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Differential Inclusions in $SBV_0(\Omega)$ and Applications to the Calculus of Variations

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We study necessary and sufficient conditions for the existence of solutions in $SBV_0(\Omega)$ of a variational problem involving only bulk energy. Related to that we study the problem of finding $u \in SBV_0(\Omega)$ such that

$$\nabla u(x) \in E, \text{ a.e. in } \Omega,$$

subject to the condition

$$\int_\Omega \nabla u = \zeta_0 |\Omega|,$$

where $E \subseteq \mathbb{R}^N$ is a given set and $\zeta_0 \in \text{int co } E$ is prescribed.

1. Preliminaries and Definitions

The problem of finding necessary and sufficient conditions for existence of solutions of the problem:

$$\inf \left\{ \int_\Omega f(\nabla u)\, dx \ : u \in W_0^{1,\infty}(\Omega) \right\}, \tag{1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ and $f : \mathbb{R}^N \to \mathbb{R}_0^+$ is a lower semicontinuous function has received considerable attention (cf. [6], [7], [11], [12] and the references therein). Recently the same question was also considered in a generalized setting, allowing for other differential operators, namely the curl (cf. [5]), and general differential forms (cf. [4]).

The purpose of this work is to extend the problem (1), to the space of functions of bounded variation with zero trace: following closely the techniques used in [5] and [4], we look for necessary and sufficient conditions for existence of solutions of

$$\inf \left\{ \int_\Omega f(\nabla u)\, dx \ u \in SBV_0(\Omega) \right\}, \tag{2}$$

where ∇u is the absolutely continuous part of the gradient Du relative to Lebesgue measure.

It is clear that, if f is convex, by Jensen's inequality, (1) has $u \equiv 0$ as a trivial solution. The main reason for considering the problem in the space of special functions of bounded variation is that, in this new setting, it is not clear if problem (2) attains solution even if f is convex.

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Notice that a more appropriate problem would take also the jump set into account. Namely we could consider:

$$(P) \quad \inf\left\{\int_\Omega f(\nabla u)\,dx + \int_{S_u}(1+|[u](x)|)dH^{N-1}(x)\ u\in SBV_0(\Omega)\right\},$$

penalizing both the existence of "jumps" and the "jump" itself (see for instance [1], [3]). The main difficulty lies in the fact that we dont know what is $\inf(P)$, in this case. However, in studying the much simpler case where we do not penalize the existence of "jumps", we hope to get some insight upon which values can be achieved by the bulk part of a functional of this type.

We start with some notations which are used throughout this paper. Although these notations are somewhat standard we mention them here for the sake of completeness.

- $\mathbb{R}_0^+$ denotes the set of all non-negative real numbers.
- For $E\subseteq\mathbb{R}^N, E\neq\emptyset$, we write span$E$ to denote the subspace spanned by E.
- Let W be a subspace of $\mathbb{R}^N$. We write dim$W$ to denote the dimension of W.
- H^k denotes the k-dimensional Hausdorff measure.
- $\mathcal{B}(\Omega)$ denotes the Borel σ-algebra of subsets of Ω.
- co C denotes the convex hull of $C\subseteq\mathbb{R}^N$ and $\overline{co}C$ its closure.
- For a convex set $C\subseteq\mathbb{R}^N$,
  1. Aff(C) denotes the affine hull of C which is the intersection of all affine subsets of $\mathbb{R}^N$ containing C.
  2. ri(C) denotes the relative interior of C which is the interior of C with respect to the topology relative to the affine hull of C.
  3. rbd(C) denotes the relative boundary of C which is $\overline{C}\setminus \mathrm{ri}(C)$.
- For a function $f:\mathbb{R}^N\to\mathbb{R}$, f^{**} denotes the convex envelope of f, that is,

$$f^{**}=\inf\{g : g \text{ convex},\ g\leq f\},$$

For reasons that will become clear further on, we consider, $\Omega\subset\mathbb{R}^N$ to be open, bounded with Lipschitz boundary.

We recall briefly some facts on functions of bounded variation which will be used in the sequel. We refer to [13], [16] and [20] for a detailed exposition on this subject.

Given a $L^1(\Omega)$ function u the *Lebesgue set* of u, Ω_u, is defined as the set of points $x\in\Omega$ such that there exists $\tilde u(x)\in\mathbb{R}$ satisfying

$$\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon^N}\int_{B(x,\varepsilon)}|u(y)-\tilde u(x)|\,dy=0.$$

The *Lebesgue discontinuity set* S_u of u is the set of points $x\in\Omega$ which are not Lebesgue points, that is $S_u:=\Omega\setminus\Omega_u$. By Lebesgue's Differentiation Theorem, S_u is H^N-negligible and $\tilde u:\Omega\to\mathbb{R}$, which coincides with u H^N-almost everywhere in Ω_u, is called the *Lebesgue representative* of u.

The *approximate upper* and *lower limits* of u are given by

$$u^+(x):=\inf\left\{t\in\mathbb{R} : \lim_{\varepsilon\to0^+}\frac{1}{\varepsilon^N}H^N(\{y\in\Omega\cap B(x,\varepsilon) : u(y)>t\})=0\right\}$$

and

$$u^-(x) := \sup\left\{t \in \mathbb{R} : \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} H^N(\{y \in \Omega \cap B(x,\varepsilon) : u(y) < t\}) = 0\right\};$$

if $u^+(x) = u^-(x)$ then $x \in \Omega_u$ and $u^+(x) = u^-(x) = \tilde{u}(x)$. The *jump set* or *singular set* of u is defined as

$$J_u := \{x \in \Omega : u^-(x) < u^+(x)\}$$

and we denote by $[u](x)$ the *jump* of u at x, i.e. $[u](x) := u^+(x) - u^-(x)$.

A function $u \in L^1(\Omega)$ is said to be of *bounded variation*, $u \in BV(\Omega)$, if for all $j = 1, ..., N$, there exists a finite Radon measure μ_j such that

$$\int_\Omega u(x) \frac{\partial \phi}{\partial x_j}(x)\, dx = -\int_\Omega \phi(x)\, d\mu_j(x)$$

for every $\phi \in C_0^1(\Omega)$. The distributional derivative Du is the vector-valued measure μ with components μ_j.

The space $BV(\Omega)$ is a Banach space when endowed with the norm

$$\|u\|_{BV} = \|u\|_{L^1} + |Du|(\Omega),$$

where $|Du|(\Omega)$ represents the total variation of the measure Du.

A set $A \subset \Omega$ is said to be of *finite perimeter* in Ω if $\chi_A \in BV(\Omega)$. The *perimeter* of A in Ω is defined by

$$\mathrm{Per}_\Omega(A) := \sup\left\{\int_A \mathrm{div}\varphi(x)\, dx : \varphi \in C_0^1(\Omega;\mathbb{R}^N),\ \|\varphi\|_\infty \le 1\right\}.$$

For a set $A \subset \Omega$ the *reduced boundary* of A in Ω, $\partial^* A$, is given by

$$\partial^* A \cap \Omega = S_{\chi_A} \cap \Omega$$

and we recall that for H^{N-1} a. e. $x \in \partial^* A$, it is possible to define a measure theoretical interior normal to A, $\nu_A(x) \in S^{N-1}$, such that

$$D\chi_A(B) = \int_{B\cap\partial^* A} \nu_A(x)\, dH^{N-1}(x)$$

for every $B \in \mathcal{B}(\Omega)$.

If $u \in BV(\Omega)$ it is well known that S_u is countably $N-1$ rectifiable, i.e.

$$S_u = \bigcup_{n=1}^{\infty} K_n \cup E,$$

where $H^{N-1}(E) = 0$ and K_n are compact subsets of C^1 hypersurfaces. Furthermore, for H^{N-1} a.e. $x \in S_u$, $u^+(x) \neq u^-(x)$ and there exists a unit vector $\nu_u(x) \in S^{N-1}$, normal to S_u at x, such that

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \int_{\{y\in B(x,\varepsilon):(y-x)\cdot\nu_u(x)>0\}} |u(y) - u^+(x)|\, dy = 0$$

and

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \int_{\{y\in B(x,\varepsilon):(y-x)\cdot\nu_u(x)<0\}} |u(y) - u^-(x)|\, dy = 0.$$

In particular, $H^N(S_u \setminus J_u) = 0$.

If $u \in BV(\Omega)$ then the distributional derivative Du may be decomposed as

$$Du = \nabla u H^N + (u^+ - u^-) \otimes \nu_u H^{N-1}\lfloor S_u + C_u, \tag{3}$$

where ∇u is the density of the absolutely continuous part of Du with respect to the Lebesgue measure and C_u is the Cantor part of Du which vanishes on all $B \in \mathcal{B}(\Omega)$ with $H^N(B) < +\infty$. The three measures appearing in (3) are mutually singular.

The space of *special functions of bounded variation*, $SBV(\Omega)$, introduced by De Giorgi and Ambrosio in [17], is the space of functions $u \in BV(\Omega)$ such that $C_u = 0$, i.e. for which

$$Du = \nabla u H^N + (u^+ - u^-) \otimes \nu_u H^{N-1}\lfloor S_u.$$

For Ω open bounded with Lipschitz boundary the outer unit normal to $\partial\Omega$ (denoted by ν) exists H^{N-1} a.e., and we can define the trace for functions in $BV(\Omega)$. Namely, there exists a bounded linear mapping:

$$T : BV(\Omega) \to L^1(\partial\Omega; H^{N-1})$$

such that

$$\int_\Omega u\mathrm{div}\phi\, dx = -\int_\Omega \phi \cdot d[Du] + \int_{\partial\Omega} (\phi\cdot\nu) Tu\, dH^{N-1},$$

for all $u \in BV(\Omega)$ and $\phi \in C^1(\mathbb{R}^N;\mathbb{R}^N)$. The function Tu is uniquely defined up to sets of $H^{N-1}\lfloor\partial\Omega$ measure zero, is called the *trace* of u on $\partial\Omega$.

We denote by $SBV_0(\Omega)$, the set of $u \in SBV(\Omega)$ such that $Tu = 0$ on $\partial\Omega$. We will need the following extension result:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ open, bounded with $\partial\Omega$ Lipschitz. Let $f_1 \in BV(\Omega), f_2 \in BV(\mathbb{R}^n\backslash\bar\Omega)$. Define*

$$\bar f(x) = \begin{cases} f_1(x) & x \in \Omega \\ f_2(x) & x \in \mathbb{R}^n\backslash\bar\Omega \end{cases}$$

Then $\bar f \in BV(\mathbb{R}^N)$, and:

$$||D\bar f||(\mathbb{R}^N) = ||Df_1||(\Omega) + ||Df_2||(\mathbb{R}^n\backslash\bar\Omega) + \int_{\partial\Omega} |Tf_1 - Tf_2|\, dH^{N-1}.$$

2. Statement of the problem

Consider the following problem:

$$(P) \qquad \inf\left\{\int_\Omega f(\nabla u)\, dx\ u \in SBV_0(\Omega)\right\},$$

where $\Omega \subset \mathbb{R}^N$ is open bounded with Lipschitz boundary and $f : \mathbb{R}^n \to \mathbb{R}$ is convex, attaining its minimum at ζ_0. Two situations may occur:

a) Suppose $\zeta_0 = 0$. Then, trivially, $w = 0$ is a (trivial) classical solution of (P).

b) Suppose $\zeta_0 \neq 0$. Since we can express Ω as a disjoint union of cubes plus a set of small measure, take Ω to be a cube, and for $\epsilon > 0$, let $\Omega_\epsilon \subset \Omega$ a smaller cube such that $|\Omega \backslash \Omega_\epsilon| \leq \epsilon$. Then we can construct an admissible sequence for (P) taking $u_{\zeta_0} \in W^{1,\infty}(\Omega)$ such that $Du_{\zeta_0} = \zeta_0$ and setting

$$u_n = \begin{cases} u_{\zeta_0} & \text{in } \Omega_{\frac{1}{n}} \\ 0 & \text{in } \Omega \backslash \Omega_{\frac{1}{n}}, \end{cases}$$

it follows that $\inf(P) = f(\zeta_0)|\Omega|$.

3. Necessary conditions

The following is a necessary condition for (P) to attain solution.

Lemma 3.1. *Suppose (P) attains a solution u. Then:*

i)

$$f(\frac{1}{|\Omega|} \int \nabla u) = f(\zeta_0), \;\; a.e. \; in \; \Omega,$$

ii)

$$f(\frac{\nabla u(x) + \zeta_0}{2}) = \frac{f(\nabla u(x)) + f(\zeta_0)}{2}, \;\; a.e. \; in \; \Omega.$$

Proof. The condition *i)* results trivially from Jensen's inequality. Regarding condition *ii)*, if there exists $u \in SBV_0(\Omega)$ such that $\int_\Omega f(\nabla u(x)) \, dx = f(\zeta_0)|\Omega|$, then:

$$\begin{aligned} f(\zeta_0)|\Omega| &= \frac{1}{2} \left[\int_\Omega f(\nabla u) \, dx + f(\zeta_0)|\Omega| \right] \\ &= \frac{1}{2} \left[\int_\Omega f(\nabla u) + f(\zeta_0) \right] \geq \int_\Omega f(\frac{\nabla u + \zeta_0}{2}) \geq f(\zeta_0)|\Omega| \end{aligned}$$

and we conclude that:

$$f(\frac{\nabla u(x) + \zeta_0}{2}) = \frac{f(\nabla u(x)) + f(\zeta_0)}{2}, \;\; \text{a.e. in } \Omega.$$

□

From condition *ii)* of the previous lemma, if u is a minimizer for (P), then either:

a)

$$\nabla u = \zeta_0 \text{ a.e. in } \Omega,$$

or

b) f is affine at ζ_0 in at least one direction.

The first question we may pose is whether there is any function $w \in SBV_0(\Omega)$ such that

$$\nabla w = \zeta_0, \;\; \text{a.e. in } \Omega.$$

For $N = 1$ one trivially can find such a solution (taking $\Omega = (0,1)$, just consider a piecewise affine function with slope ζ_0 and such that $w(0) = w(1) = 0$, with for instance a "jump" at $x_0 \in (0,1)$). However, for $N > 2$ there is not any function $w \in SBV_0(\Omega)$ such that

$$\nabla w = \zeta_0, \text{ a.e. in } \Omega.$$

This is a consequence of the following lemma:

Lemma 3.2. *Let $u \in SBV_0(\Omega)$ and suppose there exists $\lambda \in \mathbb{R}^N \backslash \{0\}$ such that*

$$< \nabla u(x), \lambda >= 0, \; a.e. \; x \in \Omega$$

and

$$< \nu_u(x), \lambda >= 0, \; H^{N-1} \; a.e. \; on \; S_u \cap \Omega.$$

Then $u = 0$ a.e. in Ω.

Proof. Since $Tu = 0$ on $\partial\Omega$, for all $\phi \in C^1(\mathbb{R}^N;\mathbb{R}^N)$, we have:

$$\int_\Omega u \operatorname{div} \phi \, dx = -\int_\Omega \phi \cdot [du] = -\int_\Omega \phi \cdot \nabla u \, dx - \int_{S_u \cap \Omega} < \phi, [u]\nu_u > \; dH^{N-1}(x)$$

Step 1. Suppose $\lambda = e_1$. Then, taking $\eta \in C^1(\mathbb{R}^N)$, setting $\phi = (\eta, 0, \dots, 0)$ in the previous equality

$$\int_\Omega u \frac{\partial \eta}{dx_1} = -\int_\Omega \eta \frac{\partial u}{dx_1} \, dx - \int_{S_u \cap \Omega} [u]\eta < e_1, \nu_u > \; dH^{N-1}(x) = 0,$$

$\forall \eta \in C^1(\mathbb{R}^N)$, according to the hypothesis. Taking in particular $\eta \in C_c^\infty(\mathbb{R}^N)$, we conclude that

$$\frac{Du}{dx_1} = 0,$$

(derivative in the sense of distributions). For $x_0 \in \Omega \backslash S_u$ and for $t \in (t^-, t^+)$, such that $x_0 + t^- e_1, x_0 + t^+ e_1 \in \partial\Omega$, and $x_0 + te_1 \in \Omega$ for $t \in (t^-, t^+)$, set:

$$\bar{u}_{x_0}(t) := u(x_0 + te_1).$$

Then, $\bar{u}_{x_0} \in SBV(\mathbb{R})$, for a.e. $x \in \Omega$, and since its derivative is zero, we conclude that $\bar{u}_{x_0} = c_{x_0}$, i.e., u is contant along lines with direction e_1. Now, by the definition of trace, considering $x_0 \in \Omega \backslash S_u$ such that ν_u is well defined at $x_0 + t^+ e_1, x_0 + t^- e_1$, we have

$$Tu(y) = \lim_{\epsilon \to 0} \frac{1}{H^N(B_\epsilon(y) \cap \Omega)} \int_{B_\epsilon(y) \cap \Omega} u(x) \, dx = 0,$$

at $y = x_0 + t^+ e_1, y = x_0 + t^- e_1$, and since a ball centered at x_0 and radius ϵ is contained in translations (along the direction e_1) and possibly dilations of, $B_\epsilon(y) \cap \Omega$ with $y = x_0 + t^+ e_1, y = x_0 + t^- e_1$, we conclude that:

$$\lim_{\epsilon \to 0} \frac{1}{B_\epsilon(x_0)} \int_{B_\epsilon(x_0)} u(x) \, dx = 0,$$

i.e., by Lebesgue-Besicovitch differentiation theorem, $u(x_0) = 0$ $\mathcal{L}^N$ a.e. $x_0 \in \Omega$.

Step 2. Just apply a rotation argument: let Q be an orthogonal matrix such that $Qe_1 = \lambda$ and define $\tilde{u}(y) = u(Qy)$ for $y \in \tilde{\Omega} := Q^T\Omega$. It is clear that $\tilde{u} \in SBV_0(\tilde{\Omega})$, with $S_{\tilde{u}} = \{y \in \tilde{\Omega} : Qy \in S_u\}$, and

$$< \nabla\tilde{u}(y), e_1 >= 0 \text{ a.e. } y \in \tilde{\Omega},$$
$$< \nu_{\tilde{u}}(y), e_1 >= 0, \ H^{N-1} \text{ a.e. on } S_{\tilde{u}} \cap \tilde{\Omega}.$$

Therefore, $\tilde{u} = 0$ in $\tilde{\Omega}$ and hence $u = 0$ in Ω. □

Corollary 3.3. *Let $N \geq 2, \Omega \subset \mathbb{R}^N$, and let $\zeta_0 \in \mathbb{R}^N\backslash\{0\}$. Then there is no $u \in SBV_0(\Omega)$ such that $\nabla w = \zeta_0$ (a.e. in Ω).*

Proof. We argue by contradiction. Suppose there exists $w \in SBV_0(\Omega)$ such that $\nabla w = \zeta_0$ (a.e.). Since $Tw = 0$ on $\partial\Omega$, we have, $\forall\phi \in C^1(\mathbb{R}^N; \mathbb{R}^N)$:

$$\int_\Omega w \operatorname{div}\phi \, dx = -\int_\Omega \phi \cdot [dw] = -\int_\Omega \phi \cdot \nabla w \, dx - \int_{S_w \cap \Omega} < \phi, [w]\nu_w > \ dH^{N-1}(x)$$

Let $\lambda \in \mathbb{R}^N\backslash\{0\}$, a vector orthogonal to ζ_0 and consider the function $\phi \equiv \lambda$, in the previous equality. We conclude that:

$$< \nabla w(x), \lambda >=< \zeta_0, \lambda >= 0, \text{ a.e. } x \in \Omega$$

and

$$< \nu_w(x), \lambda >= 0, \ H^{N-1} \text{ a.e. on } S_w \cap \Omega.$$

Hence, thanks to Lemma 3.2, $w = 0$, a contradiction. □

Therefore, for $N > 1$, a necessary condition for (P) to attain a solution is that f is affine in at least one direction at ζ_0. Notice that in fact a necessary condition is that f is affine at ζ_0. In fact, (writing for sake of simplicity $Aff\zeta_0$ instead of $Aff(\{\zeta_0\})$), suppose that dim $Aff\zeta_0 = k$ $(1 < k < N)$.Then, taking λ orthogonal to $Aff\zeta_0$ and arguing as in the previous corollary we conclude that there is no $u \in SBV_0(\Omega)$ such that $\nabla u \in Aff_{\zeta_0}$.

4. Sufficient conditions

One might ask if this condition is also sufficient.

In order to answer that question, we will need a couple of lemmas which we will not prove here. For the proof of these 2 lemmas we refer to [7], Lemma 2.11 of [12] and [14].

Lemma 4.1. *Let $E \subset \mathbb{R}^N$ with dim span $E = n \leq N$ and let $0 \in ri$ $(\text{co}E)$. Then there exist $m \geq n+1$, $z^\alpha \in E$, $t^\alpha > 0$ such that*

$$\sum_{\alpha=1}^m t^\alpha z^\alpha = 0, \ \sum_{\alpha=1} t^\alpha = 1, \ \text{dim span}\{z^\alpha | \alpha = 1, \dots, m\} = n.$$

Lemma 4.2 (Pyramids)**.** *Let $\Omega \subset \mathbb{R}^N$ be open and bounded. Let $z^\alpha \in \mathbb{R}^N$, $t^\alpha > 0$, $\alpha = 1, \dots, m$, with $m \geq N+1$, be such that*

$$\sum_{\alpha=1}^m t^\alpha z^\alpha = 0, \ \sum_{\alpha=1} t^\alpha = 1, \ \text{dim span}\{z^\alpha | \alpha = 1, \dots, m\} = N.$$

Then there exists $u \in W_0^{1,\infty}(\Omega)$ *satisfying*

$$\nabla u \in \{z^\alpha | \alpha = 1, \dots, m\}, \text{ a.e. in } \Omega.$$

In order to look for solutions of our problem, we will now adapt Lemma 4.2 to our setting. Namely, we have:

Lemma 4.3. *Let* $\Omega \subset \mathbb{R}^N$ *be open and bounded. Let* $z^\alpha \in \mathbb{R}^N,\ t^\alpha > 0,\ \alpha = 1, \dots, m$, *with* $m \geq N+1$, *be such that*

$$\sum_{\alpha=1}^{m} t^\alpha z^\alpha = \zeta_0, \ \sum_{\alpha=1} t^\alpha = 1, \ \dim \operatorname{span}\{z^\alpha | \alpha = 1, \dots, m\} = N.$$

Then there exists $u \in SBV_0(\Omega)$ *satisfying*

$$\nabla u \in \{z^\alpha | \alpha = 1, \dots, m\}, \text{ a.e. in } \Omega.$$

and

$$\int_\Omega \nabla u = \zeta_0 |\Omega|.$$

Proof. We can take w.l.o.g. Ω to be the unit cube centered at origin with side length equal to 1. Then, using translations and dilations, by Vitali's covering theorem we have the same result for a general Ω. Since $\sum_{\alpha=1}^m t^\alpha (z^\alpha - \zeta_0) = 0$, and dim span $\{z^\alpha - \zeta_0 | \alpha = 1, \dots, m\} = N$, we can apply Lemma 4.2 to $Q_1 \subset \Omega, Q_1 = (-\frac{1}{4}, \frac{1}{4})^N$ to get $v_1 \in W_0^{1,\infty}(Q_1)$ such that $\nabla v_1 \in \{z^\alpha - \zeta_0,\ \alpha = 1, \dots, m\}$. Now set $w_1 = v_1 + g_1$ on Q_1, where g_1 is affine, $\nabla g_1 = \zeta_0$ and $g_1(-\frac{1}{4}, \dots, -\frac{1}{4}) = 0$. Extending v_1 by 0 to Ω, we constructed a function $w_1 \in SBV_0(\Omega)$ such that $\nabla w_1 \in \{z^\alpha, \alpha = 1, \dots, m\}$ a.e. in Q_1, $\int_{Q_1} \nabla w_1 = \zeta_0 |Q_1|$, and since

$$S_{w_1} \subset \partial Q_1,$$

and the functions involved are linear,

$$|D^s w_1|(\Omega) \leq C(\zeta_0).2N.(\frac{1}{2})^{N-1}\frac{1}{2}. \tag{4}$$

(Note that $2N$ is the number of faces of a cube in $\mathbb{R}^N$.) Consider now the cube $Q_2 = (-\frac{3}{8}, \frac{3}{8})^N$ and set $S_2 = Q_2 - \bar{Q_1}$, and divide S_2 into $r(2)$(open) cubes of side $\frac{1}{2^3}$, i.e.

$$\bar{S_2} = \cup_{i=1}^{r(2)} \bar{Q}_2^i(x^i, \frac{1}{2^3}).$$

In each cube Q_2^i apply Lemma 4.2 to obtain $v_2^i \in W_0^{1,\infty}(Q_2^i)$, $\nabla v_2^i \in \{z^\alpha - \zeta_0, \alpha = 1, \dots, m\}$, and set $w_2^i = v_2^i + g_2^i$ in Q_2^i with g_2^i affine, $\nabla g_2^i = \zeta_0,\ g_2^i(\bar{x}_2^i) = 0$, where

$$\bar{x}_2^i = x_2^i + (-\frac{1}{2^3}, \dots, -\frac{1}{2^3}).$$

Extend each of the functions w_2^i by 0 to Ω and set

$$w_2 = w_1 + \sum_{i=1}^{r(2)} w_2^i.$$

Clearly $w_2 \in SBV_0(\Omega), \nabla w_2 \in \{z^\alpha, \alpha = 1, \dots, m\}$ a.e. in Q_2, and $\int_{Q_2} \nabla w_2 = \zeta_0 |Q_2|$. Also, since $S_{w_2} \subset \cup_{i=1,\dots,r(2)} \partial Q_2^i$, we clearly have

$$|D^s w_2|(\Omega) \leq |D^s w_1|(\Omega) + C(\zeta_0).r(2).(\frac{1}{2^3})^{N-1}.(\frac{1}{2^3}).$$

Now iterate the process. For

$$Q_n = (-(\frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^{n+1}}), \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^{n+1}}),$$

set $S_n = Q_n - \bar{Q}_{n-1}$, and divide S_n into $r(n)$ cubes of side length $\frac{1}{2^{n+1}}$. Notice that

$$r(n) \leq 2N. \left(\frac{\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n}}{\frac{1}{2^{n+1}}} \right)^{N-1} \leq 2N 2^{(N-1)(n+1)} \left(\sum_{n=1}^{\infty} \frac{1}{2^n} \right)^{N-1} \leq 2N 2^{(N-1)(n+1)}.$$

On each cube Q_n^i set $w_n^i = v_n^i + g_n^i$ with v_n^i and g_n^i as before, and

$$w_n = \sum_{j=1}^{n-1} w_j + \sum_{i=1}^{r(n)} w_n^i.$$

Fore each $n \in \mathbb{N}$ we obtain a function $w_n \in SBV_0(\Omega)$ satisfying the desired properties and such that

$$|D^s w_n|(\Omega)| \leq \sum_{j=1}^{n-1} |D^s w_j|(\Omega) + C(\zeta_0) 2N 2^{(N-1)(n+1)} \left(\frac{1}{2^{n+1}} \right)^{N-1} \frac{1}{2^{n+1}}. \tag{5}$$

We constructed a sequence $\{w_n\} \subset SBV_0(\Omega)$ that converges in $BV(\Omega)$ to

$$w = \sum_{n=1}^{\infty} w_n.$$

For $B \in \mathcal{B}(\Omega)$ such that $\mathcal{H}^N(B) = 0$, set $\hat{B} = B \backslash (B \cup S_w)$. It is clear that $|Dw_n|(\hat{B}) = 0$, $\forall n \in \mathbb{N}_1$, and so $|Dw|(\hat{B}) = 0$ and therefore $w \in SBV(\Omega)$. Also, by continuity of the trace operator, $Tw = 0$ on $\partial\Omega$, and so

$$w \in SBV_0(\Omega),$$

and

$$\int_\Omega \nabla w = \zeta_0 |\Omega|,$$

and, combining (4), (5),

$$|D^s u|(\Omega) \leq C(\zeta_0) N.$$

□

Combining the previous lemma with the hypothesis of f affine at ζ_0 we derive sufficient conditions for existence of solutions to our problem. We start with a definition:

Definition 4.4. A function $g : \mathbb{R}^N \to \mathbb{R}$ is said to be affine at ζ_0 if there exist an open set $S \in \mathbb{R}^N$ such that $\zeta_0 \in S$ and $a \in \mathbb{R}^N, b \in \mathbb{R}$ such that

$$g(\zeta) =< a, \zeta > +b, \ \forall \zeta \in S.$$

Remark 4.5. Note that in our case, the largest set S satisfying the previous definition is convex since f is convex. We denote this set (or more generally its connected component containing ζ_0), by S_{ζ_0}.

We are now in conditions to prove the following:

Theorem 4.6. *Let $f : \mathbb{R}^N \to \mathbb{R}$ be convex, and attaining its minimum at ζ_0. Suppose that f is affine at ζ_0 . Then the problem:*

$$(P) \quad \inf\left\{ \int_\Omega f(\nabla u)\, dx \; u \in SBV_0(\Omega) \right\}$$

attains solution.

Proof. By definition, dim $Aff_{\zeta_0} = N$ and $\zeta_0 \in$ int co S_{ζ_0}. By Lemmas 4.1 and 4.3, we can construct $u \in SBV_0(\Omega)$ such that $\nabla u \in E$ (a.e. in Ω), where

$$E = \partial V_{\zeta_0}.$$

Then,

$$\begin{aligned}
\int_\Omega f(\nabla u(x))\, dx &= \int_\Omega < a, \nabla u(x) > +b =< a, \int_\Omega \nabla u > +b|\Omega| \\
&= \int_\Omega f(\zeta_0)\, dx = f(\zeta_0)|\Omega| = \inf(P)
\end{aligned}$$

□

Remark 4.7. Note that, if $0 \notin S_{\zeta_0}$, then we may have existence on (nonclassical) solution for (P), with $\inf(P) = f(\zeta_0)|\Omega| < f(0)|\Omega|$.

5. Non-convex integrand

In the case where f is non-convex, consider as customary, the auxiliary problem:

$$(\bar{P}) \quad \inf\{ \int_\Omega f^{**}(\nabla u)\, dx \; u \in SBV_0(\Omega)\}.$$

where f^{**} denotes the convex-hull of f, i.e.,

$$f^{**} = \inf\{g : g \text{ convex}, \ g \le f\},$$

and suppose that f^{**} attains a minimum at ζ_0. Two situations may occur:

1)

$$\zeta_0 = 0.$$

Then, trivially $\inf(\bar{P}) = f^{**}(0)|\Omega|$, and if $f(0) = f^{**}(0), w = 0$ is a trivial solution for (P) (and $(\bar{P})$). Assuming that $f(0) > f^{**}(0)$, let:

$$E = \{\zeta \in \mathbb{R}^N : f(\zeta) = f^{**}(\zeta)\}.$$

Then, a sufficient condition for existence of solutions is:
There exists $w \in SBV_0(\Omega)$ such that:
a)

$$\nabla u(x) \in E \text{ a.e. in } \Omega,$$

b)

$$\int_\Omega f^{**}(\nabla u) = f^{**}(0)|\Omega|.$$

In fact, the previous conditions ensure that $\inf(P) \leq \int_\Omega f(\nabla u) = \inf(\bar{P})$ and since trivially $\inf(\bar{P}) \leq \inf(P)$, we are done.
An answer to this question is given by the classical setting: If dim span $E = N$, there exists $w \in W_0^{1,\infty}(\Omega)$ that satisfies the conditions above, and this together with the hypothesis of f^{**} affine at $K = E^c$, leads to the existence of solutions (cf. [6], [7], [14], [8]). The condition on the dimension of the span of E is necessary and sufficient for existence of such a $w \in W_0^{1,\infty}(\Omega)$. Also, Lemma 3.3, (and the comments that follow it), show that this condition is also necessary in $SBV_0(\Omega)$.

2)

$$\zeta_0 \neq 0.$$

In this case, $\inf(\bar{P}) = f^{**}(\zeta_0)|\Omega|$, and two situations may arise:
2.1)

$$\zeta_0 \in K = \{\zeta \in \mathbb{R}^N : \ f^{**}(\zeta) < f(\zeta)\}.$$

2.2)

$$\zeta_0 \in E = \{\zeta \in \mathbb{R}^N : f^{**}(\zeta) = f(\zeta)\}.$$

2.1) A sufficient condition for existence of solutions is:
There exists $w \in SBV_0(\Omega)$ such that:
a)

$$\nabla w \in E \text{ a.e. in } \Omega$$

b)

$$\int_\Omega f^{**}(\nabla w) = f^{**}(\zeta_0)|\Omega|.$$

This leads to the consideration of the following differential inclusion problem:
Given $E \subset \mathbb{R}^N$ and $\zeta_0 \in \mathrm{co}\bar{E}$ find $w \in SBV_0(\Omega)$ such that:

$$\nabla w \in E \text{ a.e. in } \Omega$$

$$\int_\Omega \nabla w = \zeta_0|\Omega|.$$

2.2) In this case, ($\zeta_0 \in E$), it would be sufficient to show that there exists $u \in SBV_0(\Omega)$ such that $f(\zeta_0)|\Omega| = \int_\Omega f(\nabla u)\, dx$. This problem would be easily solved if there exists a function $u \in SBV_0(\Omega)$ such that $\nabla u = \zeta_0$, but this is impossible as shown in the previous session.

Remark 5.1. By Jensen's inequality, and since

$$\bar{\mathrm{co}} E = \{\zeta \in \mathbb{R}^N |\, f(\zeta) \leq 0 \text{ for all } f; \mathbb{R}^N \to \mathbb{R}, f \text{ convex and } f|_E = 0\},$$

we have that ζ_0 is to be taken in $\bar{\mathrm{co}} E$.

Therefore, existence of solutions of the differential inclusion problem together with affinity hypothesis on f^{**} lead to existence of solutions. Taking into account the results from the previous section, a partial answer is:

Theorem 5.2. *Let K be bounded and connected and let $\zeta_0 \in K$and let f^{**} be affine on $\bar{K}$. Then (P) attains solution.*

Proof. Since K (or more generally its connected component that contains ζ_0) is open (because f is lower semicontinuos and f^{**} is continuous), we have that $\zeta_0 \in$int co∂K. Applying now the results from the previous section we have the result. □

Remark 5.3. If $\zeta_0 \in E$, but if there exist $z^\alpha \in V_{\zeta_0} \subset E, t^\alpha \in (0,1)$ in the conditions of Lemma 4.1 and if f^{**} is affine on $int\bar{co}V_{\zeta_0}$, we still are able to apply the previous procedure and obtain existence of solutions. In particular, if $\zeta_0 \in$ intco E, and f is affine at ζ_0 there is solution of the problem.

Remark 5.4. This analysis suggests that, regarding a problem with jump term, the infimum will result from some sort of compromise between $f(\zeta_0)|\Omega|$ and $f(0)|\Omega|$. In fact, applying Lemma 4.3 to any level set, $E_t = \{\zeta \in \mathbb{R}^N : f(\zeta) = t\}$, for $t \in (f(\zeta), f(0))$, one sees that we can construct functions $u_{\zeta_0,t} \in SBV_0(\Omega)$ such that $\int_\Omega f(\nabla u_{\zeta_0,t}) = f(t)|\Omega|$. By getting t "closer" to ζ_0, the bulk energy decreases, while by getting t "closer" to 0, the jump term should decrease.

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