Journal of Convex Analysis
Volume 14 (2007), No. 1, 49–68

On a Maximization Problem for the Convex Hull of Connected Systems of Segments

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Received: July 4, 2005
Revised manuscript received: October 3, 2005

Fixed a set of finitely many segments in $\mathbb{R}^d$ we look for a connected system of these segments whose convex hull has the greatest d-dimensional volume, solving the problem when d and $d+1$ segments are considered. Some remarks about the analogous maximization problem on the set of the continuous and rectifiable curves with fixed length are also proposed.

Keywords: Convex hull, system of segments

2000 Mathematics Subject Classification: Primary 52A40

1. Introduction

A still open problem posed by Bezdek and Bezdek in [2] involves the convex hull of the so called *connected system of segments*, that is connected sets given by the union of a finite number of segments: such a problem asks to rearrange, without changing their lengths, a fixed set of finitely many segments of $\mathbb{R}^d$ in order to form a connected system whose convex hull has the greatest d-dimensional volume.

In order to fix the notations and present a precise mathematical formulation of this problem, we define *segment of* $\mathbb{R}^d$ a set of the type

$$[q, q+s] := \{q + \lambda s : \lambda \in [0,1]\}, \quad q \in \mathbb{R}^d, s \in \mathbb{R}^d \setminus \{0\},$$

and, fixed $n \in \mathbb{N}$, we say that a set $\Sigma \subseteq \mathbb{R}^d$ is a *connected system of n segments of* $\mathbb{R}^d$, briefly $\Sigma \in \mathrm{CSS}(d,n)$, if Σ is connected and

$$\Sigma = \bigcup_{i=1}^{n} [q_i, q_i + s_i], \quad q_i \in \mathbb{R}^d, s_i \in \mathbb{R}^d \setminus \{0\}. \tag{1}$$

Referring to (1) we define the vector of the lengths of the edges of Σ as $L(\Sigma) = (|s_1|, \ldots, |s_n|) \in (\mathbb{R}_+)^n$ and the set of the vertices of Σ as $V(\Sigma) = \bigcup_{i=1}^{n} \{q_i, q_i + s_i\}$. Moreover, given $L = (l_1, \ldots, l_n) \in (\mathbb{R}_+)^n$ we say that $\Sigma \in \mathrm{CSS}(d, n; L)$ if $\Sigma \in \mathrm{CSS}(d,n)$ and $L(\Sigma) = (l_{\varphi(1)}, \ldots, l_{\varphi(n)})$ where φ is a suitable permutation of the indexes $\{1, \ldots, n\}$, in symbols $\varphi \in S_n$.

With these notations we can state the considered problem as follows (here $\mathcal{L}^d$ denote the d-dimensional Lebesgue measure and $\mathrm{co}(\Sigma)$ is the convex hull of the set Σ).

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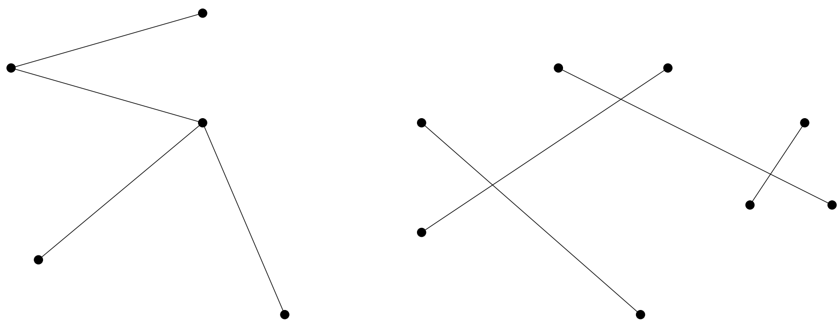

Figure 1.1: Connected systems of 4 segments in $\mathbb{R}^2$

Problem 1.1. *Let $n \geq d \geq 2$, $L \in (\mathbb{R}_+)^n$ and*

$$M(d,n;L) = \max\left\{\mathcal{L}^d(\mathrm{co}(\Sigma)) : \Sigma \in \mathrm{CSS}(d,n;L)\right\}. \tag{2}$$

Find a set $\Sigma_{\mathrm{opt}} \in \mathrm{CSS}(d,n;L)$ such that $\mathcal{L}^d(\mathrm{co}(\Sigma_{\mathrm{opt}})) = M(d,n;L)$.

Problem 1.1 has been recently solved in the two dimensional case by Siegel [11] which proved that an optimal set is given by the support of any polygonal curve whose edges have the fixed lengths and which is inscribed in a semicircle (of course the order of the edges in this polygonal curve can be arbitrary).

When higher dimensional spaces are considered the situation is instead more difficult and only few partial results have been proved (see [2, 3]). In particular Bezdek and Bezdek found, in the case of $n = d+1$, the shape of an optimal set among the connected systems of segments having each segment with a vertex in common with an other segment of the system (Figure 1.1 shows two connected systems of 4 segments in $\mathbb{R}^2$: the first satifies the conditions required by Bezdek and Bezdek, the second does not). Further to our notations, we can state this result as follows (see [2] Theorem 2).

Theorem 1.2. *Fix $L = (l_1, \dots, l_{d+1}) \in (\mathbb{R}_+)^{d+1}$ and consider the set $T(d,d+1;L) \in \mathrm{CSS}(d,d+1;L)$ given by*

$$T(d,d+1;L) = \bigcup_{i=1}^{d+1} [0,s_i], \quad s_i \in \mathbb{R}^d \setminus \{0\},$$

where $0 \in \mathrm{int}(\mathrm{co}(\{s_1,\dots,s_{d+1}\}))$ and $\langle s_i, s_j - s_k\rangle = 0$ for every $j,k \neq i$. Then given

$$\Sigma = \bigcup_{i=1}^{d+1} [q_i, q_i + s_i] \in \mathrm{CSS}(d,d+1;L)$$

such that, for every $r_a, r_b \in V(\Sigma)$, there exist $k \in \mathbb{N}$ and $\{r_j\}_{j=1}^k \subseteq V(\Sigma)$ satisfying $r_1 = r_a$, $r_k = r_b$ and, for every $j = 1,\dots,k-1$, $[r_j, r_{j+1}] = \left[q_{i(j)}, q_{i(j)} + s_{i(j)}\right]$ for a suitable $i(j) \in \{1,\dots,d+1\}$, it holds $\mathcal{L}^d(\mathrm{co}(\Sigma)) \leq \mathcal{L}^d(\mathrm{co}(T(d,d+1;L)))$.

Note that the set $T(d,d+1;L)$ considered in Theorem 1.2 is uniquely determined up to isometries and that the conditions $0 \in \mathrm{int}(\mathrm{co}(\{s_1,\dots,s_{d+1}\}))$ and $\langle s_i, s_j - s_k\rangle = 0$ for

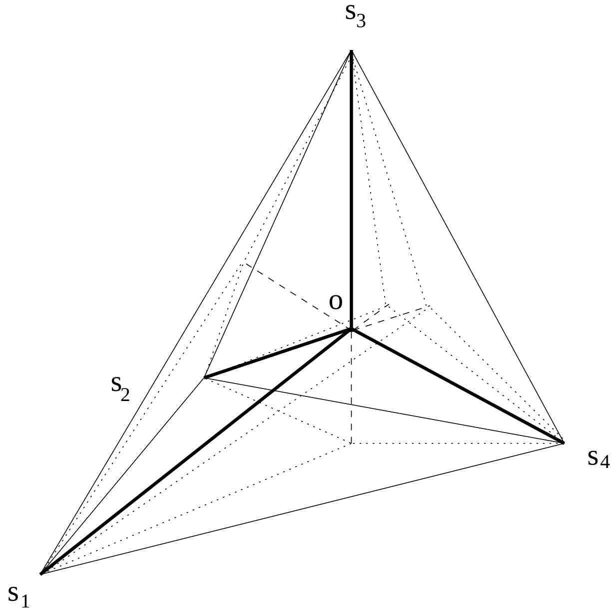

Figure 1.2: The set $T(3, 4; L)$

every $j, k \neq i$ required on $T(d, d+1; L)$, say that 0 is the center of the altitudes of the simplex $\mathrm{co}(T(d, d+1; L))$ whose vertices are the points s_i (see Figure 1.2).

While about Problem 1.1 only few studies have been carried on, on the contrary the (still unsolved) classical problem of the maximization of the convex hull among the continuous and rectifiable curves with fixed length has been considered by several authors (see in particular [5, 6, 7, 10, 12]; see also Section 7). An important result on this argument is the proof that, in the smaller class of the so called *convex curves*, which were introduced by Shoenberg [10], the equation of an optimal curve can be explicitly written down (as proved by Nudel'man [7]): indeed the measure of the convex hull of the support of these curves can be explicitly computed knowing their equations and the maximization problem can be successful treated. Moreover it is reasonable to conjecture, following Zalgaller [12], that the optimal curve among the convex ones solves also the problem among all the continuous and rectifiable curves.

Let us give now precise definitions. A function $p : [0, 1] \to \mathbb{R}^d$ is said a *curve* and the set $p([0, 1])$ is said the *support* of p. A curve $p \in \mathrm{Lip}([0, 1], \mathbb{R}^d)$ is said to be a *polygonal curve* if there exists $n \in \mathbb{N}$ and a partition $0 = t_0 < t_1 < \ldots < t_n = 1$ of $[0, 1]$ such that p is linear on every interval of the type $[t_{i-1}, t_i]$, $i = 1, \ldots, n$. A continuous curve $p(t) = (p_1(t), \ldots, p_d(t)) : [0, 1] \to \mathbb{R}^d$ is said to be *convex* if $p([0, 1])$ is not contained in any hyperplane and, for every $(\alpha_0, \alpha_1, \ldots, \alpha_d) \in \mathbb{R}^{d+1}$, the function $\alpha_0 + \sum_{i=1}^{d} \alpha_i p_i(t) : [0, 1] \to \mathbb{R}$ changes its sign at most $d + 1$ times (see [7, 10]). As proved in [6] (see Lemma 8.1 therein, see also [7]) a polygonal curve p with partition $0 = t_0 < t_1 < \ldots < t_n = 1$ is convex *if and only if*, for every $0 \leq i_0 < i_1 < \ldots < i_d \leq n$,

$$\det_{d+1}\Big((1, p(t_{i_0})), \ldots, (1, p(t_{i_d}))\Big) \tag{3}$$

has the same sign and there exists at least a case for which the determinant is not null: when the determinants are all non negative (non positive) the curve is said to be a *non negative* (*non positive*) convex polygonal curve. The following theorem is due to

Nudel'man (see [7] pg. 282).

Theorem 1.3. *Given a non negative convex polygonal curve with partition* $0 = t_0 < t_1 < \ldots < t_n = 1$, *and defined for* $i = 1, \ldots, n$ *the vectors* $s_i := p(t_i) - p(t_{i-1})$, *it holds*

$$\mathcal{L}^d(\mathrm{co}(p([0,1]))) = \frac{1}{d!} \sum_{1 \le i_1 < \ldots < i_d \le n} \det_d(s_{i_1}, \ldots, s_{i_d}). \tag{4}$$

In this paper, after having proved in Section 2 that Problem 1.1 is well defined, that is, there always exists the maximum in (2) (see Theorem 2.4), we propose in Section 3 the proof of the following formula, very similar to formula (4), which provides an upper bound of the d-dimensional volume of the convex hull of a connected system of segments.

Theorem 1.4. *Given* $n \ge d \ge 2$ *and* $\Sigma = \bigcup_{i=1}^n [q_i, q_i + s_i] \in \mathrm{CSS}(d,n)$ *it holds*

$$\mathcal{L}^d(\mathrm{co}(\Sigma)) \le \frac{1}{d!} \sum_{1 \le i_1 < \ldots < i_d \le n} \left|\det_d(s_{i_1}, \ldots, s_{i_d})\right|. \tag{5}$$

It is worth noting that the right hand side of (5) has an interesting geometrical interpretation: indeed, as proved by Shephard [9], for every $q_1, \ldots, q_n \in \mathbb{R}^d$ and $s_1, \ldots, s_n \in \mathbb{R}^d \setminus \{0\}$, it holds

$$\mathcal{L}^d\left(\sum_{i=1}^n [q_i, q_i + s_i]\right) = \sum_{1 \le i_1 < \ldots < i_d \le n} \left|\det_d(s_{i_1}, \ldots, s_{i_d})\right|, \tag{6}$$

where $\sum_{i=1}^n [q_i, q_i + s_i]$ is the usual Minkowsky sum of the segments $[q_1, q_1 + s_1], \ldots, [q_n, q_n + s_n]$.

By means of Theorem 1.4 and some further properties of the convex polygonal curves, we are able to deduce several facts about Problem 1.1. The first consequence we propose is the following theorem, proved in Section 4, which establishes sufficient conditions in order to have an optimal set for Problem 1.1 given by the support of an injective non negative convex polygonal curve: this theorem allows also to achieve in a simple way Theorem 1.6 below, whose proof can be found in Section 5. We note that the part of Theorem 1.6 concerning the case $d = 2$ has been already proved, in a slight different way, by Siegel in [11].

Theorem 1.5. *Let* $n \ge d \ge 2$ *and assume that, for every* $s_1, \ldots, s_n \in \mathbb{R}^d$, *there exist a function* $\rho : \{1, \ldots, n\} \to \{-1, 1\}$, *a permutation* $\varphi \in S_n$ *and a linear isometry* $\Psi : \mathbb{R}^d \to \mathbb{R}^d$ *such that, if we define, for every* $i = 1, \ldots, n$, $v_i = \Psi\left(\rho(\varphi(i)) s_{\varphi(i)}\right)$, *it holds*

$$\det_d\left(v_{i_1}, \ldots, v_{i_d}\right) \ge 0, \quad \forall 1 \le i_1 < \ldots < i_d \le n.$$

Then, fixed $L \in (\mathbb{R}_+)^n$, *there exists an injective non negative convex polygonal curve* $p : [0,1] \to \mathbb{R}^d$ *such that*

$$p([0,1]) \in \mathrm{CSS}(d,n;L) \quad \text{and} \quad \mathcal{L}^d(\mathrm{co}(p([0,1]))) = M(d,n;L).$$

Theorem 1.6. *If* $d = 2$, $n = d$ *or* $n = d + 1$, *then Problem* 1.1 *admits an optimal set which is the support of a suitable injective non negative convex polygonal curve.*

Subsequently we prove in Section 6 the two following theorems in which it is described, in the very special cases $n = d$ and $n = d + 1$, the shape of an optimal set for Problem 1.1. Note that Theorem 1.8 below generalizes Theorem 1.2.

Theorem 1.7. *Fixed* $L = (l_1, \ldots, l_d) \in (\mathbb{R}_+)^d$, *it holds* $M(d, d; L) = \frac{1}{d!} l_1 \ldots l_d$ *and, letting* $e_1, \ldots, e_d$ *be the standard basis of* $\mathbb{R}^d$, *the set* $T(d, d; L) \in \mathrm{CSS}(d, d; L)$ *given by*

$$T(d, d; L) = \bigcup_{i=1}^{d} [0, l_i e_i],$$

satisfies $\mathcal{L}^d(\mathrm{co}(T(d, d; L))) = M(d, d; L)$.

Theorem 1.8. *Fixed* $L = (l_1, \ldots, l_{d+1}) \in (\mathbb{R}_+)^{d+1}$, *the set* $T(d, d+1; L) \in \mathrm{CSS}(d, d+1; L)$ *given by*

$$T(d, d+1; L) = \bigcup_{i=1}^{d+1} [0, s_i], \quad s_i \in \mathbb{R}^d \setminus \{0\},$$

where $0 \in \mathrm{int}(\,\mathrm{co}(\,\{\,s_1, \ldots, s_{d+1}\,\}\,)\,)$ *and* $\langle\, s_i, s_j - s_k \,\rangle = 0$ *for every* $j, k \neq i$, *satisfies* $\mathcal{L}^d(\mathrm{co}(T(d, d+1; L))) = M(d, d+1; L)$.

We note at this point that, thanks to Theorem 1.6, it seems reasonable to think that the class of the convex polygonal curves assumes a fundamental role also in the problem of the maximization of the convex hull of connected systems of segments: for this reason we propose here the following conjecture.

Conjecture 1.9. *Problem* 1.1 *always admits an optimal set which is the support of a suitable injective non negative convex polygonal curve.*

In Section 7 we will discuss some consequences of the validity of Conjecture 1.9 in relation with the already quoted problem of the maximization of the convex hull among the continuous and rectifiable curves and with other related problems.

2. Existence of the maximum for the Problem 1.1

Let $E \subseteq \mathbb{R}^d$ be a non empty compact set, denote by $\mathcal{C}_E$ the family of all the closed non empty subsets of E and let, for every $\Sigma_1, \Sigma_2 \in \mathcal{C}_E$,

$$d_{\mathcal{H}}(\Sigma_1, \Sigma_2) := \inf \{r > 0 : \Sigma_1 \subseteq \Sigma_2 + B(0, r) \text{ and } \Sigma_2 \subseteq \Sigma_1 + B(0, r)\}.$$

The function $d_{\mathcal{H}}$ is a distance on $\mathcal{C}_E$, called Hausdorff distance, and when $\{\Sigma_h\}_{h\in\mathbb{N}} \subseteq \mathcal{C}_E$ converges to $\Sigma \in \mathcal{C}_E$ with respect to the topology induced by $d_{\mathcal{H}}$, that is, $d_{\mathcal{H}}(\Sigma_h, \Sigma) \to 0$ as $h \to \infty$, we write $\Sigma_h \xrightarrow{\mathcal{H}} \Sigma$.

In the following we need some results about Hausdorff distance: the proof of Theorem 2.1 below can be found in [1] (see Proposition 4.4.12, Theorem 4.4.15 and Theorem 4.4.17); Propositions 2.2 and 2.3 are straightforward and their proofs are omitted. $\mathcal{H}^k$ means the standard k-dimensional Hausdorff measure.

Theorem 2.1. *The metric space* $(\mathcal{C}_E, d_{\mathcal{H}})$ *is complete and compact. Moreover if* $\{\Sigma_h\}_{h\in\mathbb{N}} \subseteq \mathcal{C}_E$ *is a sequence such that* Σ_h *is connected and* $\Sigma_h \xrightarrow{\mathcal{H}} \Sigma$*, then* Σ *is connected and*

$$\mathcal{H}^1(\Sigma) \leq \liminf_{h\to\infty} \mathcal{H}^1(\Sigma_h).$$

Proposition 2.2. *Let* $\Sigma_h, \Sigma \in \mathcal{C}_E$ *such that* $\Sigma_h \xrightarrow{\mathcal{H}} \Sigma$*. Then* $\mathrm{co}(\Sigma_h) \xrightarrow{\mathcal{H}} \mathrm{co}(\Sigma)$*.*

Proposition 2.3. *Let* $\Sigma_h, \Sigma \in \mathcal{C}_E$ *such that* $\Sigma_h \xrightarrow{\mathcal{H}} \Sigma$*. Then* $\limsup_{h\to\infty} \mathcal{L}^d(\Sigma_h) \leq \mathcal{L}^d(\Sigma)$*.*

Let us prove now the existence of the maximum for the Problem 1.1.

Theorem 2.4. *For every* $n \geq d \geq 2$ *and* $L \in (\mathbb{R}_+)^n$*, the set*

$$\left\{\mathcal{L}^d(\mathrm{co}(\Sigma)) : \Sigma \in \mathrm{CSS}(d, n; L)\right\}, \tag{7}$$

admits a maximum.

Proof. Fixed $L = (l_1, \dots, l_n) \in (\mathbb{R}_+)^n$, let us consider a maximizing sequence of (7) given by

$$\Sigma_h = \bigcup_{i=1}^n [q_i^h, q_i^h + s_i^h], \quad q_i^h \in \mathbb{R}^d,\ s_i^h \in \mathbb{R}^d \setminus\{0\},\ |s_i^h| = l_i.$$

Up to isometries, we can suppose that $\Sigma_h \subseteq \overline{B(0,R)}$, where $R > 0$ is sufficiently large: this fact, together with Theorem 2.1, allows to state also that there exists a compact and connected set $\Sigma_{\mathrm{opt}} \subseteq \overline{B(0,R)}$ such that $\Sigma_h \xrightarrow{\mathcal{H}} \Sigma_{\mathrm{opt}}$ and that, for every $i = 1, \dots, n$, $q_i^h \to q_i$ and $s_i^h \to s_i$ (where $|s_i| = l_i$). A simple computation shows that, for every $i = 1, \dots, n$,

$$[q_i^h, q_i^h + s_i^h] \xrightarrow{\mathcal{H}} [q_i, q_i + s_i] \quad \text{and} \quad \Sigma_{\mathrm{opt}} = \bigcup_{i=1}^n [q_i, q_i + s_i].$$

This implies that Σ_{opt} belongs to $\mathrm{CSS}(d, n; L)$ and, by Propositions 2.2 and 2.3, we obtain that $\mathcal{L}^d(\mathrm{co}(\Sigma_{\mathrm{opt}}))$ realizes the maximum for (7). $\square$

3. Proof of Theorem 1.4

In order to achieve the proof of Theorem 1.4 we need some preliminary results: while Proposition 3.1 below is very simple and its proof can be omitted, Theorem 3.2 requires a certain effort to be proved.

Proposition 3.1. *Let* $n \geq d \geq 2$, $\Sigma = \bigcup_{i=1}^n [q_i, q_i + s_i] \in \mathrm{CSS}(d, n)$ *and* $s \in \mathbb{R}^d \setminus \{0\}$*. If* $P_s : \mathbb{R}^d \to \pi_s$ *is the projection on* $\pi_s = \{x \in \mathbb{R}^d : \langle x, s\rangle = 0\}$*, then* $P_s(\Sigma)$ *is connected,* $\mathrm{co}(P_s(\Sigma)) = P_s(\mathrm{co}(\Sigma))$ *and*

$$P_s(\Sigma) = \bigcup_{i=1}^n [P_s(q_i), P_s(q_i) + P_s(s_i)].$$

Theorem 3.2. *Let $C \subseteq \mathbb{R}^d$ be a convex and compact set, $q \in \mathbb{R}^d$ and $s \in \mathbb{R}^d \setminus \{0\}$ such that $[q, q+s] \cap C \neq \emptyset$. Then*

$$\mathcal{L}^d(\mathrm{co}(C \cup [q, q+s])) \leq \mathcal{L}^d(C) + \frac{1}{d}|s|\,\mathcal{H}^{d-1}(P_s(C)), \tag{8}$$

where $P_s : \mathbb{R}^d \to \pi_s$ is the projection on $\pi_s = \{x \in \mathbb{R}^d : \langle x, s\rangle = 0\}$.

Proof. If $q, q+s \in C$ there is nothing to prove.

Step 1. Let us consider now the two cases $q \in C, q+s \notin C$ and $q \notin C, q+s \in C$: since they are completely analogous, only the first case is proved.

For every isometry $\Psi : \mathbb{R}^d \to \mathbb{R}^d$, (8) holds if and only if

$$\mathcal{L}^d(\mathrm{co}(\Psi(C) \cup [\Psi(q), \Psi(q) + \Psi(s)])) \leq \mathcal{L}^d(\Psi(C)) + \frac{1}{d}|\Psi(s)|\,\mathcal{H}^{d-1}(P_{\Psi(s)}(\Psi(C))),$$

and then denoting with $e_1, e_2, \ldots, e_d$ the canonical basis of $\mathbb{R}^d$, we can assume $q = 0$, $s = |s|e_d$ and $\pi_s = \{x \in \mathbb{R}^d : x_d = 0\}$. Defined now $E := \mathrm{co}(C \cup [0, s]) \setminus C$, we achieve the claim of Step 1 proving

$$\mathcal{L}^d(E) \leq \frac{1}{d}|s|\,\mathcal{H}^{d-1}(P_s(C)). \tag{9}$$

In order to simplify the notations, we consider here only the case $d \geq 4$: the (simpler) proofs of the cases $d = 2, 3$ can be obtained following a similar argument and, for this reason, are left to the reader.

In order to evaluate $\mathcal{L}^d(E)$ we change the variables passing to the cylindric coordinates. Let us consider then the function

$$\phi : (0, \infty) \times (0, \pi)^{d-3} \times (0, 2\pi) \times \mathbb{R} \to \mathbb{R}^d$$

defined as

$$\begin{cases} x_1 = \phi_1(\rho, \theta_1, \ldots, \theta_{d-3}, \theta_{d-2}, z) = \rho \cos\theta_1\ , \\ x_2 = \phi_2(\rho, \theta_1, \ldots, \theta_{d-3}, \theta_{d-2}, z) = \rho \sin\theta_1 \cos\theta_2\ , \\ \vdots \\ x_{d-2} = \phi_{d-2}(\rho, \theta_1, \ldots, \theta_{d-3}, \theta_{d-2}, z) = \rho \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{d-3} \cos\theta_{d-2}\ , \\ x_{d-1} = \phi_{d-1}(\rho, \theta_1, \ldots, \theta_{d-3}, \theta_{d-2}, z) = \rho \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{d-3} \sin\theta_{d-2}\ , \\ x_d = \phi_d(\rho, \theta_1, \ldots, \theta_{d-3}, \theta_{d-2}, z) = z\ , \end{cases}$$

and its Jacobian determinant

$$J(\rho, \theta_1, \ldots, \theta_{d-3}, \theta_{d-2}) = \rho^{d-2} \sin^{d-3}\theta_1 \sin^{d-4}\theta_2 \ldots \sin\theta_{d-3}.$$

By means of this change of variables, letting $\theta = (\theta_1, \ldots, \theta_{d-3}, \theta_{d-2})$, we have

$$\mathcal{L}^d(E) = \int_{\mathbb{R}^d} \chi_E(x)dx = \int_{(0,\pi)^{d-3}\times(0,2\pi)} \int_{(0,\infty)} \left(\int_{\mathbb{R}} \chi_{\phi^{-1}(E)}(\rho, \theta, z)dz \right) J(\rho, \theta) d\rho d\theta,$$

where $\chi_E(x)$ is the characteristic function of the set E.

Let us consider now $F = \mathrm{co}(P_s(C) \cup [0, s]) = \mathrm{co}(P_s(C) \cup \{s\})$ (since $0 \in P_s(C)$). The volume of F is

$$\mathcal{L}^d(F) = \frac{1}{d}|s|\,\mathcal{H}^{d-1}(P_s(C)).$$

On the other hand, by means of the cylindric coordinates, we obtain also

$$\mathcal{L}^d(F) = \frac{1}{d}|s|\,\mathcal{H}^{d-1}(P_s(C)) = \int_{(0,\pi)^{d-3}\times(0,2\pi)} \int_{(0,\infty)} \left(\int_{\mathbb{R}} \chi_{\phi^{-1}(F)}(\rho, \theta, z)dz \right) J(\rho, \theta)d\rho d\theta,$$

so that, defined

$$H(\rho, \theta) := \int_{\mathbb{R}} \chi_{\phi^{-1}(E)}(\rho, \theta, z)dz = \mathcal{L}^1(\{z \in \mathbb{R} : \phi(\rho, \theta, z) \in E\}),$$

and

$$R(\rho, \theta) := \int_{\mathbb{R}} \chi_{\phi^{-1}(F)}(\rho, \theta, z)dz = \mathcal{L}^1(\{z \in \mathbb{R} : \phi(\rho, \theta, z) \in F\}),$$

we have proved (9) showing that, for every (ρ, θ), it is

$$0 \le H(\rho, \theta) \le R(\rho, \theta). \tag{10}$$

In order to prove (10), we work with θ fixed: this allows to consider a two dimensional problem. Let us define

$$\mu(\theta) = \phi(1, \theta, 0) \quad \text{and} \quad \pi_\theta = \{\rho\mu(\theta) + ze_d : \rho \ge 0, z \in \mathbb{R}\},$$

and consider the function, whose range is included in π_θ, defined as

$$r(\alpha, t) = s + t(\mu(\theta)\sin\alpha - e_d \cos\alpha),$$

where $\alpha \in [0, \pi], t \ge 0$. Note that with the notations introduced

$$H(\rho, \theta) = \mathcal{L}^1(\{z : \rho\mu(\theta) + ze_d \in E \cap \pi_\theta\})$$

and

$$R(\rho, \theta) = \mathcal{L}^1(\{z : \rho\mu(\theta) + ze_d \in F \cap \pi_\theta\}).$$

Let us define now

$$\alpha_\theta = \max\Big\{\alpha \in [0, \pi] : \{r(\alpha, t) : t \ge 0\} \cap C \ne \emptyset\Big\}.$$

This definition is clearly well posed as C is compact and $\alpha_\theta \in [0, \pi)$ since, by construction, $\{r(\pi, t) : t \ge 0\} \cap C = \emptyset$. Moreover, for every $0 \le \beta \le \alpha_\theta$, $\{r(\beta, t) : t \ge 0\} \cap C \ne \emptyset$ and we can consider also

$$t(\beta) = \min\{t \ge 0 : r(\beta, t) \in C\}$$

(that is well defined and strictly positive, since $s \notin C$), and $p(\theta) = r(\alpha_\theta, t(\alpha_\theta)) \in C$ (see Figure 3.1). Let us prove now that

$$E \cap \pi_\theta \subseteq \mathrm{co}(\{0, s, p(\theta)\}). \tag{11}$$

Fixed $x \in E \cap \pi_\theta$ we can suppose $x \neq s$ (otherwise there is nothing to prove): we show at first there exist $\beta_x \in [0, \alpha_\theta]$ and $0 \leq t_x \leq t(\beta_x)$ such that $x = r(\beta_x, t_x)$. Indeed we have

$$\pi_\theta = \{r(\alpha, t) : t \geq 0, \alpha \in [0, \pi]\}$$

so that $x = r(\beta_x, t_x)$ for some $\beta_x \in [0, \pi]$ and $t_x \geq 0$. Moreover $x \in \mathrm{co}(C \cup [0, s]) = \mathrm{co}(C \cup \{s\})$ and then, by the equality

$$\mathrm{co}(C \cup \{s\}) = \{\lambda s + (1 - \lambda) y : y \in C, \lambda \in [0, 1]\}$$

there exist $y \in C$ and $\lambda \in (0, 1)$ (since $x \notin C$, $x \neq s$) with $x = \lambda s + (1 - \lambda) y$. A simple computation shows that

$$y = s + \frac{t_x}{1 - \lambda}(\mu(\theta) \sin \beta_x - e_d \cos \beta_x) = r\left(\beta_x, \frac{t_x}{1 - \lambda}\right) \in \{r(\beta_x, t) : t \geq 0\},$$

that implies $C \cap \{r(\beta_x, t) : t \geq 0\} \neq \emptyset$ that is $\beta_x \leq \alpha_\theta$. Finally we have also $t_x \leq t(\beta_x)$ because $t_x < \frac{t_x}{1-\lambda}$ and if it is $t_x > t(\beta_x)$, then it follows

$$x \in \left[r\left(\beta_x, t(\beta_x)\right), r\left(\beta_x, \frac{t_x}{1 - \lambda}\right)\right] \subseteq C$$

that is a contradiction.

Let pass now to prove (11). If $\beta_x = 0$ this follows simply since $x \in [0, s] \subseteq \mathrm{co}(\{0, s, p(\theta)\})$. If instead $0 < \beta_x \leq \alpha_\theta$, we can find a unique couple $(\lambda_0, t_0) \in (0, 1] \times (0, \infty)$ solution of the equation

$$r(\beta_x, t) = \lambda p(\theta).$$

Indeed, by the definitions of the involved quantities, the previous equation is equivalent to the linear system in (λ, t) given by

$$\begin{cases} t \sin \beta_x = \lambda t(\alpha_\theta) \sin \alpha_\theta \\ |s| - t \cos \beta_x = \lambda |s| - \lambda t(\alpha_\theta) \cos \alpha_\theta, \end{cases}$$

that has the unique solution (λ_0, t_0) given by

$$\lambda_0 = \frac{|s|}{|s| + t(\alpha_\theta)(\sin \alpha_\theta \cot \beta_x - \cos \alpha_\theta)}$$

and

$$t_0 = t(\alpha_\theta) \frac{|s|}{|s| + t(\alpha_\theta)(\sin \alpha_\theta \cot \beta_x - \cos \alpha_\theta)} \frac{\sin \alpha_\theta}{\sin \beta_x},$$

where $\lambda_0 \in (0, 1]$ and $t_0 > 0$ since $t(\alpha_\theta) > 0, |s| > 0$ and, by the assumption $0 < \beta_x \leq \alpha_\theta < \pi$, $\sin(\beta_x) > 0, \sin(\alpha_\theta) > 0$ and $\sin \alpha_\theta \cot \beta_x - \cos \alpha_\theta \geq 0$. Being

$$\lambda_0 p(\theta) = r(\beta_x, t_0) \in [0, p(\theta)] \subseteq C,$$

we achieve $x \in \mathrm{co}(\{0, s, p(\theta)\})$ proving that $x \in [s, r(\beta_x, t_0)]$: but this is simple because $r(\beta_x, t_0) \in C$ implies $t_0 \geq t(\beta_x) \geq t_x$ from which is follows $x \in [s, r(\beta_x, t_0)]$ (see Figure 3.1). Hence (11) is proved.

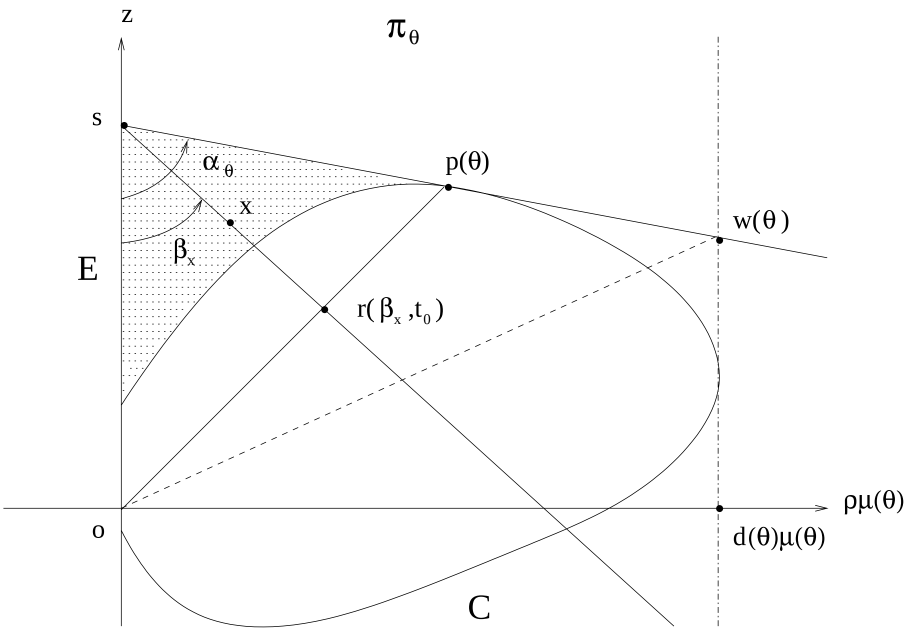

Figure 3.1: The set $C \cap \pi_\theta$

Let us consider now the set $Z = \{ze_d : z \in \mathbb{R}\}$ and define

$$d(\theta) = \max\{d(x, Z) : x \in C \cap \pi_\theta\} = \max\{d(x, Z) : x \in P_s(C) \cap \pi_\theta\} < \infty,$$

where $d(x, Z)$ is the usual distance of x from the set Z; note also that

$$P_s(C) \cap \pi_\theta = \{\rho\mu(\theta) : 0 \leq \rho \leq d(\theta)\}.$$

If $d(\theta) = 0$ we have, for every $\rho > 0$, $H(\theta, \rho) = R(\theta, \rho) = 0$ and (10) trivially holds. If instead $d(\theta) > 0$ it follows $\alpha_\theta > 0$ and we consider the unique solution (z_0, t_0) of the equation

$$d(\theta)\mu(\theta) + ze_d = r(\alpha_\theta, t),$$

given by

$$z_0 = |s| - d(\theta)\cot\alpha_\theta \in \mathbb{R} \quad \text{and} \quad t_0 = \frac{d(\theta)}{\sin(\alpha_\theta)} > 0$$

as it can be seen by solving, with respect to z and t, the equivalent linear system

$$\begin{cases} d(\theta) = t\sin\alpha_\theta, \\ z = |s| - t\cos\alpha_\theta. \end{cases}$$

If we define

$$w(\theta) = d(\theta)\mu(\theta) + (|s| - d(\theta)\cot\alpha_\theta)e_d = r\left(\alpha_\theta, \frac{d(\theta)}{\sin(\alpha_\theta)}\right), \tag{12}$$

since $p(\theta) \in [s, w(\theta)]$, we have $\{0, s, p(\theta)\} \subseteq \text{co}(\{0, s, w(\theta)\})$ and by (11) it follows

$$E \cap \pi_\theta \subseteq \text{co}(\{0, s, w(\theta)\})$$

(see Figure 3.1). By the argument till now developed, we have obtained that, for every fixed θ, if $d(\theta) = 0$ then (10) holds while if $d(\theta) > 0$, it is

$$\begin{aligned} H(\rho, \theta) &= \mathcal{L}^1(\{z : \rho\mu(\theta) + ze_d \in E \cap \pi_\theta\}) \\ &\leq \mathcal{L}^1(\{z : \rho\mu(\theta) + ze_d \in \mathrm{co}(\{0, s, w(\theta)\})\}) := R'(\theta, \rho), \end{aligned}$$

where $w(\theta)$ is defined in (12). Moreover, since $F \cap \pi_\theta = \mathrm{co}(\{0, s, d(\theta)\mu(\theta)\})$, it is also

$$R(\rho, \theta) = \mathcal{L}^1(\{z : \rho\mu(\theta) + ze_d \in \mathrm{co}(\{0, s, d(\theta)\mu(\theta)\})\}).$$

By construction we have, for every $\rho > d(\theta)$, $R(\theta, \rho) = R'(\theta, \rho) = 0$ and, if $0 < \rho \leq d(\theta)$, it can be easily computed that $R(\theta, \rho) = R'(\theta, \rho) = |s| \left(1 - \frac{\rho}{d(\theta)}\right)$. This proves (10) and then Theorem 3.2 follows in the special case of Step 1.

Step 2. Assume that $q, q + s \notin C$ and let $w \in [q, q + s] \cap C$. We apply Step 1 to the segments $[w, w + s_1]$ and $[w, w + s_2]$ where $s_1 = q - w$ and $s_2 = q - w + s$. Clearly $|s_1| + |s_2| = |s|$ and since $s_1 \parallel s_2$ it is $\pi_{s_1} = \pi_{s_2}$ and $P_{s_1} = P_{s_2} = P_s$. Then

$$\begin{aligned} &\mathcal{L}^d\left(\mathrm{co}(C \cup [q, q + s])\right) \\ &= \mathcal{L}^d\left(\mathrm{co}(\mathrm{co}(C \cup [w, w + s_1]) \cup [w, w + s_2])\right) \\ &\leq \mathcal{L}^d(\mathrm{co}(C \cup [w, w + s_1])) + \frac{1}{d}|s_2|\,\mathcal{H}^{d-1}\left(P_s(\mathrm{co}(C \cup [w, w + s_1]))\right) \\ &\leq \mathcal{L}^d(C) + \frac{1}{d}|s_1|\,\mathcal{H}^{d-1}(P_s(C)) + \frac{1}{d}|s_2|\,\mathcal{H}^{d-1}\left(P_s(\mathrm{co}(C \cup [w, w + s_1]))\right) \\ &= \mathcal{L}^d(C) + \frac{1}{d}|s|\,\mathcal{H}^{d-1}(P_s(C)), \end{aligned}$$

since, as is can be immediately verified,

$$P_s(\mathrm{co}(C \cup [w, w + s_1])) = P_s(C).$$

Then (8) is achieved in the full generality. □

Proof of Theorem 1.4. Our purpose is to prove the validity of (5) on $\mathrm{CSS}(d, n)$, for every couple (d, n) with $n \geq d \geq 2$: in order to do this we make use of the mathematical induction on the dimension $d \geq 2$. In the following, fixed $\Sigma = \bigcup_{i=1}^n [q_i, q_i + s_i] \in \mathrm{CSS}(d, n)$, we will call $\Sigma_j = \bigcup_{i=1}^j [q_i, q_i + s_i]$ with $j = 1, \ldots, n$. We note that, up to reordering the indexes, we can always suppose that every Σ_j is connected.

Step 1. We prove at first that (5) holds on $\mathrm{CSS}(2, n)$, for every $n \geq 2$: now we use the induction on $n \geq 2$ as described in the two further steps.

Step 1.1. Let $n = 2$, $q_1, q_2 \in \mathbb{R}^2$ and $s_1, s_2 \in \mathbb{R}^2 \setminus \{0\}$ such that $\Sigma_2 = \bigcup_{i=1}^2 [q_i, q_i + s_i]$ is connected. By Theorem 3.2 applied with $C = [q_1, q_1 + s_1]$, $q = q_2$ and $s = s_2$, we have

$$\mathcal{L}^2(\mathrm{co}(\Sigma_2)) \leq \mathcal{L}^2(\Sigma_1) + \frac{1}{2}|s_2|\,\mathcal{H}^1(P_{s_2}(\Sigma_1)) = \frac{1}{2}|s_2||s_1||\sin\alpha| = \frac{1}{2}\left|\det{}_2(s_1, s_2)\right|,$$

where α is the angle between s_1 and s_2.

Step 1.2. Assume now that (5) holds on CSS$(2, n-1)$ and let us prove it holds on CSS$(2, n)$ too. Consider $q_1, \dots, q_n \in \mathbb{R}^2$ and $s_1, \dots, s_n \in \mathbb{R}^2 \setminus \{0\}$ such that the sets $\Sigma_j = \bigcup_{i=1}^{j} [q_i, q_i + s_i]$ are connected for every $j = 1, \dots, n$. By the induction hypothesis on the number of segments we have

$$\mathcal{L}^2(\mathrm{co}(\Sigma_{n-1})) \le \frac{1}{2} \sum_{1 \le i < j \le n-1} |\det{}_2(s_i, s_j)| \,. \tag{13}$$

Then, by Theorem 3.2 applied with $C = \mathrm{co}(\Sigma_{n-1})$, $q = q_n$ and $s = s_n$, we have

$$\begin{aligned}
\mathcal{L}^2(\mathrm{co}(\Sigma_n)) &\le \mathcal{L}^2(\mathrm{co}(\Sigma_{n-1})) + \frac{1}{2}|s_n|\, \mathcal{H}^1(P_{s_n}(\mathrm{co}(\Sigma_{n-1}))) \\
&\le \mathcal{L}^2(\mathrm{co}(\Sigma_{n-1})) + \frac{1}{2}|s_n| \sum_{1 \le i \le n-1} |s_i| |\sin \alpha_i| \\
&= \mathcal{L}^2(\mathrm{co}(\Sigma_{n-1})) + \frac{1}{2} \sum_{1 \le i \le n-1} |\det{}_2(s_i, s_n)| \\
&\le \frac{1}{2} \sum_{1 \le i < j \le n-1} |\det{}_2(s_i, s_j)| + \frac{1}{2} \sum_{1 \le i \le n-1} |\det{}_2(s_i, s_n)| = \frac{1}{2} \sum_{1 \le i < j \le n} |\det{}_2(s_i, s_j)| \,,
\end{aligned}$$

where α_i is the angle between s_n and s_i and the inequality

$$\mathcal{H}^1(P_{s_n}(\mathrm{co}(\Sigma_{n-1}))) \le \sum_{1 \le i \le n-1} |s_i| |\sin \alpha_i|$$

follows by Proposition 3.1. Then we have proved (5) when $d = 2$.

Step 2. Let us suppose now that (5) holds on CSS$(d-1, n)$, for every $n \ge d-1$ and prove its validity on CSS(d, n), for every $n \ge d$: also in this case we use the induction on $n \ge d$.

Step 2.1. Let $n = d$, $q_1, \dots, q_d \in \mathbb{R}^d$ and $s_1, \dots, s_d \in \mathbb{R}^d \setminus \{0\}$ such that the sets $\Sigma_j = \bigcup_{i=1}^{j} [q_i, q_i + s_i]$ are connected for every $j = 1, \dots, d$. By Theorem 3.2 applied to $C = \mathrm{co}(\Sigma_{d-1})$, $q = q_d$ and $s = s_d$ we obtain

$$\mathcal{L}^d(\mathrm{co}(\Sigma_d)) \le \mathcal{L}^d(\mathrm{co}(\Sigma_{d-1})) + \frac{1}{d}|s_d|\, \mathcal{H}^{d-1}(P_{s_d}(\mathrm{co}(\Sigma_{d-1}))).$$

Note that, since Σ_{d-1} is connected, one can easily show that $\mathrm{co}(\Sigma_{d-1}) \subseteq q_1 + \mathrm{span}\langle s_1, \dots, s_{d-1} \rangle$, then $\mathcal{L}^d(\mathrm{co}(\Sigma_{d-1})) = 0$. Moreover, by Proposition 3.1,

$$P_{s_d}(\mathrm{co}(\Sigma_{d-1})) = \mathrm{co}\left(\bigcup_{i=1}^{d-1} [P_{s_d}(q_i), P_{s_d}(q_i) + P_{s_d}(s_i)] \right).$$

Let us define $v_d = s_d / |s_d|$ and let $v_1, \dots, v_d$ be an orthonormal basis of $\mathbb{R}^d$. With respect to such basis we write, for every $i = 1, \dots, d$,

$$s_i = (a_{i,1}, \dots, a_{i,d-1}, a_{i,d}) \quad (\text{in particular } s_d = (0, \dots, 0, |s_d|))$$

and then

$$P_{s_d}(s_i) = (a_{i,1}, \dots, a_{i,d-1}, 0).$$

By induction hypothesis on the dimension and by the isometry between π_{s_d} and $\mathbb{R}^{d-1}$ we have

$$\mathcal{H}^{d-1}(P_{s_d}(\mathrm{co}(\Sigma_{d-1}))) \le \frac{1}{(d-1)!}\left|\det_{d-1}\begin{pmatrix} a_{1,1} & \dots & a_{d-1,1} \\ \vdots & \ddots & \vdots \\ a_{1,d-1} & \dots & a_{d-1,d-1}\end{pmatrix}\right|.$$

In the end we have

$$\begin{aligned}\mathcal{L}^d(\mathrm{co}(\Sigma_d)) &\le \frac{1}{d}|s_d|\frac{1}{(d-1)!}\left|\det_{d-1}\begin{pmatrix} a_{1,1} & \dots & a_{d-1,1} \\ \vdots & \ddots & \vdots \\ a_{1,d-1} & \dots & a_{d-1,d-1}\end{pmatrix}\right| \\ &= \frac{1}{d!}\left|\det_d\begin{pmatrix} a_{1,1} & \dots & a_{d-1,1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{1,d-1} & \dots & a_{d-1,d-1} & 0 \\ a_{1,d} & \dots & a_{d-1,d} & |s_d|\end{pmatrix}\right| = \frac{1}{d!}\left|\det_d(s_1, \dots, s_d)\right|.\end{aligned}$$

Step 2.2. Assume now that (5) holds on $\mathrm{CSS}(d, n-1)$ and let us prove it holds on $\mathrm{CSS}(d, n)$ too. Consider $q_1, \dots, q_n \in \mathbb{R}^d$ and $s_1, \dots, s_n \in \mathbb{R}^d \setminus \{0\}$ such that the sets $\Sigma_j = \bigcup_{i=1}^{j}[q_i, q_i + s_i]$ are connected for every $j = 1, \dots, n$. By Theorem 3.2 applied to $C = \mathrm{co}(\Sigma_{n-1})$, $q = q_n$ and $s = s_n$, it holds

$$\mathcal{L}^d(\mathrm{co}(\Sigma_n)) \le \mathcal{L}^d(\mathrm{co}(\Sigma_{n-1})) + \frac{1}{d}|s_n|\,\mathcal{H}^{d-1}(P_{s_n}(\mathrm{co}(\Sigma_{n-1}))),$$

where, by Proposition 3.1, we have

$$P_{s_n}(\mathrm{co}(\Sigma_{n-1})) = \mathrm{co}\left(\bigcup_{i=1}^{n-1}[P_{s_n}(q_i), P_{s_n}(q_i) + P_{s_n}(s_i)]\right).$$

As in the previous step, define $v_d = s_n/|s_n|$ and let $v_1, \dots, v_d$ be an orthonormal basis of $\mathbb{R}^d$ and with respect to such basis we write, for every $i = 1, \dots, n$,

$$s_i = (a_{i,1}, \dots, a_{i,d-1}, a_{i,d}) \quad \text{(in particular } s_n = (0, \dots, 0, |s_n|))$$

and

$$P_{s_n}(s_i) = (a_{i,1}, \dots, a_{i,d-1}, 0).$$

By the induction hypothesis on the dimension and the isometry between π_{s_n} and $\mathbb{R}^{d-1}$ we get

$$\mathcal{H}^{d-1}(P_{s_n}(\mathrm{co}(\Sigma_{n-1}))) \le \frac{1}{(d-1)!}\sum_{1 \le i_1 < \dots < i_{d-1} \le n-1}\left|\det_{d-1}\begin{pmatrix} a_{i_1,1} & \dots & a_{i_{d-1},1} \\ \vdots & \ddots & \vdots \\ a_{i_1,d-1} & \dots & a_{i_{d-1},d-1}\end{pmatrix}\right|,$$

and by the induction hypotheses on the number of segments

$$\mathcal{L}^d(\mathrm{co}(\Sigma_{n-1})) \leq \frac{1}{d!} \sum_{1\leq i_1<\ldots<i_d\leq n-1} \left| \det_d \begin{pmatrix} a_{i_1,1} & \cdots & a_{i_d,1} \\ \vdots & \ddots & \vdots \\ a_{i_1,d} & \cdots & a_{i_d,d} \end{pmatrix} \right|.$$

Then

$$\begin{aligned} \mathcal{L}^d(\mathrm{co}(\Sigma_n)) \leq \frac{1}{d!} & \sum_{1\leq i_1<\ldots<i_d\leq n-1} \left| \det_d \begin{pmatrix} a_{i_1,1} & \cdots & a_{i_d,1} \\ \vdots & \ddots & \vdots \\ a_{i_1,d} & \cdots & a_{i_d,d} \end{pmatrix} \right| \\ & + \frac{1}{d}|s_n| \frac{1}{(d-1)!} \sum_{1\leq i_1<\ldots<i_{d-1}\leq n-1} \left| \det_{d-1} \begin{pmatrix} a_{i_1,1} & \cdots & a_{i_{d-1},1} \\ \vdots & \ddots & \vdots \\ a_{i_1,d-1} & \cdots & a_{i_{d-1},d-1} \end{pmatrix} \right| \\ \leq & \frac{1}{d!} \sum_{1\leq i_1<\ldots<i_d\leq n-1} \left| \det_d \begin{pmatrix} a_{i_1,1} & \cdots & a_{i_d,1} \\ \vdots & \ddots & \vdots \\ a_{i_1,d} & \cdots & a_{i_d,d} \end{pmatrix} \right| \\ & + \frac{1}{d!} \sum_{1\leq i_1<\ldots<i_{d-1}\leq n-1} \left| \det_d \begin{pmatrix} a_{i_1,1} & \cdots & a_{i_{d-1},1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{i_1,d-1} & \cdots & a_{i_{d-1},d-1} & 0 \\ a_{i_1,d} & \cdots & a_{i_{d-1},d} & |s_n| \end{pmatrix} \right| \\ = & \frac{1}{d!} \sum_{1\leq i_1<\ldots<i_d\leq n} \left| \det_d \begin{pmatrix} a_{i_1,1} & \cdots & a_{i_d,1} \\ \vdots & \ddots & \vdots \\ a_{i_1,d} & \cdots & a_{i_d,d} \end{pmatrix} \right| = \frac{1}{d!} \sum_{1\leq i_1<\ldots<i_d\leq n} |\det_d(s_{i_1},\ldots,s_{i_d})| \end{aligned}$$

and (5) is finally achieved. □

Theorem 1.4 allows immediately to give an alternative proof of the following theorem due to Bezdek, Brass and Harborth (see [3] Theorem 1).

Theorem 3.3. *Let $n \geq d \geq 2$ and $L = (l_1, \ldots, l_n) \in (\mathbb{R}_+)^n$ with $\sum_{i=1}^n l_i = 1$. Consider $\Sigma \in \mathrm{CSS}(d, n; L)$ such that all the segments of Σ are parallel to the standard coordinate axes of $\mathbb{R}^d$. Then $\mathcal{L}^d(\mathrm{co}(\Sigma)) \leq \frac{1}{d^d d!}$.*

Proof. Let $\Sigma = \bigcup_{i=1}^n [q_i, q_i + s_i] \in \mathrm{CSS}(d, n; L)$ and suppose that every s_i is parallel to one of the standard coordinate axes of $\mathbb{R}^d$. Letting $e_1, \ldots, e_d$ be the standard basis of $\mathbb{R}^d$, consider, for every $j = 1, \ldots, d$,

$$D_j = \{i : s_i = \lambda_i e_j \text{ with } \lambda_i \neq 0\}.$$

Then $D_1 \cup \ldots \cup D_d = \{1, \ldots, n\}$ and by Theorem 1.4 it is

$$\begin{aligned}\mathcal{L}^d(\mathrm{co}(\Sigma)) &\leq \frac{1}{d!} \sum_{1 \leq i_1 < \ldots < i_d \leq n} |\det_d(s_{i_1}, \ldots, s_{i_d})| \\ &= \frac{1}{d!} \sum_{i_1 \in D_1, \ldots, i_d \in D_d} |\det_d(s_{i_1}, \ldots, s_{i_d})| \\ &= \frac{1}{d!} \sum_{i_1 \in D_1, \ldots, i_d \in D_d} |s_{i_1}| \ldots |s_{i_d}| = \frac{1}{d!} \left(\sum_{i_1 \in D_1} |s_{i_1}| \right) \ldots \left(\sum_{i_d \in D_d} |s_{i_d}| \right).\end{aligned}$$

Then we have

$$\mathcal{L}^d(\mathrm{co}(\Sigma)) \leq \frac{1}{d!} a_1 \ldots a_d.$$

As $\sup\{a_1 \ldots a_d : a_i \geq 0, \sum_{i=1}^d a_i = 1\} = \frac{1}{d^d}$, the claimed inequality follows. □

4. Proof of Theorem 1.5

In order to prove Theorem 1.5 we need the following preliminary propositions about polygonal curves.

Proposition 4.1. *Let $s_1, \ldots, s_n \in \mathbb{R}^d \setminus \{0\}$ be such that, for every $1 \leq i_1 < \ldots < i_d \leq n$, it is $\det_d(s_{i_1}, \ldots, s_{i_d}) \geq 0$ and there exists at least a case for which the determinant is not null. Then the polygonal curve $p : [0, 1] \to \mathbb{R}^d$ defined as*

$$p(0) = 0,\ p(1/n) = s_1,\ p(2/n) = s_1 + s_2, \ldots,\ p(1) = s_1 + \ldots + s_n,$$

and extended linearly elsewhere is an open non negative convex polygonal curve.

Proof. Let's start proving that p is an open polygonal curve. Supposing by contradiction that $\sum_{i=1}^n s_i = 0$ (that is, p is closed), by hypotheses we know there exist d vectors among $s_1, \ldots, s_n$, which are linearly independent: then we can define

$$\begin{aligned} i_1 &= 1, \\ i_2 &= \min\{i : s_i \notin \mathrm{span}(s_{i_1})\}, \\ \vdots &\quad \vdots \\ i_d &= \min\{i : s_i \notin \mathrm{span}(s_{i_1}, \ldots, s_{i_{d-1}})\}. \end{aligned}$$

Clearly $\det_d(s_{i_1}, \ldots, s_{i_d}) > 0$ and since $s_{i_d} = -\sum_{i \neq i_d} s_i$, it is also

$$0 < \det_d(s_{i_1}, \ldots, s_{i_d}) = -\sum_{i \neq i_d} \det_d(s_{i_1}, \ldots, s_{i_{d-1}}, s_i). \tag{14}$$

However, by definition of $s_{i_1}, \ldots, s_{i_d}$, it is

$$\det_d(s_{i_1}, \ldots, s_{i_{d-1}}, s_i) = 0, \quad \forall i \leq i_{d-1}, \tag{15}$$

while, by hypothesis, it is

$$\det_d(s_{i_1}, \ldots, s_{i_{d-1}}, s_i) \geq 0, \quad \forall i > i_{d-1}. \tag{16}$$

Since (15) and (16) contradict (14), we conclude that p has to be open.

Using (3), we prove now that p is a non negative convex polygonal curve showing that, for every $0 \le i_0 < i_1 < \ldots < i_d \le n$, we have

$$\det_{d+1}\Big((1, p(i_0/n)), (1, p(i_1/n)), \ldots, (1, p(i_d/n))\Big) \ge 0,$$

and there exists at least a case for which the determinant does not vanish. Indeed, using the elementary properties of the determinants, it holds

$$\begin{aligned} &\det_{d+1}\Big((1, p(i_0/n)), (1, p(i_1/n)), \ldots, (1, p(i_d/n))\Big) \\ &= \det_{d+1}\Big((1, p(i_0/n)), (0, p(i_1/n) - p(i_0/n)), \ldots, (0, p(i_d/n) - p(i_{d-1}/n))\Big) \\ &= \det_d(p(i_1/n) - p(i_0/n), \ldots, p(i_d/n) - p(i_{d-1}/n)) \\ &= \det_d(s_{i_0+1} + \ldots + s_{i_1}, s_{i_1+1} + \ldots + s_{i_2}, \ldots, s_{i_{d-1}+1} + \ldots + s_{i_d}) \\ &= \sum_{j_1=i_0+1}^{i_1} \sum_{j_2=i_1+1}^{i_2} \cdots \sum_{j_d=i_{d-1}+1}^{i_d} \det_d(s_{j_1}, s_{j_2}, \ldots, s_{j_d}), \end{aligned}$$

and the proof is finally achieved. $\square$

Proposition 4.2. *Let $n \ge d \ge 2$, $L = (l_1, \ldots, l_n) \in (\mathbb{R}_+)^n$ and $p : [0,1] \to \mathbb{R}^d$ be a polygonal curve with partition $0 = t_0 < t_1 < \ldots < t_n = 1$ such that, for every $i = 1, \ldots, n$, $|p(t_i) - p(t_{i-1})| = l_{\varphi(i)}$ (where $\varphi \in S_n$). If $\mathcal{L}^d(\mathrm{co}(p([0,1]))) = M(d, n; L)$, then p is injective.*

Proof. Let us consider p as in the statement (note that, in particular, $p([0,1]) \in \mathrm{CSS}(d, n; L)$) and assume, by contradiction, that p is not injective: then there exist $x_1, x_2 \in [0,1]$ such that $x_1 < x_2$ and $p(x_1) = p(x_2)$. Since, for every $i = 1, \ldots, n$, p has to be injective on every interval $[t_{i-1}, t_i]$ (note that $p(t_{i-1}) \ne p(t_i)$), then there exist $i, j \in \{1, \ldots, n\}$ such that $i < j$ and $x_1 \in [t_{i-1}, t_i]$, $x_2 \in [t_{j-1}, t_j]$.

Step 1. Assume at first $i < j - 1$. Then the set

$$\Sigma := p([0, t_i]) \cup p([t_{i+1}, 1])$$

is compact and connected, because $p([0, t_i])$ and $p([t_{i+1}, 1])$ are compact and connected sets with $p(x_1) \in p([0, t_i]) \cap p([t_{i+1}, 1])$. Defined

$$L' = (l_1, \ldots, l_{\varphi(i+1)-1}, l_{\varphi(i+1)+1}, \ldots, l_n) \in (\mathbb{R}_+)^{n-1},$$

clearly it is $\Sigma \in \mathrm{CSS}(d, n-1; L')$ and moreover it holds

$$\mathcal{L}^d(\mathrm{co}(\Sigma)) = \mathcal{L}^d(\mathrm{co}(p([0,1]))), \tag{17}$$

since the inequality $\mathcal{L}^d(\mathrm{co}(\Sigma)) \le \mathcal{L}^d(\mathrm{co}(p([0,1])))$ follows by the hypotheses on p, while the opposite inequality by the relations

$$\mathcal{L}^d(\mathrm{co}(p([0,1]))) = \mathcal{L}^d(\mathrm{co}(\{p(t_0), \ldots, p(t_n)\})) \quad \text{and} \quad \{p(t_0), \ldots, p(t_n)\} \subseteq \Sigma.$$

If now $n = d$, then $\mathcal{L}^d(\mathrm{co}(\Sigma)) = 0$ and (17) implies $\mathcal{L}^d(\mathrm{co}(p([0,1]))) = 0$ which contradicts $M(d,d;L) > 0$. If instead $n \geq d+1$ (and $\mathcal{L}^d(\mathrm{co}(\Sigma)) > 0$, otherwise we argue as above), let us consider the set

$$\Sigma' := \Sigma \cup [q, q+s],$$

where $|s| = l_{\varphi(i+1)}$, $q \in \{p(t_0), \ldots, p(t_n)\}$ is an extremal point of $\mathrm{co}(\Sigma)$ and $q+s \notin \Sigma$. Then $\Sigma' \in \mathrm{CSS}(d,n;L)$, $\mathcal{L}^d(\mathrm{co}(\Sigma')) > \mathcal{L}^d(\mathrm{co}(p([0,1]))) = M(d,n;L)$ and the contradiction is found again.

Step 2. Assuming now $i = j-1$, it is $x_1 \in [t_{i-1}, t_i]$, $x_2 \in [t_i, t_{i+1}]$. Since it cannot be $x_1 = t_i$ or $x_2 = t_i$ (else p is constant on $[t_i, t_{i+1}]$ or $[t_{i-1}, t_i]$ respectively) it is $x_1 \in [t_{i-1}, t_i)$, $x_2 \in (t_i, t_{i+1}]$ and a simple computation shows that one of the following inclusions

$$p([t_i, t_{i+1}]) \subseteq p([t_{i-1}, t_i]), \quad p([t_{i-1}, t_i]) \subseteq p([t_i, t_{i+1}]),$$

has to hold. Assuming the validity of the first inclusion (resp. the second inclusion), we consider, if $i < n-1$ (resp. $i > 1$), the set

$$\Sigma := p([0, t_i]) \cup p([t_{i+1}, 1]) \quad (\text{resp. } \Sigma := p([0, t_{i-1}]) \cup p([t_i, 1]))\,,$$

while, if $i = n-1$ (resp. $i = 1$), the set

$$\Sigma := p([0, t_i]) \quad (\text{resp. } \Sigma := p([t_i, 1]))\,,$$

and working exactly as in the previous step we find the contradiction. □

Proof of Theorem 1.5. Let $L = (l_1, \ldots, l_n) \in (\mathbb{R}_+)^n$ and define

$$G_1(d,n;L) = \max\left\{\frac{1}{d!}\sum_{1\leq i_1<\ldots<i_d\leq n} \det{}_d(s_{i_1}, \ldots, s_{i_d}) : \vartheta \in S_n, s_i \in \mathbb{R}^d, \left|s_{\vartheta(i)}\right| = l_i\right\},$$

and

$$G_2(d,n;L) = \max\left\{\frac{1}{d!}\sum_{1\leq i_1<\ldots<i_d\leq n} \left|\det{}_d(s_{i_1}, \ldots, s_{i_d})\right| : \vartheta \in S_n, s_i \in \mathbb{R}^d, \left|s_{\vartheta(i)}\right| = l_i\right\}.$$

Clearly $G_1(d,n;L) \leq G_2(d,n;L)$: we prove that the assumptions made in the statement assure also the validity of the opposite inequality. Indeed let $s_1, \ldots, s_n \in \mathbb{R}^d$ and $\vartheta \in S_n$ such that, for every $i = 1, \ldots, n$, $\left|s_{\vartheta(i)}\right| = l_i$ and

$$\frac{1}{d!}\sum_{1\leq i_1<\ldots<i_d\leq n} \left|\det{}_d(s_{i_1}, \ldots, s_{i_d})\right| = G_2(d,n;L),$$

and consider ρ, φ, Ψ and $v_1, \ldots, v_n$ as in the statement. Then

$$\det{}_d(v_{i_1}, \ldots, v_{i_d}) \geq 0 \quad \forall\, 1 \leq i_1 < \ldots < i_d \leq n, \tag{18}$$

and considering $\vartheta_0 = \vartheta \circ \varphi^{-1} \in S_n$, we have that, for every $i = 1, \ldots, n$, $\left|v_{\vartheta_0(i)}\right| = l_i$ and

$$G_2(d,n;L) = \frac{1}{d!} \sum_{1 \le i_1 < \ldots < i_d \le n} |\det_d(s_{i_1}, \ldots, s_{i_d})| = \frac{1}{d!} \sum_{1 \le i_1 < \ldots < i_d \le n} |\det_d(v_{i_1}, \ldots, v_{i_d})|$$
$$= \frac{1}{d!} \sum_{1 \le i_1 < \ldots < i_d \le n} \det_d(v_{i_1}, \ldots, v_{i_d}) \le G_1(d,n;L),$$

that implies

$$G_1(d,n;L) = G_2(d,n;L) = \frac{1}{d!} \sum_{1 \le i_1 < \ldots < i_d \le n} \det_d(v_{i_1}, \ldots, v_{i_d}). \tag{19}$$

Applying now Proposition 4.1 and Theorem 1.3 to the vectors $v_1, \ldots, v_n$, we find an open non negative convex polygonal curve $p : [0,1] \to \mathbb{R}^d$ such that

$$p([0,1]) \in \mathrm{CSS}(d,n;L) \quad \text{and} \quad \mathcal{L}^d(\mathrm{co}(p([0,1]))) = G_2(d,n;L).$$

Since by Theorem 1.4 it follows $M(d,n;L) \le G_2(d,n;L)$ and since it is also $\mathcal{L}^d(\mathrm{co}(p([0,1]))) \le M(d,n;L)$, it has to be $\mathcal{L}^d(\mathrm{co}(p([0,1]))) = M(d,n;L)$. Finally we end applying Proposition 4.2 to the polygonal curve p in order to obtain injectivity. □

5. Proof of Theorem 1.6

Proof of Theorem 1.6. We will prove this theorem showing that, when $d = 2$, $n = d$ or $n = d + 1$, the assumptions of Theorem 1.5 are satisfied.

The case $d = 2$. Given $s_1, \ldots, s_n \in \mathbb{R}^2 \setminus \{0\}$, let us write $s_i = (s_{i,1}, s_{i,2})$ and consider $\rho : \{1, \ldots, n\} \to \{-1, 1\}$ such that, for every $i = 1, \ldots, n$, $\rho(i)s_{i,2} \ge 0$, $\varphi \in S_n$ such that the finite sequence $\frac{\rho(\varphi(i))s_{1,\varphi(i)}}{\left|\rho(\varphi(i))s_{\varphi(i)}\right|}$ decreases when i increases and Ψ equal to the identity. Then defining, for every $i = 1, \ldots, n$, $v_i = \Psi\left(\rho(\varphi(i))s_{\varphi(i)}\right)$, it holds

$$\det_2(v_i, v_j) \ge 0 \quad \forall\, 1 \le i < j \le n. \tag{20}$$

The case $n = d$. The proof is trivial and left to the reader.

The case $n = d + 1$. Given $s_1, \ldots, s_{d+1} \in \mathbb{R}^d \setminus \{0\}$, let us consider the $\binom{d+1}{d} = d + 1$ determinants

$$D_i = \det_d(s_1, \ldots, s_{i-1}, s_{i+1} \ldots, s_{d+1}),$$

where $i = 1, \ldots, d+1$, and let $I = \{i : D_i < 0\}$ and $|I|$ its cardinality. If $I = \emptyset$ there is nothing to prove. If $I \ne \emptyset$ let us define $\rho(i) = 1$, if $i \notin I$, and $\rho(i) = -1$, if $i \in I$. Considering now

$$D_i' = \det_d(\rho(1)s_1, \ldots, \rho(i-1)s_{i-1}, \rho(i+1)s_{i+1} \ldots, \rho(d+1)s_{d+1}),$$

it is $D_i' = (-1)^{|I|}D_i$, if $i \notin I$, and $D_i' = (-1)^{|I|-1}D_i$, if $i \in I$: then every D_i' has the same sign and the thesis is achieved by choosing, if they are all non negative, φ and Ψ as the identities while, if they are all negative, φ as the identity and Ψ as the reflection with respect to any coordinate axis. □

Remark 5.1. It is worth noting that, if $d = 2$, $n = d$ or $n = d+1$, and $L = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right) \in (\mathbb{R}_+)^n$, Theorem 1.6 together with Theorems II and IV in [7] allows to achieve the value of the constant $M(d,n;L)$ and the equations of an injective non negative convex polygonal curve whose support solves Problem 1.1.

6. Proof of Theorems 1.7 and 1.8

Proof of Theorem 1.7. Let $L = (l_1, \ldots, l_d) \in (\mathbb{R}_+)^d$ and consider, by means of Theorem 2.4, $\Sigma_{\rm opt} = \bigcup_{i=1}^d [q_i, q_i + s_i] \in \mathrm{CSS}(d, d; L)$ such that $\mathcal{L}^d(\mathrm{co}(\Sigma_{\rm opt})) = M(d, d; L)$. By applying Theorem 1.4 we have

$$M(d, d; L) = \mathcal{L}^d(\mathrm{co}(\Sigma_{\rm opt})) \leq \frac{1}{d!} |\mathrm{det}_d(s_1, \ldots, s_d)| \leq \frac{1}{d!} |s_1| \ldots |s_d| = \frac{1}{d!} l_1 \ldots l_d.$$

Since $\mathcal{L}^d(\mathrm{co}(T(d, d; L))) = \frac{1}{d!} l_1 \ldots l_d$, then $\mathcal{L}^d(\mathrm{co}(T(d, d; L))) = M(d, d; L)$ and we end the proof. □

Proof of Theorem 1.8. Let $L = (l_1, \ldots, l_{d+1}) \in (\mathbb{R}_+)^{d+1}$. By means of Theorem 1.6 we know there exists an injective non negative convex polygonal curve $p : [0, 1] \to \mathbb{R}^d$ whose support belongs to $\mathrm{CSS}(d, d+1; L)$ and satisfies the equality $\mathcal{L}^d(\mathrm{co}(p([0, 1]))) = M(d, d+1; L)$. Since $p([0, 1])$ satisfies also the hypotheses of Theorem 1.2 we obtain that $\mathcal{L}^d(\mathrm{co}(T(d, d+1; L))) = M(d, d+1; L)$. □

7. Some remarks about Conjecture 1.9

Given a curve γ we define its *variation* (or *length*) as

$$\mathrm{Var}(\gamma) = \sup \left\{ \sum_{i=1}^{k} |\gamma(t_i) - \gamma(t_{i-1})| : k \in \mathbb{N}, 0 = t_0 < t_1 < t_2 < \ldots < t_k = 1 \right\}, \tag{21}$$

and when $\mathrm{Var}(\gamma) < \infty$ we say that γ is *rectifiable*. Given the sets

$$\mathrm{RC}(d) = \left\{ \gamma : [0, 1] \to \mathbb{R}^d : \gamma \text{ is continuous, rectifiable and } \mathrm{Var}(\gamma) \leq 1 \right\}$$

and

$$\mathrm{CC}(d) = \{\Sigma \subseteq \mathbb{R}^d : \Sigma \text{ is connected, compact and } \mathcal{H}^1(\Sigma) \leq 1\},$$

we can consider the two following problems:

Problem 7.1. *Let $d \geq 2$ and $M_{\rm RC}(d) = \max \left\{ \mathcal{L}^d(\mathrm{co}(\gamma([0, 1]))) : \gamma \in \mathrm{RC}(d) \right\}$. Find a curve $\gamma_{\rm opt} \in \mathrm{RC}(d)$ such that $\mathcal{L}^d(\mathrm{co}(\gamma_{\rm opt}([0, 1]))) = M_{\rm RC}(d)$.*

Problem 7.2. *Let $d \geq 2$ and $M_{\rm CC}(d) = \max \left\{ \mathcal{L}^d(\mathrm{co}(\Sigma)) : \Sigma \in \mathrm{CC}(d) \right\}$. Find a set $\Sigma_{\rm opt} \in \mathrm{CC}(d)$ such that $\mathcal{L}^d(\mathrm{co}(\Sigma_{\rm opt})) = M_{\rm CC}(d)$.*

Note that, as already described in the introduction, Problem 7.1 has been considered by several authors (see [5, 6, 7, 10, 12]) while, on the contrary, Problem 7.2 seems to appear here for the first time.

By means of Theorems II' and IV' in [7], it can be proved that if Conjecture 1.9 is true then the convex curve $\gamma_{\rm opt} = (\gamma_1, \ldots, \gamma_d) \in \mathrm{RC}(d)$ defined, when $d = 2\nu + 1$, by the equations

$$\gamma_{2j-1}(t) = \frac{\cos(2\pi j t)}{2\pi j \sqrt{\nu + \frac{1}{2}}}, \quad \gamma_{2j}(t) = \frac{\sin(2\pi j t)}{2\pi j \sqrt{\nu + \frac{1}{2}}}, \quad j = 1, 2, \ldots, \nu$$

$$\gamma_{2\nu+1}(t) = \frac{t}{\sqrt{2\nu + 1}},$$

and, when $d = 2\nu$, by the equations

$$\gamma_{2j-1}(t) = \frac{\cos((2j-1)\pi t)}{(2j-1)\pi\sqrt{\nu}}, \quad \gamma_{2j}(t) = \frac{\sin((2j-1)\pi t)}{(2j-1)\pi\sqrt{\nu}}, \quad j = 1, 2, \ldots, \nu,$$

satisfies the equality

$$\mathcal{L}^d(\mathrm{co}(\gamma_{\mathrm{opt}}([0,1]))) = M_{\mathrm{CC}}(d) = \begin{cases} \frac{1}{\pi^\nu \nu!(2\nu+1)!(2\nu+1)^{\nu+\frac{1}{2}}} & \text{if } d = 2\nu+1, \\ \frac{1}{(\pi\nu)^\nu (2\nu)!(2\nu-1)!!} & \text{if } d = 2\nu. \end{cases}$$

Since, for every $\gamma \in \mathrm{RC}(d)$ it is $\gamma([0,1]) \in \mathrm{CC}(d)$, we can conclude that the validity of Conjecture 1.9 implies that γ_{opt} is an optimal curve for Problem 7.1 and its support $\gamma_{\mathrm{opt}}([0,1])$ is an optimal set for Problem 7.2.

Acknowledgements. I wish to thank Giuseppe Buttazzo and Francesco Maggi for reading a preliminary version of the paper and giving me suggestions for its improvement. I also thank Paolo Gronchi for drawing my attention to the equality (6) and the referee for useful remarks which allowed to simplify several parts of the paper.

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