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Multiscale Homogenization of Convex Functionals with Discontinuous Integrand

Marco Barchiesi
S.I.S.S.A., Via Beirut 2-4,
34014 Trieste, Italy
barchies@sissa.it

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This article is devoted to obtain the Γ-limit, as ε tends to zero, of the family of functionals

$$u \mapsto \int_\Omega f\Big(x, \frac{x}{\varepsilon}, \dots, \frac{x}{\varepsilon^n}, \nabla u(x)\Big) dx,$$

where $f = f(x, y^1, \dots, y^n, z)$ is periodic in $y^1, \dots, y^n$, convex in z and satisfies a very weak regularity assumption with respect to $x, y^1, \dots, y^n$. We approach the problem using the multiscale Young measures.

Keywords: Convexity, discontinuous integrands, iterated homogenization, periodicity, multiscale convergence, Young measures, Γ-convergence

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1. Introduction

Multiscale composites are structures constituted by two or more materials which are finely mixed on many different microscopic scales. The fact that a composite often combines the properties of the constituent materials makes these structures particularly interesting in many fields of science. There is a vast literature on the subject; we refer the reader to [22] and references therein.

Determining macroscopic behavior of these strongly heterogeneous structures when the size ε of the heterogeneity becomes "small" is the aim of homogenization theory.

In the particular case of a periodic multiscale composite, from a variational point of view, the homogenization problem is to characterize the behavior, for the parameter ε tending to zero, of functionals on $W^{1,p}(\Omega, \mathbb{R}^s)$ of the type

$$F_\varepsilon(u) = \int_\Omega f\Big(x, \langle \frac{x}{\rho_1(\varepsilon)} \rangle, \dots, \langle \frac{x}{\rho_n(\varepsilon)} \rangle, \nabla u(x)\Big) dx, \tag{1}$$

where $\langle \cdot \rangle$ denotes the fractional part of a vector componentwise, Ω is an open bounded domain in $\mathbb{R}^d$, $\Box$ is the unit cell $[0,1)^d$, ρ_k are the length scales and $f = f\left(x, y^1, \dots, y^n, z\right)$ is a non-negative function on $\Omega \times \Box^n \times \mathbb{M}^{s\times d}$.

The purpose of this paper is to analyze (1) under the following assumptions.

Assumption 1.1. f is convex in the argument z for all $x \in \Omega$ and $y^1, \dots, y^n \in \Box$.

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Assumption 1.2. f is p-coercive and with p-growth:

$$c_1 |z|^p \le f\left(x, y^1, \ldots, y^n, z\right) \le c_2 \left(1 + |z|^p\right)$$

for some $p \in (1, +\infty)$, $c_1, c_2 > 0$ and for all $(x, y^1, \ldots, y^n, z) \in \Omega \times \square^n \times \mathbb{M}^{s\times d}$.

Assumption 1.3. f is an *admissible integrand*, *i.e.*, for every $\delta > 0$ there exist a compact set $X \subseteq \Omega$ with $|\Omega \backslash X| \le \delta$ and a compact set $Y \subseteq \square$ with $|\square \backslash Y| \le \delta$, such that $f|_{X\times Y^n\times\mathbb{M}^{s\times d}}$ is continuous.

In particular we cover the following two significant cases (see Examples 4.12 and 4.13).

(i) The case of a single microscale ($n = 1$): the function $f : \Omega \times \square \times \mathbb{M}^{s\times d} \to [0, +\infty)$ is continuous in x, measurable in y and satisfies Assumptions 1.1 and 1.2. Notice that f is continuous in z uniformly with respect to x and hence is continuous in (x, z). It is possible to interchange the regularity conditions on f requiring the measurability in x and the continuity in y.

(ii) The case of a multiscale mixture of two materials: the function $f : \Omega \times \square^n \times \mathbb{M}^{s\times d} \to [0, +\infty)$ is of the type

$$\begin{aligned} &f(x, y^1, \ldots, y^n, z) \\ &= \prod_{k=1}^{n} \chi_{P_k}(y^k)\, f_1(x, y^1, \ldots, y^n, z) + \Big[1 - \prod_{k=1}^{n} \chi_{P_k}(y^k)\Big]\, f_2(x, y^1, \ldots, y^n, z), \end{aligned} \tag{2}$$

where χ_{P_k} $(k = 1, \ldots, n)$ is the characteristic function of a measurable subset P_k of $\square$ and the functions $f_1, f_2 : \Omega \times \square^n \times \mathbb{M}^{s\times d} \to [0, +\infty)$ are measurable in x, continuous in $(y^1, \ldots, y^n)$ and satisfy Assumptions 1.1 and 1.2. The regularity conditions on f_1 and f_2 can be replaced by the continuity in $(x, y^1, \ldots, y^{n-1})$ and the measurability in the fastest oscillating variable y^n.

Problems of the type (1) have captured the attention of many authors. For instance, the case of a single microscale

$$\int_\Omega f\Big(x, \langle \frac{x}{\varepsilon} \rangle, \nabla u(x)\Big) dx$$

has been studied by Braides (see [8] and also [9, Chapter 14]) under Assumption 1.1 and requiring in addition a p-growth condition on the integrand f and a uniform continuity in x, precisely

$$|f(x, y, z) - f(x', y, z)| \le \omega(|x - x'|)\big[\alpha(y) + f(x, y, z)\big] \tag{3}$$

for all $x, x' \in \mathbb{R}^d$, $y \in \square$ and $z \in \mathbb{M}^{s\times d}$, where $\alpha \in L^1(\square)$ and ω is a continuous positive function with $\omega(0) = 0$. Recently Baía and Fonseca [3] have studied this problem under Assumpion 1.2 and requiring continuity in (y, z) and measurability in x.

In [10] (see also [9, Chapter 22], [19] and [20]) Braides and Lukkassen study functionals of the form

$$\int_\Omega f\Big(\langle \frac{x}{\varepsilon} \rangle, \ldots, \langle \frac{x}{\varepsilon^n} \rangle, \nabla u(x)\Big) dx.$$

The authors provide an iterated homogenization formula for functions as in (2) with an additional request on the functions f_1 and f_2 of a uniform continuity, similar to (3), with respect to the slower oscillating variables $y^1, \dots, y^{n-1}$. The same result is obtained by Fonseca and Zappale [16] but with a continuous function f satisfying Assumptions 1.1 and 1.2.

Since the variable x describes the macroscopic heterogeneity of the constituent materials while the variables $y^1, \dots, y^n$ describe the microscopic heterogeneity of the composite structure, it is desirable to have the weakest possible regularity on them. In particular, the oscillating variables should be able to describe the discontinuity on the interfaces between different materials. At any rate the only request that f is borelian is not enough to obtain a homogenization formula, as it is shown in Examples 5.10 and 5.11 (see also [1] and [13]).

In order to weaken the continuity assumptions taken in the works cited above, we approach the problem using the multiscale Young measures as in [24] (see also [21] and [23]). The peculiarity of our work is the introduction of the concept of *admissible integrand* (Definition 4.10). The crucial point is to extend the lower semicontinuity property (7) to this kind of integrand: this is achieved in Theorem 4.14.

The paper is organized as follows. In Section 2 we recall concepts and basic facts about Young measures. In Section 3 we introduce the notion of multiscale convergence in the general framework of multiscale Young measures. In Section 4 we discuss the properties of admissible integrands. By Theorems 4.6 and 4.14 we derive, in Section 5, the upper and lower estimates for the $\Gamma(L^p)$-limit of the family F_ε (Lemmas 5.7 and 5.5). Finally, in Section 6, we give an iterated homogenization formula.

2. Young measures

We gather briefly in this section some of the main results about Young measures, for more details and proofs we refer the reader to [5], [11] and [26].

We denote with

- D a bounded and locally compact subset of $\mathbb{R}^l$, equipped with the Lebesgue σ-algebra $\mathcal{L}(D)$;
- $|A|$ the Lebesgue measure of a set $A \in \mathcal{L}(D)$;
- S a locally compact and separable metric space, equipped with the Borel σ-algebra $\mathcal{B}(S)$;
- $\mathcal{U}(D, S)$ the family of measurable functions $u : D \to S$;
- $C_0(S)$ the space $\{\phi : S \to \mathbb{R}$ continuous: $\forall \delta > 0 \ \exists K_\delta \subseteq S$ compact: $|\phi(z)| < \delta$ for $z \in S \setminus K_\delta\}$, endowed with the supremum norm;
- $\mathcal{M}(S)$ the space of real valued and finite Radon measures on S;
- $\mathcal{P}(S) := \{\mu \in \mathcal{M}(S) : \mu \geq 0$ and $\mu(S) = 1\}$ the set of probability measures on S;
- $L^1(D, C_0(S))$ the Banach space of all measurable maps $x \in D \xrightarrow{\phi} \phi_x \in C_0(S)$ such that $\|\phi\|_{L^1} := \int_D \|\phi_x\|_{C_0(S)} \, dx$ is finite;
- $L^\infty_w(D, \mathcal{M}(S))$ the Banach space of all weak* measurable maps $x \in D \xrightarrow{\mu} \mu_x \in \mathcal{M}(S)$ such that $\|\mu\|_{L^\infty_w} := \operatorname{ess\,sup}_{x \in D} \|\mu_x\|_{\mathcal{M}(S)}$ is finite;
- $\mathcal{Y}(D, S)$ the family of all weak* measurable maps $\mu : D \to \mathcal{M}(S)$ such that $\mu_x \in \mathcal{P}(S)$ *a.e.* $x \in D$.

Remark 2.1.

(i) As it is known, the dual of $C_0(S)$ may be identified with $\mathcal{M}(S)$ through the duality

$$\langle \mu, \phi \rangle = \int_S \phi \, d\mu \qquad \forall \mu \in \mathcal{M}(S) \quad \text{and} \quad \forall \phi \in C_0(S).$$

(ii) A map $\mu : D \to \mathcal{M}(S)$ is said to be *weak* measurable* if $x \to \langle \mu_x, \phi \rangle$ is measurable for all $\phi \in C_0(S)$.

(iii) More precisely, the elements of $L^1(D, C_0(S))$, $L^\infty_w(D, \mathcal{M}(S))$ and $\mathcal{Y}(D, S)$ are equivalence classes of maps that agree *a.e.*; we do not distinguish these maps from their equivalence classes.

(iv) $L^\infty_w(D, \mathcal{M}(S))$ can be identified with the dual of $L^1(D, C_0(S))$ through the duality

$$\langle \mu, \phi \rangle = \int_D \langle \mu_x, \phi_x \rangle \, dx \qquad \forall \mu \in L^\infty_w\big(D, \mathcal{M}(S)\big) \quad \text{and} \quad \forall \, \phi \in L^1\big(D, C_0(S)\big).$$

In the following we will refer to the weak* topology of $L^\infty_w(D, \mathcal{M}(S))$ as the topology induced by this duality pairing.

(v) Let $\widehat{\mathcal{Y}}(D, S) := \{\varrho \in \mathcal{M}(D \times S) : \varrho \geq 0 \text{ and } \varrho(A \times S) = |A| \ \forall A \in \mathcal{B}(D)\}$. By the Disintegration Theorem [25], the map which associates to $\mu \in \mathcal{Y}(D, S)$ the measure $\widehat{\mu} \in \widehat{\mathcal{Y}}(D, S)$ defined by

$$\widehat{\mu}(A) := \int_D \left(\int_S \chi_A(x, z) d\mu_x(z) \right) dx \quad \forall A \in \mathcal{B}(D \times S)$$

induces a bijection between $\mathcal{Y}(D, S)$ and $\widehat{\mathcal{Y}}(D, S)$. Given a function $f : D \times S \to \mathbb{R}$ $\widehat{\mu}$-integrable, it turns out that $f(x, \cdot)$ is μ_x-integrable for *a.e.* $x \in D$, $x \to \int_S f(x, z) d\mu_x(z)$ is integrable and

$$\int_{D \times S} f d\widehat{\mu} = \int_D \left(\int_S f(x, z) d\mu_x(z) \right) dx;$$

this last equality remains true if f is $\mathcal{L}(D) \otimes \mathcal{B}(S)$-measurable and non-negative.

The family $\mathcal{U}(D, S)$ can be embedded in $L^\infty_w(D, \mathcal{M}(S))$ associating to every $u \in \mathcal{U}(D, S)$ the function

$$x \xrightarrow{\delta_u} \delta_{u(x)},$$

where $\delta_{u(x)}$ is the Dirac probability measure concentrated at the point $u(x)$.

Definition 2.2. A function $\mu \in L^\infty_w(D, \mathcal{M}(S))$ is called the *Young measure* generated by the sequence u_h if $\delta_{u_h} \rightharpoonup \mu$ in the weak* topology.

Remark 2.3. This notion makes sense: by the identification of $L^\infty_w(D, \mathcal{M}(S)) \simeq L^1(D, C_0(S))^*$ and as a direct consequence of the Banach-Alaoglu theorem, every sequence u_h in $\mathcal{U}(D, S)$ admits a subsequence generating a Young measure.

The following result is a "light" version of the Fundamental Theorem on Young Measures.

Theorem 2.4. *Let u_h be a sequence in $\mathcal{U}(D,S)$ generating a Young measure μ and for which the "tightness condition" is satisfied, i.e.,*

$$\forall \delta > 0\ \exists K_\delta \subseteq S \text{ compact}: \sup_{h\in\mathbb{N}^+} \big|\{x \in D : u_h(x) \notin K_\delta\}\big| \le \delta. \tag{4}$$

The following properties hold:

(*i*) $\mu \in \mathcal{Y}(D,S)$;

(*ii*) *if $f : D \times S \to [0,+\infty)$ is a Carathéodory integrand, then*

$$\liminf_{h\to+\infty} \int_D f\big(x, u_h(x)\big)dx \ge \int_D \overline{f}(x)dx$$

where

$$\overline{f}(x) := \int_S f(x,z)d\mu_x(z);$$

(*iii*) *if $f : D\times S \to \mathbb{R}$ is a Carathéodory integrand and $f(\cdot, u_h(\cdot))$ is equi-integrable, then f is $\widehat{\mu}$-integrable and $f(\cdot, u_h(\cdot)) \rightharpoonup \overline{f}$ weakly in $L^1(D)$.*

We remember that a $\mathcal{L}(D)\otimes\mathcal{B}(S)$-measurable function f is a Carathéodory integrand if $f(x,\cdot)$ is continuous for all $x \in D$.

3. Multiscale Young measures

We introduce now the notion of multiscale convergence, an extension of the two-scale convergence carried out by Allaire ([2]) in joint work with Briane. We present it in the general framework of multiscale Young measures, following essentially the ideas exposed in [27], [4] and [21].

We start presenting an example that does not only show a fine and explicit case of Young measure, but it is a fundamental mainstay in this section. Before we add some new notations:

- Ω is a bounded open subset of $\mathbb{R}^d$, equipped with the Lebesgue σ-algebra $\mathcal{L}(\Omega)$;
- $\square$ is the unit cell $[0,1)^d$, equipped with the Lebesgue σ-algebra $\mathcal{L}(\square)$;
- n is the number of scales, a positive integer;
- $\rho_1,\dots,\rho_n$ are positive functions of a parameter $\varepsilon > 0$ which converge to 0 as ε does, for which the following *separation of scales hypothesis* is supposed to hold:

$$\lim_{\varepsilon\to0^+} \frac{\rho_{k+1}(\varepsilon)}{\rho_k(\varepsilon)} = 0 \quad \forall k \in \{1,\dots,n-1\};$$

- $\langle x\rangle \in \square$ is the fractional part of $x \in \mathbb{R}^d$ componentwise, *i.e.*,

$$\langle x\rangle_k = x_k - \lfloor x_k\rfloor \quad \text{for } k \in \{1,\dots,d\},$$

  where $\lfloor x_k\rfloor$ stands for the largest integer less than or equal to x_k;
- $p \in (1,+\infty)$ and $q \in [1,+\infty]$ (unless otherwise stated), moreover q' is the Hölderian conjugate exponent of q;

- $C_c^\infty(\Omega)$ stands for the space of the functions in $C^\infty(\Omega)$ with compact support;
- $C_{per}(\Box^k)$ is the space of the functions $u = u(y^1, \ldots, y^k)$ in $C((\mathbb{R}^d)^k)$ $\Box$-periodic in $y^1, \ldots, y^k$; corresponding definitions hold for $C^1_{per}(\Box^k)$ and $C^\infty_{per}(\Box^k)$;
- $W^{1,p}_{per}(\Box)$ denotes the space of the functions in $W^{1,p}_{loc}(\mathbb{R}^d)$ $\Box$-periodic.

We fix a sequence $\varepsilon_h \to 0^+$ of values of the parameter ε.

Example 3.1. We denote by T the set $\Box$ equipped with the topological and differential structure of the d-dimensional torus; any function on T can be identified with its periodic extension to $\mathbb{R}^d$, in particular

$$C(T) = C_0(T) \simeq C_{per}(\Box).$$

We consider the sequence $v_h : \Omega \to \Box^n$ defined by

$$v_h(x) := \left(\langle \frac{x}{\rho_1(\varepsilon_h)} \rangle, \ldots, \langle \frac{x}{\rho_n(\varepsilon_h)} \rangle \right). \tag{5}$$

For our example, we need an auxiliary ingredient concerning weak convergence. It is a particular case of [14, Proposition 3.3].

Theorem 3.2. Riemann-Lebesgue lemma: *given $\phi \in C_{per}(\Box^n)$, define $\phi_h(x) := \phi(v_h(x))$. Then $\phi_h \rightharpoonup \int_{\Box^n} \phi(y^1, \ldots, y^n)\, dy^1 \ldots dy^n$ weakly* in $L^\infty(\Omega)$.*

As consequence of Riemann-Lebesgue lemma, for all $\varphi \in L^1(\Omega)$ and $\phi \in C_{per}(\Box^n)$

$$\int_\Omega \varphi(x)\phi(v_h(x))\, dx \to \int_{\Omega\times\Box^n} \varphi(x)\phi\left(y^1, \ldots, y^n\right) dx\, dy^1 \ldots dy^n.$$

The map $\varphi\otimes\phi$ that takes every $x \in \Omega$ into $\varphi(x)\phi(\cdot) \in C_{per}(\Box^n)$ belongs to $L^1(\Omega, C_{per}(\Box^n))$. Since the space $L^1(\Omega) \otimes C_{per}(\Box^n)$, defined as the linear closure of $\{\varphi \otimes \phi : \varphi \in L^1(\Omega)$ and $\phi \in C_{per}(\Box^n)\}$, is dense in $L^1(\Omega, C_{per}(\Box^n))$, we conclude that v_h generates the Young measure $\mu \in \mathcal{Y}(\Omega, T^n)$ with

$$\mu_x = \mathcal{L}_{\llcorner\Box^n} \ \text{ for } a.e.\ x \in \Omega,$$

where $\mathcal{L}_{\llcorner\Box^n}$ is the restriction to $\Box^n$ of the Lebesgue measure on $\left(\mathbb{R}^d\right)^n$.

Definition 3.3. Let u_h be a sequence in $L^1(\Omega)$. The sequence u_h is said to be multiscale convergent to a function $u = u(x, y^1, \ldots, y^n) \in L^1(\Omega \times \Box^n)$ if

$$\begin{aligned}\lim_{h\to+\infty} &\int_\Omega \varphi(x)\phi\left(\frac{x}{\rho_1(\varepsilon_h)}, \ldots, \frac{x}{\rho_n(\varepsilon_h)}\right) u_h(x) dx \\ = &\int_{\Omega\times\Box^n} \varphi(x)\phi\left(y^1, \ldots, y^n\right) u\left(x, y^1, \ldots, y^n\right) dx\, dy^1 \ldots dy^n\end{aligned}$$

for any $\varphi \in C_c^\infty(\Omega)$ and any $\phi \in C^\infty_{per}(\Box^n)$. We simply write $u_h \rightsquigarrow u$. A sequence in $L^1(\Omega, \mathbb{R}^m)$ is called multiscale convergent if it is so componentwise.

Proposition 3.4. *Let u_h be an equi-integrable sequence in $L^1(\Omega)$ multiscale convergent to a function $u \in L^1(\Omega \times \Box^n)$. Then u_h converges weakly to u_∞ in $L^1(\Omega)$, where*

$$u_\infty(x) := \int_{\Box^n} u\left(x, y^1, \ldots, y^n\right) dy^1 \ldots dy^n.$$

Proof. An equi-integrable sequence is sequentially weakly compact in L^1, therefore it is sufficient to prove that $u_h \to u_\infty$ in distribution. But this is a direct consequence of the definition, taking $\phi \equiv 1$. ♦

Let u_h be a bounded sequence in $L^1(\Omega, \mathbb{R}^m)$; we consider the sequence $w_h : \Omega \to \square^n \times \mathbb{R}^m$ defined by

$$w_h(x) := \left(\langle \frac{x}{\rho_1(\varepsilon_h)} \rangle, \dots, \langle \frac{x}{\rho_n(\varepsilon_h)} \rangle, u_h(x) \right). \tag{6}$$

Suppose that w_h generates a Young measure μ (at any rate this is true, up to a subsequence). Thanks to the boundness hypothesis, it can be easily proved that w_h satisfies tightness condition (4), so $\mu \in \mathcal{Y}(\Omega, T^n \times \mathbb{R}^m)$. Roughly speaking, by Remark 2.1(v), it is possible to piece together μ in a measure $\widehat{\mu} \in \widehat{\mathcal{Y}}(\Omega, T^n \times \mathbb{R}^m)$. Thanks to Example 3.1, actually $\widehat{\mu} \in \widehat{\mathcal{Y}}(\Omega \times \square^n, \mathbb{R}^m)$ and so, by Remark 2.1(v) again, it is possible to dismantle this measure in a new function $\nu \in \mathcal{Y}(\Omega \times \square^n, \mathbb{R}^m)$, called the *multiscale Young measure* generated by u_h (with respect to $(\rho_1(\varepsilon_h), \dots, \rho_n(\varepsilon_h))$). In particular we have:

Theorem 3.5. *Let $\mu \in \mathcal{Y}(\Omega, T^n \times \mathbb{R}^m)$ be the Young measure generated by w_h and let $\nu \in \mathcal{Y}(\Omega \times \square^n, \mathbb{R}^m)$ be the multiscale Young measure generated by u_h. Then*

$$\begin{aligned} &\int_\Omega \left(\int_{\square^n \times \mathbb{R}^m} f(x, y^1, \dots, y^n, z) d\mu_x(y^1, \dots, y^n, z) \right) dx \\ = &\int_{\Omega \times \square^n} \left(\int_{\mathbb{R}^m} f(x, y^1, \dots, y^n, z) d\nu_{(x,y^1,\dots,y^n)}(z) \right) dx\, dy^1 \dots dy^n \end{aligned}$$

for all $f : \Omega \times \square^n \times \mathbb{R}^m \to \mathbb{R}$ $\widehat{\mu}$-integrable or non-negative $\mathcal{L}(\Omega) \otimes \mathcal{B}(T^n \times \mathbb{R}^m)$-measurable.

The next statement lights up the link between Young measures and multiscale convergence. Sometimes we will use in the sequel the shorter notation $y := (y^1, \dots, y^n)$.

Theorem 3.6. *Let u_h be a bounded sequence in $L^q(\Omega, \mathbb{R}^m)$, $q \in [1, +\infty)$, generating a multiscale Young measure ν. The following properties hold:*

(*i*) *the center of mass $\overline{\nu}$, defined by*

$$\overline{\nu}(x, y^1, \dots, y^n) := \int_{\mathbb{R}^m} z \, d\nu_{(x,y^1,\dots,y^n)}(z),$$

is in $L^q(\Omega \times \square^n, \mathbb{R}^m)$;

(*ii*) *if u_h is equi-integrable, then $u_h \rightsquigarrow \overline{\nu}$;*

(*iii*) *if $f : \Omega \times T^n \times \mathbb{R}^m \to [0, +\infty)$ is a Carathéodory integrand, i.e., $\mathcal{L}(\Omega) \otimes \mathcal{B}(T^n \times \mathbb{R}^m)$-measurable and continuous on $T^n \times \mathbb{R}^m$, then*

$$\liminf_{h \to +\infty} \int_\Omega f\big(x, w_h(x)\big) dx \geq \int_{\Omega \times \square^n} \overline{f}(x, y^1, \dots, y^n) \, dx\, dy^1 \dots dy^n, \tag{7}$$

where

$$\overline{f}(x, y^1, \dots, y^n) := \int_{\mathbb{R}^m} f(x, y^1, \dots, y^n, z) \, d\nu_{(x,y^1,\dots,y^n)}(z);$$

(*iv*) *if* $f : \Omega \times T^n \times \mathbb{R}^m \to \mathbb{R}$ *is a Carathéodory integrand and* $f(\cdot, w_h(\cdot))$ *is equi-integrable, then* $f(x, y, \cdot)$ *is* $\nu_{(x,y)}$*-integrable for a.e.* $(x, y) \in \Omega \times \Box^n$, $\overline{f}$ *is in* $L^1(\Omega \times \Box^n)$ *and* $f(\cdot, w_h(\cdot)) \rightsquigarrow \overline{f}$.

Proof. Assertion (*iii*) is a straight consequence of Theorems 2.4(*ii*) and 3.5. The integrability properties in assertion (*iv*) follow by Theorem 2.4(*iii*) and Remark 2.1(v), by noting that $\widehat{\mu} = \widehat{\nu}$. In order to prove the multiscale convergence, fixed $\varphi \in C_c^\infty(\Omega)$ and $\phi \in C_{per}^\infty(\Box^n)$, we define the function $g : \Omega \times \Box^n \times \mathbb{R}^m \to \mathbb{R}$ by

$$g(x, y, z) := \varphi(x)\phi(y)f(x, y, z).$$

The function g is a Carathéodory integrand on $\Omega \times T^n \times \mathbb{R}^m$ and $g(\cdot, w_h(\cdot))$ is equi-integrable, thus, by Theorems 2.4(*iii*) and 3.5,

$$\int_\Omega g\big(x, w_h(x)\big) \to \int_\Omega \left(\int_{\Box^n \times \mathbb{R}^m} g(x, y, z) d\mu_x(y, z)\right) dx = \int_{\Omega \times \Box^n} \varphi(x)\phi(y)\overline{f}(x, y)\, dx\, dy.$$

Assertion (*ii*) follows by applying (*iv*) with $f(x, y, z) = z_j$, $j = 1, \ldots, m$. Finally, by Jensen's inequality and (*iii*) with $f(x, y, z) = |z|^q$, we obtain assertion (*i*):

$$\int_{\Omega \times \Box^n} |\overline{\nu}(x, y)|^q \, dx\, dy \le \int_{\Omega \times \Box^n} \left(\int_{\mathbb{R}^m} |z|^q \, d\nu_{(x,y)}(z)\right) dx\, dy \le \liminf_{h \to +\infty} \int_\Omega |u_h(x)|^q \, dx < +\infty.$$

♦

Remark 3.7. Actually assertion (*ii*) is a compactness result about the multiscale convergence: for every sequence u_h equi-integrable in L^1 or bounded in L^p, there exists a subsequence u_{h_i} which generates a multiscale Young measure and therefore multiscale convergent. Remember that a bounded sequence in L^p is equi-integrable by Hölder's inequality.

We conclude with a basic result about bounded sequences in $W^{1,p}(\Omega)$.

Theorem 3.8. *Given a sequence* u_h *weakly convergent to* u *in* $W^{1,p}(\Omega)$, *we have that* $u_h \rightsquigarrow u$ *and*

$$\nabla u_h \rightsquigarrow \nabla u + \sum_{k=1}^n \nabla_{y^k} \phi_k$$

for n *suitable functions* $\phi_k(x, y^1, \ldots, y^k) \in L^p\big(\Omega \times \Box^{k-1}, W^{1,p}_{per}(\Box)\big)$.

The proof can be found in [2, Theorem 2.6] and in [4, Theorem 1.6]. In the first reference, the idea is to work on the image of $W^{1,2}(\Omega)$ under the gradient mapping, by characterizing it as the space orthogonal to all divergence-free functions. Instead in the second reference it is used its characterization as the space of all rotation-free fields. This last method is simpler and works for general p, even if only the case $p = 2$ is examined in the original statement. Another proof can be found in [24].

4. Continuity results

As first result of this section, we show that it is possible to use in the multiscale convergence a more complete system of "test functions", not merely $\psi(x,y)=\varphi(x)\phi(y)$ with $\varphi\in C_c^\infty(\Omega)$ and $\phi\in C_{per}^\infty(\square^n)$. Following Valadier [27], we introduce appropriate classes of functions.

Definition 4.1. A function $\psi:\Omega\times\square^n\to\mathbb{R}$ is said to be *admissible* if there exist a family $\{X_\delta\}_{\delta>0}$ of compact subsets of Ω and a family $\{Y_\delta\}_{\delta>0}$ of compact subsets of $\square$ such that $|\Omega\backslash X_\delta|\le\delta$, $|\square\backslash Y_\delta|\le\delta$ and $\psi|_{X_\delta\times Y_\delta^n}$ is continuous for every $\delta>0$.

Remark 4.2. It is not restrictive to suppose that the families $\{X_\delta\}_{\delta>0}$ and $\{Y_\delta\}_{\delta>0}$ are decreasing, *i.e.*, $\delta'\le\delta$ implies $X_\delta\subseteq X_{\delta'}$ and $Y_\delta\subseteq Y_{\delta'}$. Otherwise, it is sufficient to consider the new families $\{\widetilde{X}_\delta\}_{\delta>0}$ and $\{\widetilde{Y}_\delta\}_{\delta>0}$, where

$$\widetilde{X}_\delta:=\bigcap_{i\ge i_\delta}X_{2^{-i}},\quad \widetilde{Y}_\delta:=\bigcap_{i\ge i_\delta}Y_{2^{-i}}$$

and i_δ is the minimum positive integer such that $2^{1-i_\delta}\le\delta$.

Admissible functions have good measurability properties, as stated in the following lemma. We omit the easy proof.

Lemma 4.3. *If $\psi:\Omega\times\square^n\to\mathbb{R}$ is an admissible function, then there exist a Borel set $X\subseteq\Omega$ with $|\Omega\backslash X|=0$ and a Borel set $Y\subseteq\square$ with $|\square\backslash Y|=0$, such that $\psi|_{X\times Y^n}$ is borelian. In particular, for every fixed ε, the function $x\to\psi\left(x,\langle\frac{x}{\rho_1(\varepsilon)}\rangle,\dots,\langle\frac{x}{\rho_n(\varepsilon)}\rangle\right)$ is measurable.*

Definition 4.4. An admissible function ψ is said to be *q-admissible*, and we write $\psi\in\mathcal{A}dm^q$, if there exists a positive function $\alpha\in L^q(\Omega)$ such that

$$|\psi(x,y)|\le\alpha(x)\quad\forall(x,y)\in\Omega\times\square^n.$$

The next theorem proves that it is possible to use $\mathcal{A}dm^q$ as system of test functions. The proof is very close to [27, Proposition 5]. Before we state the following lemma, that can be derived by [14, Lemma 3.1] (see also [2, Remark 2.13]). We use the same definition of v_h given in (5).

Lemma 4.5. *Let A_k be a measurable subset of $\square$ for $k=1,\dots,n$ and let $A:=\prod_{k=1}^n A_k$. Denoted with χ_A the characteristic function of A, the sequence $\chi_A(v_h(\cdot))$ converges weakly* to $|A|$ in $L^\infty(\Omega)$.*

Theorem 4.6. *Let u_h be a bounded sequence in $L^q(\Omega)$, $q\in(1,+\infty]$, generating a multiscale Young measure ν and let $\psi\in\mathcal{A}dm^{q'}$. Then*

$$\lim_{h\to+\infty}\int_\Omega\psi\big(x,v_h(x)\big)u_h(x)dx=\int_{\Omega\times\square^n}\psi\,(x,y)\,\overline{\nu}\,(x,y)\,dx\,dy.$$

In particular, taking $u_h\equiv 1$, we obtain

$$\lim_{h\to+\infty}\int_\Omega\psi\big(x,v_h(x)\big)dx=\int_{\Omega\times\square^n}\psi\,(x,y)\,dx\,dy. \tag{8}$$

Proof. Let $\delta > 0$; by Lusin theorem applied to α and by definition of admissible function, there exist two compact sets $X \subseteq \Omega$ and $Y \subseteq \square$ such that $|\Omega \backslash X| \leq \delta$, $|\square \backslash Y| \leq \delta$ and $\psi|_{X \times Y^n}$, $\alpha|_X$ are continuous. Let $M := \max_X \alpha$; by Tietze-Urysohn's theorem, $\psi|_{X \times Y^n}$ can be extended to a continuous function ψ_0 on $\Omega \times T^n$ with $|\psi_0(x,y)| \leq M$ for every $(x,y) \in \Omega \times \square^n$. We define on $\Omega \times \square^n \times \mathbb{R}$ the functions

$$f(x,y,z) := \psi(x,y)z \quad \text{and} \quad f_0(x,y,z) := \psi_0(x,y)z.$$

With the same definition of w_h given in (6), the sequence $f_0(\cdot, w_h(\cdot))$ is equi-integrable because $|f_0(x, w_h(x))| \leq M\,|u_h(x)|$. By Theorem 3.6($iv$), $f_0(\cdot, w_h(\cdot)) \rightsquigarrow \overline{f_0}$ and therefore, by Proposition 3.4, $f_0(\cdot, w_h(\cdot)) \rightharpoonup \int_{\square^n} \psi_0(\cdot, y)\overline{\nu}(\cdot, y)\, dy$ weakly in $L^1(\Omega)$. This is sufficient to assert that

$$\lim_{h \to +\infty} \int_X f_0\big(x, w_h(x)\big) dx = \int_{X \times \square^n} \psi_0\,(x,y)\,\overline{\nu}\,(x,y)\,dx\,dy.$$

Now

$$\begin{aligned}
&\left| \int_{\Omega \times \square^n} \psi\,(x,y)\,\overline{\nu}\,(x,y)\,dx\,dy - \int_\Omega f\big(x, w_h(x)\big)\,dx \right| \\
&\leq \left| \int_{(\Omega\backslash X)\times\square^n} \psi\,(x,y)\,\overline{\nu}\,(x,y)\,dx\,dy \right| + \left| \int_{X\times\square^n} \Big[\psi\,(x,y) - \psi_0\,(x,y)\Big]\overline{\nu}\,(x,y)\,dx\,dy \right| \\
&\quad + \left| \int_{X\times\square^n} \psi_0\,(x,y)\,\overline{\nu}\,(x,y)\,dx\,dy - \int_X f_0\big(x, w_h(x)\big)\,dx \right| \\
&\quad + \left| \int_X \Big[f_0\big(x, w_h(x)\big) - f\big(x, w_h(x)\big)\Big]\,dx \right| + \left| \int_{\Omega\backslash X} f\big(x, w_h(x)\big)\,dx \right| \\
&= \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}.
\end{aligned}$$

We have to show that I, II, IV and V can be made arbitrarily small. Observe that the function

$$\gamma_{\mathrm{I}}(x,y) := \alpha(x) \int_{\mathbb{R}} |z|\; d\nu_{(x,y)}(z)$$

is in $L^1(\Omega \times \square^n)$ as consequence of Theorem 3.6(iii), Hölder's inequality and the L^q-boundness of the sequence u_h:

$$\begin{aligned}
\int_{\Omega\times\square^n} \gamma_{\mathrm{I}}(x,y)\,dx\,dy \;&\leq\; \liminf_{h\to+\infty} \int_\Omega \alpha(x)\,|u_h(x)|\;dx \\
&\leq\; \|\alpha\|_{L^{q'}(\Omega)} \sup_h \|u_h\|_{L^q(\Omega)} < +\infty.
\end{aligned}$$

The same for $\gamma_{\mathrm{II}}(x,y) := \int_\Omega |z|\; d\nu_{(x,y)}(z)$. By the absolute continuity of the integral and by the estimates

$$\mathrm{I} \leq \int_{(\Omega\backslash X)\times\square^n} \gamma_{\mathrm{I}}(x,y)\,dx\,dy$$

and

$$\begin{aligned}
\mathrm{II} &= \int_{X\times(\square^n\backslash Y^n)} \Big[\psi(x,y) - \psi_0(x,y)\Big]\overline{\nu}\,(x,y)\,dx\,dy \\
&\leq \int_{X\times(\square^n\backslash Y^n)} \gamma_{\mathrm{I}}(x,y)\,dx\,dy + M \int_{X\times(\square^n\backslash Y^n)} \gamma_{\mathrm{II}}(x,y)\,dx\,dy,
\end{aligned}$$

we obtain that I and II tend to 0 for $\delta \to 0$. By using again Hölder's inequality and the L^q-boundness of u_h, we get for a suitable positive constant c

$$\begin{aligned} \mathrm{IV} &\leq \left| \int_X \chi_{\square^n \setminus Y^n}(v_h(x)) \Big[f_0\big(x, w_h(x)\big) - f\big(x, w_h(x)\big) \Big] \, dx \right| \\ &\leq \int_X \chi_{\square^n \setminus Y^n}(v_h(x)) \, [\alpha(x) + M] \, |u_h(x)| \, dx \\ &\leq c \left(\int_X \chi_{\square^n \setminus Y^n}(v_h(x)) \, [\alpha(x) + M]^{q'} \, dx \right)^{\frac{1}{q'}} \end{aligned}$$

and

$$\mathrm{V} \leq c \left(\int_{\Omega \setminus X} [\alpha(x)]^{q'} \, dx \right)^{\frac{1}{q'}}.$$

By Lemma 4.5, it follows that $\chi_{\square^n \setminus Y^n}(v_h(\cdot)) = 1 - \chi_{Y^n}(v_h(\cdot))$ converges weakly* to $|\square^n \setminus Y^n|$ and therefore

$$\int_X \chi_{\square^n \setminus Y^n}(v_h(x)) \, [\alpha(x) + M]^{q'} \, dx \xrightarrow{h \to \infty} |\square^n \setminus Y^n| \int_X [\alpha(x) + M]^{q'} \, dx.$$

Hence we conclude that IV and V tend to 0 for $h \to \infty$ and $\delta \to 0$. ♦

Remark 4.7. Let $\psi = \psi(x, y^1, \ldots, y^n)$ be a real function on $\Omega \times \square^n$ either continuous in $(y^1, \ldots, y^n)$ and measurable in x or continuous in $(x, y^1, \ldots, y^{k-1}, y^{k+1}, \ldots, y^n)$ and measurable in y^k. By Scorza-Dragoni theorem (see [15]), ψ is an admissible function. This is no longer true if one removes the continuity assumption on two variables. More generally, the invocation of (8) may be invalid, as shown in the next two examples. The first covers the case $\psi = \psi(x, y)$ $(n = 1)$ while the second covers the case $\psi = \psi(y^1, y^2)$. We remark that in both examples ψ is a Borel function. See also the example in [1, Proposition 5.8].

Example 4.8. Define the Borel sets $A_i := \bigcup_{j=0}^{i-1} \{(x, y) \in [0, 1)^2 : y = i\,x - j\}$ and $A := \bigcup_{i=1}^{\infty} A_i$. Now, in the simple case $d = n = 1$, $\rho_1(\varepsilon_h) = h^{-1}$ and $\Omega = (0, 1)$, consider the function $\psi(x, y) := \chi_A(x, y)$. We have

$$\int_0^1 \psi(x, \langle h\,x \rangle) \, dx \equiv 1 \quad \text{but} \quad \int_0^1 \int_0^1 \psi(x, y) \, dx \, dy = 0.$$

Example 4.9. In the case $d = 1$, $n = 2$, $\rho_1(\varepsilon_h) = h^{-1}$, $\rho_2(\varepsilon_h) = h^{-2}$ and $\Omega = (0, 1)$, consider the function $\psi(y^1, y^2) := \chi_A(y^1, y^2)$, where A is defined as in the former example. We have

$$\int_0^1 \psi(\langle h\,x \rangle, \langle h^2\,x \rangle) \, dx \equiv 1 \quad \text{but} \quad \int_0^1 \int_0^1 \int_0^1 \psi(y^1, y^2) \, dx \, dy^1 \, dy^2 = 0.$$

Notice that the result of weak* convergence in L^∞ stated in Lemma 4.5 is not applicable to A.

So far we have considered Carathéodory functions f on $\Omega \times T^n \times \mathbb{R}^m$. As we explained in the introduction, one would like to have a minimal regularity in $(x, y^1, \ldots, y^n)$. For this reason, we introduce an appropriate class of integrands and extend to this Theorem 3.6(*iii*).

Definition 4.10. A function $f : \Omega \times \square^n \times \mathbb{R}^m \to [0, +\infty)$ is said to be an *admissible integrand* if for every $\delta > 0$ there exist a compact set $X \subseteq \Omega$ with $|\Omega \backslash X| \leq \delta$ and a compact set $Y \subseteq \square$ with $|\square \backslash Y| \leq \delta$, such that $f|_{X \times Y^n \times \mathbb{R}^m}$ is continuous.

As in the analogous case for admissible functions (Lemma 4.3), it is easy to verify the following measurability properties of admissible integrands.

Lemma 4.11. *If $f : \Omega \times \square^n \times \mathbb{R}^m \to [0, +\infty)$ is an admissible integrand, then there exist a Borel set $X \subseteq \Omega$ with $|\Omega \backslash X| = 0$ and a Borel set $Y \subseteq \square$ with $|\square \backslash Y| = 0$, such that $f|_{X \times Y^n \times \mathbb{R}^m}$ is borelian. In particular, for every fixed ε, the function $(x, z) \to f\left(x, \langle \frac{x}{\rho_1(\varepsilon)} \rangle, \dots, \langle \frac{x}{\rho_n(\varepsilon)} \rangle, z\right)$ is $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^m)$-measurable.*

Example 4.12. Let $f : \Omega \times \square \times \mathbb{R}^m \to [0, +\infty)$ be a function such that

(i) $f(\cdot, y, \cdot)$ is continuous for all $y \in \square$;
(ii) $f(x, \cdot, z)$ is measurable for all $x \in \Omega$ and $z \in \mathbb{R}^m$.

By Scorza-Dragoni theorem, f is an admissible integrand. Clearly it is possible to replace conditions (i) and (ii) with

(i)' $f(x, \cdot, \cdot)$ is continuous for all $x \in \Omega$;
(ii)' $f(\cdot, y, z)$ is measurable for all $y \in \square$ and $z \in \mathbb{R}^m$.

Example 4.13. Let $f : \Omega \times \square^n \times \mathbb{R}^m \to [0, +\infty)$ be a function of the type

$$\begin{aligned} & f(x, y^1, \dots, y^n, z) \\ &= \prod_{k=1}^{n} \chi_{P_k}(y^k)\, f_1(x, y^1, \dots, y^n, z) + \left[1 - \prod_{k=1}^{n} \chi_{P_k}(y^k)\right] f_2(x, y^1, \dots, y^n, z), \end{aligned}$$

where P_k $(k = 1, \dots, n)$ is a measurable subset of $\square$ and f_j $(j = 1, 2)$ is a non-negative function on $\Omega \times \square^n \times \mathbb{R}^m$ such that

(i) f_j is continuous in $(y^1, \dots, y^n, z)$;
(ii) f_j is measurable in x.

By Scorza-Dragoni theorem for every $\delta > 0$ there exists a compact set $X \subseteq \Omega$ such that the functions f_1 and f_2 are continuous on $X \times \square^n \times \mathbb{R}^m$. By applying Lusin theorem to each χ_{P_k}, we obtain that f is an admissible integrand. Obviously, the conditions (i) and (ii) can be replaced by

(i)' f_j is continuous in $(x, y^1, \dots, y^{k-1}, y^{k+1}, \dots, y^n, z)$;
(ii)' f_j is measurable in y^k.

Theorem 4.14. *Let u_h be a bounded sequence in $L^1(\Omega, \mathbb{R}^m)$ generating a multiscale Young measure ν and let $f : \Omega \times \square^n \times \mathbb{R}^m \to [0, +\infty)$ be an admissible integrand. Then, with the same definition of w_h given in* (6)*,*

$$\liminf_{h \to +\infty} \int_{\Omega} f\big(x, w_h(x)\big) dx \geq \int_{\Omega \times \square^n} \overline{f}(x, y^1, \dots, y^n)\, dx\, dy^1 \dots dy^n, \tag{9}$$

where as usual

$$\overline{f}(x, y^1, \dots, y^n) := \int_{\mathbb{R}^m} f(x, y^1, \dots, y^n, z)\, d\nu_{(x,y^1,\dots,y^n)}(z).$$

Proof. In addition assume initially that f is bounded from above by a constant $b > 0$:

$$f(x, y, z) \le b \quad \text{for all } (x, y, z) \in \Omega \times \Box^n \times \mathbb{R}^m. \tag{10}$$

By the admissibility condition, for every $\delta > 0$ there exist a compact set $X \subseteq \Omega$ and a compact set $Y \subseteq \Box$ such that $|\Omega \backslash X| \le \delta$, $|\Box \backslash Y| \le \delta$ and $f|_{X \times Y^n \times \mathbb{R}^m}$ is continuous. By Tietze-Urysohn's theorem, $f|_{X \times Y^n \times \mathbb{R}^m}$ can be extended to a non-negative and continuous function f_0 on $\Omega \times T^n \times \mathbb{R}^m$ such that $f_0(x, y, z) \le b$ for every $(x, y, z) \in \Omega \times \Box^n \times \mathbb{R}^m$. Obviously $f_0\left(\cdot, w_h(\cdot)\right)$ is equi-integrable and so, by Theorem 3.6(*iv*) and by Proposition 3.4,

$$\lim_{h \to +\infty} \int_\Omega f_0\big(x, w_h(x)\big)\, dx = \int_{\Omega \times \Box^n} \overline{f_0}(x, y)\, dx\, dy.$$

For a suitable subsequence h_i

$$\lim_{i \to +\infty} \int_\Omega f\big(x, w_{h_i}(x)\big) dx = \liminf_{h \to +\infty} \int_\Omega f\big(x, w_h(x)\big) dx.$$

We can write

$$\begin{aligned}
&\lim_{i \to +\infty} \int_\Omega f\big(x, w_{h_i}(x)\big)\, dx \, - \int_{\Omega \times \Box^n} \overline{f}(x, y)\, dx\, dy \\
&= \lim_{i \to +\infty} \int_\Omega \Big[f\big(x, w_{h_i}(x)\big) - f_0\big(x, w_{h_i}(x)\big)\Big] dx \\
&\qquad + \left[\lim_{i \to +\infty} \int_\Omega f_0\big(x, w_{h_i}(x)\big)\, dx - \int_{\Omega \times \Box^n} \overline{f_0}(x, y)\, dx\, dy\right] \\
&\qquad + \int_{\Omega \times \Box^n} \Big[\overline{f_0}(x, y) - \overline{f}(x, y)\Big]\, dx\, dy = \mathrm{I{+}II{+}III}.
\end{aligned}$$

Clearly, the negative part of III can be made arbitrarily small. Let us check that the same holds for I. By Lemma 4.5,

$$\begin{aligned}
\mathrm{I} = &\lim_{i \to +\infty} \int_{\Omega \backslash X} \Big[f\big(x, w_{h_i}(x)\big) - f_0\big(x, w_{h_i}(x)\big)\Big] dx \\
&+ \lim_{i \to +\infty} \int_X \chi_{\Box^n \backslash Y^n}(v_{h_i}(x)) \Big[f\big(x, w_{h_i}(x)\big) - f_0\big(x, w_{h_i}(x)\big)\Big] dx \\
\ge &- b\, |\Omega \backslash X| \,-\, \lim_{i \to +\infty} b \int_X \chi_{\Box^n \backslash Y^n}(v_{h_i}(x))\, dx \\
\ge &- b\big(|\Omega \backslash X| + |X|\, |\Box^n \backslash Y^n|\big) \ge -b\, \delta\big(1 + n\, |\Omega|\big).
\end{aligned}$$

This concludes the first part of the proof.

In order to remove assumption (10) we consider, for $k \in \mathbb{N}^+$, the functions

$$f_k(x, y, z) := \min\big\{k, f(x, y, z)\big\}.$$

By applying the first part of the theorem, we have

$$\liminf_{h\to+\infty}\int_\Omega f\big(x,w_h(x)\big)dx \geq \liminf_{h\to+\infty}\int_\Omega f_k\big(x,w_h(x)\big)dx \geq \int_{\Omega\times\square^n}\overline{f_k}(x,y)\,dx\,dy.$$

By noting that f_k is increasing and that $f_k(x,y,\cdot)\to f(x,y,\cdot)$ *a.e.* in $\mathbb{R}^m$ for every fixed $(x,y)\in\Omega\times\square^n$, we deduce from the monotone convergence theorem that $\overline{f_k}\to\overline{f}$ *a.e.* in $\Omega\times\square^n$. The sequence $\overline{f_k}$ is increasing so, again from monotone convergence theorem,

$$\int_{\Omega\times\square^n}\overline{f_k}(x,y)\,dx\,dy \xrightarrow{k\to\infty} \int_{\Omega\times\square^n}\overline{f}(x,y)\,dx\,dy.$$

♦

Remark 4.15. Lower semicontinuity property (9) is not true if f is only borelian. For instance, consider the function $f(x,y,z) := [1-\psi(x,y)]\,|z|$, where ψ is defined as in Example 4.8.

5. Gamma-convergence

In the present section we examine the multiscale homogenization of nonlinear convex functionals by means of the Γ-convergence combined with the multiscale Young measures.

Before we recall the definition of Γ-convergence, referring to [9] and [12] for an exposition of the main properties.

Definition 5.1. Let (U,τ) be a topological space satisfying the first countability axiom and F_h, F functionals from U to $[-\infty,+\infty]$; we say that F is the $\Gamma(\tau)$-limit of the sequence F_h or that F_h $\Gamma(\tau)$-converges to F, and write

$$F = \Gamma(\tau)\text{-}\lim_{h\to+\infty} F_h,$$

if for every $u\in U$ the following conditions are satisfied:

$$F(u) \leq \inf\Big\{\liminf_{h\to+\infty} F_h(u_h) : u_h \xrightarrow{\tau} u\Big\} \tag{11}$$

and

$$F(u) \geq \inf\Big\{\limsup_{h\to+\infty} F_h(u_h) : u_h \xrightarrow{\tau} u\Big\}. \tag{12}$$

We can extend the definition of Γ-convergence to families depending on a parameter $\varepsilon>0$.

Definition 5.2. For every $\varepsilon>0$, let F_ε be a functional from U to $[-\infty,+\infty]$. We say that F is the $\Gamma(\tau)$-limit of the family F_ε as $\varepsilon\to 0^+$, and write

$$F = \Gamma(\tau)\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon,$$

if we have for every sequence $\varepsilon_h\to 0^+$

$$F = \Gamma(\tau)\text{-}\lim_{h\to+\infty} F_{\varepsilon_h}.$$

Throughout this section, we work in the space $L^p(\Omega, \mathbb{R}^s)$ endowed with the strong topology. As pointed out in the introduction, we consider a non-negative function $f = f(x, y^1, \dots, y^n, z)$ on $\Omega \times \Box^n \times \mathbb{M}^{s\times d}$ satisfying Assumptions 1.1, 1.2 and 1.3.

We fully characterize the $\Gamma(L^p)$-limit of the family $F_\varepsilon : L^p(\Omega, \mathbb{R}^s) \to [0, +\infty]$ where the functionals are defined by

$$F_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega f\Big(x, \langle \frac{x}{\rho_1(\varepsilon)} \rangle, \dots, \langle \frac{x}{\rho_n(\varepsilon)} \rangle, \nabla u(x)\Big) dx & \text{if } u \in W^{1,p}(\Omega, \mathbb{R}^s), \\ +\infty & \text{otherwise.} \end{cases}$$

Precisely, this is our main result.

Theorem 5.3. *The family F_ε $\Gamma(L^p)$-converges and its $\Gamma(L^p)$-limit $F_{hom} : L^p(\Omega, \mathbb{R}^s) \to [0, +\infty]$ is given by*

$$F_{hom}(u) = \begin{cases} \displaystyle\int_\Omega f_{hom}\big(x, \nabla u(x)\big) dx & \text{if } u \in W^{1,p}(\Omega, \mathbb{R}^s), \\ +\infty & \text{otherwise,} \end{cases}$$

where f_{hom} is obtained by the following cell problem

$$f_{hom}(x, z) := \inf_{\phi \in \Phi} \int_{\Box^n} f\Big(x, y, z + \sum_{k=1}^n \nabla_{y^k} \phi_k(y^1, \dots, y^k)\Big) dy$$

with the space Φ defined by

$$\Phi := \prod_{k=1}^n \Phi_k \quad \text{and} \quad \Phi_k := L^p\big(\Box^{k-1}, W^{1,p}_{per}(\Box, \mathbb{R}^s)\big).$$

Remark 5.4. (i) Using the p-growth condition of f and a density argument, it can be shown that

$$f_{hom}(x, z) = \inf_{\phi \in \Phi_{reg}} \int_{\Box^n} f\Big(x, y, z + \sum_{k=1}^n \nabla_{y^k} \phi_k(y^1, \dots, y^k)\Big)\, dy,$$

where

$$\Phi_{reg} := \prod_{k=1}^n \Phi_{k,reg} \quad \text{and} \quad \Phi_{k,reg} := C^1\Big(\overline{\Box}^{k-1}, C^1_{per}(\Box, \mathbb{R}^s)\Big). \tag{13}$$

(ii) For every $\delta > 0$ there exists a compact set $X \subseteq \Omega$ with $|\Omega \backslash X| \leq \delta$ such that the restriction of f to $X \times \Box^n \times \mathbb{M}^{s\times d}$ is continuous in (x, z) for *a.e.* $(y^1, \dots, y^n) \in \Box^n$ and so f_{hom} is lower semicontinuous on $X \times \mathbb{M}^{s\times d}$. In particular f_{hom} is $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{M}^{s\times d})$-measurable.
(iii) The convexity, the p-coerciveness and the p-growth condition on f give the corresponding properties for the function f_{hom}. In particular F_{hom} is continuous on $W^{1,p}(\Omega, \mathbb{R}^s)$, endowed with the strong topology.

Before proving the theorem, we state a series of lemmas. Only for simplicity of notations, we restrict ourselves to the case $s=1$. Fixed a sequence $\varepsilon_h \to 0^+$, we use for v_h the same definition given in (5).

Lemma 5.5. *Let u_h be a sequence converging weakly in $W^{1,p}(\Omega)$ to a function u. Then*

$$\liminf_{h\to+\infty} \int_\Omega f\big(x, v_h(x), \nabla u_h(x)\big)dx \geq \int_\Omega f_{hom}\big(x, \nabla u(x)\big)dx.$$

Proof. For a suitable subsequence h_i,

$$\lim_{i\to+\infty} \int_\Omega f\big(x, v_{h_i}(x), \nabla u_{h_i}(x)\big)dx = \liminf_{h\to+\infty} \int_\Omega f\big(x, v_h(x), \nabla u_h(x)\big)dx.$$

Refining the subsequence if necessary, we can suppose that ∇u_{h_j} generates a multiscale Young measure $\nu \in \mathcal{Y}(\Omega \times \square^n, \mathbb{R}^d)$. By Theorem 4.14 and Jensen's inequality

$$\begin{aligned}\lim_{i\to+\infty} \int_\Omega f\big(x, v_{h_i}(x), \nabla u_{h_i}(x)\big)dx &\geq \int_{\Omega\times\square^n} \left(\int_{\mathbb{R}^d} f(x,y,z)\, d\nu_{(x,y)}\right) dx\, dy \\ &\geq \int_{\Omega\times\square^n} f\Big(x, y, \int_{\mathbb{R}^d} z\, d\nu_{(x,y)}\Big)\, dx\, dy\end{aligned}$$

and by Theorems 3.6(ii) and 3.8

$$\begin{aligned}&\geq \int_{\Omega\times\square^n} f\Big(x, y, \nabla u(x) + \sum_{k=1}^n \nabla_{y^k}\phi_k(x, y^1, \ldots, y^k)\Big)\, dx\, dy \\ &\geq \int_\Omega f_{hom}\big(x, \nabla u(x)\big)dx.\end{aligned}$$

♦

Lemma 5.6. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function, such that for every $z \in \mathbb{R}^d$*

$$|f(z)| \leq c\,(b + |z|)^p\,, \tag{14}$$

where b and c are positive constants. Then, for all $z_1, z_2 \in \mathbb{R}^d$

$$|f(z_1) - f(z_2)| \leq c\, d\,(1 + 2^p)\big(b + |z_1| + |z_2|\big)^{p-1} |z_1 - z_2|\,. \tag{15}$$

The proof can be derived from [17, Lemma 5.2]. We observe that in (15) the estimate depends only by the costants b, c of growth condition (14) and not by the particular function f.

Lemma 5.7. *Let $u \in W^{1,p}(\Omega) \cap C^1(\Omega)$. Then*

$$\inf_{u_h\to u}\Big\{\limsup_{h\to+\infty} F_{\varepsilon_h}(u_h)\Big\} \leq \inf_{\psi\in\Psi} \int_{\Omega\times\square^n} f\Big(x, y, \nabla u(x) + \sum_{k=1}^n \nabla_{y^k}\psi_k(x, y^1, \ldots, y^k)\Big)\, dx\, dy, \tag{16}$$

where the inf*'s are made respectively on the sequences u_h that converge strongly in $L^p(\Omega)$ to u and on the space Ψ defined by*

$$\Psi := \prod_{k=1}^n \Psi_k \quad \textit{and} \quad \Psi_k := C^1\left(\overline{\Omega} \times \overline{\square}^{k-1}, C^1_{per}(\square)\right).$$

Proof. Given an arbitrary function $\psi = (\psi_1, \dots, \psi_n) \in \Psi$, consider the sequence

$$u_h(x) := u(x) + \sum_{k=1}^{n} \rho_k(\varepsilon_h)\psi_k\big(x, v_h^k(x)\big),$$

where we used the short notation $v_h^k(x) := \Big(\langle \frac{x}{\rho_1(\varepsilon_h)} \rangle, \dots, \langle \frac{x}{\rho_k(\varepsilon_h)} \rangle\Big)$. We have $u_h \to u$ strongly in $L^p(\Omega)$ and $\nabla u_h = \nabla u + \sum_{k=1}^n \nabla_{y^k}\psi_k + r_h$, with $r_h \to 0$ strongly in $L^p(\Omega, \mathbb{R}^d)$. The function $g : \Omega \times \square^n \to \mathbb{R}$ defined by

$$g(x, y) := f\Big(x, y, \nabla u(x) + \sum_{k=1}^{n} \nabla_{y^k}\psi_k(x, y^1, \dots, y^k)\Big)$$

is admissible. Actually $g \in \mathcal{A}dm^1$, as evident by the estimate obtained through the p-growth condition:

$$|g(x, y)| \le c_2\Big[1 + (n+1)^{p-1}\Big(|\nabla u(x)|^p + \sum_{k=1}^{n} M_k^p\Big)\Big],$$

where $M_k := \sup_{\Omega\times\square^k} |\nabla_{y^k}\psi_k|$.

By Lemma 5.6, the following inequality holds for some positive constants b, c:

$$\Big|g\big(x, v_h(x)\big) - f\big(x, v_h(x), \nabla u_h(x)\big)\Big| \le c\,|r_h(x)|\,\Big(b + |\nabla u(x)|^{p-1} + |r_h(x)|^{p-1}\Big).$$

By integrating over Ω, from Hölder's inequality we obtain, for another positive constant c',

$$\int_\Omega \Big|g\big(x, v_h(x)\big) - f\big(x, v_h(x), \nabla u_h(x)\big)\Big|\,dx \le c' \int_\Omega |r_h(x)|^p\,dx$$

and thus Theorem 4.6 gives

$$\lim_{h\to+\infty} \int_\Omega f\big(x, v_h(x), \nabla u_h(x)\big)\,dx = \lim_{h\to+\infty} \int_\Omega g\big(x, v_h(x)\big)dx$$

$$= \int_{\Omega\times\square^n} g\,(x, y)\,dx\,dy = \int_{\Omega\times\square^n} f\Big(x, y, \nabla u(x) + \sum_{k=1}^{n} \nabla_{y^k}\psi_k(x, y^1, \dots, y^k)\Big)\,dx\,dy.$$

♦

Definition 5.8. We say that $\Lambda \subseteq L^1(\Omega)$ is an *inf-stable family* if, given $\{\lambda_1, \dots, \lambda_N\} \subseteq \Lambda$ and $\{\varphi_1, \dots, \varphi_N\} \subseteq C^1\big(\overline{\Omega}, [0,1]\big)$, with $\sum_{j=1}^N \varphi_j = 1$ and $N \in \mathbb{N}^+$, there exists a $\lambda \in \Lambda$ such that

$$\lambda \le \sum_{j=1}^{N} \varphi_j\lambda_j.$$

Lemma 5.9. *Let Λ be an inf-stable family of non-negative integrable functions on Ω. If for every $\delta > 0$ there exists a compact set $X_\delta \subseteq \Omega$ such that $|\Omega \backslash X_\delta| \leq \delta$ and $\lambda|_{X_\delta}$ is continuous for each $\lambda \in \Lambda$, then the function $\inf_{\lambda\in\Lambda} \lambda$ is measurable and the following commutation property holds:*

$$\inf_{\lambda\in\Lambda} \int_\Omega \lambda(x)\, dx = \int_\Omega \inf_{\lambda\in\Lambda} \lambda(x)\, dx. \tag{17}$$

This lemma can be derived by [6, Lemma 4.3] (see also [18]), by noting that for every $\delta > 0$ $\inf_{\lambda\in\Lambda} \lambda = \operatorname{ess\,inf}_{\lambda\in\Lambda} \lambda$ on X_δ. Anyway, we prefer to give a simple direct proof.

Proof. Firstly we observe that for every $\delta > 0$ the function $\inf_{\lambda\in\Lambda} \lambda$ is lower semicontinuous on X_δ. In particular $\inf_{\lambda\in\Lambda} \lambda$ is measurable. By applying the Lindelöf theorem to each family $\{E^\lambda_\delta\}_{\lambda\in\Lambda}$, where

$$E^\lambda_\delta := \big\{(x,t) \in X_\delta \times \mathbb{R} : \lambda(x) < t\big\},$$

we can find a sequences λ_i in Λ such that

$$\inf_{\lambda\in\Lambda} \lambda(x) = \inf_i \lambda_i(x) \quad \text{for } a.e.\ x \in \Omega.$$

Fixed $N \in \mathbb{N}^+$ and $\zeta > 0$, we choose a $\delta > 0$ such that $\sum_{j=1}^N \int_{\Omega\backslash X_\delta} \lambda_j \leq \zeta$. By the continuity property of the elements $\lambda \in \Lambda$, the sets

$$A_i := \left\{x \in X_\delta : \lambda_i(x) < \inf_{1\leq j\leq N} \lambda_j(x) + \zeta\right\}$$

are open in X_δ. Notice that $X_\delta = \bigcup_{i=1}^\infty A_i$. For every $i \in \mathbb{N}^+$, let B_i be a open subset of $\mathbb{R}^d$ for which $B_i \cap X_\delta = A_i$ and let $\{\varphi_i\}_i \subseteq C^1\big(\overline{\Omega}, [0,1]\big)$ be a partition of unity subordinate to $\{B_i\}_i$. By the inf-stability property, there exists a $\lambda \in \Lambda$ such that $\lambda \leq \sum_{j=1}^N \varphi_j \lambda_j$. We have

$$\begin{aligned}
\int_\Omega \lambda(x)\, dx &= \int_{\Omega\backslash X_\delta} \lambda(x)\, dx + \int_{X_\delta} \lambda(x)\, dx \\
&\leq \sum_{j=1}^N \int_{\Omega\backslash X_\delta} \varphi_j(x)\, \lambda_j(x)\, dx + \sum_{i=1}^\infty \int_{X_\delta} \varphi_i(x)\lambda_i(x)\, dx \\
&\leq \zeta + \int_\Omega \inf_{1\leq j\leq N} \lambda_j(x)\, dx + \zeta\, |\Omega|\, .
\end{aligned}$$

Being N and ζ arbitrary, the claim follows. ♦

Proof of Theorem 5.3. We are now ready to provide a proof of Theorem 5.3.

Let $u_h \to u$ strongly in $L^p(\Omega)$. We want to show that $\liminf F_{\varepsilon_h}(u_h) \geq F_{hom}(u)$. In this way inequality (11) will be proved. If $\liminf F_{\varepsilon_h}(u_h) = +\infty$, there is nothing to prove, so we can assume $\liminf F_{\varepsilon_h}(u_h) < +\infty$. For a suitable subsequence h_i,

$$\lim_{i\to+\infty} F_{\varepsilon_{h_i}}(u_{h_i}) = \liminf_{h\to+\infty} F_{\varepsilon_h}(u_h).$$

For i large enough, $F_{\varepsilon_{h_i}}(u_{h_i})$ is finite and therefore, by the definition of F_ε, $u_{h_i} \in W^{1,p}(\Omega)$. Thanks to the p-coerciveness hypothesis on f, we can infer that u_{h_i} is bounded in $W^{1,p}(\Omega)$. Refining the subsequence if necessary, we can suppose that u_{h_j} converges weakly in $W^{1,p}(\Omega)$ to u and thus we can apply Lemma 5.5.

It remains to check inequality (12). If $u \in L^p(\Omega) \setminus W^{1,p}(\Omega)$, then $F_{hom}(u) = +\infty$ and the inequality is obvious, while if $u \in W^{1,p}(\Omega)$, then we can apply Lemma 5.7 and, as in [7, Theorem 3.3], Lemma 5.9. In view of the density of $W^{1,p}(\Omega) \cap C^1(\Omega)$ in $W^{1,p}(\Omega)$ and of the continuity of F_{hom}, by a standard diagonalization argument, it is not restrictive to assume that $u \in W^{1,p}(\Omega) \cap C^1(\Omega)$.

For every $\psi = (\psi_1, \dots, \psi_n) \in \Psi$, define the function

$$\lambda_\psi(x) := \int_{\square^n} f\Big(x, y, \nabla u(x) + \sum_{k=1}^n \nabla_{y^k} \psi_k(x, y^1, \dots, y^k)\Big)\, dy.$$

We claim that the family $\Lambda := \{\lambda_\psi : \psi \in \Psi\}$ satisfies the hypotheses of Lemma 5.9. In fact, from the p-growth condition on f, it is easy to show that each function in Λ is integrable on Ω. Moreover, by Remark 5.4(ii), for every $\delta > 0$ there exists a compact set $X \subseteq \Omega$ with $|\Omega \backslash X| \leq \delta$ such that λ_ψ is continuous on X for each $\psi \in \Psi$. It remains to prove the inf-stability.

Given $\{\psi^{(1)}, \dots, \psi^{(N)}\} \subseteq \Psi$ and $\{\varphi_1, \dots, \varphi_N\} \subseteq C^1\big(\overline{\Omega}, [0,1]\big)$, with $\sum_{j=1}^N \varphi_j = 1$ and $N \in \mathbb{N}^+$, consider the function

$$\psi := \left(\sum_{j=1}^N \varphi_j \psi_1^{(j)}, \dots, \sum_{j=1}^N \varphi_j \psi_n^{(j)}\right) \in \Psi.$$

Thanks to the convexity of f, we have $\lambda_\psi \leq \sum_{j=1}^N \varphi_j \lambda_{\psi^{(j)}}$:

$$\begin{aligned}
\lambda_\psi(x) &= \int_{\square^n} f\Big(x, y, \nabla u(x) + \sum_{j=1}^N \sum_{k=1}^n \nabla_{y^k} \varphi_j(x) \psi_k^{(j)}(x, y^1, \dots, y^k)\Big)\, dy \\
&= \int_{\square^n} f\left(x, y, \sum_{j=1}^N \varphi_j(x) \Big(\nabla u(x) + \sum_{k=1}^n \nabla_{y^k} \psi_k^{(j)}(x, y^1, \dots, y^k)\Big)\right) dy \\
&\leq \sum_{j=1}^N \varphi_j(x) \int_{\square^n} f\Big(x, y, \nabla u(x) + \sum_{k=1}^n \nabla_{y^k} \psi_k^{(j)}(x, y^1, \dots, y^k)\Big)\, dy = \sum_{j=1}^N \varphi_j(x) \lambda_{\psi^{(j)}}(x).
\end{aligned}$$

Finally, by inequality (16), equality (17) and Remark 5.4(i),

$$\begin{aligned}
&\inf_{u_h \to u} \Big\{ \limsup_{h \to +\infty} F_{\varepsilon_h}(u_h) \Big\} \leq \int_\Omega \inf_{\psi \in \Psi} \left(\int_{\square^n} f\Big(x, y, \nabla u(x) + \sum_{k=1}^n \nabla_{y^k} \psi_k(x, y^1, \dots, y^k)\Big)\, dy \right) dx \\
&\leq \int_\Omega \inf_{\phi \in \Phi_{reg}} \left(\int_{\square^n} f\Big(x, y, \nabla u(x) + \sum_{k=1}^n \nabla_{y^k} \phi_k(y^1, \dots, y^k)\Big)\, dy \right) dx = \int_\Omega f_{hom}\big(x, \nabla u(x)\big)\, dx.
\end{aligned}$$

The proof is complete. ♦

Assumption 1.3 cannot be weakened too much: even if f is a Borel function, Γ-convergence Theorem 5.3 may be not applicable, as shown in the following examples (see also [13, Example 3.1]).

Example 5.10. Let A_i be the Borel sets defined as in Example 4.8 and let $B := \bigcup_{i=1}^{\infty} A_{2i}$. Notice that $\bigcup_{i\neq j}(A_i \cap A_j)$ is countable. In the case $d = n = s = 1$, $\rho_1(\varepsilon_h) = h^{-1}$ and $\Omega = (0,1)$, consider the Borel function $f(x,y,z) := [2 - \chi_B(x,y)]\,|z|^p$. We remark that f satisfies only Assumptions 1.1 and 1.2. We have for every $u \in W^{1,p}((0,1))$

$$F_h(u) = \int_0^1 f\big(x, \langle h\, x\rangle, \nabla u(x)\big)\, dx = \begin{cases} \displaystyle\int_0^1 |\nabla u(x)|^p \, dx & \text{if } h \equiv 0 \bmod 2, \\ \displaystyle 2\int_0^1 |\nabla u(x)|^p \, dx & \text{if } h \equiv 1 \bmod 2. \end{cases}$$

Clearly the sequence F_h is not Γ-convergent in $L^p((0,1))$ with respect to the strong topology.

Example 5.11. Let $d = s = 1$, $n = 2$, $\rho_1(\varepsilon_h) = h^{-1}$, $\rho_2(\varepsilon_h) = h^{-2}$ and $\Omega = (0,1)$. Consider the Borel function $f(x, y^1, y^2, z) := [2 - \chi_B(y^1, y^2)]\,|z|^p$, where B is defined as in the former example. Even if f does not depend by x and satisfies Assumptions 1.1 and 1.2, the sequence F_h is not $\Gamma(L^p)$-convergent.

6. Iterated homogenization

The homogenized function f_{hom} can be obtained also by the following iteration:

$$f_{hom}^{[n]}\left(x, y^1, \dots, y^{n-1}, z\right) := \inf_{\phi \in W^{1,p}_{per}(\square, \mathbb{R}^s)} \int_{\square} f\left(x, y^1, \dots, y^n, z + \nabla\phi(y^n)\right) dy^n,$$

$$f_{hom}^{[n-1]}\left(x, y^1, \dots, y^{n-2}, z\right) := \inf_{\phi \in W^{1,p}_{per}(\square, \mathbb{R}^s)} \int_{\square} f_{hom}^{[n]}\big(x, y^1, \dots, y^{n-1}, z + \nabla\phi(y^{n-1})\big)\, dy^{n-1},$$

$$\vdots$$

$$f_{hom}\,(x,z) = f_{hom}^{[1]}\,(x,z) := \inf_{\phi \in W^{1,p}_{per}(\square, \mathbb{R}^s)} \int_{\square} f_{hom}^{[2]}\big(x, y^1, z + \nabla\phi(y^1)\big)\, dy^1.$$

Remark 6.1. (i) The convexity, the p-coerciveness and the p-growth condition on f give the corresponding properties for the function $f_{hom}^{[n]}$. Moreover, $f_{hom}^{[n]}$ is still an admissible integrand. In fact, for every $\delta > 0$ there exist a compact set $X \subseteq \Omega$ with $|\Omega \backslash X| \leq \delta$ and a compact set $Y \subseteq \square$ with $|\square \backslash Y| \leq \delta$, such that the restriction of f to $X \times Y^{n-1} \times \square \times \mathbb{M}^{s\times d}$ is continuous in $(x, y^1, \dots, y^{n-1}, z)$ for *a.e.* $y^n \in \square$. Consequently, following closely [16, Lemma 4.1], it can be proved that $f_{hom}^{[n]}$ is continuous on $X \times Y^{n-1} \times \mathbb{M}^{s\times d}$.
(ii) Clearly, the properties of $f_{hom}^{[n]}$ give the corresponding ones for $f_{hom}^{[n-1]}$ and so on.

We prove only the inequality $f_{hom} \leq f_{hom}^{[1]}$, since the opposite inequality comes directly. Fixed $(x,z) \in \Omega \times \mathbb{M}^{s\times d}$ and $\phi_k \in \Phi_{k,reg}$ (as defined in (13)) for $k = 1, \dots, n-1$, by using a commutation argument as Lemma 5.9, we get

$$\begin{aligned}
&\inf_{\phi_n\in\Phi_{n,reg}}\int_{\Box^n} f\Big(x,y,z+\sum_{k=1}^{n}\nabla_{y^k}\phi_k(y^1,\dots,y^k)\Big)\,dy\\
&=\int_{\Box^{n-1}}\inf_{\phi_n\in\Phi_{n,reg}}\left(\int_{\Box} f\Big(x,y,z+\sum_{k=1}^{n}\nabla_{y^k}\phi_k(y^1,\dots,y^k)\Big)\,dy^n\right)dy^1\dots dy^{n-1}\\
&\le\int_{\Box^{n-1}} f^{[n]}_{hom}\Big(x,y^1,\dots,y^{n-1},z+\sum_{k=1}^{n-1}\nabla_{y^k}\phi_k(y^1,\dots,y^k)\Big)\,dy^1\dots dy^{n-1}.
\end{aligned}$$

By repeating the commutation procedure, we obtain

$$\begin{aligned}
&\inf_{\phi_{n-1}\in\Phi_{n-1,reg}}\int_{\Box^{n-1}} f^{[n]}_{hom}\Big(x,y^1,\dots,y^{n-1},z+\sum_{k=1}^{n-1}\nabla_{y^k}\phi_k(y^1,\dots,y^k)\Big)\,dy^1\dots dy^{n-1}\\
&\le\int_{\Box^{n-2}} f^{[n-1]}_{hom}\Big(x,y^1,\dots,y^{n-2},z+\sum_{k=1}^{n-2}\nabla_{y^k}\phi_k(y^1,\dots,y^k)\Big)\,dy^1\dots dy^{n-2}
\end{aligned}$$

and so on. Then

$$f_{hom}(x,z)\le\inf_{\phi_1\in\Phi_{1,reg}}\dots\inf_{\phi_n\in\Phi_{n,reg}}\int_{\Box^n} f\Big(x,y,z+\sum_{k=1}^{n}\nabla_{y^k}\phi_k(y^1,\dots,y^k)\Big)\,dy\le f^{[1]}_{hom}(x,z).$$

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