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A Varifolds Representation of the Relaxed Elastica Functional

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We prove a new representation result for the L^1-lower semicontinuous envelope of the elastica functional in terms of a minimum problem over a suitable class of varifolds. We also show a representation result in a suitable class of Sobolev-type submanifolds.

1. Introduction

Let us consider the functional

$$\mathcal{F}(E) := \int_{\partial E} [1 + |\kappa|^p] \, d\mathcal{H}^1, \tag{1}$$

where $E \subset \mathbb{R}^2$ is a bounded open subset of class $\mathcal{C}^2$, $p > 1$ is a real number, $\kappa = \kappa_{\partial E}$ is the curvature of ∂E and $\mathcal{H}^1$ is the one-dimensional Hausdorff measure in $\mathbb{R}^2$. Extend the functional $\mathcal{F}$ with the value $+\infty$ to the class $\mathcal{M}$ of all measurable subsets of $\mathbb{R}^2$. Let us denote by $\overline{\mathcal{F}} : \mathcal{M} \to [0, +\infty]$ the $L^1(\mathbb{R}^2)$-lower semicontinuous envelope of $\mathcal{F}$, i.e.,

$$\overline{\mathcal{F}}(E) = \inf\{\liminf_{h\to\infty} \mathcal{F}(E_h) : E_h \to E \text{ in } L^1(\mathbb{R}^2) \text{ as } h \to \infty\}, \quad E \in \mathcal{M}. \tag{2}$$

The study of $\mathcal{F}$ and $\overline{\mathcal{F}}$ is of interest in variational models for the image segmentation and image inpainting [2], [3]. It is also related to the problem of reconstructing the occluded boundaries and the ordering of the objects in a digital image; indeed (see [17]) it has been observed that curvature energies can be used to recover those parts of the objects occluded by other objects closer to the observer. Connections of $\mathcal{F}$ with Γ-convergence, phase transitions and geometric evolutions were pointed out in [9]. From the mathematical point of view, the study of $\overline{\mathcal{F}}$ was initiated in [4] and further developed in [5]. In particular in [4, Theorem 7.3] it is proved that $\overline{\mathcal{F}}$ can not be represented in integral form. Then in [5, Proposition 6.1] it is proved that

$$\overline{\mathcal{F}}(E) = \min\{\mathcal{F}(\Gamma) : \Gamma \in \mathcal{A}^o(E)\}, \tag{3}$$

where elements of $\mathcal{A}^o(E)$ are $H^{2,p}$-immersions of a finite number of copies of the unit circle, having E as "interior" (see Section 2 for the details). Both in [4] and [5] these immersions

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describe the limit behavior of sequences of smooth boundaries ∂E_h with equibounded energy and keep some information that would be lost when looking only at the L^1-limit of the sets E_h.

In this paper we prove a new representation formula for the functional $\overline{\mathcal{F}}$. Precisely, our result (Theorem 4.3) reads as follows: $\overline{\mathcal{F}}$ can be represented as

$$\overline{\mathcal{F}}(E) = \min\{\mathcal{F}(f) : f \in HV_{\rm tg}(\partial E)\}, \quad E \in \mathcal{M}. \tag{4}$$

Here $HV_{\rm tg}(\partial E)$ is the class of Hutchinson's curvature varifolds without boundary, having a unique tangent line at *every* point and such that the associated weight measure has odd density exactly on the essential boundary $\partial^* E$ of E. This result is much in the spirit of [11, Corollary 5.4], where a related partial result was established in any space dimension using the so-called generalized Gauss graphs.

The key point to obtain (4) relies on the following property (see Proposition 4.7 and Corollary 4.10): given $f \in HV_{\rm tg}(\partial E)$ there exists $\Gamma \in \mathcal{A}^o(E)$ such that $\mathcal{F}(f) = \mathcal{F}(\Gamma)$. To prove this result we need to adapt some of the techniques developed in [5], once we know that the elements of $HV_{\rm tg}(\partial E)$ belong to the more regular space $S^p(\mathbb{R}^2)$ of *Sobolev-type submanifolds*. Inspired by [7] and [8], Sobolev-type submanifolds were introduced in [1] (see Definition 3.7 below) and consist in one-dimensional sets which can be locally covered by finite unions of graphs of functions of class $H^{2,p}$. As a consequence, we obtain also a representation of $\overline{\mathcal{F}}$ in terms of $S^p(\mathbb{R}^2)$, namely

$$\overline{\mathcal{F}}(E) = \min\left\{\mathcal{F}(f) : f \in S^p(\mathbb{R}^2), f \equiv\ 0 \pmod 2 \text{ out of } \partial^* E, f \equiv\ 1 \pmod 2 \text{ on } \partial^* E\right\}.$$

There are various reasons to develop new representation results for $\overline{\mathcal{F}}$, in particular using the varifolds approach. Firstly varifolds describe in a rather natural way the limits of sequences of smooth boundaries with equibounded energy. In fact given one of these sequences $\{E_h\}$ converging to a set E in $L^1(\mathbb{R}^2)$, there exists a unique varifold associated with the limit of the sequence $\{\partial E_h\}$. On the contrary there are infinitely many different $H^{2,p}$-systems of curves that parametrize the limit of the sequence $\{\partial E_h\}$. Another reason is that varifolds seem to play a rôle when trying to approach a conjecture in [9] concerning the Γ-approximation of the Willmore functional (see [6] and [16] for partial results in this direction). Finally a representation formula based on varifolds or on Sobolev-type submanifolds could be the first step for a representation formula in arbitrary space dimension; in such a (considerably more difficult) case the parametrizations approach probably cannot be easily pursued. Concerning the arbitrary dimension n, we point out that, in a recent paper [18], Schätzle proved that the functional in (1) (where κ stands for the mean curvature of ∂E and $p = 2$) is $L^1(\mathbb{R}^n)$-lower semicontinuous on smooth sets. Observe that there is another generalization of $\mathcal{F}$ in higher space dimension, where the curvature term is replaced by the L^p-norm of the second fundamental form. In both cases varifolds can be used to study the properties of the corresponding L^1-relaxed energies (see [2], [16], [11]). At least in the second case and under the assumption $p > n$, one could expect that varifolds that are limits of sequences of smooth boundaries with equibounded energy should be locally representable as finite unions of $H^{2,p}$-graphs (see [12]). This property characterizes Sobolev-type submanifolds, but it is not fulfilled by 2-dimensional Hutchinson's curvature varifolds in $\mathbb{R}^3$ for any $p > 2$ (see [1, Example 5.9]).

The paper is organized as follows. In Section 2 we give some definitions and fix some notation. Recalling some results from [1], in Section 3 we introduce the class $HV^p(\mathbb{R}^2)$ of curvature varifolds in the sense of Hutchinson [13], [14], the class $S^p(\mathbb{R}^2)$ of one-dimensional Sobolev-type submanifolds and its subclass $S^p_{\mathrm{tg}}(\mathbb{R}^2)$ consisting of elements of $S^p(\mathbb{R}^2)$ having only tangential intersections. In Section 4 we prove Theorem 4.3, which is the main result of the paper. Firstly in Definition 4.1 we define the class $HV_{\mathrm{tg}}(\partial E)$ in (4) and through some examples we show that the result of Theorem 4.3 is in some sense optimal. In Lemma 4.6 we show that if $f \in HV^p(\mathbb{R}^2)$ has unique tangent line at every point then it can be locally represented as a finite union of graphs of class $H^{2,p}$ (and not only of class $\mathcal{C}^{1,1-1/p}$). The proof of Theorem 4.3 is a consequence of Lemma 4.6, Proposition 4.7, Corollary 4.10 and Theorem 2.9.

2. Notation and preliminaries

Throughout the paper $p > 1$ is a real number. For any $z_0 \in \mathbb{R}^2$, $\rho > 0$, $B_\rho(z_0) := \{z \in \mathbb{R}^2 : |z - z_0| < \rho\}$ is the ball centered at z_0 with radius ρ. The letter U indicates an open subset of $\mathbb{R}^2$. In the sequel we adopt the convention on repeated indices.

We denote by $G(1,2)$ the set of unoriented 1-dimensional subspaces of $\mathbb{R}^2$, and use the same notation P for an element of $G(1,2)$, for the orthogonal projection $P : \mathbb{R}^2 \to \mathbb{R}^2$ which maps $\mathbb{R}^2$ onto P, and for the associated matrix, whose entries are denoted by P_{jk}. The distance $\|P - Q\|$ between $P, Q \in G(1,2)$ is the one induced on $G(1,2)$ as a subset of $\mathbb{R}^4$.

For every set $E \subseteq \mathbb{R}^2$ we denote by χ_E the characteristic function of E, that is $\chi_E(z) = 1$ if $z \in E$, $\chi_E(z) = 0$ if $z \notin E$. We define $E^* := \{z \in \mathbb{R}^2 : \exists\, r > 0 : |B_r(z) \setminus E| = 0\}$, $|\cdot|$ the Lebesgue measure; $\mathrm{int}(E)$, $\overline{E}$ and ∂E are respectively the interior, the closure and the topological boundary of E. All sets we consider are assumed to belong to $\mathcal{M}$, the class of all measurable subsets of $\mathbb{R}^2$.

If $E \in \mathcal{M}$ is a finite perimeter subset of $\mathbb{R}^2$ by $\partial^* E$ we denote the essential boundary of E.

We say that $E \subset \mathbb{R}^2$ is of class $H^{2,p}$ (resp. $\mathcal{C}^k$, $k \geq 1$) if E is open and can be locally represented as the subgraph of a function of class $H^{2,p}$ (resp. $\mathcal{C}^k$).

If $E \subset \mathbb{R}^2$ is of class $H^{2,p}$ we denote by $\kappa_{\partial E} \in L^p(\partial E, \mathcal{H}^1)$ the curvature of ∂E; we often write κ in place of $\kappa_{\partial E}$ when no confusion is possible.

Let $S \subset \mathbb{R}^2$. We say that S is *countably 1-rectifiable* if there exists a sequence $\{M_i\}$ of 1-dimensional submanifolds of class $\mathcal{C}^1$ such that $\mathcal{H}^1(S \setminus \cup_{i \in \mathbb{N}} M_i) = 0$.

We recall some definitions from [4] and [5].

Definition 2.1. Let $S \subset \mathbb{R}^2$ be a countably 1-rectifiable set. Let $z_0 \in S$ be a point where S admits tangent line. Let $\tau(z_0)$ be a unit tangent vector to S at z_0, and $\tau^\perp(z_0)$ be the rotation of $\tau(z_0)$ of $\pi/2$ around the origin in counterclockwise order. We say that $R(z_0)$ is a nice rectangle for S at z_0 if

$$R(z_0) = \{z \in \mathbb{R}^2 : z = z_0 + l\tau(z_0) + d\tau^\perp(z_0),\ |l| \leq a, |d| \leq b\},$$

where $a > 0$ and $b > 0$ are selected in such a way that $S \cap R(z_0)$ is the union of the cartesian graphs, with respect to the tangent line $T_{z_0}S$ to S at z_0, of a finite number of functions $\{g_1, \ldots, g_r\}$ such that g_l is of class $\mathcal{C}^1$ and $\mathrm{graph}(g_l)$ does not intersect the two sides of $R(z_0)$ which are parallel to $T_{z_0}S$ for every $l \in \{1, \ldots, r\}$.

Let $z_0 \in S$; when we write a nice rectangle $R(z_0)$ for S at z_0 in the form $R(z_0) = [-a,a] \times [-b,b]$, we implicitely assume that z_0 is the origin of the coordinates, that $T_{z_0}S$ is the x-axis, and that $\tau^\perp(z_0)$ agrees with the vector $e_2 = (0,1)$. In this case we also set $R^+(z_0) := [0,a] \times [-b,b]$ and $R^-(z_0) := [-a,0] \times [-b,b]$.

Definition 2.2. Let $M \subset \mathbb{R}^2$ be an immersed $\mathcal{C}^1$-curve and let $z_0 \in M$. We say that z_0 is a regular point for M, and we write $z_0 \in \mathrm{Reg}_M$, if there exists a neighborhood $U_{z_0} \subset \mathbb{R}^2$ of z_0 such that $M \cap U_{z_0}$ is the graph of a function of class $\mathcal{C}^1$ with respect to $T_{z_0}M$. We say that $z_0 \in M$ is a singular point of M, and we write $z_0 \in \mathrm{Sing}_M$, if z_0 is not a regular point of M.

By an arc of regular points of M we mean a relative connected component of Reg_M.

If $z_0 \in M$ is a point where the tangent line to M is uniquely defined, by $B^+_\rho(z_0)$ (resp. $B^-_\rho(z_0)$) we mean $\{z \in B_\rho(z_0) : (z-z_0)\cdot\tau(z_0) \geq 0\}$ (resp. $\{z \in B_\rho(z) : (z-z_0)\cdot\tau(z_0) \leq 0\}$), where $\tau(z_0) \in T_{z_0}M$ is a unit vector.

Definition 2.3. We say that $z_0 \in \mathrm{Sing}_M$ is a node of M, and we write $z_0 \in \mathrm{Nod}_M$, if there exists $N_{z_0} \in \mathbb{N}$, $N_{z_0} > 1$, such that for any $\rho > 0$ sufficiently small either $B^+_\rho(z_0) \cap M \setminus \{z_0\}$ or $B^-_\rho(z_0) \cap M \setminus \{z_0\}$ consists of the union of N_{z_0} arcs of regular points of M which do not intersect each other.

Proposition 2.4. *Let $M \subset \mathbb{R}^2$ be such that $0 \in M$ and $T_0M = e_1 \otimes e_1$. Suppose that there exists a nice rectangle $R = [-a,a] \times [-b,b]$ for M at 0 such that $M \cap R = \cup_{l=1}^r \mathrm{graph}(g_l)$ and $g_l \in \mathcal{C}^1([-a,a])$. Then*

$$\overline{\mathrm{Nod}_{M\cap R}} = \mathrm{Sing}_{M\cap R}, \qquad \overline{\mathrm{Reg}_{M\cap R}} = M \cap R. \tag{5}$$

Moreover there exists $\delta \in [0,a[$ such that $R_\delta := [-a+\delta, a-\delta] \times [-b+\delta, b-\delta]$ is a nice rectangle for M at 0 and

$$M \cap \partial R_\delta \subset \mathrm{Reg}_{M\cap R} \tag{6}$$

Proof. It is enough to repeat the proof of [5, Proposition 3.1 and Corollary 3.4] □

2.1. Curves and systems of curves

A curve $\gamma : [0,1] \to \mathbb{R}^2$ of class $\mathcal{C}^1$ is said to be regular if $\frac{d\gamma(t)}{dt} \neq 0$ for every $t \in [0,1]$. By $(\gamma) = \{\gamma(t) : t \in [0,1]\}$ we denote the trace of γ and by $l(\gamma)$ its length; if $z \in \mathbb{R}^2 \setminus (\gamma)$, $\mathcal{I}(\gamma, z)$ is the index of γ with respect to z. If γ is a curve of class $H^{2,p}$ we set

$$\mathcal{F}(\gamma) := l(\gamma) + \|\kappa(\gamma)\|^p_{L^p},$$

where $\kappa(\gamma)$ is the curvature of γ.

Definition 2.5. A system of curves is a finite family $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ of closed regular curves of class $\mathcal{C}^1$ such that $|\frac{d\gamma_i}{dt}|$ is constant on $[0,1]$ for any $i = 1, \ldots, m$. We say that Γ is of class $H^{2,p}$ if each γ_i is of class $H^{2,p}$.

Denoting by $\mathbb{S}$ the disjoint union of m oriented circles $S^1_1, \ldots, S^1_m$ of unitary length, we identify Γ with the map $\Gamma : \mathbb{S} \mapsto \mathbb{R}^2$ defined by $\Gamma_{|S^1_i} := \gamma_i$ for $i = 1, \ldots, m$. The trace (Γ) of Γ is defined as $(\Gamma) := \cup_{i=1}^m (\gamma_i)$. When Γ is of class $H^{2,p}$ we write $\Gamma \in H^{2,p}(\mathbb{S})$.

Definition 2.6. We say that a system of curves $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ is without crossings if $\frac{d\gamma_i(t_1)}{dt}$ and $\frac{d\gamma_j(t_2)}{dt}$ are parallel, whenever $\gamma_i(t_1) = \gamma_j(t_2)$ for some $i, j \in \{1, \ldots, m\}$ and $t_1, t_2 \in [0, 1]$.

If $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ is a system of curves of class $H^{2,p}$ we define

$$\mathcal{F}(\Gamma) := \sum_{i=1}^{m} \mathcal{F}(\gamma_i). \tag{7}$$

We set

$$A^o_\Gamma := \{z \in \mathbb{R}^2 \setminus (\Gamma) : \mathcal{I}(\Gamma, z) \equiv 1 \ (\mathrm{mod}\ 2)\}, \tag{8}$$

where $\mathcal{I}(\Gamma, z) := \sum_{i=1}^m \mathcal{I}(\gamma_i, z)$ is the index of $z \in \mathbb{R}^2 \setminus (\Gamma)$ with respect to Γ.

We define the density function θ_Γ of Γ as

$$\theta_\Gamma : (\Gamma) \to \mathbb{N}, \qquad \theta_\Gamma(z) := \sharp\{\Gamma^{-1}(z)\},$$

$\sharp$ the counting measure. As noted in [5, Lemma 2.16], if $\Gamma \in H^{2,p}(\mathbb{S})$ we have $\|\theta_\Gamma\|_{L^\infty} < +\infty$. We denote by $f_\Gamma : \mathbb{R}^2 \to \mathbb{N}$ the function defined as

$$f_\Gamma := \theta_\Gamma \ \chi_{(\Gamma)}. \tag{9}$$

Definition 2.7. Let $E \subset \mathbb{R}^2$. We denote by $\mathcal{A}^o(E)$ the collection of all systems of curves Γ of class $H^{2,p}$ without crossings satisfying

$$(\Gamma) \supseteq \partial E^*, \qquad E^* = \mathrm{int}\left(A^o_\Gamma \cup (\Gamma)\right). \tag{10}$$

Remark 2.8. In [4] it is proved that

(i) for every $E \subset \mathbb{R}^2$ such that $\overline{\mathcal{F}}(E) < +\infty$ we have $|E \Delta E^*| = 0$. In the following we always identify E with E^*.

(ii) if $\Gamma \in \mathcal{A}^o(E)$ then θ_Γ is odd on $\mathrm{Reg}_{\partial E}$ and even on $\mathbb{R}^2 \setminus \partial E$. In addition, from [4, p. 269] it follows that θ_Γ is odd on $\partial^* E$ and even on $\mathbb{R}^2 \setminus \partial^* E$.

The following results [5, Theorem 5.1, Corollary 5.2 and Proposition 6.1] will be used in the sequel.

Theorem 2.9. *Given $E \subset \mathbb{R}^2$ we have $\overline{\mathcal{F}}(E) < +\infty$ if and only if $\mathcal{A}^o(E) \neq \emptyset$. In this case* (3) *holds.*

3. Varifolds with curvature and Sobolev-type submanifolds

$\mathbb{N}$ denotes the set of all nonnegative integer numbers. Given $f : U \to \mathbb{N}$ we set $S_f := \{z \in U : f(z) \neq 0\}$.

Definition 3.1. Let $f : U \to \mathbb{N}$. We say that f is $\mathcal{H}^1$-rectifiable if $f^{-1}(i)$ is countably $\mathcal{H}^1$-rectifiable for every $i \geq 1$ and $\int_U f d\mathcal{H}^1 < +\infty$.

If $f : U \to \mathbb{N}$ is $\mathcal{H}^1$-rectifiable, then for $\mathcal{H}^1$-almost every $z_0 \in S_f$ there is a unique $P^f(z_0) = (P^f_{jk}(z_0)) \in G(1,2)$, called the approximate tangent line to f at z_0, such that

$$\lim_{\rho \to 0^+} \frac{1}{\rho} \int_{S_f} \varphi\left(\frac{z - z_0}{\rho}\right) f(z) \, d\mathcal{H}^1 = f(z_0) \int_{P^f(z_0)} \varphi \, d\mathcal{H}^1, \quad \forall \varphi \in \mathcal{C}^1_c(\mathbb{R}^2). \tag{11}$$

If S (or equivalently f) has approximate tangent line at z_0 and φ is a $\mathcal{C}^1$ function defined in a neighborhood of z_0, we define the tangential gradient $\delta^f\varphi(z_0) = \left(\delta_1^f\varphi(z_0), \delta_2^f\varphi(z_0)\right)$ of φ to S_f at the point z_0 as the orthogonal projection of $\nabla\varphi(z_0)$ on $P^f(z_0)$.

With every $\mathcal{H}^1$-rectifiable function $f : U \to \mathbb{N}$ we associate the Radon measure $V_f := v(S_f, f)$ on $U \times G(1,2)$ defined by

$$\begin{aligned}\int_{U\times G(1,2)} \phi(z,P)\, dV_f &:= \int_{U\times G(1,2)} \phi(z,P)\, d[f\mathcal{H}^1 \llcorner S_f \otimes \delta_{P^f}] \\ &= \int_U \phi(z,P^f) f\, d\mathcal{H}^1, \quad \forall \phi \in \mathcal{C}_c^0(U\times G(1,2)),\end{aligned}$$

that is the one-rectifiable integer varifold associated with the pair (S_f, f), see [19].

We denote by $\pi : \mathbb{R}^2 \times G(1,2) \to \mathbb{R}^2$ the projection map, $\pi_\sharp V_f = f\mathcal{H}^1 \llcorner S_f$ the image measure of V_f on $\mathbb{R}^2$ through π,

$$\theta^{1*}(f,z) = \limsup_{\rho\to 0^+} \frac{\pi_\sharp V_f(B_\rho(z))}{2\rho}, \qquad \theta^1_*(f,z) = \liminf_{\rho\to 0^+} \frac{\pi_\sharp V_f(B_\rho(z))}{2\rho},$$

and if $\theta^{1*}(f,z) = \theta^1_*(f,z) < +\infty$ we set $\theta^1(f,z) := \theta^{1*}(f,z)$.

For any $z \in \mathbb{R}^2$ we define

$$\mathrm{Tan}_z(f) := \{P \in G(1,2) : (z,P) \in \mathrm{spt}(V_f)\}.$$

If f is $\mathcal{H}^1$-rectifiable then for $\mathcal{H}^1$-a.e. $z \in S_f$ we have $\mathrm{Tan}_z(f) = \{P^f(z)\}$ and for any $z \notin \overline{S_f}$ we have $\mathrm{Tan}_z(f) = \emptyset$.

Let $f : U \to \mathbb{N}$ be $\mathcal{H}^1$-rectifiable, let $z_0 \in S_f$ and suppose that $R(z_0) = [-a,a]\times[-b,b]+z_0$ is a nice rectangle for S_f at z_0. We say that f verifies the *train track property* in $R(z_0)$ if the function

$$t \mapsto \sum_{z\in\{z_0+(\{t\}\times[-b,b])\}} f(z)$$

is constant for every $t \in [-a,a]$.

Let $\psi : S_f \to \mathbb{R}$ be an $f\mathcal{H}^1 \llcorner S_f$-measurable function. We say that ψ is approximately differentiable at $z_0 \in S_f$ with approximate gradient $\nabla^{S_f}\psi(z_0) = v$ if $\mathrm{Tan}_{z_0}(f) = \{P^f(z_0)\}$, $v \in P^f(z_0)$, and for every $\varepsilon > 0$ the set $L^\varepsilon_\psi := \left\{z \in \overline{S_f}\setminus\{z_0\} : \frac{|\psi(z)-\psi(z_0)-v\cdot(z-z_0)|}{|z-z_0|} > \varepsilon\right\}$ satisfies

$$\lim_{\rho\to 0^+} \frac{1}{\rho}\int_{L^\varepsilon_\psi \cap B_\rho(z_0)} f d\mathcal{H}^1 = 0. \tag{12}$$

3.1. The class $HV^p(U)$

Definition 3.2. Let $f : U \to \mathbb{N}$ be a $\mathcal{H}^1$-rectifiable function. We say that f is a Hutchinson's varifold with second fundamental form in L^p, and we write $f \in HV^p(U)$, if for any $i,j,k \in \{1,2\}$ there are Borel functions $A^f_{ijk} : U \to \mathbb{R}$ such that

$$\|A^f\|^p_{p,U} := \int_U \sum_{l,m,n=1}^2 |A^f_{lmn}|^p\, f\, d\mathcal{H}^1 < +\infty, \tag{13}$$

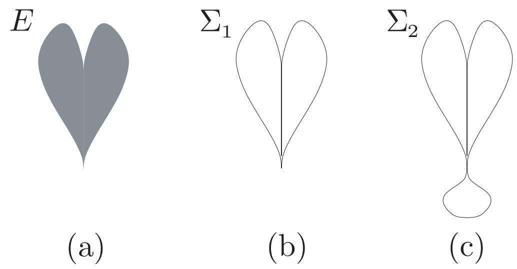

Figure 3.1: Example 3.3

and

$$\int_U \left(\delta_i^f \phi(z, P^f) + \frac{\partial \phi(z, P^f)}{\partial P_{mn}} A_{imn}^f \right) f \, d\mathcal{H}^1 = - \int_U \phi(z, P^f) A_{mim}^f \, f \, d\mathcal{H}^1, \tag{14}$$

for any $\phi \in \mathcal{C}^1(U \times G(1,2))$ with compact support in U.

To illustrate the meaning of Definition 3.2 let us consider the following example.

Example 3.3. Let us consider the set E in in grey in Figure 3.1(a). Let Σ_1 (resp. Σ_2) be the 1-dimensional immersed curve in (b) (resp. in (c)). Then $f_1 := 2\chi_{\Sigma_1} - \chi_{\partial E}$ is not an element of $HV^p(\mathbb{R}^2)$, as V_{f_1} has generalized boundary (in the sense of varifolds, see [19]) concentrated in the lower cusp point. On the contrary $f_2 := 2\chi_{\Sigma_2} - \chi_{\partial E} \in HV^p(\mathbb{R}^2)$.

Remark 3.4. In [15, Theorem 5.4] it is proved that if $f \in HV^p(U)$ then the functions P_{jk}^f are approximately differentiable $f\mathcal{H}^1 \llcorner S_f$-almost everywhere for every $j, k \in \{1, 2\}$, with approximate gradient

$$\delta_i^f P_{jk}^f = \delta_i^f \delta_j^f x_k = A_{ijk}^f, \quad i, j, k \in \{1, 2\}, \tag{15}$$

where $z = (x_1, x_2)$.

Given $f \in HV^p(U)$ we set

$$\mathcal{F}(f, U) := \int_U f d\mathcal{H}^1 + \|A^f\|_{p,U}^p. \tag{16}$$

In case $U = \mathbb{R}^2$ we set $\mathcal{F}(\cdot) := \mathcal{F}(\cdot, \mathbb{R}^2)$. With a small abuse of notation, with the same symbol $\mathcal{F}$ in (1), (7) and (16) we denote a functional defined on $\mathcal{M}$, on $H^{2,p}$-systems of curves, and on $HV^p(\mathbb{R}^2)$.

Remark 3.5. By varifolds theory (see [19, Remark 17.9]) if $f \in HV^p(U)$ then $\theta^{1*}(f, z) = \theta_*^1(f, z) \in \mathbb{N} \setminus \{0\}$ for every $z \in \overline{S_f}$. If we set $f^*(z) = \theta^1(f, z)$ then $f^* = f$ $\mathcal{H}^1$-almost everywhere on S_f and $V_{f^*} = V_f$ as Radon measures. Moreover by Remark 3.4 we have $\mathcal{F}(f^*) = \mathcal{F}(f)$.

In the sequel when $f \in HV^p(U)$ we always identify f with f^*. In particular S_f is a closed set.

Let $f_k, f \in HV^p(U)$. We write $f_k \rightharpoonup f$ if the sequence $\{V_{f_k}\}$ converges to V_f in the sense of Radon measures on $\mathbb{R}^2 \times G(1,2)$.

Combining the results contained in [14] and [10] we have the following

Theorem 3.6. *Let $f \in HV^p(U)$ be such that*

$$\sharp \mathrm{Tan}_z(f) = 1 \quad \forall z \in S_f. \tag{17}$$

Suppose that $0 \in S_f$ and $\mathrm{Tan}_0(f) = e_1 \otimes e_1$. Then there exist $a, b > 0$ such that setting $R := [-a, a] \times [-b, b]$, we have $R \subset U$ and

(A) *the map $t \in [-a, a] \to \sum_{z \in \{t\} \times [-b,b]} f(z)$ is a positive constant $L \in \mathbb{N}$ on $[-a, a]$;*

(B) *there exist a positive natural number $r \le L$, distinct functions $g_1, \dots, g_r$ of class $\mathcal{C}^{1,1-1/p}([-a, a])$ and natural numbers $\mu_1, \dots, \mu_r$ such that*

$$f = \sum_{l=1}^{r} \mu_l \chi_{\mathrm{graph}(g_l)} \quad \text{in } R; \tag{18}$$

$$z = (t, g_l(t)) \Longrightarrow P^f(z) = \frac{v}{|v|} \otimes \frac{v}{|v|}, \ v = (1, g_l'(t)), \quad t \in \ ]-a, a[, \tag{19}$$

$$-b < g_1(t) \le g_2(t) \le \cdots \le g_r(t) < b, \quad t \in [-a, a]; \tag{20}$$

(C) *f is constant on every relative connected component of $\mathrm{Reg}_{S_f} \cap R$. If $z_0 = (t_0, g_l(t_0)) \in \mathrm{Sing}_{S_f} \cap R$ then*

$$\limsup_{t \to t_0} f(t, g_l(t)) \le f(z_0). \tag{21}$$

Proof. In [14, Theorem 3.7] it is proved that (A) holds. Moreover it is also proved that $S_f \cap (]-a, a[\times \mathbb{R})$ is given by the graph of an L-valued function of class $\mathcal{C}^{1,1-1/p}$ in the sense of Almgren. Using (17) we can apply [10, Theorem 4.2 and Corollary 3.13] and find a positive natural number $r' \le L$ and (not necessarily distinct) functions $\widetilde{g}_1, \dots, \widetilde{g}_{r'}$ of class $\mathcal{C}^{1,1-1/p}([-a, a])$ such that

$$\begin{aligned} f &= \sum_{l=1}^{r'} \chi_{\mathrm{graph}(\widetilde{g}_l)} \qquad \text{in } R; \\ z = (t, g_l(t)) \Longrightarrow P^f(z) &= \frac{\tilde v}{|\tilde v|} \otimes \frac{\tilde v}{|\tilde v|}, \ \tilde v = (1, \tilde g_l'(t)), \quad t \in \ ]-a, a[. \end{aligned} \tag{22}$$

As a consequence for every $z \in S_f \cap R$ there exists a nice rectangle $R(z)$ for S_f at z and f verifies the train tracks property in $R(z)$. Hence we can repeat the proof of [5, Proposition 3.7] and obtain (C).

In order to prove (B) we proceed as follows. Let

$$\mathcal{E} := \{g \in \mathcal{C}^0([-a, a]) : \ \mathrm{graph}(g) \subset S_f\},$$

and define $g_1(t) := \inf\{g(t) : \ g \in \mathcal{E}\}$. By (22) and (17) we have that $g_1 \in \mathcal{C}^{1,1-1/p}([-a, a])$ and g_1 verifies (19). In addition $\mu_1 := \min\{f(z) : z \in \mathrm{graph}(g_1)\} \in \mathbb{N} \setminus \{0\}$. From (21) and (5), it follows that we can find $z_1 \in \mathrm{Reg}_{S_f} \cap \mathrm{graph}(g_1)$ such that $f(z_1) = \mu_1$. From (C) it follows that $f \equiv \mu_1$ on the connected component C_1 of $\mathrm{Reg}_{S_f} \cap R$ containing z_1. Then consider the function

$$f_1 : R \to \mathbb{N}, \qquad f_1 := \begin{cases} f - \mu_1 & \text{on graph}(g_1), \\ f & \text{otherwise in } R. \end{cases}$$

Observe that S_{f_1} is a finite union of graphs of elements of $\mathcal{C}^{1,1-1/p}([-a,a])$, that the map $t \in [-a,a] \to \sum_{z\in\{t\}\times[-b,b]} f_1(z) \equiv L - \mu_1$, f_1 verifies (C) and that $\mathrm{Sing}_f \supseteq \mathrm{Sing}_{f_1}$. Repeating the argument above replacing f with f_1 and g_1 with $g_2(t) := \inf\{g \in \mathcal{C}^0([-a,a]) : \mathrm{graph}(g) \subset S_{f_1}\}$, we obtain $\mu_2 := \min\{f_1(z) : z \in \mathrm{graph}(g_2)\} \in \mathbb{N}\setminus\{0\}$ and a connected component C_2 of $\mathrm{Reg}_{S_f} \cap \mathrm{graph}(g_2)$. Repeating this construction, after $r \le L$ steps we obtain that $f_{r+1} \equiv 0$. In this way we construct a family $Y := \{(g_1,\mu_1),\ldots,(g_r,\mu_r)\}$ such that

$$g_l \neq g_j \text{ for every } l \neq j, \text{ since if } l < j \text{ then } \mathrm{graph}(g_j) \cap C_l = \emptyset \text{ and } C_l \subset \mathrm{graph}(g_l),$$

and satisfying (18), (19) and (20). □

3.2. The classes $S^p(\mathbb{R}^2)$ and$S^p_{\mathrm{tg}}(\mathbb{R}^2)$

Definition 3.7. Let $f : \mathbb{R}^2 \to \mathbb{N}$. We say that f is a Sobolev-type submanifold, and we write $f \in S^p(\mathbb{R}^2)$, if S_f is closed and for any $z_0 \in S_f$ there exist a neighborhood $U_{z_0} \subset \mathbb{R}^2$ of z_0, a positive integer q and $M_1,\ldots,M_q$ one-dimensional (not necessarily distinct) embedded $H^{2,p}$-manifolds without boundary in U_{z_0} such that

$$f = \sum_{i=1}^{q} \chi_{M_i} \quad \text{in } U_{z_0}. \tag{23}$$

As noted in [1, Remark 2.3], if $f \in S^p(\mathbb{R}^2)$ then f is $\mathcal{H}^1$-rectifiable in $\mathbb{R}^2$ and $\int_K f d\mathcal{H}^1 < +\infty$ for any compact set $K \subset \mathbb{R}^2$.

Definition 3.8. We define

$$S^p_{\mathrm{tg}}(\mathbb{R}^2) := \left\{ f \in S^p(\mathbb{R}^2) : \sharp\mathrm{Tan}_z(f) = 1 \text{ for any } z \in S_f \text{ and } \int_{\mathbb{R}^2} f d\mathcal{H}^1 < +\infty \right\}.$$

The next observations show in which sense the above classes generalize the notion of a manifold with curvature without boundary.

Remark 3.9.

(i) If $f \in S^p(\mathbb{R}^2)$ then $f \in HV^p(\mathbb{R}^2)$, see [1, Remark 2.10]. In addition $\mathrm{Tan}_z(f)$ coincides with the (finite) union of the tangent lines to the branches of S_f at z, see [14].

(ii) If $f \in HV^p(\mathbb{R}^2)$ then by Theorem 3.6 we have that f is a $\mathcal{C}^{1,1-1/p}(\mathbb{R}^2)$-type submanifold (that is, f satisfies Definition 3.7 with $H^{2,p}$ replaced by $\mathcal{C}^{1,1-1/p}$).

(iii) If Γ is an $H^{2,p}$-system of curves, then $f_\Gamma \in S^p(\mathbb{R}^2)$ (see [5]). In addition, from [15, Proposition 2.3] and (i) we have

$$\mathcal{F}(\Gamma) = \mathcal{F}(f_\Gamma). \tag{24}$$

Remark 3.10. By Theorem 3.6 and Remark 3.9 it follows that for every $f \in S^p_{\mathrm{tg}}(\mathbb{R}^2)$ and every $z \in S_f$ there exists a nice rectangle $R(z)$ for S_f at z (see also [1, Theorems 3.4, 5.4]) and f verifies the train tracks property in $R(z)$. Hence by Proposition 2.4 and repeating the proof of [5, Lemma 3.11] we obtain that

(i) Nod_{S_f} is at most countable, Sing_{S_f} has empty interior, and

$$\mathrm{Sing}_{S_f} = \overline{\mathrm{Nod}_{S_f}}, \qquad \overline{\mathrm{Reg}_{S_f}} = S_f. \tag{25}$$

(ii) We can choose $R(z)$ in such a way that

$$S_f \cap \partial R(z) \subset \mathrm{Reg}_{S_f}. \tag{26}$$

(iii) If $f, \widehat{f} \in S^p_{\mathrm{tg}}(\mathbb{R}^2)$, $\mathrm{Reg}_{S_f} = \mathrm{Reg}_{S_{\widehat{f}}}$ and $f = \widehat{f}$ in Reg_{S_f}, then $f = \widehat{f}$.

4. Representation formulas

In order to state our main result (Theorem 4.3) we need the following

Definition 4.1. Let $E \subset \mathbb{R}^2$. We define $HV_{\mathrm{tg}}(\partial E)$ as the class of all $f \in HV^p(\mathbb{R}^2)$ such that

$$\sharp \mathrm{Tan}_z(f) = 1 \quad \forall z \in S_f, \tag{27}$$

$$f(z) \equiv\ 0 \ (\mathrm{mod}\ 2) \quad \forall z \notin \partial^* E, \tag{28}$$

$$f(z) \equiv\ 1 \ (\mathrm{mod}\ 2) \quad \forall z \in \partial^* E. \tag{29}$$

Remark 4.2. Clearly we have $\mathrm{Reg}_{\partial E} \subseteq \partial^* E$. Therefore from (29) it follows that $S_f \supseteq \mathrm{Reg}_{\partial E}$ for every $f \in HV_{\mathrm{tg}}(\partial E)$.

Theorem 4.3. *Let $E \subset \mathbb{R}^2$ be a bounded open set. Then*

$$\overline{\mathcal{F}}(E) < +\infty \iff HV_{\mathrm{tg}}(\partial E) \neq \emptyset. \tag{30}$$

In this case

$$\overline{\mathcal{F}}(E) = \min\{\mathcal{F}(f) : f \in HV_{\mathrm{tg}}(\partial E)\}. \tag{31}$$

Before proving Theorem 4.3, we show with two examples that if we drop one of the conditions (27), (28), (29) in the definition of $HV_{\mathrm{tg}}(\partial E)$, then Theorem 4.3 is false.

Example 4.4. Let $p = 2$ and $E \subset \mathbb{R}^2$ be the set in grey in Figure 4.1 having four cusps $p_1 = (-1, 0) = -p_2$, $q_1 = (0, 1) = -q_2$.

Let $f := \chi_{\mathrm{Reg}_{\partial E}} + 2\chi_{\{0\}\times[-1,1]} + 2\chi_{[-1,1]\times\{0\}}$. Then $f \in HV^2(\mathbb{R}^2)$ satisfies (28) and (29), but not (27) since $0 \in S_f$ and $\sharp\mathrm{Tan}_0(f) = 2$.

We claim that

$$\overline{\mathcal{F}}(E) > \int_{\mathrm{Reg}_{\partial E}} [1 + \kappa^2]\, d\mathcal{H}^1 + 8 = \mathcal{F}(f), \tag{32}$$

where $\mathrm{Reg}_{\partial E} = \partial E \setminus \{p_1, p_2, q_1, q_2\}$ are the regular points of ∂E. To prove (32) we recall that in view of [5, Theorem 8.6] we have

$$\overline{\mathcal{F}}(E) = \int_{\mathrm{Reg}_{\partial E}} [1 + \kappa^2]\, d\mathcal{H}^1 + 2 \min_{(\sigma_1,\sigma_2)\in\Sigma(E)} [\mathcal{F}(\sigma_1) + \mathcal{F}(\sigma_2)], \tag{33}$$

where $\Sigma(E)$ consists of those pairs of constant speed curves $(\sigma_1, \sigma_2) \in (H^{2,2}(]0, 1[, \mathbb{R}^2))^2$ with $\sigma_1(0) = p_1$ and $\{p_2, q_1, q_2\} = \{\sigma_1(1), \sigma_2(0), \sigma_2(1)\}$, such that:

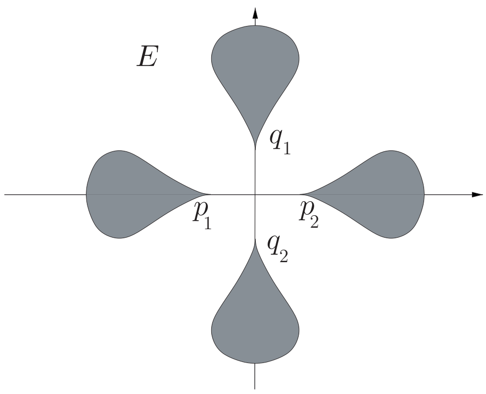

Figure 4.1: Example 4.4

if $\sigma_i(t_1) = \sigma_j(t_2)$ for some $t_1, t_2 \in [0,1]$ then $d\sigma_i(t_1)/dt$ and $d\sigma_j(t_2)/dt$ are parallel; moreover if $\sigma_i(t) \in \partial E$ for some $t \in [0,1]$, then $d\sigma_i(t)/dt$ is parallel to the tangent line $T_{\sigma_i(t)}(\partial E)$ to ∂E at $\sigma_i(t)$;

$\frac{d\sigma_i}{dt}(0)$ (resp. $\frac{d\sigma_i}{dt}(1)$) points in the direction of $R^-(\sigma_i(0))$ (resp. of $R^+(\sigma_i(1))$), where $R^-(p_i) \cap E = R^-(q_i) \cap E = \emptyset$ (resp. $R^+(p_i) \cap E \neq \emptyset,\ R^+(q_i) \cap E \neq \emptyset$) with $i = 1, 2$.

We have two cases.

Case 1. There exists $(\sigma_1, \sigma_2) \in \Sigma(E)$ solution of the minimum problem (33) such that $\sigma_1(1) \neq p_2$.

In this case due to the geometry of the set E and the definition of $\Sigma(E)$ it is always possible to find a constant speed curve $\lambda \in H^{2,2}(S^1)$ such that $\mathcal{F}(\lambda) = 2[\mathcal{F}(\sigma_1) + \mathcal{F}(\sigma_2)]$. For example if $\sigma_1(1) = q_1$, $\sigma_2(0) = p_2$ and $\sigma_i(t) := (\sigma_{i,1}(t), \sigma_{i,2}(t))$, then

$$\begin{aligned}(\lambda) :=& \{(\sigma_{1,1}(t)+1, \sigma_{1,2}(t)-1) :\ t \in\ ]0,1[\} \cup \{(\sigma_{1,1}(t)+1, 1-\sigma_{1,2}(t)) :\ t \in\ ]0,1[\} \\ &\cup \{(\sigma_{2,1}(t)-1, \sigma_{2,1}(t)+1) :\ t \in\ ]0,1[\} \cup \{(\sigma_{2,1}(t)-1, -(1+\sigma_{2,2}(t))) :\ t \in\ ]0,1[\}.\end{aligned}$$

Moreover setting

$$\widetilde{\sigma}_1(t) := \Big(\cos(\frac{\pi}{2}(t-1)+2\pi) - 1, \sin(\frac{\pi}{2}(t-1)+2\pi) + 1\Big),$$

$$\widetilde{\sigma}_2(t) := \Big(\cos(\frac{\pi}{2}(t-1)+\pi) + 1, \sin(\frac{\pi}{2}(t-1)+\pi) - 1\Big),$$

we have $(\widetilde{\sigma}_1, \widetilde{\sigma}_2) \in \Sigma(E)$ and

$$2[\mathcal{F}(\widetilde{\sigma}_1) + \mathcal{F}(\widetilde{\sigma}_2)] = 4\pi = 2\int_{S^1} |\kappa(\lambda)|\, ds \leq \mathcal{F}(\lambda),$$

where we used the Young's inequality. Hence

$$\begin{aligned}\overline{\mathcal{F}}(E) = \int_{\mathrm{Reg}_{\partial E}} [1+\kappa^2]\, d\mathcal{H}^1 + \mathcal{F}(\lambda) &\geq \int_{\mathrm{Reg}_{\partial E}} [1+\kappa^2]\, d\mathcal{H}^1 + 4\pi \\ &> \int_{\mathrm{Reg}_{\partial E}} [1+\kappa^2]\, d\mathcal{H}^1 + 8 = \mathcal{F}(f),\end{aligned}$$

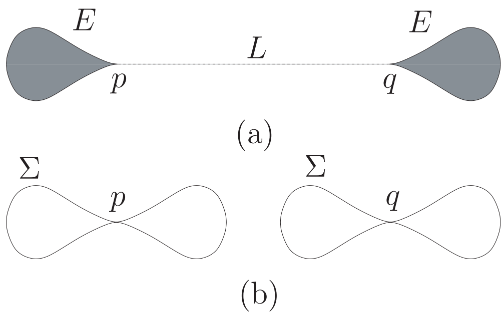

Figure 4.2: In grey the set E of Example 4.5 (fig. (a)). The dotted line appears when considering the limit of the boundaries of a minimizing sequence. In fig. (b) the set Σ

that is (32) holds.

Case 2. For every $(\sigma_1, \sigma_2) \in \Sigma(E)$ solution of the minimum problem (33) we have $\sigma_1(1) = p_2$.

In this case we have $(\sigma_1) \cap (\sigma_2) = \emptyset$. In fact if $(\sigma_1) \cap (\sigma_2) \neq \emptyset$ by the tangential crossing property of the elements of $\Sigma(E)$ and a reparametrization argument we would obtain $(\widehat{\sigma_1}, \widehat{\sigma_2}) \in \Sigma(E)$ solution of (33) and such that $\widehat{\sigma_1}(1) \neq p_2$, a contradiction.

As $(\sigma_1) \cap (\sigma_2) = \emptyset$ we have

$$\begin{aligned}
\overline{\mathcal{F}}(E) &= \int_{\mathrm{Reg}_{\partial E}} [1+\kappa^2]\, d\mathcal{H}^1 + 2[\mathcal{F}(\sigma_1) + \mathcal{F}(\sigma_2)] \\
&\geq \int_{\mathrm{Reg}_{\partial E}} [1+\kappa^2]\, d\mathcal{H}^1 + 2[l(\sigma_1) + l(\sigma_2)] \\
&> \int_{\mathrm{Reg}_{\partial E}} [1+\kappa^2]\, d\mathcal{H}^1 + 2[|p_1 - p_2| + |q_1 - q_2|] \\
&= \int_{\mathrm{Reg}_{\partial E}} [1+\kappa^2]\, d\mathcal{H}^1 + 8 = \mathcal{F}(f),
\end{aligned}$$

which is again (32).

Example 4.5. Let $p = 2$, E be the grey set in Figure 4.2(a) and let Σ be the 1-dimensional immersed nonconnected curve of $\mathbb{R}^2$ depicted in Figure 4.2(b). Obviously $\partial E \subset \Sigma$. We choose

$$L := |p - q| > \frac{3}{2} \int_{\mathrm{Reg}_{\partial E}} [1+\kappa^2]\, d\mathcal{H}^1, \tag{34}$$

and define

$$f_1, f_2 : \mathbb{R}^2 \to \mathbb{N}, \qquad f_1 := \chi_\Sigma, \quad f_2 := 2\chi_\Sigma.$$

Then $f_1, f_2 \in S^p_{\mathrm{tg}}(\mathbb{R}^2)$. Hence $f_1, f_2 \in HV^p(\mathbb{R}^2)$ by Remark 3.9(i), f_1, f_2 verify (27) and $S_{f_i} \supset \partial E$. Moreover f_1 fulfills (28) but not (29), while f_2 fulfills (29) but not (28). By [4, Theorem 7.2] (see also [5, Theorem 8.6]) we know that

$$\overline{\mathcal{F}}(E) = \int_{\mathrm{Reg}_{\partial E}} [1+\kappa^2]\, d\mathcal{H}^1 + 2L.$$

From (34) we deduce

$$2\mathcal{F}(f_1) = \mathcal{F}(f_2) = 4\int_{\mathrm{Reg}_{\partial E}} [1+\kappa^2]\, d\mathcal{H}^1 < \overline{\mathcal{F}}(E)$$

and (31) does not hold.

The proof of Theorem 4.3 is based on the combination of Proposition 4.7, Corollary 4.10 and Theorem 2.9. More precisely from Proposition 4.7 it follows that $HV_{\mathrm{tg}}(\partial E) \subset S^p_{\mathrm{tg}}(\mathbb{R}^2)$. Hence every $f \in HV_{\mathrm{tg}}(\partial E)$ can be locally represented using a finite number of $H^{2,p}$-functions. This observation allows to gain enough regularity to adapt some of the localization techniques developed in [5] and prove Proposition 4.8 and Lemma 4.9. As a consequence we obtain Corollary 4.10, which establishes that for every $f \in HV_{\mathrm{tg}}(\partial E)$ we can find $\Gamma \in A^o(E)$ such that $\mathcal{F}(\Gamma) = \mathcal{F}(f)$. Hence Theorem 4.3 follows by Theorem 2.9 and (9).

The following lemma is a crucial step toward the proof of Proposition 4.7.

Lemma 4.6. *Let $f \in HV^p(U)$, $R = [-a,a]\times[-b,b]$, $g_1,\dots,g_r$ be as in Theorem 3.6 and suppose that*

$$\max_{1\le l\le r} \|g_l'\|_{L^\infty(]-a,a[)} \le \frac{1}{2}. \tag{35}$$

Then $g_l \in H^{2,p}(]-a,a[)$ for every $l \in \{1,\dots,r\}$.

Proof. Let

$$T_1(\xi) := \frac{1}{\sqrt{1+\xi^2}}, \qquad T_2(\xi) := \xi T_1(\xi), \qquad \xi \in \mathbb{R}.$$

Given $l \in \{1,\dots,r\}$ we will prove that

$$(T_i(g_l'))' = A^f_{mim}(\cdot, g_l)(1+(g_l')^2)^{1/2p} \in L^p(]-a,a[), \quad i = 1,2, \tag{36}$$

where $A^f_{mim}(\cdot, g_l)$ denotes the restriction to the graph of g_l of the generalized second fundamental form of f. For every $z \in \mathrm{graph}(g_l)$, recalling (19) we have that the i-th row $(P^f_{i1}(z), P^f_{i2}(z))$ of the matrix P^f can be written as

$$\Big(P^f_{i1}(z), P^f_{i2}(z)\Big) = T_i(g_l'(t))\frac{(1, g_l'(t))}{\sqrt{1+(g_l'(t))^2}}, \qquad z := (t, g_l(t)),\ i = 1,2. \tag{37}$$

The idea is to rewrite the integration by parts formula (14) defining the curvature of a varifold (and which considers all branches of the varifold at once) on the basis interval $[-a,a]$, and try to obtain regularity informations on each branch separately. This is not immediate in presence of more than two graphs with a possibly complicated (tangential) intersection. We proceed by induction on r.

If $r = 1$ then, setting $g = g_1$, we have $S_f \cap R = \mathrm{graph}(g) \subset \mathrm{Reg}_{S_f}$ and $f \equiv L$ on $\mathrm{graph}(g)$. Let us show that $T_i(g) \in H^{1,p}(]-a,a[)$.

Let $\psi \in \mathcal{C}^1_c(]-a,a[)$. Since $S_f \cap R = \mathrm{graph}(g)$ we can find $\varepsilon > 0$ such that setting

$$K_\varepsilon := \left\{ z \in R : \mathrm{dist}(z, \mathrm{graph}(g_{|\mathrm{spt}(\psi)})) < \varepsilon \right\}$$

we have $K_\varepsilon \subset\subset R$ and $\mathrm{graph}(g_{|\mathrm{spt}(\psi)}) \subset\subset K_\varepsilon \cap S_f \subset\subset \mathrm{graph}(g)$. Since $g \in \mathcal{C}^{1,1-1/p}([-a,a])$ we can find a function $\varphi \in \mathcal{C}^1_c(\mathbb{R}^2)$ such that

$$\mathrm{spt}(\varphi) \subset\subset K_\varepsilon, \qquad \varphi(t, g(t)) = \psi(t) \quad \forall t \in [-a,a].$$

Hence, using also (37), we have for $i = 1, 2$

$$\begin{aligned} L \int_{]-a,a[} T_i(g')\psi' \, dt &= L \int_{]-a,a[} T_i(g') \left[\nabla\varphi(t,g) \frac{(1,g')}{\sqrt{1+(g')^2}} \right] \sqrt{1+(g')^2} \, dt \\ &= L \int_{]-a,a[} \left(P^f_{i1}(t,g), P^f_{i2}(t,g) \right) \nabla\varphi(t,g) \sqrt{1+(g')^2} \, dt = \int_{\mathrm{graph}(g)} \delta^f_i \varphi f \, d\mathcal{H}^1 \\ &= -\int_{\mathrm{graph(g)}} A^f_{mim} \varphi f \, d\mathcal{H}^1, \end{aligned} \tag{38}$$

where in the last equality we use (14) (applied with $\varphi = \phi$ independent of P^f). By (38) and (35) we have

$$\begin{aligned} \Big| \int_{]-a,a[} T_i(g')\psi' \, dt \Big| &\le L^{-1} \left(\int_{\mathrm{graph}(g)} |A^f_{mim}|^p f \, d\mathcal{H}^1 \right)^{1/p} \left(\int_{\mathrm{graph}(g)} |\varphi|^{p'} f \, d\mathcal{H}^1 \right)^{1/p'} \\ &\le C \left(\int_{]-a,a[} |A^f_{mim}|^p \sqrt{1+(g')^2} \, dt \right)^{1/p} \|\psi\|_{L^{p'}(]-a,a[)} \le C' \|\psi\|_{L^{p'}(]-a,a[)}, \end{aligned}$$

where $1/p + 1/p' = 1$ and C, C' are positive constants. Hence $T_i(g') \in H^{1,p}(]-a,a[)$ and (36) holds. Since T_2 is a bi-Lipschitz, smooth function on every open subset compactly contained in $]-a,a[$ and (35) holds we conclude that $g \in H^{2,p}(]-a,a[)$ from $g' = T_2^{-1}(T_2(g')) \in H^{1,p}(]-a,a[)$.

Let us show how the proof works in case $r = 2$. We first prove that $g_2 \in H^{2,p}(]-a,a[)$; the idea is to construct a sequence $\{f_h\} \subset HV^p(U)$ such that $f_h \rightharpoonup \widehat{f}$ in $HV^p(U)$ where $\widehat{f}$ is concentrated on $\mathrm{graph}(g_2)$ (hence $S_{\widehat{f}} \cap R = \mathrm{graph}(g_2)$) and $\widehat{f}$ has a constant density. By the previous case $r = 1$ we deduce $g_2 \in H^{2,p}(]-a,a[)$. The proof is concluded by showing that also $g_1 \in H^{2,p}(]-a,a[)$.

By Proposition 2.4

$$\begin{gathered} \left\{ t \in\ ]-a,a[: (t, g_1(t)) \in \mathrm{Reg}_{S_f \cap R} \right\} = \{ t \in\ ]-a,a[: g_1(t) < g_2(t) \} = \bigcup_{h\in\mathbb{N}} I_h, \\ \overline{\bigcup_{h\in\mathbb{N}} I_h} = [-a,a], \end{gathered} \tag{39}$$

where $I_h = \ ]t_h^-, t_h^+[$ are pairwise disjoint intervals and $H_h := \{(t, g_1(t)) : t \in I_h\}$ is a relative connected component of $\mathrm{Reg}_{S_f \cap R}$ for every $h \in \mathbb{N}$.

In order to construct the sequence $\{f_h\}$ we use a recursive algorithm. Let $f_0 := f$ and suppose that $f_{h-1} \in HV^p(U)$ has been defined. Then f_h is obtained modifying f_{h-1} only on $I_h \times [-b,b]$, in particular $f_h = f$ on $U \setminus \cup_{j=1}^h (I_j \times [-b,b])$, in such a way that

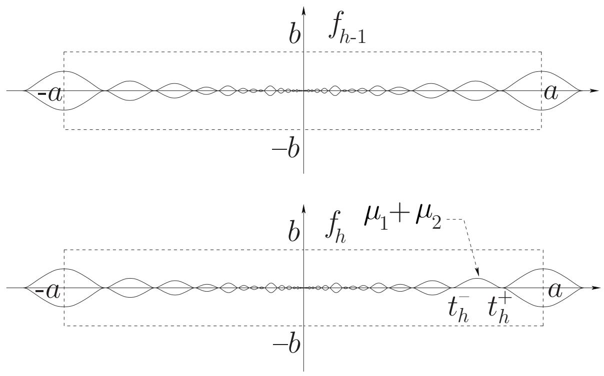

Figure 4.3: The construction leading to f_h starting from f_{h-1} in case $r = 2$ of Lemma 4.6

(i) $S_{f_h} \cap (I_h \times [-b, b]) = \text{graph}(g_2)$;
(ii) $\mathcal{F}(f_h) \le L\,\mathcal{F}(f)$.

Note that as $S_{f_h} \subset S_{f_{h-1}} \subset S_f$ we have $\sharp\text{Tan}_z f_h = 1$ for every $h \in \mathbb{N}$ and every $z \in S_{f_h}$.

Let us construct f_h. For every $\delta > 0$ sufficiently small (depending on h) let

$$I_h^\delta := \;]t_h^- + \delta, t_h^+ - \delta[, \qquad U_h^\delta := \left\{(t,y) \in R : \; t \in I_h^\delta, \; g_1(t) + \delta < y < b\right\}.$$

We have $f_{|U_h^\delta} \in HV^p(I_h^\delta \times]-b, b[)$ and $S_{f_{|U_h^\delta}} = \text{graph}(g_2) \cap (I_h^\delta \times]-b, b[)$. Using the $\mathcal{C}^{1,1-1/p}$ regularity of g_2 we can subdivide I_h^δ in $J_{h,1}^\delta, \dots, J_{h,k}^\delta$ disjoint intervals, with $k := k(h,\delta) \in \mathbb{N}$ and $|g_2'(t) - g_2'(s)| < \frac{1}{2}$ for any $t, s \in J_{h,i}^\delta$, so that by case $r = 1$ we have $g_2 \in H^{2,p}(J_{h,i}^\delta)$ and (36) holds in $J_{h,i}^\delta$.

Since

$$\|(T_i(g_2'))'\|_{L^p(I_h^\delta)} \le L\mathcal{F}(f), \quad i = 1, 2,$$

and $\sharp\text{Tan}_z(f_{h-1}) = 1$ for any $z \in S_f$, letting $\delta \to 0^+$ we conclude that $g_2 \in H^{2,p}(I_h)$ (see [5, Lemma 4.1]) and that (36) holds in I_h. With a similar argument we have that $g_1 \in H^{2,p}(I_h)$ and g_1 fulfills (36) in I_h.

Let us define f_h as follows:

$$f_h := \begin{cases} (\mu_1 + \mu_2)\chi_{\text{graph}(g_2)} & \text{in } I_h \times [-b, b], \\ f_{h-1} & \text{otherwise}, \end{cases} \tag{40}$$

where we recall that μ_1, μ_2 are given in Theorem 3.6(B). We remark that by $\sharp\text{Tan}_z(f_{h-1}) = 1$ for any $z \in S_{f_{h-1}}$, (39), (20) and Theorem 3.6(C), for $\sigma = \pm$ we have

$$g_1(t_h^\sigma) = g_2(t_h^\sigma), \qquad g_1'(t_h^\sigma) = g_2'(t_h^\sigma), \qquad f_{h-1}(z_h^\sigma) = f_h^\sigma(z_h), \tag{41}$$

where $z_h^\sigma := (t_h^\sigma, g_1(t_h^\sigma))$.

Next we prove that

$$f_h \in HV^p(U), \qquad A_{ijk}^{f_h} = A_{ijk|_{S_{f_h}}}^{f_{h-1}}. \tag{42}$$

For every $\phi \in \mathcal{C}^1(U \times G(1,2))$ with compact support in U, since $g_1, g_2 \in H^{2,p}(I_h)$ and $f(t, g_l(t)) = \mu_l$ for any $t \in I_h$ and $l = 1, 2$, we have

$$\begin{aligned}
&\int_{\mathrm{graph}(g_{l|I_h})} \left(\delta_i^f \phi(z, P^f) + \frac{\partial \phi(z, P^f)}{\partial P_{mn}} A^f_{imn} \right) \, d\mathcal{H}^1 \\
&= -\int_{\mathrm{graph}(g_{l|I_h})} \phi(z, P^f) A^f_{mim} \, d\mathcal{H}^1 \\
&\qquad + T_i(g_l'(t_h^+))\phi(z_h^+, P^f(z_h^+)) - T_i(g_l'(t_h^-))\phi(z_h^-, P^f(z_h^-)).
\end{aligned} \tag{43}$$

Since by definition (40) we have $f_h \equiv f_{h-1}$ on $U \setminus (I_h \times [-b, b])$ it follows that

$$\begin{aligned}
&\int_U \left(\delta_i^{f_h} \phi(z, P^{f_h}) + \frac{\partial \phi(z, P^{f_h})}{\partial P_{mn}} A^{f_h}_{imn} \right) \, f_h \, d\mathcal{H}^1 \\
&= \int_{U \setminus (I_h \times [-b,b])} \left(\delta_i^{f_{h-1}} \phi(z, P^{f_{h-1}}) + \frac{\partial \phi(z, P^{f_{h-1}})}{\partial P_{mn}} A^{f_{h-1}}_{imn} \right) \, f_{h-1} \, d\mathcal{H}^1 \\
&\quad + \int_{I_h \times [-b,b]} \left(\delta_i^{f_h} \phi(z, P^{f_h}) + \frac{\partial \phi(z, P^{f_h})}{\partial P_{mn}} A^{f_h}_{imn} \right) \, f_h \, d\mathcal{H}^1 =: \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned}$$

Observe that in $U \setminus (I_h \times [-b, b])$ we are not yet allowed to integrate by parts, because we still do not have enough regularity. On the other hand in U we can use (14). We have

$$\begin{aligned}
\mathcal{I}_1 =& \int_U \left(\delta_i^{f_{h-1}} \phi(z, P^{f_{h-1}}) + \frac{\partial \phi(z, P^{f_{h-1}})}{\partial P_{mn}} A^{f_{h-1}}_{imn} \right) \, f_{h-1} \, d\mathcal{H}^1 \\
&- \int_{I_h \times [-b,b]} \left(\delta_i^{f_{h-1}} \phi(z, P^{f_{h-1}}) + \frac{\partial \phi(z, P^{f_{h-1}})}{\partial P_{mn}} A^{f_{h-1}}_{imn} \right) \, f_{h-1} \, d\mathcal{H}^1 =: \mathcal{I}_1^{(1)} + \mathcal{I}_1^{(2)} \\
=& \int_U \phi(z, P^{f_{h-1}}) A^{f_{h-1}}_{mim} f_{h-1} \, d\mathcal{H}^1 + \mathcal{I}_1^{(2)} \\
=& -\int_{U \setminus (I_h \times [-b,b])} \phi(z, P^{f_{h-1}}) A^{f_{h-1}}_{mim} \, f_{h-1} \, d\mathcal{H}^1 \\
&- \int_{I_h \times [-b,b]} \phi(z, P^{f_{h-1}}) A^{f_{h-1}}_{mim} \, f_{h-1} \, d\mathcal{H}^1 + \mathcal{I}_1^{(2)}.
\end{aligned}$$

By (40), (14), (41) and (43) we have

$$\begin{aligned}
\mathcal{I}_2 =& -(\mu_1 + \mu_2) \int_{\mathrm{graph}(g_{2|I_h})} \phi(z, P^f) A^f_{mim} \, d\mathcal{H}^1 \\
&+ \sum_{l=1}^{2} \mu_l \left[T_i(g_l'(t_h^+))\phi(z_h^+, P^f(z_h^+)) - T_i(g_l'(t_h^-))\phi(z_h^-, P^f(z_h^-)) \right],
\end{aligned}$$

so that, integrating by parts,

$$\begin{aligned}
\mathcal{I}_1 + \mathcal{I}_2 =& -\int_{U \setminus (I_h \times [-b,b])} \phi(z, P^{f_{h-1}}) A^{f_{h-1}}_{mim} \, f_{h-1} \, d\mathcal{H}^1 \\
&- (\mu_1 + \mu_2) \int_{\mathrm{graph}(g_{2|I_h})} \phi(z, P^f) A^f_{mim} \, d\mathcal{H}^1,
\end{aligned}$$

that is (42) holds. Moreover (i) and (ii) hold by construction.

As $\sup_h \mathcal{F}(f_h) < +\infty$ we can apply [13, Theorem 4.4.2] and find a subsequence weakly converging in the sense of varifolds to a certain $\widehat{f} \in HV^p(U)$. Since $(\mathrm{Reg}_{S_f} \cap \mathrm{graph}(g_1)) \cap S_{\widehat{f}} = \emptyset$, we can conclude that $S_{\widehat{f}} \cap R = \mathrm{graph}(g_2)$. Hence, applying the result proved in case $r = 1$ to $\widehat{f}$, we get $g_2 \in H^{2,p}(]-a,a[)$ and (36). Repeating the same argument with g_2 replaced by g_1 we obtain that $g_1 \in H^{2,p}(]-a,a[)$ and that g_1 satisfies (36). This concludes the proof in case $r = 2$.

We suppose by induction that the thesis holds if $S_f \cap R = \cup_{l=1}^{r'} \mathrm{graph}(g_l)$ with $r' \leq (r-1)$, and we prove the assertion when $S_f \cap R = \cup_{l=1}^{r} \mathrm{graph}(g_l)$.

Again we construct recursively a sequence $\{f_h\} \subset HV^p(U)$ such that $f_h \rightharpoonup \widehat{f}$ in $HV^p(U)$, with $S_{\widehat{f}} \cap R = \cup_{l=2}^{r} \mathrm{graph}(g_l)$, so that we can apply the induction hypothesis and get $g_l \in H^{2,p}(]-a,a[)$ for $l = 2, \ldots, r$ and (36) holds. Then by a symmetric argument we prove that $g_l \in H^{2,p}(]-a,a[)$ for $l = 1, \ldots, r-1$ and this concludes the proof.

By Proposition 2.4 it follows $\left\{t \in \ ]-a,a[: (t, g_1(t)) \in \mathrm{Reg}_{S_f \cap R}\right\} = \cup_{h\in\mathbb{N}} I_h$, with $\overline{\cup_{h\in\mathbb{N}} I_h} = [-a,a]$, where $I_h = \ ]t_h^-, t_h^+[$ are pairwise disjoint intervals, $H_h := \{(t, g_1(t)) : t \in I_h\}$ are relative connected components of $\mathrm{Reg}_{S_f \cap R}$ and $f \equiv L_h \leq L$ on H_h for every $h \in \mathbb{N}$.

Let $f_0 := f$ and suppose that $f_{h-1} \in HV^p(U)$ has been defined. Then f_h is obtained modifying f_{h-1} only on $I_h \times [-b,b]$ in such a way that (i) holds with $\mathrm{graph}(g_2)$ replaced by $\cup_{l=2}^{r} \mathrm{graph}(g_l)$, and (ii) holds. Let $j_h := \min\{l : \ 2 \leq l \leq r, \ g_l \neq g_1 \text{ on } I_h\}$, and

$$f_h := \begin{cases} (\mu_{j_h} + L_h)\chi_{\mathrm{graph}(g_{j_h})} + \displaystyle\sum_{l=j_h+1}^{r} \mu_l \chi_{\mathrm{graph}(g_l)} & \text{in } I_h \times [-b,b], \\ f_{h-1} & \text{otherwise.} \end{cases} \tag{44}$$

We remark that, since $(t_h^\sigma, g_1(t_h^\sigma)) \in \mathrm{Sing}_{S_f} \cap R$ with $\sigma = \pm$, by (20) and the hypothesis $\mathrm{Tan}_z(f) \equiv 1$ for any $z \in S_f$ we have

$$g_1(t_h^\sigma) = g_{j_h}(t_h^\sigma), \qquad g_1'(t_h^\sigma) = g_{j_h}'(t_h^\sigma), \qquad \sigma = \pm. \tag{45}$$

Moreover by construction and using Theorem 3.6(C) we have, for $\sigma = \pm$,

$$\begin{gathered} S_{f_{h-1}} \cap (\{t_h^\sigma\} \times [-b,b]) = S_{f_h} \cap (\{t_h^\sigma\} \times [-b,b]), \\ f_{h-1}(z) = f_h(z), \quad \text{for } z \in \{t_h^\sigma\} \times [-b,b]. \end{gathered} \tag{46}$$

With the same arguments used in case $r = 2$ and by the induction hypothesis we obtain that $g_1, \ldots, g_r \in H^{2,p}(I_h)$ and (36) holds on I_h; moreover by (46) and the analog of (43), we have $f_h \in HV^p(U)$. Again (i) and (ii) hold by construction. The conclusion of the proof is then the same as the one in case $r = 2$. □

Proposition 4.7. *We have*

$$S^p_{\mathrm{tg}}(\mathbb{R}^2) = \left\{f \in HV^p(\mathbb{R}^2) : \sharp\mathrm{Tan}_z(f) = 1 \textit{ for any } z \in S_f\right\}. \tag{47}$$

Proof. In view of Remark 3.9(i), we only need to prove that the right hand side on (47) is contained in the left hand side. Let $f \in HV^p(\mathbb{R}^2)$ be such that $\sharp\mathrm{Tan}_z(f) = 1$ for any $z \in S_f$. By Remark 3.5 we can suppose that S_f is closed. Fix $\widetilde{z} \in S_f$. Without loss of generality we can suppose that $\widetilde{z} = 0$ and $\mathrm{Tan}_0(f) = e_1 \otimes e_1$. Let $a, b \in \mathbb{R}^+$, $R := [-a, a] \times [-b, b]$ and $g_1, \ldots, g_r$ be as in Theorem 3.6(B). Possibly restricting R we can suppose that (35) holds. Hence the thesis directly follows from Lemma 4.6. □

The following proposition shows that the local property (23) defining the elements of $S^p_{\mathrm{tg}}(\mathbb{R}^2)$ can be converted into a global property (i.e., a global parametrization of a system Γ with the same density) at least assuming the finiteness of the singular set Sing_{S_f}. Removing this assumption will be the aim of Corollary 4.10.

Proposition 4.8. *Let $f \in S^p_{\mathrm{tg}}(\mathbb{R}^2)$ be such that $\sharp\mathrm{Sing}_{S_f} < +\infty$. Then there exists a system of curves $\Gamma \in H^{2,p}(\mathbb{S})$ without crossings such that*

$$f = \theta_\Gamma\ \chi_{(\Gamma)}.$$

Proof. The proof is based on the construction in [5, Theorem 8.6] with minor modifications. Let $\mathrm{Sing}_{S_f} = \{z_1, \ldots, z_q\}$. For every $\zeta \in \mathrm{Sing}_{S_f}$, we denote by $R(\zeta)$ a nice rectangle for S_f at ζ such that

$$R(\zeta) \cap \mathrm{Sing}_{S_f} = \{\zeta\}, \tag{48}$$

and we make an arbitrary choice of the normal unit vector to S_f at ζ so that $R^+(\zeta)$ and $R^-(\zeta)$ are defined. From now on with the symbol δ_j we denote $+1$ or -1. Accordingly we write $R^{\delta_j}(\zeta)$ in place of $R^{\pm}(\zeta)$ when $\delta_j = \pm 1$. Let us construct the first curve γ_1 of the system Γ.

Construction of γ_1. Fix $z_1 \in \mathrm{Sing}_{S_f}$. Set $(f_0, (\zeta_0, \delta_0)) := (f, (z_1, +1))$. The construction of $(f_1, (\zeta_1, \delta_1))$ is the same as in [5, Theorem 8.6], and is such that $(f_1, (\zeta_1, \delta_1))$ satisfies properties (a)-(e) listed below, with $i = 1$. Suppose we have defined $(f_i, (\zeta_i, \delta_i))$ for some $i \geq 1$, with:

(a)

$$f_i : \mathbb{R}^2 \to \mathbb{N}, \qquad f_i := \begin{cases} f_{i-1} - 1 & \text{on } H_i, \\ f_{i-1} & \text{on } \mathbb{R}^2 \setminus H_i, \end{cases} \tag{49}$$

where $H_i \subset S_{f_{i-1}}$ is a connected component of $\mathrm{Reg}_{S_{f_{i-1}}}$ such that: $f_{i-1} \geq 1$ is constant on H_i; $\zeta_i \in \mathrm{Sing}_{S_f}$ is a point of the relative boundary of H_i (which is composed either by ζ_{i-1} itself, and in this case we understand that $\zeta_{i-1} = \zeta_i$, or by two points $\{\zeta_{i-1}, \zeta_i\} \subset \mathrm{Sing}_{S_f}$); H_i crosses $R^{-\delta_{i-1}}(\zeta_{i-1})$ and reaches ζ_i crossing $R^{\delta_i}(\zeta_i)$;

(b) f_i verifies the train tracks property in $R(\zeta)$ for every $\zeta \in S_{f_i} \cap (\mathrm{Sing}_{S_f} \setminus \{z_1, \zeta_i\})$;

(c) if $\zeta_i \neq z_1$ we have $\sum_{z \in \partial R^+(z_1)} f_i(z) = \sum_{z \in \partial R^-(z_1)} f_i(z) + 1$; moreover $\delta_i = \pm 1$ implies $\sum_{z \in \partial R^+(\zeta_i)} f_i(z) = \sum_{z \in \partial R^-(\zeta_i)} f_i(z) \mp 1$;

(d) if $\zeta_i = z_1$ and $\delta_i = -1$ then $\sum_{z \in \partial R^+(z_1)} f_i(z) = \sum_{z \in \partial R^-(z_1)} f_i(z) + 2$;

(e) if $\zeta_i = z_1$ and $\delta_i = +1$ then $\sum_{z \in \partial R^+(z_1)} f_i(z) = \sum_{z \in \partial R^-(z_1)} f_i(z)$.

The construction of $(f_{i+1}, (\zeta_{i+1}, \delta_{i+1}))$ when $(f_i, (\zeta_i, \delta_i))$ is given is the same as the one in [5, Theorem 8.6]. The only difference is that in our case the algorithm stops when $\{z \in \partial R^{-\delta_i}(\zeta_i) : f_i(z) \geq 1\} = \emptyset$. By (c) and (d) this can happen only if we are in case (e).

We now define $(\gamma_1) := \overline{H_1} \cup \cdots \cup \overline{H_n}$. Since H_i and H_{i+1} have ζ_i as a boundary point and belong to opposite half planes with respect to the normal line to S_{f_i} at ζ_i, using [5, Lemma 4.1] we can find $\gamma_1 \in H^{2,p}(S^1_1)$ parametrized with constant speed. Note that by construction we have $\theta_{\gamma_1}\chi_{(\gamma_1)} \leq f$.

Construction of γ_{i+1} $(i \geq 1)$. Suppose we have defined $\Gamma_i = (\gamma_1, \dots, \gamma_i) \in H^{2,p}(\mathbb{S}_i)$, where $\mathbb{S}_i := S^1_1 \times \cdots \times S^1_i$, verifying $f_{\Gamma_i} := \theta_{\Gamma_i}\chi_{(\Gamma_i)} \leq f$. If $f - f_{\Gamma_i}$ is not identically zero, $\Gamma_{i+1} \in H^{2,p}(\mathbb{S}_i \times S^1)$ is constructed as in the previous step starting from $f_i := f - f_{\Gamma_i} \in S^p_{\mathrm{tg}}(\mathbb{R}^2)$. Since $\sharp\mathrm{Sing}_{S_f} < +\infty$ the algorithm stops in a finite number n of steps $n \geq 1$. Hence $f - f_{\Gamma_n} \equiv 0$ on $\mathbb{R}^2$ which is the thesis. □

Lemma 4.9. *Let $f \in S^p_{\mathrm{tg}}(\mathbb{R}^2)$. There exists a sequence $\{f_k\} \subset S^p_{\mathrm{tg}}(\mathbb{R}^2)$ such that*

$$S_{f_k} \subseteq S_f, \qquad \sharp\mathrm{Sing}_{S_{f_k}} < +\infty, \qquad \mathcal{F}(f_k) \leq \mathcal{F}(f) \qquad \forall k \in \mathbb{N}, \tag{50}$$

and

$$f_k \rightharpoonup f \quad \text{in } HV^p(\mathbb{R}^2), \qquad \lim_{k\to\infty} \mathcal{F}(f_k) = \mathcal{F}(f). \tag{51}$$

Proof. This lemma can be proved almost along the same lines as in [5, Theorem 5.1]. We sketch the main ideas of the proof for the reader's convenience.

Step 1. Fix $k \in \mathbb{N}$. For any $z \in \mathrm{Sing}_{S_f}$ let $R(z)$ be a nice rectangle for f centered at z, with diameter strictly smaller than 2^{-k}. Since Sing_{S_f} is compact (see (25)), there are $z_1, ..., z_{m(k)}$ points of Sing_{S_f} such that $\mathrm{Sing}_{S_f} \subset \bigcup_{i=1}^{m(k)} R(z_i)$, and $S_f \cap \partial R(z_i) \subset \mathrm{Reg}_{S_f}$ for any $i \in \{1, \dots, m(k)\}$ (see (26)). In order to construct $f_k \in S^p_{\mathrm{tg}}(\mathbb{R}^2)$ we use a recursive algorithm consisting of $m(k)$ steps. Let $f^k_0 := f$, let $1 \leq i \leq m(k)$, and suppose that f^k_{i-1} has been defined. Then f^k_i is obtained by modifying f^k_{i-1} only on $\mathrm{int}(R(z_i))$ as follows. Let us suppose for simplicity that $z_i = 0$, that $P^{f^k_{i-1}}(z_i) = e_1 \otimes e_1$ and that $R(z_i) = [-a, a] \times [-b, b]$. Moreover let $\mathcal{F}(\cdot, R(z_i))$ be the localization of $\mathcal{F}$ on $R(z_i)$. Repeating the proof of [5, Lemmata 4.3, 4.5, 4.11] it is possible to show the existence of functions $g_1, \dots, g_r \in H^{2,p}(]-a, a[)$ and integer numbers $\lambda_1, \dots, \lambda_r$ (see *Step 1* in the proof of [5, Theorem 5.1] for further details) such that

- for any function $\varphi \in \mathcal{C}^1(]-a, a[)$ such that $\mathrm{graph}(\varphi) \subset S_{f^k_{i-1}} \cap R(z_i)$ and $\varphi(\pm a) = g_l(\pm a)$ we have
$$\mathcal{F}(\chi_{\mathrm{graph}(g_l)}, R(z_i)) \leq \mathcal{F}(\chi_{\mathrm{graph}(\varphi)}, R(z_i)),$$
- the set $\mathrm{Sing}_{\cup_{l=1}^r \mathrm{graph}(g_l)}$ is finite and
$$\sum_{l=1}^r \lambda_l \mathcal{F}((\chi_{\mathrm{graph}(g_l)}, R(z_i)) \leq \mathcal{F}(f^k_{i-1}, R(z_i));$$
- defining $f^k_i : \mathbb{R}^2 \to \mathbb{N}$ as $f^k_i := f^k_{i-1}\chi_{\mathbb{R}^2 \setminus R(z_i)} + \chi_{R(z_i)} \sum_{l=1}^r \lambda_l \chi_{\mathrm{graph}(g_l)}$, we have $f^k_i \in S^p_{\mathrm{tg}}(\mathbb{R}^2)$.

Once f_i^k are defined it results that

$$S_{f_i^k} \subseteq S_{f_{i-1}^k}, \qquad \sharp(\mathrm{Sing}_{S_{f_i^k}} \cap \mathrm{int}(R(z_i))) < +\infty, \qquad \mathcal{F}(f_i^k) \leq \mathcal{F}(f_{i-1}^k). \tag{52}$$

We now define $f_k := f_{m(k)}^k$. From (52) we have $\mathcal{F}(f_k) = \mathcal{F}(f_{m(k)}^k) \leq \mathcal{F}(f_{m(k)-1}^k) \leq \cdots \leq \mathcal{F}(f)$. By construction we have that $S_{f_i^k}$ is closed for every $i \in \{0, \ldots, m(k)\}$ and

$$S_{f_k} = S_{f_{m(k)}^k} \subseteq S_{f_{m(k)-1}^k} \subseteq \cdots \subseteq S_{f_0^k} = S_f.$$

Furthermore, since $\sharp(\mathrm{Sing}_{S_{f_k}} \cap \cup_{i=1}^{m(k)} R(z_i)) < +\infty$, with $\cup_{i=1}^{m(k)} R(z_i) \supseteq \mathrm{Sing}_{S_f}$ and $S_{f_k} \subseteq S_f$, we have that $\mathrm{Sing}_{S_{f_k}}$ is a finite set. So we defined $\{f_k\} \subset S^p_{\mathrm{tg}}(\mathbb{R}^2)$ satisfying (50) and this concludes the proof of *Step 1.*

Step 2. We prove that for $\{f_k\}$ defined in *Step 1* (51) holds.

Since

$$\sup_{k \in \mathbb{N}} \mathcal{F}(f_k) \leq \mathcal{F}(f), \tag{53}$$

we can find a subsequence (still denoted by $\{f_k\}$) which converges in $HV^p(\mathbb{R}^2)$ as $k \to +\infty$ to a certain $\widehat{f} \in S^p_{\mathrm{tg}}(\mathbb{R}^2)$ (see [13]) such that $S_{\widehat{f}} \subseteq S_f$ by construction.

We want to prove that $\widehat{f} = f$. To this aim we use Remark 3.10(iii). We start by proving that $S_{\widehat{f}} = S_f$. Let $p \in \mathrm{Reg}_{S_f}$. Since $\mathrm{Sing}_{S_f} = S_f \setminus \mathrm{Reg}_{S_f}$ is compact, we have $\mathrm{dist}(p, \mathrm{Sing}_{S_f}) > 0$. So, for every k with $1/2^k < \mathrm{dist}(p, \mathrm{Sing}_{S_f})$, the point p is outside the region where we made our modifications and therefore there is a whole neighborhood of p where $f_k = f$. Therefore $p \in \mathrm{Reg}_{S_{\widehat{f}}}$ and $\mathrm{Reg}_{S_f} \subseteq \mathrm{Reg}_{S_{\widehat{f}}}$. Hence, recalling (25) and the inclusion $S_{\widehat{f}} \subseteq S_f$, we get $S_f = \overline{\mathrm{Reg}_{S_f}} \subseteq \overline{S_{\widehat{f}}}$. So $S_f = \overline{S_{\widehat{f}}}$ and therefore $\mathrm{Reg}_{S_f} = \mathrm{Reg}_{S_{\widehat{f}}}$.

By construction we have $f = \widehat{f}$ on $\mathrm{Reg}_{S_f} = \mathrm{Reg}_{S_{\widehat{f}}}$. Hence $f = \widehat{f}$ by Remark 3.10(iii).

Finally, using (53) and the lower semicontinuity of $\mathcal{F}$ on $S^p(\mathbb{R}^2)$, we have

$$\mathcal{F}(f) \leq \liminf_{k \to \infty} \mathcal{F}(f_k) \leq \limsup_{k \to \infty} \mathcal{F}(f_k) \leq \mathcal{F}(f)$$

□

Using Proposition 4.8 and Lemma 4.9 we obtain the following strenghtened version of Proposition 4.8.

Corollary 4.10. *Let $f \in S^p_{\mathrm{tg}}(\mathbb{R}^2)$. Then there exists a system of curves $\Gamma \in H^{2,p}(\mathbb{S})$ without crossings such that*

$$f = \theta_\Gamma \chi_{(\Gamma)}.$$

Proof. Let $\{f_k\} \subset S^p_{\mathrm{tg}}(\mathbb{R}^2)$ be the sequence built in the proof of Lemma 4.9. By Proposition 4.8, for every $k \in \mathbb{N}$ we can find a system of curves $\Gamma_k \in H^{2,p}(\mathbb{S}_k)$ such that $f_k = \theta_{\Gamma_k} \chi_{(\Gamma_k)}$. From (50) it follows that

$$\mathcal{F}(\Gamma_k) = \mathcal{F}(f_k) \leq \mathcal{F}(f) < +\infty.$$

Since $(\Gamma_k) = S_{f_k} \subseteq S_f$ and S_f is compact, we can apply [4, Theorem 3.1], hence we can find a subsequence of systems of curves $\{\Gamma_{k_j}\}$ all defined on same parameter space $\mathbb{S}$ and such that $\{\Gamma_{k_j}\}$ converges in the $H^{2,p}$-weak topology to a system of curves $\Gamma \in H^{2,p}(\mathbb{S})$. Since by construction we have $(\Gamma) = S_f$ and $\theta_\Gamma = f$ on Reg_{S_f}, we deduce $f = \theta_\Gamma \chi_{(\Gamma)}$. □

Remark 4.11. Lemma 4.8 defines a bijective map between $S^p_{\mathrm{tg}}(\mathbb{R}^2)$ and the quotient of the set of all $H^{2,p}$-systems of curves without crossings with respect to the equivalence relation defined in [5, Definition 2.30].

We are now in the position to conclude the proof.

Proof of Theorem 4.3. . As a consequence of Theorem 2.9 and Proposition 4.7 and Corollary 4.10 we obtain (30).

Let us prove (31). Let $\Gamma \in \mathcal{A}^o(E)$; as noted in Remark 3.9(iii) and using also (30), $f_\Gamma := \theta_\Gamma \chi_{(\Gamma)} \in S^p_{\mathrm{tg}}(\mathbb{R}^2)$ and $\mathcal{F}(\Gamma) = \mathcal{F}(f)$. Moreover from Definition 2.7 we have $S_f \supseteq \partial E$, and (28), (29) are a consequence of Remark 2.8(ii). From Theorem 2.9 we have

$$\overline{\mathcal{F}}(E) = \min\{\mathcal{F}(\Gamma) : \Gamma \in \mathcal{A}^o(E)\} \geq \inf\{\mathcal{F}(f) : f \in HV_{\mathrm{tg}}(\partial E)\}. \tag{54}$$

Let us prove

$$\overline{\mathcal{F}}(E) \leq \inf\{\mathcal{F}(f) : f \in HV_{\mathrm{tg}}(\partial E)\}. \tag{55}$$

Let $f \in HV_{\mathrm{tg}}(\partial E)$. Thanks to Corollary 4.10 there exists a system of curves Γ_f of class $H^{2,p}$ without crossings such that

$$f = \theta_{\Gamma_f}\chi_{(\Gamma_f)}, \qquad \mathcal{F}(\Gamma_f) = \mathcal{F}(f).$$

By (28), (29) applying [5, Proposition 3.13] we deduce that $\Gamma_f \in \mathcal{A}^o(E)$. Hence (55) follows again from Theorem 2.9.

To conclude we have to prove that the infimum of $\mathcal{F}$ is achieved on $HV_{\mathrm{tg}}(\partial E)$, but this is a consequence of (24) with Γ a minimizer of $\mathcal{F}$ on $\mathcal{A}^o(E)$. □

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