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High Order Smoothness and Asymptotic Structure in Banach Spaces

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In this paper we study the connections between moduli of asymptotic convexity and smoothness of a Banach space, and the existence of high order differentiable bump functions or equivalent norms on the space. The existence of a high order uniformly differentiable bump function is related to an asymptotically uniformly smooth renorming of power type. On the other hand, the asymptotic uniform convexity of power type is related to the existence of high order rough norms. Finally, we also obtain some applications to the best order smoothness of Nakano sequence spaces.

1. Introduction

We are concerned in this paper with smoothness of Banach spaces, in the sense of the existence of smooth *bump* functions, that is, real functions with bounded nonempty support and with a given degree of differentiability. The existence of such smooth bump functions on a Banach space has a deep impact on the geometry and structure of the space, and we refer to [6] for an extensive account about this topic. Here we are mainly interested in the connections between high order smoothness and the asymptotic structure of the space, more precisely the relations with moduli of asymptotic convexity and smoothness.

Along the paper we will use two concepts of high order smoothness, which are defined in terms of Taylor expansions. Let X be a Banach space and consider $\varphi : X \to \mathbb{R}$. For $1 \le p < \infty$, the function φ is said to be T^p-*smooth* on a subset S of X if for each $x \in S$ there is a polynomial P of degree $\le p$, with $P(0) = 0$, verifying that

$$|\varphi(x+h) - \varphi(x) - P(h)| = o(\|h\|^p).$$

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In the case of a *norm* we always understand T^p-smoothness on the set $X \setminus \{0\}$.

On the other hand, for $1 < p < \infty$, the function φ is said to be *U^p-smooth* on a subset S of X if for each $x \in S$ there is a polynomial P of degree $< p$, with $P(0) = 0$, verifying that

$$|\varphi(x+h) - \varphi(x) - P(h)| = O(\|h\|^p),$$

uniformly on $x \in S$. In the case of a *norm* we always understand U^p-smoothness on the unit sphere S_X of X.

As usual, we say that the space X is *T^p-smooth* (respectively, *U^p-smooth*) if there is a bump function on X which is T^p-smooth (resp. U^p-smooth) on X.

Using Taylor's theorem, it is easily seen that if φ is m-times differentiable on S, then φ is T^m-smooth on S. On the other hand, let $1 < p < \infty$ and consider k the greatest integer strictly less than p. If φ is C^k-smooth and its k^{th}-derivative is uniformly $(p-k)$-Hölder continuous on S, it is said that φ is *UH^p-smooth* on S (see [29]). Then, as an application of Taylor's formula with integral remainder, we obtain that if φ is UH^p-smooth on S then it is U^p-smooth on S.

It is well known (see [9] or [6]) that for $1 < p \leq 2$ the existence of an UH^p-smooth bump function is equivalent to the existence of an equivalent norm with modulus of smoothness of power type p. Of course, this result cannot be directly extended to $p > 2$ since the modulus of smoothness of a norm is at most of power type 2. In this direction, in [18] it is proved that, for spaces with symmetric basis which are not isomorphic to ℓ_{2k} (for $k \in \mathbb{N}$), the existence of a UH^p-smooth bump function implies that the space admits an upper p-estimate. This result cannot be generalized to spaces with unconditional basis, as the example $(\bigoplus_{n=1}^{\infty} \ell_4^n)_{\ell_6}$ shows. Indeed, such space is UH^6-smooth (see [7]) and it does not admit an upper 6-estimate. To extend this kind of results we consider here the moduli of *asymptotic* uniform convexity and smoothness of the space. These moduli were first introduced by Milman [30] under different notations and names. We refer to [23] for a survey about the most relevant known results concerning these moduli.

The contents of the paper are as follows. In Section 2 we recall the definitions of the the moduli of asymptotic uniform convexity and smoothness, and we give some technical basic results about them.

Section 3 is devoted to study the connections between the modulus of asymptotic smoothness and the existence of smooth norms or bump functions. In [24] it is proved that if the norm of X is Fréchet differentiable then X is asymptotically smooth. We generalize this result to high order smoothness, using power type estimates of modulus of smoothness. On the other hand, the main result in Section 3 is that, assuming weak sequential continuity of polynomials up to degree p, U^p-smoothness of the space implies the existence of an equivalent norm with modulus of asymptotic smoothness of power type p. The converse of this result is not true in general, even for superreflexive spaces, as we also see.

While modulus of asymptotic smoothness of a certain power type is related to high order smoothness up to certain degree, we show in Section 4 that for not very smooth spaces (that is, spaces without separating polynomials), modulus of asymptotic convexity of power type is strongly related to some kind of high order non-smoothness. In this direction, high order roughness for norms was introduced in [13] generalizing the well known notion of rough norms (see e.g. [6]). We extend this notion by introducing *rough* *functions of order* p. Such functions, in particular, are not T^p-smooth at any point, even in an uniform way. Especial interest has the case $p = 2$ of second order roughness of convex functions, since it provides examples where Alexandrov theorem [1] strongly fails. Namely, this gives examples of norms which are twice Fréchet differentiable at no point. We prove that if a not very smooth space has modulus of asymptotic convexity of power type p, then its norm is rough of order p. We apply these results to study the smoothness of the space in the same lines as [13]. In this way we prove that the existence of a continuous function which is rough of order p implies the non existence of T^p-smooth bump functions in spaces with Radon-Nikodym property. To finish the Section we obtain that, for a large class of spaces, having modulus of asymptotic convexity of power type p is in fact equivalent to a certain kind of sequential roughness.

Finally, Section 5 is devoted to the study of the best order smoothness of a certain kind of modular spaces, namely Nakano sequence spaces $\ell_{(p_n)}$. In [26] the smoothness of these spaces is studied. In particular it is proved there that if $\lim_{n\to\infty} p_n = 2k$ and $p_n \geq 2k$, for every $,n \in \mathbb{N}$, then there exits a $\mathcal{C}^{2k}$-smooth bump function if and only if $\ell_{(p_n)}$ is isomorphic to ℓ_{2k}. In the case of $p_n \leq 2k$ for every $n \in \mathbb{N}$, the same result is obtained where $k = 1, 2$. As an application of the preceding sections, we are able to extend this last result for every $k \in \mathbb{N}$.

2. Moduli of asymptotic uniform convexity and smoothness.

As we mentioned before, in [23] the authors survey most of known results concerning these moduli. Along the paper we will use the same notation and terminology as in [23].

For a Banach space $(X, \|\cdot\|)$, the *modulus of asymptotic pointwise convexity* is defined for $\|x\| = 1$ and $0 < t \leq 1$ by:

$$\overline{\delta}(t;x) = \sup_{dim(X/Y)<\infty} \ \inf_{y\in Y,\ \|y\|\geq t} \|x+y\| - 1,$$

and the *modulus of asymptotic uniform convexity* is defined for $0 < t \leq 1$ by:

$$\overline{\delta}(t) = \inf_{\|x\|=1} \overline{\delta}(t;x).$$

The *modulus of asymptotic pointwise smoothness* is defined for $\|x\| = 1$ and $0 < t \leq 1$ by:

$$\overline{\rho}(t;x) = \inf_{dim(X/Y)<\infty} \ \sup_{y\in Y,\ \|y\|\leq t} \|x+y\| - 1,$$

and the *modulus of asymptotic uniform smoothness* is defined for $0 < t \leq 1$ by:

$$\overline{\rho}(t) = \sup_{\|x\|=1} \overline{\rho}(t;x).$$

The space X is said to be *asymptotically uniformly convex* if $\overline{\delta}(t) > 0$ for every $0 < t \leq 1$, and *asymptotically uniformly smooth* if $\overline{\rho}(t)/t \to 0$ as $t \to 0$. Along the paper, we will use power type estimates of moduli. More precisely, for $1 \leq p < \infty$ we say that the space has modulus of *asymptotic convexity of power type* p if there exists $C > 0$ such that $\overline{\delta}(t) \geq Ct^p$, for all $0 < t \leq 1$. In the same way, the space has modulus of *asymptotic smoothness of power type* p if there exists $C > 0$ so that $\overline{\rho}(t) \leq Ct^p$, for all $0 < t \leq 1$. For example, in the case of space ℓ_p with $1 \leq p < \infty$, we have that $\overline{\rho}(t) = \overline{\delta}(t) = (1+t^p)^{1/p} - 1$; and for the space c_0 we have $\overline{\rho}(t) = \overline{\delta}(t) = 0$. As a consequence, ℓ_p has modulus of asymptotic convexity and smoothness of power type p for every $1 \leq p < \infty$. Note in particular that ℓ_1 is uniformly asymptotically convex and c_0 is asymptotically uniformly smooth.

In order to obtain estimates for the moduli of asymptotic uniform convexity and smoothness, we give the following simple Lemma:

Lemma 2.1. *Let X be a Banach space, and let D be a dense subset of the unit sphere S_X. Then, for each $x \in S_X$:*

$$\overline{\delta}(t,x) = \sup_{dim(X/Y)<\infty} \inf_{y\in Y,\ \|y\|=t} \|x+y\| - 1; \quad \overline{\delta}(t) = \inf_{x\in D} \overline{\delta}(t,x).$$

$$\overline{\rho}(t,x) = \inf_{dim(X/Y)<\infty} \sup_{y\in Y,\ \|y\|=t} \|x+y\| - 1; \quad \overline{\rho}(t) = \sup_{x\in D} \overline{\rho}(t,x);$$

Proof. Let $x \in S_X$ and consider a norming functional $x^* \in X^*$ with $x^*(x) = 1$ and $\|x^*\| = 1$. First note that, if Y is a finite codimensional subspace of X and $y \in H = Y \cap \ker x^*$, the function $\phi(t) = \|x + ty\|$ is convex and non-decreasing on $t \in [0,\infty)$ and consequently,

$$\sup_{y\in H,\|y\|=t} \|x+y\| - 1 = \sup_{y\in Y,\|y\|\leq t} \|x+y\| - 1$$

and

$$\inf_{y\in H,\|y\|=t} \|x+y\| - 1 = \inf_{y\in H,\|y\|\geq t} \|x+y\| - 1.$$

On the other hand, if $x \in S_X$ and $\varepsilon > 0$ are given, it is possible to find $z \in D \cap B(x,\varepsilon)$, and then for all $y \in X$

$$\|x+y\| - 1 - \varepsilon \leq \|z+y\| - 1 \leq \|x+y\| - 1 + \varepsilon$$

$\square$

Remark 2.2. If X is Banach space with a finite dimensional decomposition (E_n), we consider the subspaces $H_n = \bigoplus_{i=1}^n E_i$ and $H^n = \overline{\bigoplus_{i=n+1}^\infty E_i}$ the closure of $\bigoplus_{i=n+1}^\infty E_i$ and the dense subset of the unit sphere $D = \cup_{n=1}^\infty H_n \cap S_X$. Using the above Lemma, it is easy to see that:

$$\overline{\rho}(t) \leq \sup\{\|x+y\| - 1 : x \in H_n; \|x\| = 1; y \in H^n; \|y\| = t; n \in \mathbb{N}\}$$

$$\overline{\delta}(t) \geq \inf\{\|x+y\| - 1 : x \in H_n; \|x\| = 1; y \in H^n; \|y\| = t; n \in \mathbb{N}\}.$$

As a consequence we have that if (E_n) is a sequence of finite-dimensional spaces, for $1 \leq p < \infty$ the usual norm of the space $(\bigoplus_{n=1}^\infty E_n)_{\ell_p}$ has modulus of asymptotic smoothness and convexity of power type p.

Let X be a Banach space with a finite dimensional decomposition (E_n). A function $\Phi : X \to \mathbb{R}$ is said to be *orthogonally additive* with respect to (E_n) if for every $x, y \in X$ with $x \in E_n$, $y \in E_m$ and $n \neq m$, we have:

$$\Phi(x+y) = \Phi(x) + \Phi(y).$$

Lemma 2.3. *Let $1 \leq p < \infty$, and let X be a Banach space with a finite dimensional decomposition (E_n). Suppose that $\Phi : X \to \mathbb{R}$ is a continuous convex function, orthogonally additive with respect to (E_n), such that $\Phi(0) = 0$ and*

$$\|x\| = \inf\left\{\lambda > 0 : \Phi(\frac{x}{\lambda}) \leq 1\right\}.$$

If there exists $C > 0$ such that $\Phi(h) \leq C\|h\|^p$ whenever $\|h\| \leq 1$, then $\|\cdot\|$ has modulus of asymptotic smoothness of power type p.

Proof. Consider again $H_n = \bigoplus_{i=1}^n E_i$ and $H^n = \overline{\bigoplus_{i=n+1}^\infty E_i}$. Let $x \in H_n$ with $\|x\| = 1$, and consider a norming functional $x^* \in X^*$ with $x^*(x) = 1$ and $\|x^*\| = 1$. If $h \in \ker x^*$, then $\|x + h\| \geq 1$. Consequently, if $h \in H^n \cap \ker x^*$ and $\|h\| = t \leq 1$, using the convexity of Φ we have that

$$\|x + h\| - 1 \leq \Phi(x + h) - \Phi(x) = \Phi(h) \leq Ct^p.$$

Then, from Lemma 2.1 we obtain that the norm has modulus of asymptotic smoothness of power type p. $\square$

We can apply these results to give estimates of the modulus of asymptotic smoothness of modular spaces. Recall that if $\{M_n\}_{n=1}^\infty$ is a sequence of Orlicz functions, the *modular sequence space* $h_{\{M_n\}}$ (see e.g. [25]) is defined as the space of all real sequences $x = (x_n)_{n=1}^\infty$ such that for every $\rho > 0$,

$$\sum_{n=1}^\infty M_n(\frac{|x_n|}{\rho}) < \infty.$$

It is equipped with the Luxemburg norm:

$$\|x\| = \inf\left\{\rho > 0 : \sum_{n=1}^\infty M_n\left(\frac{|x_n|}{\rho}\right) \leq 1\right\}.$$

In what follows we will consider the function Φ defined on $h_{\{M_n\}}$ by:

$$\Phi\left((x_n)_{n=1}^\infty\right) = \sum_{n=1}^\infty M_n(|x_n|).$$

Note that Φ is a continuous convex function, orthogonally additive with respect to the usual basis of $h_{\{M_n\}}$, and the Minkowsky functional of the set $G = \{x \in h_{\{M_n\}} : \Phi(x) \leq 1\}$ is precisely the Luxemburg norm.

Theorem 2.4. *Let $1 \leq p < \infty$ and let $h_{\{M_n\}}$ be a modular space endowed with the Luxemburg norm. If*

$$\sup_{n\in\mathbb{N}} \sup_{0\leq u,v\leq 1} \frac{M_n(uv)}{u^p M_n(v)} < \infty,$$

then $h_{\{M_n\}}$ has modulus of asymptotic smoothness of power type p.

Proof. Let $C > 0$ be such that $M_n(uv) \le CM_n(u)v^p$ for all $n \in \mathbb{N}$ and $0 \le u, v \le 1$. Let $x = (x_n) \in h_{\{M_n\}}$ with $\|x\| \le 1$. Since $|x_n| \le \|x\|$ we have that

$$\Phi(x) = \sum_{n=1}^{\infty} M_n(|x_n|) = \sum_{n=1}^{\infty} M_n\left(\frac{|x_n|}{\|x\|}\|x\|\right) \le C\|x\|^p \sum_{n=1}^{\infty} M_n\left(\frac{|x_n|}{\|x\|}\right) = C\|x\|^p.$$

The result now follows from Lemma 2.3. □

As a consequence we have that an Orlicz space h_M has modulus of asymptotic uniform smoothness of power type p for every $1 \le p < \alpha_M$, where the index α_M is defined by (see e.g. [25]):

$$\alpha_M = \sup\left\{p \ge 1;\ \sup_{0<u,v\le 1} \frac{M(uv)}{u^p M(v)} < \infty\right\}.$$

3. Smooth norms and modulus of asymptotic uniform smoothness.

Let X be a Banach space, and consider a function $\varphi : X \to \mathbb{R}$. We say that φ is *weakly sequentially continuous* (in short, w.s.c.) if it takes weakly convergent sequences in X into convergent sequences in $\mathbb{R}$. We say that φ is *weakly continuous on bounded sets* if, for each bounded subset B of X, the restriction of φ to B is continuous with respect to the topology induced on B by the weak topology of X, that is for every $x \in B$ and every $\varepsilon > 0$ there exist $x_1^*, \dots, x_n^* \in X^*$ and $\delta > 0$ such that if $y \in B$ satisfies $|x_i^*(x - y)| < \delta$ for every $i = 1, \dots, n$, then $|\varphi(x) - \varphi(y)| < \varepsilon$. If Y is an infinite dimensional subspace of X, we denote by:

$$\|\varphi\|_Y = \sup\{|\varphi(y)| :\ \|y\| = 1,\ y \in Y\}.$$

Lemma 3.1. *Let $1 \le p < \infty$ and let X be a Banach space which contains no subspace isomorphic to ℓ_1. Let $\varphi : X \to \mathbb{R}$ be a w.s.c. function with $\varphi(0) = 0$. Then for every $\varepsilon > 0$ there exists a finite codimensional subspace H of X such that $\|\varphi\|_H < \varepsilon$.*

Proof. Since X contains no copy of ℓ_1 and φ is weakly sequentially continuous, by [10] we have that φ is weakly continuous in on bounded sets. Then, given $\varepsilon > 0$ there are $x_1^*, \dots, x_n^* \in X^*$ and $\delta > 0$ such that if $x \in B_X$ and $|\,x_i^*(x)| < \delta$ for $i = 1, \dots, n$, we have $|\,\varphi(x)| < \varepsilon$. Now if we consider the finite codimensional subspace $H = \cap_{i=1}^n \ker x_i^*$ we have that $\|\varphi\|_H < \varepsilon$. □

Following [24] X is said to be *asymptotically smooth* if $\overline{\rho}(t;x)/t \to 0$ as $t \to 0$, for every $x \in S_X$. In [24] it is proved that if the norm of a space is Fréchet differentiable then the space is asymptotically smooth. Here we extend this result:

Proposition 3.2. *Let $1 \le p < \infty$ and let X be a Banach space such that all polynomials of degree $\le p$ are w.s.c. If X has T^p-smooth norm then for every $x \in S_X$*

$$\lim_{t \to 0} \frac{\overline{\rho}(t;x)}{t^p} = 0.$$

Proof. Since the space is T^p-smooth for some $p \ge 1$ it contains no subspace isomorphic to ℓ_1. Let $\varepsilon > 0$ and $x \in S_X$ be fixed. Since the norm is T^p-smooth on x there is a polynomial of degree k with $k \leq p$ and $0 < \delta \leq 1$ such that if $0 < t \leq \delta$ and $\|h\| = t$ then

$$\frac{\|x + h\| - 1 - P(h)}{t^p} < \varepsilon.$$

Let $P = P_1 + \cdots + P_k$, where each P_j is j-homogeneous, for $1 \leq j \leq k$. Let t be fixed with $0 < t \leq \delta$. Since the space is T^p-smooth for some $p \geq 1$ it contains no subspace isomorphic to ℓ_1. By applying Lemma 3.1 we obtain a finite codimensional subspace $H = H_{(t,x)}$ such that $\max\{\|P_1\|_H, \ldots, \|P_k\|_H\} < k^{-1}\varepsilon t^p$. If $h \in H$ and $\|h\| = t$ we have:

$$\begin{aligned} |P(h)| &\leq |P_1(h)| + \cdots + |P_k(h)| \leq |tP_1\left(\frac{h}{\|h\|}\right)| + \cdots + |t^k P_k\left(\frac{h}{\|h\|}\right)| \\ &\leq t\|P_1\|_H + \cdots + t^k\|P_k\|_H \leq \varepsilon\frac{t.t^p}{k} + \cdots + \varepsilon\frac{t^k t^p}{k} \leq \varepsilon t^p. \end{aligned}$$

Therefore $\overline{\rho}(t;x) \leq 2\varepsilon t^p$ if $0 < t \leq \delta$. □

Using the same techniques we obtain the following uniform version:

Proposition 3.3. *Let $1 < p < \infty$ and let X be a Banach space such that all polynomials of degree $< p$ are w.s.c. If the norm of X is U^p-smooth then X has modulus of asymptotic smoothness of power type p.*

Remark 3.4. Note that in the case of $1 \leq p < 2$, every polynomial of degree $\leq p$ is w.s.c., and therefore if the norm is T^p-smooth then $\lim_{t\to 0} \dfrac{\overline{\rho}(t;x)}{t^p} = 0$ on the unit sphere. Concerning the case $p = 2$, it would be interesting to know whether the norm in the 2-convexified *Tsirelson* space $\mathbf{T^{(2)}}$ from [11] verifies $\lim_{t\to 0} \dfrac{\overline{\rho}(t;x)}{t^2} = 0$ on the unit sphere. Indeed, it is not known if there is a $\mathcal{C}^2$-smooth bump function on $\mathbf{T^{(2)}}$ (see [6], p. 228). This space has modulus of smoothness of power type 2 and consequently it admits a bump function with Lipschitz continuous derivative. Since $\mathbf{T^{(2)}}$ contains no isomorphic copy of ℓ_2 it follows that all polynomials of degree ≤ 2 are weakly sequentially continuous (see [16]). Thus, if the space $\mathbf{T^{(2)}}$ had a T^2-smooth norm then $\lim_{t\to 0} \dfrac{\overline{\rho}(t,x)}{t^2} = 0$. We do not know whether this is true.

It is proved in [9] that, for $1 < p \leq 2$, every U^p-smooth space admits an equivalent norm with modulus of smoothness of power type p. Next, using a convexification procedure as in [9], we obtain the main result in this section:

Theorem 3.5. *Let $1 < p < \infty$ and let X be a Banach space such that all polynomials of degree $< p$ are w.s.c. If X is U^p-smooth, there exists an equivalent norm on X with modulus of asymptotic uniform smoothness of power type p.*

Proof. Since X is U^p-smooth, we can find a bump function $b : X \to \mathbb{R}$ which is U^p-smooth, and such that in addition $-1 \leq b \leq 0$; $b(0) = -1$; $\mathrm{Supp}(b) \subset B(0; \frac{1}{2})$ and b is symmetric, that is, $b(x) = b(-x)$ for every $x \in X$. We first prove the following fact:

Fact: There exist $K > 0$ and $0 < \delta \leq 1$ such that for each $0 < t \leq \delta$ and each $x \in X$ there is a finite codimensional subspace $H_{(t,x)}$ of X, such that if $h \in H_{(t,x)}$ and $\|h\| \leq t$

then

$$|b(x+h)-b(x)| \leq Kt^p.$$

Indeed, since b is U^p-smooth, there exist $C>0$ and $0<\delta\leq 1$ such that for every $x\in X$ there is a polynomial P of degree k with $k<p$, such that $P(0)=0$, and if $\|h\|\leq\delta$ then

$$|b(x+h)-b(x)-P(h)| \leq C\|h\|^p.$$

Let $P=P_1+\cdots+P_k$, where each P_j is j-homogeneous, for $1\leq j\leq k$. Note that since is X is U^p-smooth, it contains no subspace isomorphic to ℓ_1. Then by Lemma 3.1, for each $0<t\leq\delta$ and each $x\in X$, there exists a finite codimensional subspace $H=H_{(t,x)}$ of X such that $\max\{\|P_1\|_H,\ldots,\|P_k\|_H\}<k^{-1}t^{p-1}$. Then, if $h\in H$ with $\|h\|\leq t$ proceeding as in Proposition 3.2 we have:

$$|P(h)| \leq |P_1(h)|+\cdots+|P_k(h)| \leq \frac{t.t^{p-1}}{k}+\ldots\frac{t^k t^{p-1}}{k} \leq t^p$$

and consequently, $|b(x+h)-b(x)|\leq Ct^p+t^p$. This shows the fact.

Now we consider the function Φ defined on the unit ball B_X by:

$$\Phi(x)=\inf\left\{\sum_{i=1}^n \lambda_i b(x_i) : \lambda_i\geq 0, \sum_{i=1}^n\lambda_i=1, \sum_{i=1}^n \lambda_i x_i = x, x_i\in B_X, n\in\mathbb{N}\right\}.$$

Let $G=\{x\in B_X : \Phi(x)<-1/2\}$ and $\delta'=\min\{\delta/2,1/4\}$. We claim that there exists $K'>0$ such that for each $x\in G$ and each $0<t\leq\delta'$ there exists a finite codimensional subspace $Y_{(t,x)}$ such that if $h\in Y_{(t,x)}$ with $\|h\|\leq t$ then

$$\Phi(x+h)-\Phi(x)\leq K'\cdot t^p.$$

Indeed, let $x\in G$ and $0<t\leq\delta'$ be given. Consider $0<\varepsilon<1$ such that $\Phi(x)+\varepsilon<-1/2$ and choose $x_i\in B_X$ and $\lambda_i\geq 0$ (for $i=1,\cdots,n$) such that $\sum_{i=1}^n\lambda_i=1$ and $\sum_{i=1}^n\lambda_i b(x_i)<\Phi(x)+\varepsilon t^p$. Assume that $b(x_i)<0$ if $i=1,\cdots,m$ and $b(x_{m+1})=\cdots=b(x_n)=0$. Then,

$$-1/2\geq\Phi(x)+\varepsilon t^p\geq\sum_{i=1}^m\lambda_i b(x_i)\geq-\sum_{i=1}^m\lambda_i.$$

Thus $1\geq\lambda=\sum_{i=1}^m\lambda_i\geq 1/2$. Note that if $\|h\|\leq t$ we have then that $\|\frac{1}{\lambda}h\|\leq 2t\leq\min\{\delta,1/2\}$ and, as a consequence:

$$\|x_i+\frac{1}{\lambda}h\|\leq\|x_i\|+\|\frac{1}{\lambda}h\|\leq 1 \quad \text{for } i=1,\ldots,m.$$

By the above Fact, for each $0<t\leq\delta'$ and each $i=1,\ldots,m$, there exists a finite codimensional subspace $H_{(t,x_i)}$ such that if $h\in H_{(t,x_i)}$ and $\|h\|\leq t$, then:

$$b(x_i+\frac{1}{\lambda}h)-b(x_i)\leq K(2t)^p.$$

Consider the finite codimensional subspace $Y_{(t,x)} = \cap_{i=1}^{m} H_{(t,x_i)}$. If $h \in Y$ and $\|h\| \leq t$ we have:

$$\Phi(x+h) - \Phi(x) \leq \sum_{i=1}^{m} \lambda_i b(x_i + \frac{1}{\lambda} h) - \sum_{i=1}^{m} \lambda_i b(x_i) + \varepsilon t^p \leq (K \cdot 2^p + 1) t^p.$$

This establishes the claim.

Define now the function $\Psi = 4(\Phi + 1)$. It is not difficult to see that Ψ is a convex continuous function on B_X with $\Psi(0) = 0$. Consider the convex set $Q = \{x \in B_X : \Phi(x) \leq -3/4\} = \{x \in B_X : \Psi(x) \leq 1\}$, and we have that the Minkowski functional associated to this set is an equivalent norm $|||\cdot|||$ on X. We are going to prove that $|||\cdot|||$ has modulus of smoothness of power type p. Indeed, consider $0 < t \leq \delta'$ and $x \in X$ with $|||x||| = 1$. Then $\Psi(x) = 1$ and $x \in G$. By the claim, there exists a finite codimensional subspace $Y_{(t,x)}$ such that if $h \in Y$ with $\|h\| \leq t$ then

$$\Phi(x+h) - \Phi(x) \leq K' \cdot t^p.$$

Let $x^* \in X^*$ be a $|||\cdot|||$-norming functional of x, that is $|||x^*||| = 1$ and $x^*(x) = 1$. Then for every $h \in Y_{(t,x)} \cap \ker x^*$ with $|||h||| = t$ we have that $\|h\| \leq |||h||| \leq t$ and:

$$0 \leq |||x+h||| - 1 \leq \Psi(x+h) - \Psi(x) \leq 4(\Phi(x+h) - \Phi(x)) \leq 4K' \cdot t^p.$$

Consequently the norm $|||\cdot|||$ has modulus of asymptotic smoothness $\overline{\rho}(t) \leq 4K' t^p$ for $0 < t \leq \delta'$. Since $\overline{\rho}(t)$ is a non-decreasing function for $\delta' \leq t \leq 1$ (see e.g. [23]), the conclusion follows. $\square$

Remark 3.6. It is interesting to recall here that there are Banach spaces which admit a $\mathcal{C}^\infty$-smooth bump functions but without Gateaux-smooth equivalent renorming. Namely, on one hand it is proved in [21] that for any tree Υ, the space $C_0(\Upsilon)$ admits a $\mathcal{C}^\infty$-smooth bump function, and on the other hand in [20] it is given an example of a tree Υ such that the space $C_0(\Upsilon)$ admits no equivalent Gateaux-smooth renorming.

Remark 3.7. We note that having a norm with modulus of asymptotic smoothness of power type, in general does not imply high order smoothness of the space. Indeed, the space $(\bigoplus_{n=1}^{\infty} \ell_1^n)_{\ell_6}$ has modulus of asymptotic smoothness of power type 6 and it is not even superreflexive. In the superreflexive case, the space $(\bigoplus_{n=1}^{\infty} \ell_3^n)_{\ell_6}$ has modulus of asymptotic smoothness of power type 6 and nevertheless by [7] it does not admit an U^p-smooth bump function for $p > 3$.

We have seen in the above Remark that the condition about weak sequential continuity of polynomials cannot be avoided in Theorem 3.5. To finish this Section, we obtain sufficient condition of geometrical nature. We say that a sequence $\{x_n\}$ in a Banach space X is $\mathcal{P}_k$-null if, for every m-homogeneous polynomial P with $1 \leq m \leq k$ we have $P(x_n) \to 0$. By using the results of [19] and [16] we have:

Proposition 3.8. *Let $1 < p < \infty$ and let X be a U^p-smooth Banach space. Suppose that X contains no subspace isomorphic to ℓ_k with k even integer and $k < p$. Then, every polynomial on X of degree strictly less than p is weakly sequentially continuous.*

Proof. Assume the contrary. Then, we can find a weakly null sequence (x_n) and $k < p$ such that (x_n) is $\mathcal{P}_{k-1}$-null but not $\mathcal{P}_k$-null. Without loss of generality we can suppose that there is a k-homogeneous polynomial P such that $P(x_n) \geq \alpha > 0$ for all $n \in \mathbb{N}$. Using the same arguments as in Lemma 1.6 in [16] we obtain a subsequence of (x_n), which we denote in the same way, and a constant $c > 0$ such that for every $n \in \mathbb{N}$ and every $a_1, \ldots, a_n \in \mathbb{R}$ with $a_i \geq 0$ for $i = 1, \ldots, n$,

$$c \left(\sum_{i=1}^{n} a_i^k \right)^{1/k} \leq \| \sum_{i=1}^{n} a_i x_i \|.$$

On the other hand by Proposition 1.1 in [16] there is a subsequence of (x_n), that we still denote in the same way, with an upper k-estimate. Then, there exists $C > 0$ such that for every $n \in \mathbb{N}$ and every $a_1, \ldots, a_n \in \mathbb{R}$ with $a_i \geq 0$ for $i = 1, \ldots, n$:

$$c \left(\sum_{i=1}^{n} a_i^k \right)^{1/k} \leq \| \sum_{i=1}^{n} a_i x_i \| \leq C \left(\sum_{i=1}^{n} a_i^k \right)^{1/k} .$$

By using the results in [19] it follows that $\ell_k \subset X$. Since the space is U^p-smooth, k must be an even integer, and this is not possible. □

Using the above result we have:

Corollary 3.9. *Let $1 < p < \infty$ and let X be a Banach space which contains no subspace isomorphic to ℓ_k where k is an even integer and $k < p$. If X is U^p-smooth then there is an equivalent norm on X with modulus of asymptotic smoothness of power type p.*

4. Roughness of norms and modulus of asymptotic convexity

Let $1 \leq p < \infty$ and consider a Banach space X. According to [13], we say that a function $\varphi : X \to \mathbb{R}$ is *rough of order* p on a subset S of X if there exists $\varepsilon > 0$ such that, for every $x \in S$ and every polynomial P of degree $\leq p$ with $P(0) = 0$:

$$\overline{\lim_{\|h\| \to 0}} \frac{\varphi(x+h) - \varphi(x) - P(h)}{\|h\|^p} \geq \varepsilon.$$

In the same way, we say that φ is *pointwise rough of order* p on S if for every $x \in S$ there is $\varepsilon(x) > 0$ such that for every polynomial P of degree $\leq p$ with $P(0) = 0$ we have:

$$\overline{\lim_{\|h\| \to 0}} \frac{\varphi(x+h) - \varphi(x) - P(h)}{\|h\|^p} \geq \varepsilon(x).$$

In the case of a *norm* we always understand roughness on the unit sphere S_X of X. In fact, it is easy to see that a norm is rough on S_X if, and only if, it is rough on every annulus $S_{(r,R)} = \{x \in X : r < \|x\| < R\}$. In the case $p = 1$ a norm is rough of order 1 if, and only if, it rough in the usual sense (see e.g. [6]). More generally, as we see in the next proposition, in the case $1 \leq p < 2$ the notion of roughness of order p of a continuous *convex* function coincides with the notion of rough functions introduced in [22]. Recall that if φ is a convex function on an open convex set U and $x \in U$ the subdifferential of φ at x is the set:

$$\partial\varphi(x) = \{x^* \in X^* : x^*(y-x) \leq \varphi(y) - \varphi(x) \quad \text{for every } y \in U\}.$$

It is well-known that, for a continuous convex function φ defined on an open set U, the subdifferential $\partial\varphi(x)$ is non empty for every $x \in U$ (see e.g. [6]).

Proposition 4.1. *Let X be a Banach space and $1 \leq p < 2$. Consider a continuous convex function $\varphi : X \to \mathbb{R}$ and a subset S of X. The following are equivalent:*

(*1*) *There exists $\varepsilon > 0$ such that for every $x \in S$,*

$$\overline{\lim_{\|h\|\to 0}} \frac{\varphi(x+h) + \varphi(x-h) - 2\varphi(x)}{\|h\|^p} \geq \varepsilon.$$

(*2*) *There exists $\varepsilon > 0$ such that, for every $x \in S$ and every $x^* \in X^*$,*

$$\overline{\lim_{\|h\|\to 0}} \frac{\varphi(x+h) - \varphi(x) - x^*(h)}{\|h\|^p} \geq \varepsilon.$$

Proof. To prove (*1*) $\Rightarrow$ (*2*), consider $x \in S$ and $x^* \in X^*$. Since:

$$\begin{aligned}
\varepsilon &\leq \overline{\lim_{\|h\|\to 0}} \frac{\varphi(x+h) + \varphi(x-h) - 2\varphi(x)}{\|h\|^p} \\
&\leq \overline{\lim_{\|h\|\to 0}} \frac{\varphi(x+h) - \varphi(x) - x^*(h)}{\|h\|^p} + \overline{\lim_{\|h\|\to 0}} \frac{\varphi(x-h) - \varphi(x) - x^*(-h)}{\|-h\|^p},
\end{aligned}$$

we obtain that

$$\overline{\lim_{\|h\|\to 0}} \frac{\varphi(x+h) - \varphi(x) - x^*(h)}{\|h\|^p} \geq \frac{\varepsilon}{2}$$

To prove (*2*) $\Rightarrow$ (*1*), for each $x \in S$, consider $x^* \in \partial\varphi(x)$. Then for every $h \in X$,

$$\varphi(x+h) - \varphi(x) - x^*(h) \geq 0.$$

Thus,

$$\begin{aligned}
&\frac{\varphi(x+h) + \varphi(x-h) - 2\varphi(x)}{\|h\|^p} \\
= &\frac{\varphi(x+h) - \varphi(x) - x^*(h)}{\|h\|^p} + \frac{\varphi(x-h) - \varphi(x) - x^*(-h)}{\|h\|^p} \\
\geq &\frac{\varphi(x+h) - \varphi(x) - x^*(h)}{\|h\|^p}.
\end{aligned}$$

Consequently,

$$\overline{\lim_{\|h\|\to 0}} \frac{\varphi(x+h) + \varphi(x-h) - 2\varphi(x)}{\|h\|^p} \geq \varepsilon.$$

$\square$

Next we give a useful sufficient condition for high order roughness. First recall that a polynomial P on a Banach space X is said to be *separating* if $P(0) = 0$ and $\inf_{\|x\|=1} P(x) > 0$. See e.g. [17] for further information about separating polynomials. Using the same ideas as in [13] we obtain the following:

Lemma 4.2. *Let $1 \leq p < \infty$, let X be a Banach space which does not admit a separating polynomial of degree $\leq p$, and let S be a subset of X. Consider a function $\varphi : X \to \mathbb{R}$, and suppose that there exist $\varepsilon > 0$ and $\delta > 0$, such that for each $0 < t \leq \delta$ and each $x \in S$ there exists a finite codimensional subspace $H_{(t,x)}$ of X such that if $h \in H_{(t,x)}$ with $\|h\| = t$ we have*

$$\varphi(x + h) - \varphi(x) \geq \varepsilon t^p. \tag{1}$$

Then the function φ is rough of order p on S.

Proof. Let $x \in S$ and $0 < t \leq \delta$ be given, and consider a finite codimensional subspace $H = H_{(t,x)}$ of X verifying (1). Note that, since H has finite codimension, then H does not admit a separating polynomial of degree $\leq p$ (see e.g. [8]). Thus we have that $\inf\{P(h) : h \in H; \|h\| = t\} < \frac{\varepsilon}{2} t^p$. Then there exists $h \in H_{(t,x)}$ with $\|h\| = t$ such that $P(h) < \frac{\varepsilon}{2} t^p$. Therefore

$$\varphi(x + h) - \varphi(x) - P(h) \geq \varepsilon t^p - \frac{\varepsilon}{2} t^p = \frac{\varepsilon}{2} \|h\|^p.$$

As a consequence, φ is rough of order p on S. □

We say that a Banach space has *modulus of asymptotic pointwise convexity of power type* p if, for each $x \in S_X$, there is a constant $C_x > 0$ such that $\overline{\delta}(t; x) \geq C_x t^p$, for every $0 < t \leq 1$. From Lemma 4.2 it follows at once:

Theorem 4.3. *Let $1 \leq p < \infty$, and let X be a Banach space which does not admit a separating polynomial of degree $\leq p$. If X has modulus of asymptotic convexity of power type p (resp. modulus of asymptotic pointwise convexity), then the norm of X is rough of order p (resp. pointwise rough).*

Remark 4.4. In the case $p = 1$ we have that if a Banach space has modulus of asymptotic smoothness of power type 1, then it has a rough norm. In the case of spaces not isomorphic to Hilbert spaces, if the space has modulus of asymptotic convexity of power type 2, then norm is rough of second order.

Remark 4.5. The above theorem generalizes Theorem 1.3 in [13] where it is proved that if the space $(\bigoplus_{n=1}^{\infty} E_n)_{\ell_p}$, where E_n is finite dimensional for all $n \in \mathbb{N}$, does not admit a separating polynomial, then its norm is rough of order p. This is the case for example of the space $X = (\bigoplus_{n=1}^{\infty} \ell_4^n)_{\ell_2}$ whose usual norm is differentiable with Lipschitz continuous derivative, but it is rough of second order. In particular, the norm has no point of second order Fréchet differentiability, so this is an example where Alexandrov theorem strongly fails (compare with [4]). Moreover we will see in Lemma 4.6 that, in fact, in this case the derivative of the norm has no point of ε-Fréchet differentiability on the unit sphere.

Recall that, according to [23], a mapping f from an open subset U of a Banach space X into a Banach space Y is said to be *ε-Fréchet differentiable* at $x \in U$ for some $\varepsilon > 0$ if there exist a bounded linear operator $T : X \to Y$ and $\delta > 0$ such that:

$$\|f(x+h) - f(x) - T(h)\| < \varepsilon\|h\| \quad \text{whenever } 0 < \|h\| < \delta.$$

Lemma 4.6. *Let U be an open subset of a Banach space X, let $\varepsilon > 0$ and suppose that $\varphi : U \to \mathbb{R}$ is Fréchet differentiable on U and ε-rough of second order on U. Then the derivative $\varphi' : U \to X^*$ has no point of ε-Fréchet differentiability on U.*

Proof. Suppose that $x \in U$ is a point of ε-Fréchet differentiability of φ'. Then there exist a bounded linear operator $T : X \to X^*$ and $\delta > 0$ such that for $0 < \|h\| < \delta$:

$$\|\varphi'(x+h) - \varphi'(x) - T(h)\| < \varepsilon\|h\|.$$

Consider now the polynomial P of degree 2 on X defined by:

$$P(h) = \varphi'(x)(h) + \frac{1}{2}T(h)(h).$$

We are going to see that

$$\overline{\lim_{\|h\|\to 0}} \frac{\varphi(x+h) - \varphi(x) - P(h)}{\|h\|^2} \leq \frac{\varepsilon}{2}.$$

Indeed, let $u \in X$ be fixed with $\|u\| = 1$. For every $0 \leq t < \delta$ consider

$$\phi(t) = \varphi(x+tu) - \varphi(x) - \varphi'(x)(tu) - \frac{t^2}{2}T(u)(u).$$

Then $\phi(0) = 0$ and, using the Cauchy Mean Value Theorem, for each $0 < t < \delta$ we can find $0 < s < t$ such that

$$\frac{\phi(t)}{t^2} = \frac{\phi'(s)}{2s}.$$

Then if $0 < t < \delta$, we have:

$$\begin{aligned} \frac{|\varphi(x+tu) - \varphi(x) - P(tu)|}{\|tu\|^2} &= \frac{|\phi(t)|}{t^2} = \frac{|\phi'(s)|}{2s} \\ = & \frac{|\varphi'(x+su)(u) - \varphi'(x)(u) - T(su)(u)|}{2s} \\ \leq & \frac{\|\varphi'(x+su) - \varphi'(x) - T(su)\|}{2\|su\|} < \frac{\varepsilon}{2}. \end{aligned}$$

□

Remark 4.7. Even in the case of a norm $\|\cdot\|$, we do not know if the non-existence of points of ε-Fréchet differentiability of $\|\cdot\|'$ implies that the norm is rough of second order. Related to this, it would be interesting to know if the norm of the 2-convexified Tsirelson space $\mathbf{T}^{(2)}$ from [11] is rough of second order, since if this was true this space would be a counterexample to the following question asked in [23]: suppose that X^* and Y are separable spaces, Y with Radon-Nikodym property and such that every linear map from X into Y is compact. Does then every Lipschitz map from X to Y have points of ε-Fréchet differentiability for every $\varepsilon > 0$? At the view of Remark 3.4, all linear maps from the 2-convexified Tsirelson space $\mathbf{T}^{(2)}$ into its dual are compact, and the derivative of the norm is Lipschitz continuous on the unit sphere.

In what follows, we give some applications of the above results to obtain a certain kind of high order subdifferentiability and superdifferentiability of continuous functions. Our next Theorem is inspired by [7] and extends some results of [13].

Theorem 4.8. *Let $1 \leq p < \infty$ and let X be a T^p-smooth Banach space with the Radon-Nikodym property. Then for every open subset U of X and every continuous function $\varphi : U \to \mathbb{R}$, there exists $x \in U$ and a polynomial P of degree $\leq p$ with $P(0) = 0$ such that*

$$\varphi(x+h) - \varphi(x) - P(h) \leq o(\|h\|^p).$$

Proof. We may assume that φ is bounded on U. Let b be a T^p-smooth bump function on X with $\mathrm{Supp}(b) \subset U$. Let $\psi : X \to \mathbb{R} \cup \{+\infty\}$ be the function defined by $\psi(x) = 1/b(x)^2$ if $b(x) \neq 0$ and $\psi(x) = +\infty$ otherwise. The function $\psi - \varphi$ is lower semicontinuous, bounded below, and identically equal to $+\infty$ outside U. Since X has the Radon-Nikodym property, according to the Stegall variational principle [32], there exists $g \in X^*$ such that $\psi - \varphi + g$ attains its minimum at some point $x \in U$. So for every $h \in X$:

$$\psi(x+h) - \varphi(x+h) + g(x+h) \geq \psi(x) - \varphi(x) + g(x),$$

and consequently,

$$\psi(x+h) - \psi(x) + g(h) \geq \varphi(x+h) - \varphi(x).$$

Since the function ψ is T^p-smooth at x, there exits a polynomial Q of degree $\leq p$ with $Q(0) = 0$, such that

$$|\psi(x+h) - \psi(x) - Q(h)| = o(\|h\|^p).$$

Then the desired polynomial is $P(h) = Q(h) + g(h)$. □

Remark 4.9. By the results in [5] if a Banach space has the Radon-Nikodym property and admits a T^p-smooth bump function for $1 \leq p < 2$, then it is superreflexive and it admits an equivalent norm with modulus of smoothness of power type p. In this case, proceeding as in [6] it is possible to prove that every continuous convex function defined on an open convex subset U has points of T^p-smoothness on a dense set of U.

As a direct consequence of the above Theorem, we now obtain the following:

Corollary 4.10. *Let $1 \leq p < \infty$, let X be a Banach space with the Radon-Nikodym property, and let U be an open subset of X. Suppose that there is a function $\varphi : X \to \mathbb{R}$ which is continuous on U and pointwise rough of order p on U. Then X does not admit a T^p-smooth bump function.*

Remark 4.11. In the analogous result given in Proposition 1.1 of [13], it should be said that the space admits the *Radon-Nikodym property.* In fact, we do not know whether the result remains true without the assumption of Radon-Nikodym property. In this sense, we note that there exist spaces with modulus of asymptotic uniform convexity of power type p and without the Radon-Nikodym property. Namely, in [12] it is proved that predual JT_* of the James tree space JT has modulus of asymptotic uniform convexity of power type 3 and nevertheless, this space fails the Radon-Nikodym property.

Remark 4.12. Note that by Theorem II.5.3 in [6] and Corollary 4.10 it follows that, in the case $p = 1$, if X is a separable space with the *Radon-Nikodym property* the following are equivalent:

1. There exists an equivalent rough norm on X.
2. There exist an open subset U of X and a function $\varphi : X \to \mathbb{R}$ which is continuous on U and rough of first order on U.
3. There is no T^1-smooth bump function on X.

It would be interesting to know if this equivalence remains true for $p = 2$. More precisely, if the non-existence of a T^2-smooth bump function (or norm) implies the existence of an equivalent norm which is rough of second order.

Combining the above results, we obtain the following Corollary, which follows the lines of [7]:

Corollary 4.13. *Let $1 \leq p < \infty$ and let X be a Banach space with the Radon-Nikodym property. If X is T^p-smooth and has modulus of asymptotic pointwise convexity of power type p, then it admits a separating polynomial of degree $\leq p$.*

In general, having a norm rough of order p is not equivalent to the fact that the space admits a renorming with modulus of asymptotic uniform convexity of power type p. For instance, in [13] it is proved that, for $p \geq 1$, the norm of the Lorentz sequence space $d(w;p)$ is rough of order p. Nevertheless, this space cannot be equivalently renormed with modulus of asymptotic uniform convexity of power type p; otherwise by [23], since this space is reflexive and has modulus of smoothness of power type p, it would be isomorphic to a subspace of ℓ_p. This example motivates the introduction of some variants of the notion of roughness.

Consider a Banach space and a subset S of X. Let $1 \leq p < \infty$ and $\varepsilon > 0$ be given. We say that a function $\varphi : X \to \mathbb{R}$ is *ε-sequentially rough of order p* on S if, given $x \in S$, given $0 < t \leq 1$, and a polynomial P of degree $\leq p$ with $P(0) = 0$, *there exists* a weakly null sequence (h_n) in X with $\|h_n\| = 1$,

$$\overline{\lim_{n\to\infty}} \frac{\varphi(x + th_n) - \varphi(x) - P(th_n)}{t^p} > \varepsilon.$$

In the same way, we say that φ is *ε-uniformly sequentially rough of order p* on S if given $x \in S$, given $0 < t \leq 1$, and a polynomial P of degree $\leq p$ with $P(0) = 0$, *for every* weakly null sequence (h_n) in X with $\|h_n\| = 1$,

$$\overline{\lim_{n\to\infty}} \frac{\varphi(x + th_n) - \varphi(x) - P(h_n)}{t^p} > \varepsilon.$$

As usual, we say that φ is sequentially (resp. uniform sequentially) rough of order p on S if it ε-sequentially (resp. uniform sequentially) rough of order p on S, for some $\varepsilon > 0$. Both conditions imply roughness of order p on S. On the other hand, it is obvious that a function which is uniform sequentially rough of order p is sequentially rough of order p. As before, in the case of a *norm*, we understand that S is the unit sphere S_X.

In the following result we show that, under some assumptions on the space, the property of modulus of asymptotic convexity of power type is equivalent to a certain kind of sequential roughness of high order.

Recall that a family $\{e_\alpha\}_{\alpha\in T}$ of vectors of a Banach space X is called an *M-basis* of X if there exist functionals $e_\alpha^* \in X^*$, such that $e_\beta^*(e_\alpha) = 1$ if $\beta = \alpha$ and $e_\beta^*(e_\alpha) = 0$ if $\beta \neq \alpha$, and $\{e_\alpha\}_{\alpha\in T}$ is linearly dense in X. If in addition $\{e^*_\alpha\}_{\alpha\in T}$ is linearly dense in X^*, then the M-basis $\{e_\alpha\}_{\alpha\in T}$ is said to be *shrinking*. Let us point out that every reflexive space has a shrinking M-basis [31].

Proposition 4.14. *Let $1 < p < \infty$ and let X be a Banach space with a shrinking M-basis, such that every polynomial of degree $\leq p$ on X is w.s.c. Then, the following are equivalent:*

(*1*) *The norm is uniformly sequentially rough of order p.*

(*2*) *The norm has modulus of asymptotic uniform convexity of power type p.*

Proof. In [27] and [24] it is shown that if the space X has a shrinking M-basis, then for each each $x \in S_X$ and each $0 < t \leq 1$:

$$\overline{\delta}(t;x) = \inf\{\sup_{n\in\mathbb{N}} \|x + th_n\| - 1\}$$

where the infimum is taken over all weakly null sequences (h_n) in S_X. It is easy to check that the supremum can be replaced by the limsup.

In order to prove (*1*) implies (*2*), assume that the norm is ε-uniformly sequentially rough of order p. Then it is enough to consider $P = 0$ in the definition to obtain that $\overline{\delta}(t;x) \geq \varepsilon t^p$, for every $x \in S_X$ and $0 < t \leq 1$.

To prove (*2*) implies (*1*), assume that $\overline{\delta}(t;x) \geq Ct^p$ for every $x \in S_X$ and $0 < t \leq 1$. Consider $x \in S_X$, $0 < t \leq 1$, a polynomial P of degree $\leq p$ with $P(0) = 0$, and a weakly null normalized sequence (h_n). Since P is w.s.c. then $\lim_{n\to\infty} P(th_n) = 0$. Thus,

$$\overline{\lim_{n\to\infty}} \frac{\|x + th_n\| - 1 - P(th_n)}{t^p} = \overline{\lim_{n\to\infty}} \frac{\|x + th_n\| - 1}{t^p} \geq \frac{C}{2} t^p.$$

Then the norm is $\frac{C}{2}$-uniformly sequentially rough of order p. □

Sequential roughness of order p provides in this way new examples of functions which are rough of order p. Namely, in the case of Orlicz sequence spaces we have the following.

Proposition 4.15. *Let $1 \leq p < \infty$, and let M be an Orlicz function. Suppose that $\alpha_M = p$ is reached at p, and there exists $\varepsilon > 0$ such that for every $\delta > 0$:*

$$\overline{\lim_{t\to 0}} \frac{M(\delta t)}{M(t)\delta^p} > \varepsilon.$$

Then, the function $\Phi : h_M \to \mathbb{R}$ defined by $\Phi((x_n)) = \sum_{n=1}^{\infty} M(|x_n|)$ is $\frac{\varepsilon}{2}$-sequentially rough of order p.

Proof. We will show first the following fact:

Fact: "Let $\delta > 0$ be fixed. There are $\alpha > 0$ and $k \in \mathbb{N}$ such that for every $a \in h_M$ there is $n_0 \in \mathbb{N}$ such that if $u^\alpha_n = \sum_{i=n+1}^{n+k} \alpha e_i$, then $\delta \leq \|u^\alpha_n\| \leq \frac{3}{2}\delta$ and for $n \geq n_0$:

$$\Phi(a + u^\alpha_n) - \Phi(a) \geq \frac{\varepsilon}{2}\delta."$$

We proceed in several steps. We start with the case $a = 0$. Let $0 < t_1 \leq \frac{1}{2}$ be such that $M(t_1\delta) > \varepsilon M(t_1)$, and consider $\alpha = \delta t_1$. Then $M(\alpha) = M(\frac{\alpha}{\delta}\delta) > \varepsilon M(\frac{\alpha}{\delta})$. Consider now $k = \max\{j \in \mathbb{N};\quad \|\sum_{i=1}^{j} \alpha e_i\| \leq \frac{3}{2}\delta\}$. By construction, since $\alpha \leq \frac{\delta}{2}$ we have that $\|\sum_{i=1}^{k} \alpha e_i\| \geq \delta$. Now define $u_n^\alpha = \sum_{i=n+1}^{n+k} \alpha e_i$. Then u_n^α verifies $\delta \leq \|u_n^\alpha\| \leq \frac{3}{2}\delta$ for all $n \in \mathbb{N}$ and since M is non-decreasing and $\|u^\alpha\| \geq \delta$ we have $1 = \Phi\left(\frac{u_n^\alpha}{\|u_n^\alpha\|}\right) \leq \Phi\left(\frac{u_n^\alpha}{\delta}\right)$. Therefore, for all $n \in \mathbb{N}$,

$$\Phi(u_n^\alpha) = kM(\alpha) \geq \varepsilon k M(\frac{\alpha}{\delta}) = \varepsilon \Phi(\frac{u_n^\alpha}{\delta}) \geq \varepsilon.$$

Consider now the case $0 \neq a = \sum_{n=1}^{\infty} a_i e_i \in h_M$. Let $\alpha > 0$, let $k \in \mathbb{N}$ and u_n^α as above with $\delta \leq \|u_n^\alpha\| \leq \frac{3}{2}\delta$. Denote by:

$$[a]_n = \sum_{i=1}^{n} a_i e_i, \qquad [a]^n = \sum_{i=n+1}^{\infty} a_i e_i, \qquad [a]_n^m = \sum_{i=n+1}^{m} a_i e_i.$$

Since Φ is orthogonally additive with respect to the basis, if $m = k + n$ we have:

$$\begin{aligned} \Phi(a + u_n^\alpha) - \Phi(a) &= \Phi([a]_n) + \Phi([a]_n^m + u_n^\alpha) + \Phi([a]^m) - \Phi([a]_n) - \Phi([a]^n) \\ &= \Phi([a]_n^m + u_n^\alpha) + \Phi([a]^m) - \Phi([a]^n). \end{aligned}$$

Using again that M is continuous at α we may choose $r > 0$ small enough to assure that $|M(\alpha + t) - M(\alpha)| \leq \frac{\varepsilon}{4k}$ if $0 < t \leq r$. Then, we choose $n_0 \in \mathbb{N}$ enough large to assure that if $k \geq n_0$, $\Phi([a]_k) \leq \frac{\varepsilon}{16}$ and $|a_k| \leq r$. Then, if $n \geq n_0$ we have:

$$\begin{aligned} \Phi(a + u_n^\alpha) - \Phi(a) &\geq \sum_{i=n+1}^{n+k} M(\alpha) - \sum_{i=n+1}^{n+k} |M(a_i + \alpha) - M(\alpha)| - |\Phi([a]^m)| - |\Phi([a]^n)| \\ &\geq \Phi(u_n^\alpha) - \frac{\varepsilon}{2} \geq \frac{\varepsilon}{2}. \end{aligned}$$

Note that since α_M is reached at p the basis has an upper p-estimate ([14]). Then if h_M is not isomorphic to ℓ_p, the basis does not admit any subsequence with a lower k-estimate with $k = [p]$, where $[p]$ is the greatest integer less or equal than p. Consequently by [16] the basis is $\mathcal{P}_k$-null and $\{u_n^{(\alpha)}\}$ is also $\mathcal{P}_k$-null and then the function Φ is $\frac{\varepsilon}{2}$-weakly rough of order p. □

Remark 4.16. As a consequence of the above result if $M(t) \sim t^p \log|t|$ at 0 the function Φ on the space ℓ_M is rough of order p.

5. Best order of smoothness of Nakano spaces.

In this Section we give some applications of our previous results, in order to compute the best order of smoothness of Nakano spaces $\ell_{(p_n)}$. These are the modular spaces associated to the sequence of Orlicz functions $M_n(t) = t^{p_n}$ where $p_n \geq 1$ for all $n \in \mathbb{N}$. The smoothness of these spaces has been studied in [26], where in particular the following result is obtained:

Proposition 5.1. *Let $p_n \geq 2k$, with $\lim_{n\to\infty} p_n = 2k$. If $\ell_{(p_n)}$ admits a C^{2k}-smooth bump, then $\ell_{(p_n)}$ is isomorphic to ℓ_{2k}.*

In the case $p_n \leq 2k$, an analogous result is given in [26], only for $k = 1, 2$. Here we extend this result for every $k \in \mathbb{N}$. In order to do it we need the following lemma, where we consider the function $\Phi : \ell_{(p_n)} \to \mathbb{R}$ defined by:

$$\Phi((x_n)_{n=1}^{\infty}) = \sum_{n=1}^{\infty} |\, x_n|^{p_n}.$$

Lemma 5.2. *Let $1 \leq p_n \leq 2k$, with $\lim_{n\to\infty} p_n = 2k$. Suppose that there exist $0 < \delta \leq 1$ and a polynomial P of degree $\leq 2k$ such that $|P(x)| \geq \Phi(x)$ for every $x \in \ell_{(p_n)}$ with $\|x\| \leq \delta$. Then, the space $\ell_{(p_n)}$ is isomorphic to ℓ_{2k}.*

Proof. First note that, since $p_n \leq 2k$, if $(x_n)_{n=1}^{\infty}$ verifies $\sum_{n=1}^{\infty} |\, x_n|^{p_n} < \infty$ then $\sum_{n=1}^{\infty} |\, x_n|^{2k} < \infty$. This means that the inclusion mapping $T : \ell_{(p_n)} \to \ell_{2k}$ is continuous. Assume that T is not an isomorphism. Then, there exists a sequence $(h_n)_{n=1}^{\infty} \in \ell_{2k}$ with $0 \leq h_n < \delta$, such that $(h_n)_{n=1}^{\infty} \notin \ell_{(p_n)}$. We may choose an increasing sequence of integers (k_n) such that if $D_n = \{k_n + 1, \ldots, k_{n+1}\}$ then $\sum_{j\in D_n} h_j^{2k} \to 0$ if $n \to \infty$, $\sum_{j\in D_n} h_j^{p_j} \geq 1$ and $\| \sum_{j\in D_n} h_j e_j\| \leq 2$. Then, if $j \in D_n$:

$$|P_1(h_j e_j)| + \cdots + |P_{2k-1}(h_j e_j) + |P_{2k}(h_j e_j)| \geq h_j^{p_j},$$

and consequently,

$$\sum_{j\in D_n} \sum_{m=1}^{2k} |h_j^m P_m(e_j)| \geq \sum_{j\in D_n} h_j^{p_j} \geq 1.$$

Without loss of generality we may assume $2k - 1 < p_n$ for all $n \in \mathbb{N}$. Then the basis of $\ell(p_n)$ is p-convex for $2k - 1 < p < 2k$. This implies that every weakly null sequence in the space admits an upper ℓ_p estimate and consequently by [15] all polynomials of degree less or equal than $2k - 1$ are weakly sequentially continuous. Then, proceeding as in Lemma 3.1, for each $m = 1, \ldots 2k - 1$ it follows that $\|P_m\|_{H_n} \to 0$ as $n \to \infty$, where $H_n = \overline{[e_k : k \geq n + 1]}$. Consider now the complexified space $\tilde{X}$ of X as in [16] and the complexified polynomials $\tilde{P_m}$. Then, $\|\tilde{P_m}\|_{H_n} \to 0$ as $n \to \infty$ for each $m = 1, \ldots 2k - 1$. Let $1 \leq m \leq 2k - 1$ be fixed and consider the m- generalized Rademacher functions introduced in [2]. Using the same techniques as in [16] we have:

$$\begin{aligned} \sum_{j\in D_n} |h_j^m P_m(e_j)| &= \int_0^1 \tilde{P_m}\left(\sum_{j\in D_n} \epsilon_j h_j r_j(t) e_j\right) dt \\ &\leq C\|\tilde{P_m}\|_{H_n}\| \sum_{j\in D_n} h_j e_j\|^m \leq C2^m \|\tilde{P_m}\|_{H_n} \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

where C is the unconditional constant of the basis $\{e_n\}$ in the complexified space, and $|\epsilon_j| = 1$ for $j \in D_n$ are complex numbers chosen in such a way that $\tilde{P_m}(h_j \epsilon_j e_j) = |P_m(h_j e_j)|$.

On the other hand for $m = 2k$ we have:

$$\sum_{j\in D_n} |h_j|^{2k}|P_{2k}(e_j)| \leq \|P_{2k}\| \left(\sum_{j\in D_n} h_j^{2k}\right) \to 0 \quad \text{if } n \to \infty.$$

Consequently,

$$\sum_{j\in D_n} |\sum_{i=1}^{2k} h_j^i P_i(e_j)| \leq \sum_{i=1}^{2k} \sum_{j\in D_n} |h_j|^i |P_i(e_j)| \to 0 \quad \text{if } n \to \infty$$

which is a contradiction. □

Proposition 5.3. *Let $1 \leq p_n \leq 2k$, with $\lim_{n\to\infty} p_n = 2k$. Then there is a T^{2k}-smooth bump function on $\ell_{(p_n)}$ if, and only if, $\ell_{(p_n)}$ is isomorphic to ℓ_{2k}.*

Proof. The "if" part is trivial. For the converse, we apply Theorem 4.8 to the function Φ and we find $x \in \ell_{(p_n)}$ and a polynomial Q of degree $\leq 2k$ such that:

$$\Phi(x + h) - \Phi(x) - Q(h) \leq o(\|h\|^{2k}).$$

Summing this inequality with the analogue for $-h$, we obtain that the polynomial $P(x) = Q(x) + Q(-x)$ has degree $\leq 2k$ and satisfies:

$$\Phi(x + h) + \Phi(x - h) - 2\Phi(x) \leq P(h) + o(\|h\|^{2k}). \tag{2}$$

Since the case $k = 1$ is considered in [26], we restrict ourselves to the case $k \geq 2$. In this case we can assume without loss of generality that $p_n \geq 2$ for every $n \in \mathbb{N}$, and it can be proved that there exists $C > 0$ such that

$$\Phi(x + h) + \Phi(x - h) - 2\Phi(x) \geq C\Phi(h),$$

whenever $\|h\| \leq 1$. Furthermore, it is easy to see that $\Phi(h) \geq \|h\|^{2k}$ for $\|h\| \leq 1$. Then there exists $\delta > 0$ such that, if $\|h\| \leq \delta$:

$$P(h) \geq \Phi(x + h) + \Phi(x - h) - 2\Phi(x) + o(\|h\|^{2k}) \geq C\Phi(h) - \frac{C}{2}\|h\|^{2k} = \frac{C}{2}\Phi(h).$$

Thus by Lemma 4.2 we obtain the conclusion. □

Corollary 5.4. *Let $1 \leq p_n \leq p = 2k$, and $\lim_{n\to\infty} p_n = p$. If the function Φ has a point of T^p-smoothness then $\ell_{(p_n)}$ is isomorphic to ℓ_{2k}.*

Proof. If Φ has a point of T^p-smoothness then Φ verifies (2) and then we get the conclusion in the same way as before. □

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