 4.1). Therefore $a \in \partial g^*(x^*) \cap \partial h^*(y^*)$. We define the affine function $q$ as in the proof of Proposition 4.2 and obtain that the line segment $J$ with endpoints $w_0 = (0, x^*, g^*(x^*))$ and $w_1 = (1, y^*, h^*(y^*))$ lies in $\partial B$. Since

$$h^*(y^*) - g^*(x^*) = g(a) - h(a) - \langle a, x^* - y^*\rangle = f(a) - \langle a, f'(a)\rangle = 0,$$

the segment $J$ is parallel to $S$. We have also

$$\Phi_1\left(\frac{w_1 - w_0}{\|w_1 - w_0\|}\right) = \Phi_1(w_1 - w_0) = (x^* - y^*, g(a) - h(a) - \langle a, x^* - y^*\rangle) = (f'(a), 0).$$

Now suppose that $H$ is a supporting hyperplane to $B$, that is parallel to $S$, and $J$ meets $B \cap H$ at a point $z$ in the interior of $B \cap H$ relative to $H$. Then $H$ is the unique supporting hyperplane to $B$ at $z$. Since $\operatorname{epi} q \supset B$ and $\operatorname{graph} q \supset \operatorname{conv}\{w_0, w_1\}$, we obtain $H = \operatorname{graph} q$. Therefore $a = 0$, which is a contradiction.

So we have proved $\frac{w_1 - w_0}{\|w_1 - w_0\|} \in D_S(B) \cap P$, which implies (2). Since $\Phi_1$ is locally Lipschitz, (2) implies that $\tilde{T}_0(f)$ has zero $(n-2)$-dimensional Hausdorff measure. Observing that there is at most one hyperplane in $T_0(f) \setminus \tilde{T}_0(f)$, we obtain the assertion of the theorem. □

**Proposition 4.10.** *Let $n \geq 3$, $f : \mathbb{R}^{n-2} \to \mathbb{R}$ be a d.c. function and let $L$ be a line in $\mathbb{R}^{n-1}$. Denote by $T_L(f)$ the set of all tangent hyperplanes to the graph of $f$ that are parallel to $L$. Then $T_L(f)$ has zero $(n-2)$-dimensional Hausdorff measure.*

**Proof.** Using a suitable change of the coordinates, we can assume that $L$ is parallel to $((1,0,\ldots,0),\alpha)$ for some $\alpha \in \mathbb{R}$. We can also assume $\alpha = 0$. Indeed, denoting by $L^*$ a line parallel to $((1,0,\ldots,0),0)$, we easily see that $(a,c) \in \mathbb{R}^{n-2} \times \mathbb{R}$ belongs to $T_L(f)$, if and only if $(a - (\alpha,0,\ldots,0), c)$ belongs to $T_{L^*}(g)$, where $g(u_1,\ldots,u_{n-2}) = f(u_1,\ldots,u_{n-2}) - \alpha \cdot u_1$. So

$$T_L(f) = \{(f'(a), f(a) - \langle a, f'(a)\rangle) : \ a \in \mathbb{R}^{n-2}, f'(a) \text{ exists }, f_1'(a) = 0\}$$

(where $f_1'$ denotes the first partial derivative).

As in the proof of Proposition 4.9, we can suppose that $f = g - h$, where $g, h$ are convex functions and $g^*, h^*$ are finite. Let $P$, $\Phi_1$ and $B$ have again the same meaning as in the proof of Proposition 4.2, implication $(a) \Rightarrow (b)$. Set

$$S := \{(u_1,\ldots,u_n) \in \mathbb{R}^n : \ u_2 = 0\}.$$

We shall prove that $\Phi_1(D_S(B)) \supset T_L(f)$.

Let $a \in \mathbb{R}^{n-2}$ with $f_1'(a) = 0$. There exist $x^* = (x_1^*,\ldots,x_{n-2}^*) \in \partial g(a)$ and $y^* = (y_1^*,\ldots,y_{n-2}^*) \in \partial h(a)$ such that $x^* - y^* = f'(a)$. Obviously, $x_1^* = y_1^*$. The line segment with endpoints $w_0 = (0, x^*, g^*(x^*))$ and $w_1 = (1, y^*, h^*(y^*))$ lies in $\partial B$, is parallel to $S$ and $\Phi_1(w_1 - w_0) = (f'(a), f(a) - \langle a, f'(a)\rangle)$. Obviously, there is no supporting hyperplane to $B$ parallel to $L$. So we esily see that $T_L(f) \subset \Phi_1(D_S(B) \cap P)$. Since $\Phi_1$ is locally Lipschitz, Theorem 4.7 implies that $T_P(f)$ has zero $(n-2)$-dimensional Hausdorff measure. $\square$

**Remark 4.11.** For $n = 3$ we can assert in Proposition 4.9 and Proposition 4.10 that the corresponding sets of tangent hyperplanes are even of zero 1/2-dimensional Hausdorff measure. This fact easily follows from Theorem 2.7. We could also use, in the above proofs, Theorem 6.2 (cf. Remark 6.3) instead of Theorem 4.7.

For $n = 4$, we could deduce the above-mentioned propositions from Theorem 2.6 as in Remark 4.6.

## 5. McMinn's result is close to the best possible one

Let us consider the space $BVC([0,1])$ of all continuous functions on $[0,1]$ of bounded variation equipped with the norm defined by

$$\|g\| := \sup_{[0,1]} g + \mathrm{V}_0^1\, g,$$

where $\mathrm{V}_0^1\, g$ means the variation of $g$ on $[0,1]$. It is well known that $BVC([0,1])$ is a Banach space. (It follows easily from the fact that the space $BV([0,1])$ equipped with the norm

$$\|g\|_1 := |g(0_+)| + \mathrm{V}_0^1\, g$$

is a Banach space; see [4, Chap. IV.12]. Indeed, these two norms are equivalent on $BVC([0,1])$ and $BVC([0,1])$ is closed in $BV([0,1])$.)

**Lemma 5.1.** *There exist a dense subset $A$ of $(0,1)$ and a function $g \in BVC([0,1])$ such that, for each $a \in A$ and each $\varepsilon > 0$, there exist $e, e' \in (a, a+\varepsilon) \cap (0,1)$ such that $e < e'$ and*

$$\begin{aligned}
&(i) && g(a) = g(e),\\
&(ii) && \int_a^e (g(x) - g(a))\,\mathrm{d}x < 0,\\
&(iii) && g(x) > g(a) \textit{ for each } x \in (e, e'],\\
&(iv) && g(e') - g(a) \ge |g(a) - g(x)| \textit{ for each } x \in (a, e').
\end{aligned}$$

**Proof.** For each $a \in (0,1)$ and $n \in \mathbb{N}$, denote by $Q(a,n)$ the set of all functions $g \in BVC([0,1])$ such that there exist $e, e' \in (a, a + \frac{1}{n}) \cap (0,1)$, $e < e'$, for which $(i)$, $(ii)$, $(iii)$ and $(iv)$ hold.

Now we shall prove that $\operatorname{int} Q(a,n)$ is dense in $BVC([0,1])$.

Let $h \in BVC([0,1])$ and $\delta > 0$ be given. Choose $0 < \delta_2 < \frac{1}{6}\min\{\frac{1}{n}, 1-a\}$ so small that the function $f$ defined by

$$\begin{aligned} f(x) &= h(x) \text{ for } x \geq a + 6\delta_2 \text{ or } x \leq a, \\ f(a+\delta_2) &= f(a) - \delta_2, \\ f(a+4\delta_2) &= f(a) + 2\delta_2, \\ f(x) & \text{ is affine on } [a, a+\delta_2], [a+\delta_2, a+4\delta_2], [a+4\delta_2, a+6\delta_2], \end{aligned}$$

fulfils $\|h-f\| \leq \delta$. Set $\delta_1 := \frac{\delta_2}{24}$ and suppose that $g \in BVC([0,1])$ with $\|g-f\| < \delta_1$ be given. We shall prove $g \in Q(a,n)$.

Obviously, $g(a+\delta_2) < g(a)$ and $g(x) > g(a)$ for all $x \in [a+3\delta_2, a+4\delta_2]$. Set

$$e := \sup\{x \in [a+\delta_2, a+4\delta_2] : \; g(x) = g(a)\}$$

and choose $e' \in [a, a+4\delta_2]$ such that $g(e') = \sup\{g(x) : \; x \in [a, a+4\delta_2]\}$. We easily get $e \in [a+\delta_2, a+3\delta_2), e' \in [a+3\delta_2, a+4\delta_2]$ and $(i)$, $(iii)$ and $(iv)$ clearly hold. Since

$$\begin{aligned} \int_a^e (g(x)-g(a))\,\mathrm{d}x &\leq \int_a^e (f(x)-f(a))\,\mathrm{d}x + 2\delta_1(e-a) \\ &\leq \int_a^{a+\delta_2} (f(x)-f(a))\,\mathrm{d}x + 2\delta_1 \cdot 3\delta_2 = -\frac{1}{2}\delta_2^2 + \frac{1}{4}\delta_2^2 < 0, \end{aligned}$$

the condition $(ii)$ holds too.

So the set $Q := \bigcap Q(a,n)$, where the intersection is taken over all $a \in \mathbb{Q} \cap (0,1)$ and $n \in \mathbb{N}$, is residual. Now, to prove the lemma, it is sufficient to choose $g \in Q$ and to put $A = \mathbb{Q} \cap (0,1)$. □

If $M \subset \mathbb{R}^n$ and $a \in \mathbb{R}^n$, we denote by $\operatorname{Tan}(M,a)$ the tangent (contingent) cone of $M$ at $a$ (see [6, 3.1.21]). Recall that $0 \in \operatorname{Tan}(M,a)$ if and only if $a \in \overline{M}$ and $b \in \mathbb{R}^n \setminus \{0\}$ belongs to $\operatorname{Tan}(M,a)$ if and only if there exists a sequence $(a_n)$ in $M \setminus \{a\}$ with

$$a_n \to a \quad \text{and} \quad \frac{a_n - a}{\|a_n - a\|} \to \frac{b}{\|b\|}.$$

For a d.c. function $G : \mathbb{R} \to \mathbb{R}$ of class $C^1$ we define the mapping $\tilde{G}(x) = (G'(x), G(x) - xG'(x))$, $x \in \mathbb{R}$.

**Lemma 5.2.** *There exists a d.c. function $G : \mathbb{R} \to \mathbb{R}$ of class $C^1$ such that, in every non-empty open interval $I \subset (0,1)$, there exists $a \in I$ such that $\operatorname{Tan}(\tilde{G}(I), \tilde{G}(a))$ contains infinitely many half-lines starting at 0.*

**Proof.** Take $g$ and $A$ as in Lemma 5.1. Choose increasing continuous functions $g_1$, $g_2$ on $\mathbb{R}$ such that $g = g_1 - g_2$ on $[0,1]$. Define $G(x) := \int_0^x (g_1(t) - g_2(t))\,\mathrm{d}t$, $x \in \mathbb{R}$. Obviously, $G$ is a d.c. function of class $C^1$ on $\mathbb{R}$.

Given an open interval $I \subset (0,1)$, choose a point $a \in I \cap A$ and put

$$H(x) = \frac{(G(x) - xG'(x)) - (G(a) - aG'(a))}{G'(x) - G'(a)},$$

whenever $x \in \mathbb{R}$ with $G'(x) \neq G'(a)$.

Observe that $(1, u) \in \mathrm{Tan}(\tilde{G}(I), \tilde{G}(a))$, whenever there exists a sequence $a_n \to a$ such that $G'(a_n) > G'(a)$ and $H(a_n) = u$.

A simple computation gives

$$H(x) = \frac{\int_a^x (g(t) - g(a))\,\mathrm{d}t}{g(x) - g(a)} - x.$$

Let $\varepsilon > 0$. Choose $e, e' \in (a, a+\varepsilon) \cap (0,1)$, which correspond to $g$, $A$ and $\varepsilon$ according to Lemma 5.1. Then, by (*iii*) and (*iv*),

$$H(e') + e' \geq -\frac{\int_a^{e'} |g(t) - g(a)|\,\mathrm{d}t}{g(e') - g(a)} \geq a - e',$$

and, by (*i*), (*ii*) and (*iii*), $H(e_+) = -\infty$. Therefore $H((e, e']) \supset (-\infty, a - 2e']$. Therefore, since $\varepsilon > 0$ was arbitrary and (*iii*) holds, for each $u \in (-\infty, -a)$ there exists a sequence $a_n \to a$ as above. Hence, $\{1\} \times (-\infty, -a) \subset \mathrm{Tan}(\tilde{G}(I), \tilde{G}(a))$, which completes the proof. □

**Proposition 5.3.** *Let $\mathcal{I}_1$ be the $\sigma$-ideal generated by all closed sets $C \subset S^2$ such that, for all $c \in C$, $\mathrm{Tan}(C, c)$ consists of points of finitely many half-lines starting at 0. Then there exists a convex set $K$ such that $D(K) \notin \mathcal{I}_1$.*

**Proof.** Denote by $\mathcal{I}_2$ the $\sigma$-ideal generated by all closed sets $D \subset \mathbb{R}^2$ such that, for all $d \in D$, $\mathrm{Tan}(D, d)$ consists of points of finitely many half-lines starting at 0.

Take $G$ from Lemma 5.2. By Proposition 4.2 and Lemma 4.1, it suffices to prove $T(G) = \{(G'(a), G(a) - aG'(a)) : a \in \mathbb{R}\} \notin \mathcal{I}_2$.

Suppose, on the contrary, $T(G) \in \mathcal{I}_2$. Then $\tilde{G}([0,1]) \subset \bigcup_{i=1}^{\infty} D_i$, where $D_i$ are closed sets such that, for all $d \in D_i$, $\mathrm{Tan}(D_i, d)$ consists of points of finitely many half-lines starting at 0. The sets $M_i := (\tilde{G})^{-1}(D_i)$, $i \in \mathbb{N}$, are closed and $\bigcup_{i=1}^{\infty} M_i \supset [0,1]$. By the Baire Category Theorem, for some $i \in \mathbb{N}$ there is a non-empty open interval $I \subset M_i \cap [0,1]$. Since $\tilde{G}(I) \subset D_i$, we obtain that, for each $a \in I$, $\mathrm{Tan}(D_i, \tilde{G}(a))$ consists of finitely many half-lines starting at 0. But this contradicts the choice of $G$. □

We shall say that $\varphi : [0,1] \to \mathbb{R}^n$ has the half-tangents at $a \in (0,1)$, if the limits

$$\lim_{t \to a+} \frac{\varphi(t) - \varphi(a)}{\|\varphi(t) - \varphi(a)\|}, \quad \lim_{t \to a-} \frac{\varphi(t) - \varphi(a)}{\|\varphi(t) - \varphi(a)\|}$$

exist. We shall say that $\varphi$ has the half-tangent at $a \in \{0,1\}$, if one of the limits exists.

As a consequence of Proposition 5.3, we get the following proposition.

**Proposition 5.4.** *There exists a convex body $K \subset \mathbb{R}^3$ such that $D(K)$ cannot be covered by countably many images of simple curves $\varphi_i : [0,1] \to S^2$ having the half-tangents at all points of $[0,1]$.*

**Proof.** If $\varphi : [0,1] \to \mathbb{R}^n$ is a simple curve having half-tangents at all points of $[0,1]$, then, for all $a \in [0,1]$, $\mathrm{Tan}(\varphi([0,1]), \varphi(a))$ contains at most two half-lines starting at 0. Thus Proposition 5.4 follows from Proposition 5.3. □

Let us note that the set $N(K)$ of all non-smooth points of the boundary of a convex body $K \subset \mathbb{R}^3$ can be covered by countably many simple curves having half tangents at all points ([19, Theorem 3]).

**Remark 5.5.** It is possible to prove (see [14] for a stronger example) that there exists even a convex body $K \subset \mathbb{R}^3$ such that $D(K)$ cannot be covered by images of continuous curves $\varphi_k : [0,1] \to S^3$, $k \in \mathbb{N}$, having half-tangents at all points except a countable set. We have proved only the weaker result, since the proof in [14] is much more complicated and a full characterization of the $\sigma$-ideal generated by the sets $D(K)$ is not known; it is even possible that this $\sigma$-ideal is that of all $\sigma$-1-rectifiable sets.

## 6. The directions of special $(n-2)$-dimensional and $(n-3)$-dimensional convex subsets of $\partial K$

Remember that, for $S \in G(n,s)$, we denote by $F_r^S(K)$ the set of all $V \in G(n,r)$ such that $V \subset S$ and $V$ is parallel to an $r$-dimensional ball $B \subset \partial K$ for which $(B+S) \cap \mathrm{int}\, K \neq \emptyset$.

**Lemma 6.1.** *Let $f : \mathbb{R}^{n-1} \to \mathbb{R}$ be a convex function, $n \geq 3$, and let $u_i \in \mathbb{R}^{n-2}$, $i = 1, \dots, n-1$, be affinely independent vectors. Denote by $\mathcal{F}$ the set of all $(n-2)$-dimensional convex subsets $F$ of $\partial K$, where $K = \mathrm{epi}\, f$, that are parallel to $S := \{(y_1, \dots, y_n) \in \mathbb{R}^n : y_{n-1} = 0\}$ and for which*

$$\{(u_i, v, y) : \ v, y \in \mathbb{R}\} \cap \mathrm{ri}\, F \neq \emptyset, \quad i = 1, \dots, n-1.$$

*Then the set $\mathcal{V}$ of all $V \in G(n, n-2)$ that are parallel to an $F \in \mathcal{F}$ has zero 1/2-dimensional Hausdorff measure.*

**Proof.** For $i \in \{n-2, n-1\}$, denote by $\mathcal{F}_i$ the set of all $F \in \mathcal{F}$ such that $m(F,K) = i$ and denote by $\mathcal{V}_i$ the set of all $V \in G(n, n-2)$ parallel to an $F \in \mathcal{F}_i$.

Then $\mathcal{V}_{n-1}$ is countable since there exist only countably many $V \in G(n, n-1)$ parallel to an $(n-1)$-dimensional ball in $\partial K$ and in each such $V$ there exists exactly one $V' \in G(n, n-2)$ parallel to $S$.

We shall prove that $\mathcal{V}_{n-2}$ has zero 1/2-dimensional Hausdorff measure.

Let $F \in \mathcal{F}_{n-2}$ and $V \in \mathcal{V}_{n-2}$ be parallel to $F$. Put $f_i(v) = f(u_i, v)$, $v \in \mathbb{R}$. We can assume (cf. the proof of Lemma 2.4) that $f_i^*$ are finite, $i = 1, \dots, n-1$. Choose $v_1, \dots, v_{n-1} \in \mathbb{R}$ such that $(u_i, v_i, f(v_i)) \in \mathrm{ri}\, F$, $i = 1, \dots, n-1$. Since $F$ is parallel to $S$, $v_i = v$, $i = 1, \dots, n-1$, for some $v \in \mathbb{R}$. By Lemma 2.1, there exists an affine function $\varphi$ on $\mathbb{R}^{n-1}$ such that $F \subset \mathrm{graph}\, \varphi$ and $\varphi \leq f$. Then clearly $t := \varphi'_{n-1}(x) \in \partial f_i(v)$ for $x \in \mathbb{R}^{n-1}$. Thus $v \in \partial f_i^*(t)$ for $i = 1, \dots, n-1$.

For each $1 \leq i \leq n-1$, the function $f_i^*$ is differentiable at $t$. Otherwise there exists $v \neq v' \in \partial f_i^*(t)$. Then $f_i$ is affine on $\mathrm{conv}\{v, v'\}$ and thus $f$ is affine on $\{u_i\} \times \mathrm{conv}\{v, v'\}$. Therefore $I := \mathrm{conv}\{(u_i, v, f(u_i, v)), (u_i, v', f(u_i, v'))\} \subset \mathrm{graph}\, f$. Then, by Lemma 2.2, $\mathrm{conv}(I \cup F) \subset \mathrm{graph}\, f$, which contradicts $F \in \mathcal{F}_{n-2}$.

We have $V = G(w_2, \dots, w_{n-1})$, where

$$w_i := (u_i - u_1, 0, f_i(v) - f_1(v)) = (u_i - u_1, 0, f_1^*(t) - f_i^*(t)).$$

For each $i \in \{2, \dots, n-1\}$, define the mapping $z_i(s) := (u_i - u_1, 0, f_1^*(s) - f_i^*(s)),\ s \in \mathbb{R}$. Since the mapping $z = (z_2, \dots, z_{n-1})$ is a d.c. mapping from $\mathbb{R}$ to $\mathbb{R}^{3(n-2)}$, Theorem 2.7 implies that the set

$$A := \{(z_2(s), \dots, z_{n-1}(s)) :\ s \in \mathbb{R},\ (f_1^* - f_i^*)'(s) = 0,\ i = 2, \dots, n-1\}$$

has zero 1/2-dimensional Hausdorff measure. Obviously, $(f_1^* - f_i^*)'(t) = 0,\ i = 2, \dots, n-1$. Consequently $V \in G(A)$ and Lemma 2.8 implies the assertion of the lemma. □

Denote by $H_S(K)$ the set of all points $x \in \partial K$ such that $(x+S) \cap \operatorname{int} K \neq \emptyset$. Then clearly $F_r^S(K)$ is the set of all $V \in G(n,r)$ such that $V \subset S$ and $V$ is parallel to an $r$-dimensional ball $B \subset H_S(K)$.

**Theorem 6.2.** *Let $K \subset \mathbb{R}^n$, $n \geq 3$, be an $n$-dimensional convex closed set and $S \in G(n, n-1)$. Then $F_{n-2}^S(K)$ has zero 1/2-dimensional Hausdorff measure.*

**Proof.** For each $x \in H_S(K)$, we can choose a system $(y_1, \dots, y_n)$ of cartesian coordinates, an open neighbourhood $U$ of $x$ and a convex function $f$ defined on an open convex subset of $\mathbb{R}^{n-1}$ such that $U \cap \partial K = U \cap H_S(K)$ is described by the equation $y_n = f(y_1, \dots, y_{n-1})$. (For a proof choose $z \in (x+S) \cap \operatorname{int} K$ and put $u_n := \frac{z-x}{\|z-x\|}$. Further, choose a unit vector $u_{n-1}$ orthogonal to $S$ and $u_1, \dots, u_{n-2}$ such that $u_1, \dots, u_n$ form an orthonormal basis. Take the system of cartesian coordinates $(y_1, \dots, y_n)$ with respect to $u_1, \dots, u_n$ and take a suitable small open neighbourhood $U$ of $x$.) Therefore it is sufficient to prove the theorem for $K = \operatorname{epi} f$, where $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is convex, and $S = \{(y_1, \dots, y_n) :\ y_{n-1} = 0\}$.

Let $F$ be an $(n-2)$-dimensional convex subset of $\partial K$ parallel to $S$. Consider the projection $\pi(y_1, \dots, y_n) := (y_1, \dots, y_{n-2})$. Then $\dim \pi(F) = n-2$ and thus there exist affinely independent $u_1, \dots, u_{n-1} \in \mathbb{Q}^{n-2} \cap \operatorname{int} \pi(F)$. Obviously, $\pi^{-1}(u_i) \cap \operatorname{ri} F \neq \emptyset,\ \ i = 1, \dots, n-1$. Since the set $(\mathbb{Q}^{n-2})^{n-1}$ is countable, the assertion of Theorem 6.2 follows from Lemma 6.1. □

**Remark 6.3.** For $n = 3$, the above theorem improves (LR) (or, eqivalently, Theorem 4.7).

Further, note that the result of Theorem 6.2 is the best possible in the sense that we can write "$s$-dimensional Hausdorff measure" instead of "1/2-dimensional Hausdorff measure" in the assertion of Theorem 6.2 for no $s < 1/2$. For the proof we can use the construction from the proof of Proposition 4.2, the implication $(a) \Rightarrow (b)$, and the fact (see [13, Remark 3]) that Theorem 2.7 does not hold with $s < 1/2$ instead of 1/2.

**Lemma 6.4.** *Let $f : \mathbb{R}^{n-1} \to \mathbb{R}$ be a convex function, $n \geq 4$, and let $u_i \in \mathbb{R}^{n-3}$, $i = 1, \dots, n-2$, be affinely independent vectors. Denote by $\mathcal{F}$ the set of all $(n-3)$-dimensional convex subsets $F$ of $\partial K$, where $K = \operatorname{epi} f$, that are parallel to $S := \{(y_1, \dots, y_n) \in \mathbb{R}^n : y_{n-2} = y_{n-1} = 0\}$ such that*

$$\{(u_i, v_1, v_2, y) :\ v_1, v_2, y \in \mathbb{R}\} \cap \operatorname{ri} F \neq \emptyset, \quad i = 1, \dots, n-2.$$

*Then the set $\mathcal{V}$ of all $V \in G(n, n-3)$ that are parallel to an $F \in \mathcal{F}$ has zero 1-dimensional Hausdorff measure.*

**Proof.** For $i \in \{n-3, n-2, n-1\}$, denote by $\mathcal{F}_i$ the set of all $F \in \mathcal{F}$ such that $m(F, K) = i$ and by $\mathcal{V}_i$ the set of all $V \in G(n, n-3)$ parallel to an $F \in \mathcal{F}_i$.

First we shall prove that $\mathcal{V}_{n-2} \cup \mathcal{V}_{n-1}$ has zero 1-dimensional Hausdorff measure.

Let $F \in \mathcal{F}_{n-2} \cup \mathcal{F}_{n-1}$ and $V \in G(n, n-3)$ be parallel to $F$. There exists a convex set $E \subset \operatorname{graph} f$, $E \supset F$, $\dim E = n-2$. Let us consider the projection $\pi_1(y_1, \dots, y_n) := (y_{n-2}, y_{n-1})$. Then $\dim \pi_1(E) \geq 1$ and so there exist $r \in \mathbb{Q}$ and $i \in \{n-2, n-1\}$ such that $\{u_i = r\} \cap \operatorname{ri} E \neq \emptyset$. Choose $x$ in this intersection. Then there exists $\tilde{F} \subset E$, $\dim \tilde{F} = n-3$, parallel to $F$ such that $x \in \tilde{F}$. Thus $\tilde{F} \subset \{u_i = r\}$.

Now denote $\pi_2(y_1, \dots, y_n) := (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)$. The set $\tilde{F}$ is parallel to $S$ and therefore $\pi_2(V)$ is parallel both to $\pi_2(\tilde{F})$ and to $\pi_2(S)$. Thus $\pi_2(V) \in F^{S_1}_{n-3}(K_1)$, where $S_1 := \pi_2(S)$ and $K_1 := \pi_2(K \cap \{u_i = r\})$. Therefore Theorem 6.2 implies that $\mathcal{V}_{n-2} \cup \mathcal{V}_{n-1}$ has zero 1-dimensional Hausdorff measure.

Now we shall prove that $\mathcal{V}_{n-3}$ has zero 1-dimensional Hausdorff measure. Let $F \in \mathcal{F}_{n-3}$ and $V \in \mathcal{V}_{n-3}$ be parallel to $F$. Put $f_i(v_1, v_2) = f(u_i, v_1, v_2)$. We can assume that $f_i^*$ are finite, $i = 1, \dots, n-1$. Choose $v_1, v_2 \in \mathbb{R}$ such that $(u_i, v_1, v_2, f_i(v_1, v_2)) \in \operatorname{ri} F$, $i = 1, \dots, n-1$. By Lemma 2.1 there exists an affine function $\varphi$ on $\mathbb{R}^{n-1}$ such that $F \subset \operatorname{graph} \varphi$ and $\varphi \leq f$. Then $t := (\varphi'_{n-2}(x), \varphi'_{n-1}(x)) \in \partial f_i(v_1, v_2)$ for each $x \in \mathbb{R}^{n-1}$, and so $(v_1, v_2) \in \partial f_i^*(t)$, $\quad i = 1, \dots, n-1$.

As in the proof of Lemma 6.1, we get, by Lemma 2.2, that each $f_i^*$, $i = 1, \dots, n-2$, is differentiable at $t$, since $F \in \mathcal{F}_{n-3}$.

If we define $z_i(s) := (u_i - u_1, 0, 0, f_1^*(s) - f_i^*(s))$, $i = 2, \dots, n-2$, and

$$A := \{(z_2(s), \dots, z_{n-1}(s)) : \ s \in \mathbb{R}^2,\ (f_1^* - f_i^*)'(s) = 0,\ i = 2, \dots, n-1\},$$

then $V \in G(A)$. It follows from the Morse-Sard Theorem on d.c. mappings (Theorem 2.6) that the set $A$ has zero 1-dimensional Hausdorff measure. So Lemma 2.8 implies that $G(A)$ has zero 1-dimensional Hausdorff measure. □

**Theorem 6.5.** *Let $K \subset \mathbb{R}^n$, $n \geq 4$, be an $n$-dimensional convex closed set and $S \in G(n, n-2)$. Then $F^S_{(n-3)}$ has zero 1-dimensional Hausdorff measure.*

**Proof.** The assertion of the theorem follows from Lemma 6.4 in the same way as Theorem 6.2 follows from Lemma 6.1. □

**Acknowledgements.** The research was partially supported by the grant GAČR 201/03/0931 from the Grant Agency of Czech Republic and partially supported by the grant MSM 0021620839 from the Czech Ministry of Education.

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