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Local Uniform Rotundity in Calderón-Lozanovskiĭ Spaces*

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We find criteria for local uniform rotundity of Calderón-Lozanovskiĭ spaces solving problem XII from [8] and generalizing several theorems, which give only the sufficient (or necessity) conditions (see [18], cf. also [5]). In particular we obtain the respective criteria for Orlicz-Lorentz spaces which has been proved directly in [19] and [4].

Keywords: Köthe space, Calderón-Lozanovskiĭ space, Orlicz-Lorentz space, local uniform rotundity, monotonicity properties

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1. Preliminaries

Throughout this paper R, R_+ and N denote the sets of reals, nonnegative reals and natural numbers, respectively. A triple (T, Σ, μ) stands for a positive, complete and σ-finite measure space and $L^0 = L^0(\mu)$ denotes the space of all (equivalence classes of) Σ-measurable functions $x : T \to R$. For every $x \in L^0$, we denote $\operatorname{supp} x = \{t \in T : x(t) \neq 0\}$ and by $|x|$ the absolute value of x, that is, $|x|(t) = |x(t)|$ for μ-a.e. $t \in T$.

By $E = (E, \leq, \|\cdot\|_E)$ we denote a Köthe space over the measure space (T, Σ, μ), that is, E is a Banach subspace of L^0 which satisfies the following conditions (see [28] and [31]):

(i) if $x \in E$, $y \in L^0$ and $|y| \leq |x|$ (that is, $|y(t)| \leq |x(t)|$ for μ-a.e. $t \in T$), then $y \in E$ and $\|y\|_E \leq \|x\|_E$,
(ii) there exists a function x in E that is positive on the whole T.

In particular, if we consider the Köthe space E over the non-atomic measure space (T, Σ, μ), then we shall say that E is a Köthe function space. If we replace the measure space (T, Σ, μ) by the counting measure space $(N, 2^N, m)$, then we shall say that E is a Köthe sequence space (denoted by e). In the last case the symbol e_i stands for the i-th unit vector. The symbol E_+ stands for the positive cone of E.

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An element $x \in E$ is said to be order continuous if $\|x_n\|_E \to 0$ for any sequence (x_n) in E_+ with $0 \le x_n \le |x|$ and $x_n \to 0$ μ-a.e.. The subspace E_a of all order continuous elements in E is an order ideal of E. A Banach space E is called order continuous ($E \in (\mathbf{OC})$ for short) if $E_a = E$ (see [28] and [31]).

A Banach lattice E with a partial order $\le$ is strictly monotone ($E \in (\mathbf{SM})$ for short) if the conditions $0 \le y \le x \in E$ and $y \ne x$ imply that $\|y\|_E < \|x\|_E$ (see [2]). As usual, E is said to be lower (upper) locally uniformly monotone ($E \in (\mathbf{LLUM})$ ($E \in (\mathbf{ULUM})$) for short), see [21], whenever for any $x \in E_+$ with $\|x\|_E = 1$ and any $\varepsilon \in (0,1)$ (resp. $\varepsilon > 0$) there is $\delta = \delta(x,\varepsilon) \in (0,1)$ (resp. $\delta = \delta(x,\varepsilon) > 0$) such that the conditions $0 \le y \le x$ (resp. $y \ge 0$) and $\|y\|_E \ge \varepsilon$ imply $\|x-y\|_E \le 1-\delta$ (resp. $\|x+y\|_E \ge 1+\delta$).

It is useful to formulate the local uniform monotonicity properties sequentially. Clearly, $E \in (\mathbf{LLUM})$ (resp. $E \in (\mathbf{ULUM})$) if and only if for any $x \in E_+$, $x \ne 0$, and each sequence (x_n) in E_+ such that $x_n \le x$ (resp. $x \le x_n$) and $\|x_n\|_E \to \|x\|_E$, there holds $\|x_n - x\|_E \to 0$.

For any Banach space X we denote by $B(X)$ its closed unit ball and by $S(X)$ - the unit sphere of X. Recall that X is said to be rotund ($X \in (\mathbf{R})$) if for every $x, y \in S(X)$ with $x \ne y$ we have $\|x+y\| < 2$. A Banach space X is said to be locally uniformly rotund ($X \in (\mathbf{LUR})$) if for each $x \in B(X)$ and $\varepsilon > 0$ there is $\delta = \delta(x,\varepsilon) > 0$ such that for any $y \in B(X)$ the inequality $\|x-y\| \ge \varepsilon$ implies that $\|x+y\|_E \le 2(1-\delta)$. This property has been intensively investigated in many classes of Banach spaces (see [4], [10], [18], [19], [38]).

In the whole paper φ denotes an Orlicz function, that is, $\varphi : R \to [0,\infty]$, it is convex, even, vanishing and continuous at zero, left continuous on $[0,\infty)$ and not identically equal to zero. Denote

$$a_\varphi = \sup\{u \ge 0 : \varphi(u) = 0\} \quad \text{and} \quad b_\varphi = \sup\{u \ge 0 : \varphi(u) < \infty\}.$$

We write $\varphi > 0$ when $a_\varphi = 0$ and $\varphi < \infty$ if $b_\varphi = \infty$. Let φ_r be the restriction of φ to the set G_φ, where

$$G_\varphi = \begin{cases} [a_\varphi, b_\varphi] & \text{if } \varphi(b_\varphi) < \infty, \\ [a_\varphi, b_\varphi) & \text{otherwise.} \end{cases}$$

The function φ is said to be strictly convex on the interval $[a,b]$, where $0 \le a < b < \infty$ if $\varphi((u+v)/2) < (\varphi(u) + \varphi(v))/2$ for all $u, v \in [a,b]$ with $u \ne v$.

Given any Orlicz function φ, we define on L^0 a convex modular ϱ (see [37]) by

$$\varrho(x) = \begin{cases} \|\varphi \circ x\|_E & \text{if } \varphi \circ x \in E, \\ \infty & \text{otherwise,} \end{cases}$$

where $(\varphi \circ x)(t) = \varphi(x(t))$, $t \in T$, and the Calderón-Lozanovskiĭ space

$$E_\varphi = \{x \in L^0 : \ \varphi \circ \lambda x \in E \text{ for some } \lambda > 0\}$$

(see [5], [17] and [36]), which becomes a normed space under the Luxemburg norm

$$\|x\|_\varphi = \inf\{\lambda > 0 : \varrho(x/\lambda) \le 1\}.$$

Considering the space E_φ we shall assume in the whole paper that E has the Fatou property ($E \in (\mathbf{FP})$ for short), that is, for any $x \in L^0$ and $(x_n)_{n=1}^\infty$ in E_+ such that $x_n \nearrow x$ μ-a.e. and $\sup_n \|x_n\|_E < \infty$, we have $x \in E$ and $\|x\|_E = \lim_n \|x_n\|_E$ (see [28] and [31]). Then for any Orlicz function φ the modular ϱ is left continuous, that is, $\sup\{\varrho(\lambda x) : |\lambda| \le \lambda_0\} = \varrho(\lambda_0 x)$ for any $\lambda_0 > 0$. We also have $\varrho(x) \le \|x\|_\varphi \le 1$ whenever $\|x\|_\varphi \le 1$ or $\varrho(x) \le 1$ and $1 \le \|x\|_\varphi \le \varrho(x)$ whenever $\|x\|_\varphi > 1$ or $\varrho(x) \ge 1$ (see [6]). Consequently $E_\varphi \in (\mathbf{FP})$ (see [13] and [14]), whence E_φ is a Banach space (see [34]). For the theory of Calderón-Lozanovskiĭ spaces we refer to [3], [5], [12], [13], [14], [17], [18], [20], [23], [29], [32], [33], [35], [36] and [39].

If $E = L^1$ ($e = l^1$), then E_φ (e_φ) is the Orlicz function (sequence) space equipped with the Luxemburg norm. If E (e) is a Lorentz function (sequence) space Λ_ω (λ_ω) (see page 406), then E_φ (e_φ) is the corresponding Orlicz-Lorentz function (sequence) space $(\Lambda_\omega)_\varphi = \Lambda_{\varphi,\omega}$ ($(\lambda_\omega)_\varphi = \lambda_{\varphi,\omega}$) equipped with the Luxemburg norm (see [4], [17], [19], [26], [27] and [29]).

We say an Orlicz function φ satisfies condition $\Delta_2(0)$ ($\varphi \in \Delta_2(0)$ for short) if there exist $K > 0$ and $u_0 > 0$ such that $\varphi(u_0) > 0$ and the inequality $\varphi(2u) \le K\varphi(u)$ holds for all $u \in [0, u_0]$. We say an Orlicz function φ satisfies condition $\Delta_2(\infty)$ ($\varphi \in \Delta_2(\infty)$ for short) if there exist $K > 0$, $u_0 > 0$ such that $\varphi(u_0) < \infty$ and the inequality $\varphi(2u) \le K\varphi(u)$ holds for all $u \ge u_0$. If there exists $K > 0$ such that $\varphi(2u) \le K\varphi(u)$ for all $u \ge 0$, then we say that φ satisfies condition $\Delta_2(R_+)$ ($\varphi \in \Delta_2(R_+)$ for short).

For a Köthe space E and an Orlicz function φ we say that φ satisfies condition Δ_2^E ($\varphi \in \Delta_2^E$ for short) if:

1) $\varphi \in \Delta_2(0)$ whenever $E \hookrightarrow L^\infty$,
2) $\varphi \in \Delta_2(\infty)$ whenever $L^\infty \hookrightarrow E$,
3) $\varphi \in \Delta_2(R_+)$ whenever neither $L^\infty \hookrightarrow E$ nor $E \hookrightarrow L^\infty$,

where the symbol $E \hookrightarrow F$ stands for the continuous embedding of E into F (see [5] and [17]). Clearly, if $\varphi \in \Delta_2(0)$, then $\varphi > 0$ and if $\varphi \in \Delta_2(\infty)$, then $\varphi < \infty$.

It is easy to show that if E is a Köthe function space such that $E \in (\mathbf{FP})$ and $\operatorname{supp} E_a = T$, then $E \not\subset L^\infty$.

If $e \hookrightarrow l^\infty$ and $\varphi(b_\varphi)\inf_i \|e_i\|_e = 1$, we define a new function ψ by the formula

$$\psi(u) = \begin{cases} \varphi(u) & \text{if } 0 \le u \le b_\varphi, \\ u + k & \text{for } u > b_\varphi, \end{cases}$$

where $k = 1/\inf_i \|e_i\|_e - b_\varphi$. Notice that e_φ and e_ψ are isomorphically isometric. However, ψ is convex on $[0, b_\varphi]$, nondecreasing on R_+ and not necessarily convex on the whole R_+. In the whole paper, if $e \hookrightarrow l^\infty$ and $\varphi(b_\varphi)\inf_i \|e_i\|_e = 1$, we always consider ψ and e_ψ in place of φ and e_φ, respectively.

Sufficient conditions for various properties of Calderón-Lozanovskiĭ spaces have been presented in [5], [13], [14], [17] and [18]. However, in those papers necessity of some among those conditions was only proved and it was concluded that some of sufficient conditions are not necessary. It has been shown in [5, Remark 3] that geometry of Calderón-Lozanovskiĭ space E_φ can be "good" even if geometry neither of E nor of φ is "good". For example, there exists a couple of E and φ such that φ is not strictly convex and E is not rotund, but E_φ is locally uniformly rotund. On the base of this phenomena, we shall find criteria for local uniform rotundity of Calderón-Lozanovskiĭ spaces (it refers to problem XII posed in [8]). In such a way it has been received an essential generalization of the criteria from the papers concerning Orlicz-Lorentz spaces ([4] and [19]) and some improvements of theorems on Calderón-Lozanovskiĭ spaces ([18], cf. also [5]). It is worth mentioning that problem XII from [8] has been solved for rotundity and uniform rotundity in [29] and for extreme and SU points in [20]. We shall take an inspiration and a general idea from [29].

2. Introductory results

We start with the fundamental lemma.

Lemma 2.1. *Suppose that E is a Köthe space and φ is an Orlicz function. Then for any $x \in E_\varphi$ and for any sequence (x_n) in E_φ we get:*

(*i*) *If $\varrho(x) = 1$, then $\|x\|_\varphi = 1$.*
(*ii*) *If $\varrho(x_n) \to 1$, then $\|x_n\|_\varphi \to 1$.*
(*iii*) *If $\|x_n\|_\varphi \to 0$, then $\varrho(x_n) \to 0$.*

Lemma 2.2 (see [5], [13], [17] and [29]). *Suppose that E is a Köthe function space such that $\operatorname{supp} E_a = T$ and φ is an Orlicz function. Then the following assertions are true:*

(*i*) *For any $x \in E_\varphi$ the equality $\|x\|_\varphi = 1$ implies that $\varrho(x) = 1$ if and only if $\varphi \in \Delta_2^E$ and $\varphi < \infty$.*
(*ii*) *For any sequence (x_n) in E_φ we have $\varrho(x_n) \to 1$ whenever $\|x_n\|_\varphi \to 1$ if and only if $\varphi \in \Delta_2^E$ and $\varphi < \infty$.*
(*iii*) *For any sequence (x_n) in E_φ we have $\|x_n\|_\varphi \to 0$ whenever $\varrho(x_n) \to 0$ if and only if $\varphi \in \Delta_2^E$ and $\varphi > 0$.*

Lemma 2.3 (see [5], [14] and [29]). *Suppose that e is a Köthe sequence space with $e \subset c_0(\|e_n\|_e)$ and φ is an Orlicz function such that $\varphi > 0$.*

(*i*) *For any $x \in e_\varphi$ the equality $\|x\|_\varphi = 1$ implies that $\varrho(x) = 1$ if and only if $\varphi \in \Delta_2^e$ and $\varphi(b_\varphi)\inf_i \|e_i\|_e \geq 1$.*
(*ii*) *For any sequence (x_n) in e_φ we have $\varrho(x_n) \to 1$ whenever $\|x_n\|_\varphi \to 1$ if and only if $\varphi \in \Delta_2^e$ and $\varphi(b_\varphi)\inf_i \|e_i\|_e \geq 1$.*
(*iii*) *For any sequence (x_n) in e_φ we have $\|x_n\|_\varphi \to 0$ whenever $\varrho(x_n) \to 0$ if and only if $\varphi \in \Delta_2^e$.*

In investigations on local uniform rotundity of Calderón-Lozanovskiĭ spaces E_φ the essential role play its restriction to couples of comparable elements from the positive cone, which leads to the notions of **LLUM** and **ULUM** (see [18]). These properties have been introduced in [21] and investigated in Calderón-Lozanovskiĭ spaces in [12], [14] and [23]. We shall present criteria for **LLUM** and **ULUM** of these spaces basing on several partial results from those papers. Althogh we often apply similar technics to those elaborated already we present the whole proof for the sake of completness.

Proposition 2.4.

(*i*) *Let E be a Köthe function space. Then $E_\varphi \in$ (**LLUM**) if and only if $E \in$ (**LLUM**),*

$\varphi \in \Delta_2^E$, $\varphi > 0$ *and* $\varphi < \infty$.

(ii) *Let* e *be a Köthe sequence space. Then* $e_\varphi \in$ (**LLUM**) *if and only if* $e \in$ (**LLUM**), $\varphi \in \Delta_2^e$, $\varphi > 0$ *and* $\varphi(b_\varphi) \inf_{i\in N} \|e_i\|_e \geq 1$.

Proof. (i). *Sufficiency.* Assume that the assumptions are satisfied, $x \in E_\varphi$ and (x_n) is a sequence in E_φ such that $0 \leq x_n \leq x$ for any $n \in N$, $\|x\|_\varphi = 1$ and $\|x_n\|_\varphi \to 1$. By $\varphi \in \Delta_2^E$ and $\varphi < \infty$, we get $\varrho(x) = 1$ and $\varrho(x_n) \to 1$ (see Lemma 2.2 (i) and (ii)). Since $E \in$ (**LLUM**), the superadditivity of φ on R_+ implies that $\varrho(x - x_n) \to 0$, whence, by the fact that $\varphi > 0$, Lemma 2.2(iii) yields $\|x - x_n\|_\varphi \to 0$. Thus $E_\varphi \in$ (**LLUM**).

Necessity. Assume that $E_\varphi \in$ (**LLUM**). Let $x \in (S(E))_+$ and (x_n) be a sequence in E such that $0 \leq x_n \leq x$ for any $n \in N$ and $\|x_n\|_E \to 1$. By Lemma 2.5 and Proposition 2.1(ii) in [29], we have $x \leq \varphi(b_\varphi)\chi_T$ when $\varphi(b_\varphi) < \infty$. Denote $y = \varphi_r^{-1} \circ x$ and $y_n = \varphi_r^{-1} \circ x_n$, where φ_r is defined on page 396. By Lemma 2.1, we have $y \in (S(E_\varphi))_+$, $0 \leq y_n \leq y$ and $\|y_n\|_\varphi \to 1$. Since E_φ is **LLUM**, we get $\|y - y_n\|_\varphi \to 0$. Then we find a subsequence (y_{n_k}) of (y_n) such that $y_{n_k}(t) \to y(t)$ for μ-a.e. $t \in T$ (see [28, p. 138]). Then $\varphi(y_{n_k}(t)) \to \varphi(y(t))$ for μ-a.e. $t \in T$. Since $0 \leq \varphi \circ y - \varphi \circ y_{n_k} \leq x$ and $x \in E_a$ (see Lemma 6 and 7 in [23]), we get $\|x - x_{n_k}\|_E = \|\varphi \circ y - \varphi \circ y_{n_k}\|_E \to 0$. By the double extract subsequence theorem, we obtain $\|x - x_n\|_E \to 0$, so $E \in$ (**LLUM**).

Suppose now that $b_\varphi < \infty$. Since **LLUM** $\Rightarrow$ **OC** (see Proposition 2.1 in [11]), by Lemma 2.5 and Proposition 2.1(i) in [29], we get $\varphi(b_\varphi) = \infty$. Let (A_n) be a sequence of sets such that $A_n \in \Sigma$, $A_n \cap A_m = \emptyset$ for $n \neq m$ and $0 < \|\chi_{A_n}\|_E \leq 1/(2^n\varphi((1-1/2n)b_\varphi))$ for $n \in N$. Denoting $x = \sum_{n=2}^{\infty}(1 - 1/2n)b_\varphi\chi_{A_n}$ and $y = (1 - 1/2)b_\varphi\chi_{A_1}$, we have $0 \leq x \leq x + y$, $x \neq x + y$ and $\|x\|_\varphi = \|x + y\|_\varphi = 1$, so $E_\varphi \notin$ (**SM**).

If $a_\varphi > 0$, then, by Lemma 2.5 in [29], E_φ is not (**SM**). If $\varphi \notin \Delta_2^E$, then E_φ contains an order isomorphically isometric copy of l^∞ (see [17]), so $E_\varphi \notin$ (**SM**).

Part (ii) we prove analogously as (i), applying Lemma 2.9 from [29] and Lemma 2.4 from [14].

Proposition 2.5.

(i) *Let* E *be a Köthe function space such that* $E \in$ (**OC**). *Then* E_φ *is* **ULUM** *if and only if* $E \in$ (**ULUM**), $\varphi \in \Delta_2^E$, $\varphi > 0$ *and* $\varphi < \infty$.

(ii) *Let* e *be a Köthe sequence space such that* $e \in$ (**OC**). *Then* e_φ *is* **ULUM** *if and only if* $e \in$ (**ULUM**), $\varphi \in \Delta_2^e$, $\varphi > 0$ *and* $\varphi(b_\varphi) \inf_{i\in N} \|e_i\|_e \geq 1$.

Proof. (i). *Sufficiency.* Let x and (x_n) in E_φ be such that $0 \leq x \leq x_n$ for each $n \in N$, $\|x\|_\varphi=1$ and $\|x_n\|_\varphi \to 1$. Then, by Lemma 2.2 ((i) and (ii)), we have $\varrho(x) = 1$ and $\varrho(x_n) \to 1$. Since $E \in$ (**ULUM**), the superadditivity of φ on R_+ implies that $\varrho(x - x_n) \to 0$, whence, by Lemma 2.2(iii), we get $\|x - x_n\|_\varphi \to 0$, that is, $E_\varphi \in$ (**ULUM**).

Necessity. Since $E \in$ (**OC**), we obtain that $\varphi \in \Delta_2^E$, $\varphi > 0$ and $\varphi < \infty$ analogously as in the proof of Proposition 2.4. We show the implication $E_\varphi \in$ (**ULUM**) $\Rightarrow E \in$ (**ULUM**). Let $x \in E$ and (x_n) be a sequence in E such that $0 \leq x \leq x_n$ for any $n \in N$ and $\|x_n\|_E \to \|x\|_E = 1$. Denoting $y = \varphi_r^{-1} \circ x$ and $y_n = \varphi_r^{-1} \circ x_n$, we have $0 \leq y \leq y_n$ and, by Lemma 2.1, $\|y_n\|_\varphi \to \|y\|_\varphi = 1$. Since E_φ is **ULUM**, we get $\|y_n - y\|_\varphi \to 0$. Proceeding in the same way as in the proof of Proposition 4 in [23], we obtain $\|x - x_n\|_E =$

$\|\varphi \circ y - \varphi \circ y_n\|_E \to 0$, so $E \in (\mathbf{ULUM})$.

Part (ii) we prove analogously as (i).

In the next theorem we shall consider a Köthe function spaces E over the Lebesgue measure space $([0,\alpha),\Sigma,\mu)$ with $0 < \alpha \le \infty$ and μ being the Lebesgue measure. Recall that a Köthe space E is called a symmetric space if E is rearrangement invariant in the sense that if $x \in E$ $y \in L^0$ and $x^* = y^*$, then $y \in E$ and $\|x\|_E = \|y\|_E$ (see [9]). Here, x^* denotes the nonincreasing rearrangement of x given by

$$x^*(t) = \inf\{s \ge 0 : d_x(s) \le t\},$$

where d_x is the distribution function defined by

$$d_x(t) = \mu(\{s \in T : |x(s)| > t\}), \quad t \ge 0.$$

For basic properties of symmetric spaces and rearrangements, we refer to [1], [30] and [31].

In Section 4 we shall apply the following two theorems.

Theorem 2.6. *Suppose that E is a symmetric function space. Then $E \in (\mathbf{LLUM})$ if and only if $E \in (\mathbf{SM})$ and $E \in (\mathbf{OC})$.*

By Proposition 2.1 in [11], we need to prove only sufficiency. It is known that if E is a separable symmetric space in which an equivalent symmetric norm $\|\cdot\|_o$ exists which is **LLUM**, then $E \in (\mathbf{LLUM})$ if and only if $E \in (\mathbf{SM})$ (see [18, Theorem 4]). Consequently, applying Theorem 4.8 in [10], one can get the proof of sufficiency. We also present an independent proof.

Let $0 \le x_n \le x$ and $\|x_n\|_E \to \|x\|_E = 1$. Define

$$A_n^k = \{t \in \operatorname{supp} x : x_n(t) < (1 - 1/k)x(t)\}$$

for each $n, k \in N$. We claim that for each $k \in N$

$$x\chi_{A_n^k} \to 0 \text{ globally in measure as } n \to \infty. \tag{1}$$

Suppose that this is not the case, that is, there is a number $k \in N$ such that passing to a subsequence and relabelling if necessary, one gets that there are positive numbers ε and δ such that $\mu(B_n) > \varepsilon$ for any $n \in N$, where $B_n = \{t \in [0,\alpha) : x\chi_{A_n^k}(t) > \delta\}$. We shall prove that

$$a = \liminf_{n\to\infty} \|x - \tfrac{\delta}{k}\chi_{B_n}\|_E < \|x\|_E. \tag{2}$$

Denoting $y_n = x - \frac{\delta}{k}\chi_{B_n}$, we have $0 \le y_n \le x$. Hence $y_n^* \le x^*$. Applying Helly's Theorem and passing to a subsequence and relabelling if necessary, we may assume that the sequence (y_n^*) converges to some non-increasing function y almost everywhere. Then $y \le x^*$ and consequently $\|y_n^* - y\|_E \to 0$, because $E \in (\mathbf{OC})$. Suppose now that condition (2) does not hold. Then $\|y_n^*\|_E = \|y_n\|_E \to \|x^*\|_E = \|x\|_E$. Since $\|y_n^*\|_E \to \|y\|_E$ and $y \le x^*$ we conclude that $y = x^*$, because $E \in (\mathbf{SM})$. Thus $\|y_n^* - x^*\|_E \to 0$. Proceeding analogously as in proof of implication $(iii) \Rightarrow (ii)$ of Theorem 3.2 in [9], we get that

$y_n \to x$ globally in measure. But $y_n - x = -\frac{\delta}{k}\chi_{B_n}$ and $\mu(B_n) > \varepsilon$. This contradiction proves (2). Hence, taking an appropriate subsequence and denoting $(A_n^k)^c = [0,\alpha)\backslash A_n^k$, we get

$$\begin{aligned}\|x_n\|_E = \|x_n\chi_{A_n^k} + x_n\chi_{(A_n^k)^c}\|_E &\leq \|(1-1/k)x\chi_{A_n^k} + x\chi_{(A_n^k)^c}\|_E \\ &\leq \|x - \tfrac{\delta}{k}\chi_{B_n}\|_E \to a < \|x\|_E.\end{aligned}$$

This contradiction together with the fact that $\|x_n\|_E \to \|x\|_E$ finishes the proof of the claim (1). Since $x\chi_{A_n^k} \leq x$, the order continuity of E implies that $\|x\chi_{A_n^k}\|_E \to 0$ as $n \to \infty$ for each $k \in N$. Furthermore, the inequality $x_n\chi_{(A_n^k)^c} \geq (1-1/k)x\chi_{(A_n^k)^c}$ yields that $(x - x_n)\chi_{(A_n^k)^c} \leq (1/k)x\chi_{(A_n^k)^c}$ and, in consequence, $\|(x-x_n)\chi_{(A_n^k)^c}\|_E \leq (1/k)$. Let $\varepsilon > 0$ and $k_0 > 2/\varepsilon$. Taking n_0 such that $\|x\chi_{A_n^{k_0}}\|_E < \varepsilon/2$ for $n \geq n_0$, we get

$$\|x - x_n\|_E \leq \|(x-x_n)\chi_{A_n^{k_0}}\|_E + \|(x-x_n)\chi_{(A_n^{k_0})^c}\|_E < \varepsilon$$

for each $n \geq n_0$.

In the case of Köthe sequence spaces it is easy to get more general result. Namely,

Theorem 2.7. *For any Köthe sequence space e the following conditions are equivalent:*

(i) *The space e is strictly monotone and order continuous.*

(ii) *The space e is lower locally uniformly monotone.*

Proof. Since **LLUM** $\Rightarrow$ **OC** (see Proposition 2.1 in [11]), we need only to prove the implication $(i) \Rightarrow (ii)$. Let $x \in e_+$ and (x_n) be a sequence in e such that $0 \leq x_n \leq x$ for any $n \in N$ and $\|x_n\|_e \to \|x\|_e$. Consequently, all sequences of coordinates $(x_n(i))_{n=1}^{\infty}$ are bounded for $i = 1, 2, \ldots$. Using the diagonal method we can find a subsequence (x_{n_k}) of (x_n) and $y \in l^0$ such that $x_{n_k}(i) \to y(i)$ for all $i \in N$. We have $0 \leq y \leq x$. Since $e \in (\mathbf{SM})$, so $y = x$. Moreover, $0 \leq x - x_{n_k} \leq x$ and $x - x_{n_k} \to 0$ coordinatewise. By $e \in (\mathbf{OC})$, we get $\|x - x_{n_k}\|_e \to 0$. In virtue of the double extract subsequence theorem we finish the proof.

3. Main results

Set $r \vee s = \max\{r,s\}$ and $r \wedge s = \min\{r,s\}$ for any $r, s \in R$.

Theorem 3.1. *Let E be a Köthe function space. Then $E_\varphi \in$ (**LUR**) if and only if:*

(a) *$E \in$ (**LLUM**), $\varphi > 0$, $\varphi < \infty$, $\varphi \in \Delta_2^E$ and*

(b) *for each $u \in (S(E))_+$ and any $\varepsilon > 0$ there is $\delta = \delta(u,\varepsilon) \in (0,1)$ such that for every $v \in (S(E))_+$ with $\|u - v\|_E \geq \varepsilon$ one has:*

$$\|u + v(1-w)\|_E \leq 2(1-\delta) \quad \text{or} \quad \|uw\|_E \geq \delta,$$

where $x = \varphi_r^{-1} \circ u$, $y = \varphi_r^{-1} \circ v$,

$$w(t) = \begin{cases} 1 - \frac{2\varphi((x(t)+y(t))/2)}{\varphi(x(t))+\varphi(y(t))} & \text{if } t \in B_\delta(u,v) \\ 0 & \text{otherwise} \end{cases}$$

and

$$B_\delta(u,v) = \{t \in \operatorname{supp} u \cup \operatorname{supp} v : u(t) \wedge v(t) \leq (1-\delta)(u(t) \vee v(t))\}.$$

The idea of the approach of Theorems 3.1 and 3.2: Note that if $E \notin (\mathbf{LUR})$ then $S(E)$ contains "almost flat areas" denoted by $\mathrm{Flat}(S(E))$. Then $E_\varphi \in (\mathbf{LUR})$ if and only if $E_\varphi \in (\mathbf{LLUM})$ (condition (a)) and either $E \in (\mathbf{LUR})$ or φ improves (brings into relief) the set $\mathrm{Flat}(S(E))$ because of its appropriate convexity on the set $\varphi_r^{-1}(\mathrm{Flat}(S(E))$ (condition (b)).

Proof. *Sufficiency.* Applying Theorem 3 in [18] we only need to prove that $(E_\varphi)_+ \in (\mathbf{LUR})$. Let $x \in (S(E_\varphi))_+$ and $\varepsilon > 0$. Take arbitrary $y \in (S(E_\varphi))_+$ with $\|x-y\|_\varphi \geq \varepsilon$. Denoting $\varphi \circ x = u$ and $\varphi \circ y = v$ we get $u, v \in (S(E))_+$ (see Lemma 2.2(i)). By Lemma 2.2(iii) we find $\eta = \eta(\varepsilon) > 0$ such that $\|\varphi \circ (x-y)\|_E \geq \eta$. By superadditivity of the function φ on R_+, we get

$$\|u-v\|_E = \|\varphi \circ x - \varphi \circ y\|_E \geq \|\varphi \circ (x-y)\|_E \geq \eta.$$

Applying assumption (b) with $\delta = \delta(\varphi \circ x, \eta(\varepsilon)) \in (0,1)$ we need to consider two cases.

I. Suppose $\|uw\|_E \geq \delta$. Then, using the definition of the function w, we get

$$\begin{aligned} \varphi \circ \left(\frac{x+y}{2}\right) &\leq \frac{1}{2}(\varphi \circ x + \varphi \circ y) - \frac{w}{2}(\varphi \circ x + \varphi \circ y)\chi_{B_\delta(u,v)} \\ &\leq \frac{1}{2}(\varphi \circ x - w\,\varphi \circ x) + \frac{1}{2}\,\varphi \circ y. \end{aligned}$$

Since $E \in (\mathbf{LLUM})$, we conclude that $\|\varphi \circ ((x+y)/2)\|_E \leq (1-p)/2 + 1/2 = 1 - p/2$, where $p = p(\varphi \circ x, \delta) > 0$ is from the definition of lower local uniform monotonicity. Finally, it follows from Lemma 2.2(ii) that $\|(x+y)/2\|_\varphi \leq 1 - r_1$, where $r_1 = r_1(p/2) > 0$.

II. If $\|u + v(1-w)\|_E \leq 2(1-\delta)$, then $\varrho\,((x+y)/2) \leq 1-\delta$, whence $\|(x+y)/2\|_\varphi \leq 1 - r_2$, where $r_2 = r_2(\delta)$ depends only on δ (see Lemma 2.2(ii)).

Thus $\|(x+y)/2\|_\varphi \leq 1 - r$ with $r = \min\{r_1, r_2\}$.

Necessity. If $E_\varphi \in (\mathbf{LUR})$, then $E_\varphi \in (\mathbf{LLUM})$ (see [18, Theorem 1]). Hence, by Proposition 2.4, $E \in (\mathbf{LLUM})$, $\varphi < \infty$, $\varphi > 0$ and $\varphi \in \Delta_2^E$.

Suppose now that condition (b) is not satisfied. Then there exist $u \in (S(E))_+$, $\varepsilon > 0$ and a sequence $(v_n)_{n=1}^\infty$ in $S(E_+)$ such that, taking $x = \varphi_r^{-1} \circ u$, $y_n = \varphi_r^{-1} \circ v_n$, we have

$$\|u - v_n\|_E \geq \varepsilon, \quad \|u + v_n(1-w_n)\|_E > 2(1-1/n), \quad \text{and} \quad \|uw_n\|_E < 1/n \tag{3}$$

for every $n \in N$, where

$$w_n(t) = \begin{cases} 1 - \frac{2\varphi((x(t)+y_n(t))/2)}{\varphi(x(t)) + \varphi(y_n(t))} & \text{if } t \in B_n \\ 0 & \text{otherwise} \end{cases}$$

and

$$B_n = \{t \in \operatorname{supp} u \cup \operatorname{supp} v_n : u(t) \wedge v_n(t) \leq (1-1/n)(u(t) \vee v_n(t))\}.$$

First we claim that

$$\|\varphi \circ ((x+y_n)/2)\|_E \to 1 \quad \text{as } n \to \infty. \tag{4}$$

Since $u(t) \wedge v_n(t) > (1-1/n)(u(t) \vee v_n(t))$ for any $t \in (\operatorname{supp} u \cup \operatorname{supp} v_n) \backslash B_n$, denoting $C_n = (\operatorname{supp} u \cup \operatorname{supp} v_n) \backslash B_n$, we conclude that $(u - v_n)\chi_{C_n} \to 0$ pointwisely. Then

$(x-y_n)\chi_{C_n} \to 0$ pointwisely, because φ_r^{-1} is subadditive and continuous. Consequently $(\varphi\circ(\frac{x+y_n}{2}) - \frac{\varphi\circ x+\varphi\circ y_n}{2})\chi_{C_n} \to 0$ pointwisely. Moreover,

$$\left|\varphi\circ\left(\frac{x+y_n}{2}\right)\chi_{C_n} - \left(\frac{\varphi\circ x+\varphi\circ y_n}{2}\right)\chi_{C_n}\right| \le 3\varphi\circ x\chi_{C_n} \le 3u.$$

Since $E \in (\mathbf{OC})$, we conclude that

$$\left\|\varphi\circ\left(\frac{x+y_n}{2}\right)\chi_{C_n} - \left(\frac{\varphi\circ x+\varphi\circ y_n}{2}\right)\chi_{C_n}\right\|_E \to 0.$$

Then, passing to a subsequence, if necessary, and applying (3) we get

$$\begin{aligned}
\left\|\varphi\circ\left(\frac{x+y_n}{2}\right)\right\|_E &= \left\|\varphi\circ\left(\frac{x+y_n}{2}\right)\chi_{B_n} + \varphi\circ\left(\frac{x+y_n}{2}\right)\chi_{C_n}\right\|_E \\
&\ge \left\|\varphi\circ\left(\frac{x+y_n}{2}\right)\chi_{B_n} + \frac{\varphi\circ x+\varphi\circ y_n}{2}\chi_{C_n}\right\|_E - \frac{1}{n} \\
&= \left\|\frac{\varphi\circ x+\varphi\circ y_n}{2}(1-w_n)\chi_{B_n} + \frac{\varphi\circ x+\varphi\circ y_n}{2}\chi_{C_n}\right\|_E - \frac{1}{n} \\
&\ge 1-\frac{2}{n}-\frac{1}{n} = 1-\frac{3}{n}.
\end{aligned}$$

It proves condition (4) and consequently, by Lemma 2.1, $\|(x+y_n)/2\|_\varphi \to 1$ as $n\to\infty$. Clearly, since $\|\varphi\circ x\|_E = \|\varphi\circ y_n\|_E = 1$, so $\|x\|_\varphi = \|y_n\|_\varphi = 1$ $(n\in N)$. Proceeding in the same way as in the proof of Theorem 2.11 in [29], we find $\eta>0$ such that $\|x-y_n\|_\varphi \ge \eta$ for infinitely many $n\in N$, i.e. $E_\varphi \notin$ **LUR**.

Theorem 3.2. *Let e be a Köthe sequence space. Then $e_\varphi \in$ (**LUR**) if and only if:*

(a) $e \in$ (**LLUM**), $\varphi>0$, $\varphi(b_\varphi)\inf_{i\in N}\|e_i\|_e \ge 1$, $\varphi\in\Delta_2^e$ *and*

(b) *for each $u\in(S(e))_+$ and any $\varepsilon>0$ there is $\delta=\delta(u,\varepsilon)\in(0,1)$ such that for every $v\in(S(e))_+$ with $\|u-v\|_e\ge\varepsilon$ one has:*

$$\|u+v(1-w)\|_e \le 2(1-\delta) \quad \text{or} \quad \|uw\|_e \ge \delta,$$

where $x=\varphi_r^{-1}\circ u$, $y=\varphi_r^{-1}\circ v$,

$$w(i) = \begin{cases} 1-\frac{2\varphi((x(i)+y(i))/2)}{\varphi(x(i))+\varphi(y(i))} & \text{if } i\in B_\delta(u,v) \\ 0 & \text{otherwise} \end{cases}$$

and

$$B_\delta(u,v) = \{i\in \operatorname{supp} u\cup\operatorname{supp} v : u(i)\wedge v(i) \le (1-\delta)(u(i)\vee v(i))\}.$$

Proof. We proceed analogously as in the proof of Theorem 3.1. However, in the proof of the necessity, to show that there is a number $\eta>0$ such that $\|x-y_n\|_\varphi\ge\eta$ for infinitely many $n\in N$, we need to proceed as in the proof Theorem 2.12 in [29]. Note that if $e_\varphi\in$ (**LUR**), then $e_\varphi\in$ (**LLUM**) and $e_\varphi\in$ (**ULUM**). Consequently, by Propositions 2.4(ii) and 2.5(ii), we conclude that $e\in$ (**LLUM**) and $e\in$ (**ULUM**).

Then we may imitate the proof of Theorem 2.12 in [29] (necessity) replacing $e\in$ (**UM**) by $e\in$ (**LLUM**) or by $e\in$ (**ULUM**). Namely, since $e\in$ (**LLUM**), so for any $x\in e_+$ and each $q\in(0,1)$ there is $p=p(x,q)\in(0,1)$ such that for each $y\in e_+$ with $y\le x$ the condition $\|x\|_e - \|y\|_e < p$ implies that $\|x-y\|_e < q$. Similarly, from $e \in$ (**ULUM**), we conclude that for any $x \in e_+$ and every $q \in (0,1)$ there is $p = p(x,q) \in (0,1)$ such that for each $y \in e_+$ with $y \geq x$ the condition $\|y\|_e - \|x\|_e < p$ implies that $\|x-y\|_e < q$.

Then we follow as in the proofs of Theorems 2.11 and 2.12 in [29] considering two cases:

1. $\|(\varphi \circ x - \varphi \circ y_n)\chi_{A_n}\|_e \geq \varepsilon/2$ for infinitely many $n \in N$ or
2. $\|(\varphi \circ x - \varphi \circ y_n)\chi_{N\setminus A_n}\|_e \geq \varepsilon/2$ for infinitely many $n \in N$,

where $A_n = \{i \in N : \varphi(x(i)) \geq \varphi(y_n(i))\}$. In Case 1 we apply the fact that $e \in$ (**LLUM**), and in Case 2 we use $e \in$ (**ULUM**), respectively. Notice also that we get in this way that the number η depends only on x and ε and η does not depend on the sequence (y_n).

From the Theorems 3.1 and 3.2 one can get immediately.

Corollary 3.3.

(i) *Suppose that E is a Köthe function space. If $E \in$ (**LUR**), $\varphi > 0$, $\varphi < \infty$ and $\varphi \in \Delta_2^E$, then $E_\varphi \in$ (**LUR**).*

(ii) *Suppose that e is a Köthe sequence space. If e $\in$ (**LUR**), $\varphi > 0$, $\varphi(b_\varphi)\inf_{i\in N}\|e_i\|_e \geq 1$ and $\varphi \in \Delta_2^e$, then $e_\varphi \in$ (**LUR**).*

It has been shown in [18] that $E_\varphi \in$ (**LUR**) whenever $E \in$ (**UM**), $\varphi \in \Delta_2^E$ and φ is a strictly convex function. We shall show below that one can replace the assumption that $E \in$ (**UM**) by the two essentially weaker ones that $E \in$ (**LLUM**) and $E \in$ (**ULUM**). We shall present the example of the space that is both **ULUM** and **LLUM** and is not **UM** in Section 4 (see Example 4.3, page 407).

Corollary 3.4.

(i) *Suppose that E is a Köthe function space. If $E \in$ (**LLUM**), $E \in$ (**ULUM**), $\varphi \in \Delta_2^E$ and φ is strictly convex, then $E_\varphi \in$ (**LUR**).*

(ii) *Suppose that e is a Köthe sequence space. If $e \in$ (**LLUM**), $e \in$ (**ULUM**), $\varphi(b_\varphi)\inf_{i\in N}\|e_i\|_e \geq 1$, $\varphi \in \Delta_2^e$ and φ is strictly convex on the interval $[0, \varphi_r^{-1}(1/\inf_{i\in N}\|e_i\|_e))$, then $e_\varphi \in$ (**LUR**).*

The proof of Corollary 3.4 will be preceeded by two lemmas.

Lemma 3.5. *Suppose that E is a Köthe space. If $E \in$ (**ULUM**), then for each $u \in (S(E))_+$ and any $\varepsilon > 0$ there is $\delta = \delta(u,\varepsilon) \in (0,1)$ such that for every $v \in (S(E))_+$ with $\|(u-v)\chi_{A(u,v)}\|_E \geq \varepsilon$ there holds $\|(u-v)\chi_{T\setminus A(u,v)}\|_E \geq \delta$, where $A(u,v) = \{t \in T : u(t) \leq v(t)\}$.*

Proof. Suppose for the contrary that $E \in$ (**ULUM**) and there are $u \in (S(E))_+$, $\varepsilon > 0$ and sequence (v_n) in $(S(E))_+$ such that $\|(u-v_n)\chi_{A(u,v_n)}\|_E \geq \varepsilon$ and $\|(u-v_n)\chi_{T\setminus A(u,v_n)}\|_E < 1/n$ for each $n \in N$. Denote $A_n = A(u,v_n)$ for simplicity. Since $E \in$ (**ULUM**), there is a number $p = p(u,\varepsilon) > 0$ such that $\|u + (v_n - u)\chi_{A_n}\|_E \geq 1 + p$ for each $n \in N$. Since

$$\|u\chi_{T\setminus A_n} + v_n\chi_{A_n}\|_E - \|v_n\chi_{T\setminus A_n} + v_n\chi_{A_n}\|_E \leq \|u\chi_{T\setminus A_n} - v_n\chi_{T\setminus A_n}\|_E < 1/n,$$

we get

$$\begin{aligned} 1 &= \|v_n\|_E = \|v_n\chi_{T\setminus A_n} + v_n\chi_{A_n}\|_E \geq \|u\chi_{T\setminus A_n} + v_n\chi_{A_n}\|_E - 1/n \\ &= \|u + (v_n - u)\chi_{A_n}\|_E - 1/n \geq 1 + p - 1/n \end{aligned}$$

for each $n \in N$. This contradiction for sufficiently large n finishes the proof.

The following easy observation will be useful in the next lemma.

Remark 3.6. Let E be a Köthe space and $\varepsilon > 0$ be given. Then for any $u, v \in (S(E))_+$ such that $\|(u-v)\chi_{A_0}\|_E \geq \varepsilon$, we have $\|(u-v)\chi_{A_\varepsilon}\|_E \geq \varepsilon/2$, where $A_0 = \{t \in T : v(t) < u(t)\}$ and $A_\varepsilon = \{t \in A_0 : v(t) \leq (1-\varepsilon/3)u(t)\}$.

Lemma 3.7. *Suppose that E is a Köthe space. If $E \in$ (**ULUM**), then for each $u \in (S(E))_+$ and any $\varepsilon > 0$ there is $\delta = \delta(u,\varepsilon) \in (0,1)$ such that for every $v \in (S(E))_+$ with $\|u-v\|_E \geq \varepsilon$ there holds $\|(u-v)\chi_{A_\delta(u,v)}\|_E \geq \delta$, where $A_\delta(u,v) = \{t \in T : v(t) \leq (1-\delta)u(t)\}$.*

Proof. Take arbitrary $u \in (S(E))_+$ and $\varepsilon > 0$. Let $v \in (S(E))_+$ be such that $\|u-v\|_E \geq \varepsilon$. Denote $A_0 = \{t \in T : v(t) < u(t)\}$. We will consider two cases.

1. If $\|(u-v)\chi_{A_0}\|_E \geq \varepsilon/2$, then $\|(u-v)\chi_{A_\varepsilon}\|_E \geq \varepsilon/4$, where $A_\varepsilon = \{t \in A_0 : v(t) < (1-(\varepsilon/6))u(t)\}$ (see Remark 3.6).

2. Suppose that $\|(u-v)\chi_{T\setminus A_0}\|_E \geq \varepsilon/2$. Then $\|(u-v)\chi_{A_0}\|_E \geq \delta_1$, where $\delta_1 = \delta(u,\varepsilon/2)$ is from Lemma 3.5. Thus $\|(u-v)\chi_{A_{\delta_1}}\|_E \geq \delta_1/2$, where $A_{\delta_1} = \{t \in A_0 : v(t) < (1-(\delta_1/3))u(t)\}$.

Combining Cases 1 and 2, we get $\|(u-v)\chi_{A_\delta(u,v)}\|_E \geq \delta$ with $\delta = \min\{\frac{\varepsilon}{6}, \frac{\delta_1}{3}\}$.

Proof of Corollary 3.4. (i). Since strict convexity of φ gives that $\varphi > 0$ and $\varphi < \infty$, it is enough to show that our assumptions guarantee that condition (b) in Theorem 3.1 is satisfied. Assuming that condition (b) in Theorem 3.1 does not hold we shall show that φ must be affine on some interval. Suppose that condition (b) is not satisfied. Then there exist an element $u \in (S(E))_+$, a number $\varepsilon > 0$ and a sequence $(v_n)_{n=1}^\infty$ in $(S(E))_+$ such that, taking $x = \varphi_r^{-1} \circ u$, $y_n = \varphi_r^{-1} \circ v_n$, we have

$$\|u - v_n\|_E \geq \varepsilon, \tag{5}$$

$$\|u + v_n(1-w_n)\|_E > 2(1-1/n), \quad \text{and} \quad \|uw_n\|_E < \frac{1}{n} \tag{6}$$

for every $n \in N$, where

$$w_n(t) = \begin{cases} 1 - \frac{2\varphi((x(t)+y_n(t))/2)}{\varphi(x(t))+\varphi(y_n(t))} & \text{if } t \in B_n \\ 0 & \text{otherwise} \end{cases}$$

and

$$B_n = \{t \in \operatorname{supp} u \cup \operatorname{supp} v_n : u(t) \wedge v_n(t) \leq (1-1/n)(u(t) \vee v_n(t))\}.$$

Following the proof of necessity of Theorem 2.11 in [29] and passing to a subsequence, if necessary, we conclude that

$$\|x - y_n\|_\varphi \geq \eta(\varepsilon) \tag{7}$$

for each n, where $\eta = \eta(\varepsilon) > 0$ depends only on ε. Since $E \in (\mathbf{OC})$ (see Proposition 2.1 in [11]), $E \in (\mathbf{ULUM})$, $\varphi > 0$, $\varphi < \infty$ and $\varphi \in \Delta_2^E$, by Proposition 2.5, $E_\varphi \in (\mathbf{ULUM})$. It follows from condition (7) and Lemma 3.7 that there is $\delta = \delta(x, \eta) \in (0, 1)$ such that $\|(x - y_n)\chi_{A_\delta(x,y_n)}\|_\varphi \geq \delta$ for each n, where $A_\delta(x, y_n) = \{t \in T : y_n(t) \leq (1 - \delta)x(t)\}$. Put $A_n = A_\delta(x, y_n)$ for simplicity. The convexity of φ yields that $v_n(t) \leq (1 - \delta)u(t)$ for each $t \in A_n$. Hence $A_n \subset B_n$ for each $n > 1/\delta$. Moreover, $\|x\chi_{A_n}\|_\varphi \geq \delta/2$ for each n. Thus $\|u\chi_{A_n}\|_E \geq \delta_1$ for each n, where $\delta_1 > 0$ depends only on δ. Since $E \in (\mathbf{OC})$, we find a number $C > 0$ with $\|u\chi_{T\setminus T_0}\|_E < \delta_1/2$, where $T_0 = \{t \in T : 1/C \leq u(t) \leq C\}$. Hence $\|u\chi_{T_0\cap A_n}\|_E \geq \delta_1/2$ for each n. We claim that for each $k \in N$ there is $n_k \in N$ and $t_k \in T_0 \cap A_{n_k}$ with $w_{n_k}(t_k) < 1/k$. Indeed, if not, we find a number $k_0 \in N$ such that $\|w_n u\chi_{T_0\cap A_n}\|_E \geq 1/k_0\|u\chi_{T_0\cap A_n}\|_E \geq \delta_1/2k_0$ for each n, but this contradicts inequality (6) for sufficiently large n. Note that we can take the sequence $(n_k)_{k=1}^\infty$ that is strictly increasing. By the definition of the function w_{n_k} we get

$$\varphi\left(\frac{x(t_k) + y_{n_k}(t_k)}{2}\right) > \frac{1 - 1/k}{2}\{\varphi(x(t_k)) + \varphi(y_{n_k}(t_k))\}$$

for each k. Moreover, since $t_k \in T_0$, $\varphi_r^{-1}(1/C) \leq x(t_k) \leq \varphi_r^{-1}(C)$ and consequently the sequence $(x(t_k))_{k=1}^\infty$ contains a convergent subsequence $(x(t_{k_l}))_{l=1}^\infty$. Similarly, the sequence $(y_{n_{k_l}}(t_{k_l}))_{l=1}^\infty$ contains a convergent subsequence $(y_{n_{k_{l_p}}}(t_{k_{l_p}}))_{p=1}^\infty$. Denoting these subsequences by x_k, y_k, we get $x_k \to x_0$, $y_k \to y_0$ and

$$\varphi\left(\frac{x_k + y_k}{2}\right) > \frac{1 - 1/k}{2}\{\varphi(x_k) + \varphi(y_k)\}$$

for each k. Passing to the limit we obtain $\varphi\left(\frac{x_0+y_0}{2}\right) = \frac{1}{2}\{\varphi(x_0) + \varphi(y_0)\}$. Hence it is enough to show that $x_0 \neq y_0$. By the definition of the set A_n we get $x(t) - y_n(t) \geq \delta x(t)$ for each $t \in A_n$. Hence, $x_k - y_k \geq \delta\varphi_r^{-1}(1/C)$ for each k. Passing to the limit, we get $x_0 - y_0 \geq \delta\varphi_r^{-1}(1/C)$.

(ii). The proof goes in the same way as in Case (i). Note only that the number η in inequality (7) depends only on ε and x (see the proof of Theorem 3.2).

4. Applications to Orlicz-Lorentz spaces

In this section we shall consider the Lebesgue measure space $([0, \alpha), \Sigma, \mu)$ with $0 < \alpha \leq \infty$ and μ being the Lebesgue measure or the counting measure space $(N, 2^N, m)$. Let $\omega : [0, \alpha) \to R_+$ (respectively $\omega : N \to R_+$) be a nonincreasing, nonnegative, locally integrable function (resp. nonincreasing, nonnegative sequence), called a weighted function (resp. a weighted sequence). Then the Lorentz function space Λ_ω (resp. Lorentz sequence space λ_ω) is defined as follows (see [30] and [31])

$$\Lambda_\omega = \{x \in L^0 : \|x\|_\omega = \int_0^\alpha x^*(t)\omega(t)dt < \infty\}$$

$$(\text{resp. } \lambda_\omega = \{x \in l^0 : \|x\|_\omega = \sum_{i=1}^\infty x^*(i)\omega(i) < \infty\}). \tag{8}$$

Recall that if $E = \Lambda_\omega$ (resp. $e = \lambda_\omega$), then the Calderón-Lozanovskiĭ space E_φ (resp. e_φ) is the corresponding Orlicz-Lorentz function (resp. sequence) space $\Lambda_{\varphi,\omega}$ (resp. $\lambda_{\varphi,\omega}$) (see [4], [17], [19], [26], [27] and [29]).

We say that E has the Kadec-Klee property for global convergence in measure if for any $x \in E$ and any sequence (x_m) in E such that $\|x_m\|_E \to \|x\|_E$ and $x_m \to x$ globally in measure, we have $\|x_m - x\|_E \to 0$ (see [9]).

We shall need in sequel the following results.

Proposition 4.1. *The following conditions are equivalent:*

(*i*) ω *is positive on* $[0, \alpha)$ *and* $\int_0^\infty \omega(t)dt = \infty$ *whenever* $\alpha = \infty$.
(*ii*) *The Lorentz function space* Λ_ω *is strictly monotone.*
(*iii*) *The Lorentz function space* Λ_ω *is lower locally uniformly monotone.*
(*iv*) *The Lorentz function space* Λ_ω *is upper locally uniformly monotone.*

Proof. The equivalence $(i) \Leftrightarrow (ii)$ has been proved in [29, Lemma 3.1]. By Lemma 3.2 in [29] and Theorem 2.6, we have the equivalence $(ii) \Leftrightarrow (iii)$. Since Λ_ω has the Kadec-Klee property for global convergence in measure (see Corollary 1.3 in [9], cf. also the proof of Theorem 1 in [22]), by Theorem 3.2 in [9] and Lemma 3.2 in [29], we conclude that $(ii) \Leftrightarrow (iv)$.

Proposition 4.2. *The following conditions are equivalent:*

(*i*) $\sum_{i=1}^\infty \omega(i) = \infty$.
(*ii*) *The Lorentz sequence space* λ_ω *is strictly monotone.*
(*iii*) *The Lorentz sequence space* λ_ω *is lower locally uniformly monotone.*
(*iv*) *The Lorentz sequence space* λ_ω *is upper locally uniformly monotone.*

Proposition 4.2 we prove analogously as Proposition 4.1, applying Theorem 2.7. We mention only the proof of the equivalence $(ii) \Leftrightarrow (iv)$. First note that Theorem 3.2 in [9] can be proved analogously replacing the symmetric function space by the symmetric sequence space. Furthermore, the Lorentz sequence space λ_ω has the Kadec-Klee property for global convergence in measure (one can show it using similar techniques as in the proof of Theorem 1 in [22]).

The below example shows that in Lorentz spaces uniform monotonicity is essentially stronger than lower and upper local uniform monotonicity.

Example 4.3. Let $\omega(t) = \frac{1}{n}$ for $t \in [n-1, n)$ and $n \in N$ ($\omega(i) = \frac{1}{i}$ for $i \in N$). Then, by Proposition 4.1 (4.2), the Lorentz space Λ_ω (λ_ω) is **LLUM** and **ULUM**. However, by Theorem 1 in [16], Λ_ω (λ_ω) is not **UM**.

The criteria for strict monotonicity of Orlicz-Lorentz spaces can be deduced from Corollary 1, Theorems 7 and 8 in [24]. Furthermore, from the Propositions 2.4, 2.5, 4.1 and 4.2 one can get immediately two stronger results.

Corollary 4.4. *The following conditions are equivalent:*

(*i*) ω *is positive on* $[0, \alpha)$ *and* $\int_0^\infty \omega(t)dt = \infty$ *whenever* $\alpha = \infty$, $\varphi \in \Delta_2^{\Lambda_\omega}$ *and* $\varphi > 0$.
(*ii*) *The Orlicz-Lorentz function space* $\Lambda_{\varphi,\omega}$ *is strictly monotone.*
(*iii*) *The Orlicz-Lorentz function space* $\Lambda_{\varphi,\omega}$ *is lower locally uniformly monotone.*
(*iv*) *The Orlicz-Lorentz function space* $\Lambda_{\varphi,\omega}$ *is upper locally uniformly monotone.*

Corollary 4.5. *The following conditions are equivalent:*

(*i*) $\sum_{i=1}^{\infty}\omega(i)=\infty$, $\varphi\in\Delta_2(0)$ *and* $\varphi(b_\varphi)\omega(1)\geq 1$.
(*ii*) *The Orlicz-Lorentz sequence space* $\lambda_{\varphi,\omega}$ *is strictly monotone.*
(*iii*) *The Orlicz-Lorentz sequence space* $\lambda_{\varphi,\omega}$ *is lower locally uniformly monotone.*
(*iv*) *The Orlicz-Lorentz sequence space* $\lambda_{\varphi,\omega}$ *is upper locally uniformly monotone.*

Finally, we will present new proofs of two theorems that has been already obtained in the papers [4] and [19]. These new proofs are based on the general result from this paper and they are much more simpler than the original ones.

Theorem 4.6 ([19], Theorem 12). *For the Orlicz-Lorentz function space* $\Lambda_{\varphi,\omega}$ *the following conditions are equivalent:*

(*i*) ω *is positive on* $[0,\alpha)$, $\int_0^\infty\omega(t)dt=\infty$ *whenever* $\alpha=\infty$, $\varphi\in\Delta_2^{\Lambda_\omega}$ *and* φ *is strictly convex on* R_+.
(*ii*) $\Lambda_{\varphi,\omega}$ *is locally uniformly rotund.*
(*iii*) $\Lambda_{\varphi,\omega}$ *is rotund.*

Proof. By Proposition 4.1 and Corollary 3.4(*i*), we get the implication $(i)\Rightarrow(ii)$. The implication $(ii)\Rightarrow(iii)$ is obvious. Finally, $(iii)\Rightarrow(i)$ has been proved in Corollary 3.3 in [29] (originally it was shown in [26]).

For any Orlicz function φ, by φ^* we denote its complementary function, that is, $\varphi^*(v)=\sup_{u\geq 0}\{u|v|-\varphi(u)\}$ for $v\in R$. If $\varphi(b_\varphi)\omega(1)\geq 1$, we define $\gamma_1=\varphi_r^{-1}(1/\omega(1))$ and $\gamma_2=\varphi_r^{-1}(1/(\omega(1)+\omega(2)))$.

Theorem 4.7 ([4], Theorem 11). *The Orlicz-Lorentz sequence space* $\lambda_{\varphi,\omega}$ *is* **LUR** *if and only if the following two conditions are satisfied:*

1. $\sum_{i=1}^{\infty}\omega(i)=\infty$, $\varphi\in\Delta_2(0)$, $\varphi(b_\varphi)\omega(1)\geq 1$ *and*
2. (*i*) φ *is strictly convex on* $[0,\gamma_1]$ *or*
   (*ii*) $\varphi^*\in\Delta_2(0)$ *and* φ *is strictly convex on* $[0,\gamma_2]$.

Proof. *Sufficiency.* Since $\sum_{i=1}^{\infty}\omega(i)=\infty$, by Proposition 4.2, we get $\lambda_\omega\in$ (**LLUM**) and $\lambda_\omega\in$ (**ULUM**). Simultaneosly $\lambda_\omega\hookrightarrow c_o$ and condition $\Delta_2(0)$ means condition $\Delta_2^{\lambda_\omega}$. Since $\varphi\in\Delta_2(0)$, we have $\varphi>0$. It is enough to show that our assumptions guarantee that condition (*b*) in Theorem 3.2 is satisfied. Suppose that condition (*b*) does not hold. Then there exist an element $u\in(S(\lambda_\omega))_+$, a number $\varepsilon>0$ and a sequence $(v_n)_{n=1}^\infty$ in $(S(\lambda_\omega))_+$ such that, taking $x=\varphi_r^{-1}\circ u$, $y_n=\varphi_r^{-1}\circ v_n$, we have

$$\|u-v_n\|_\omega\geq\varepsilon,\quad \|u+v_n(1-w_n)\|_\omega>2(1-1/n)\quad\text{and}\quad\|uw_n\|_\omega<1/n \tag{9}$$

for every $n\in N$, where

$$w_n(i)=\begin{cases}1-\frac{2\varphi((x(i)+y_n(i))/2)}{\varphi(x(i))+\varphi(y_n(i))} & \text{if } i\in B_n\\ 0 & \text{otherwise}\end{cases}$$

and

$$B_n=\{i\in\operatorname{supp}u\cup\operatorname{supp}v_n: u(i)\wedge v_n(i)\leq(1-1/n)(u(i)\vee v_n(i))\}.$$

Following the proof of Corollary 3.4 we conclude that there exist numbers $\delta\in(0,1)$ and $y_0,x_0\geq 0$ with $y_0\leq(1-\delta)x_0$ such that φ is affine on the interval $[y_0,x_0]$. We can assume that $\gamma_2 \leq y_0$ and $\varphi^* \in \Delta_2(0)$. Let (i_n) be a sequence from the proof of Corollary 3.4 for which $\lim_{n\to\infty} x(i_n) = x_0$ and $\lim_{n\to\infty} y_n(i_n) = y_0$. Since $\gamma_2 < x_0$, there is exactly one $i_o \in N$ such that $x(i_o) = x(i_n)$ for each $n \geq n_o$ with some $n_o \in N$. Furthermore $x_0 = x(i_o) = x^*(1)$.

First we assume that there exist a number $\eta > 0$, a subsequence of natural numbers (n_m) and a sequence (i_m), $i_m \neq i_o$ for any $m \in N$, such that $y_{n_m}(i_m) \geq \eta$ and $(y_{n_m}(i_m) \wedge x(i_m)) \leq (1-\eta)(y_{n_m}(i_m) \vee x(i_m))$. In virtue of Lemmas 5 or 6 in [25] there exists $p = p(\eta) \in (0,1)$ such that for all $m \in N$ we have

$$\frac{2\varphi\left(\frac{x(i_m)+y_{n_m}(i_m)}{2}\right)}{\varphi(x(i_m)) + \varphi(y_{n_m}(i_m))} < 1-p.$$

Let k be the smallest natural number for which $(1-p)\varphi(\eta)\sum_{i=1}^{k}\omega(i) \geq 1$. Then, for $n_m > 1/\eta$, we get $i_m \in B_{n_m}$ and

$$\begin{aligned}\|u + v_{n_m}(1-w_{n_m})\|_\omega &\leq \|u\|_\omega + \|v_{n_m}(1-w_{n_m})\|_\omega \\ &\leq \|u\|_\omega + \|v_{n_m}\chi_{N\backslash\{i_m\}} + (1-p)v_{n_m}\chi_{\{i_m\}}\|_\omega \\ &\leq \|u\|_\omega + \|v_{n_m}\|_\omega - p\varphi(\eta)\omega(k) = 2 - p\varphi(\eta)\omega(k),\end{aligned}$$

which contradicts to (9) for suffitiently large m.

Let now for each $k \in N$ there exists m_k such that for all $n \geq m_k$ and each $i \in N\backslash\{i_o\}$ we have $(y_n(i) \wedge x(i)) > (1-1/k)(y_n(i) \vee x(i))$ whenever $y_n(i) \geq 1/k$. Denoting $b = \varphi(x_0)\omega(1) - \varphi(y_0)\omega(1)$, we have $b \in (0,1)$. Without loss of generality we can assume that $y_n(i_o) = y_n^*(1)$ and $\varphi(y_n(i_o))\omega(1) \leq \varphi(x_0)\omega(1) - b/2$ for any $n \in N$. We consider two cases.

1. First we suppose that $m(\operatorname{supp} x) = s < \infty$. If $s = 1$, then $\sum_{i=2}^{\infty} v_n^*(i)\omega(i) \geq b/4$. If $s > 1$ we find $k \in N$ such that $x^*(s) \geq 1/k$ and (see Lemma 1.1 in [15])

$$\varphi\left(\frac{k}{k-1}t\right) \leq \left(1 + \frac{b}{4}\right)\varphi(t)$$

for $t \in [0,\gamma_2]$. It is easy to show that $y_n(i) < 1/k$ for each $n \geq m_k$, $i \in N\backslash\operatorname{supp} x$ and $y_n^*(i) < \frac{k}{k-1}x^*(i)$ for $n \geq m_k$ and $i = 2,3,\ldots,s$. Therefore

$$\begin{aligned}\sum_{i=1}^{s}\varphi(y_n^*(i))\omega(i) &= \varphi(y_n^*(1))\omega(1) + \sum_{i=2}^{s}\varphi(y_n^*(i))\omega(i) \qquad (10)\\ &\leq \varphi(x^*(1))\omega(1) - b/2 + \sum_{i=2}^{s}\varphi\left(\frac{k}{k-1}x^*(i)\right)\omega(i) \\ &\leq \varphi(x^*(1))\omega(1) - b/2 + \left(1+\frac{b}{4}\right)\sum_{i=2}^{s}\varphi(x^*(i))\omega(i) \leq 1 - \frac{b}{4}\end{aligned}$$

for $n \geq m_k$. Since $\sum_{i=s+1}^{\infty}\varphi(y_n^*(i))\omega(i) \geq b/4$ for $n \geq m_k$, we have $\sum_{i=1}^{\infty}\varphi(z_n^*(i))\omega(i+s) \geq b/4$ for $n \geq m_k$, where $z_n = y_n\chi_{N\backslash\operatorname{supp} x}$. Since $\varphi^* \in \Delta_2(0)$ and φ is strictly convex on $[0,\gamma_2]$, by Lemma 1.1 in [7], we have $2\varphi(t/2)/\varphi(t) \leq 1-q$ for some $q \in (0,1)$ and any $t \in [0,\gamma_2]$. Hence

$$\|u + v_n(1-w_n)\|_\omega \leq \|u\|_\omega + \|v_n\chi_{\operatorname{supp} x} + (1-q)v_n\chi_{N\backslash\operatorname{supp} x}\|_\omega \leq 2 - qb/4$$

for $n \geq m_k$, which contradicts to (9) for suffitiently large n.

2. Let now $m(\operatorname{supp} x) = \infty$. Defining $A_k = \{i \in \operatorname{supp} x : x(i) \geq 1/k\}$ and $p_k = m(\operatorname{supp} x\chi_{A_k})$, we have $p_k < \infty$ for any $k \in N$ and $\lim_{k\to\infty} \sum_{i=1}^{\infty} \varphi((x\chi_{N\setminus A_k})^*(i))\omega(i + p_k) = 0$. Since, by (9), $\lim_{k\to\infty} \|v_{m_k}(1 - w_{m_k})\|_\omega = 1$, proceeding analogously as in (10), we get $\sum_{i=1}^{\infty}(v_{m_k}(1 - w_{m_k})\chi_{N\setminus A_k})^*(i)\omega(i + p_k) \geq b/8$ beginnig from some $k = k_1$. Since $\varphi^* \in \Delta_2(0)$ and φ is strictly convex on $[0, \gamma_2]$ there exist $a, r \in (0, 1)$ such that

$$\varphi\left(\frac{t+s}{2}\right) \leq \frac{1-r}{2}(\varphi(t) + \varphi(s))$$

for all $t, s \in [0, \gamma_2]$, whenever $s \leq at$ (see Example 1.7 in [7]). Let n be the smallest natural number for which $1/a \leq 2^n$. Then there exists $k_2 \geq k_1$ such that $2^n a/(k-1) \leq \gamma_1$, $a < 1 - 1/m_k$ and $\sum_{i=1}^{\infty} \varphi((x\chi_{N\setminus A_k})^*(i))\omega(i + p_k) \leq b/(16K^n)$, for $k \geq k_2$, where K is the constant from the $\Delta_2(0)$ condition for the function φ. Denoting $D_k = \{i \in (\operatorname{supp} x \cup \operatorname{supp} y_{m_k})\setminus A_k : x(i) \leq a y_{m_k}(i)\}$ and $C_k = N\setminus(A_k \cup D_k)$, we have

$$\varphi(y_{m_k}(i)) \leq \varphi(2^n a y_{m_k}(i)) \leq K^n \varphi(a y_{m_k}(i)) \leq K^n \varphi((x(i))$$

for $i \in C_k$ and $k \geq k_2$. Therefore $\sum_{\{i:\sigma(i)\in C_k\}}(v_{m_k}(1-w_{m_k})\chi_{N\setminus A_k})^*(i)\omega(i+p_k) \leq b/16$ and, in consequence, $\sum_{\{i:\sigma(i)\in D_k\}}(v_{m_k}(1 - w_{m_k})\chi_{N\setminus A_k})^*(i)\omega(i + p_k) \geq b/16$ whenever $k \geq k_2$, where σ is a bijection from N to $N_0 \subset N$ such that $(v_{m_k}(1 - w_{m_k})\chi_{N\setminus A_k})^* = v_{m_k}(1 - w_{m_k})\chi_{N\setminus A_k} \circ \sigma$. Hence, for $k \geq k_2$, we get

$$\begin{aligned} & \|u + v_{m_k}(1 - w_{m_k})\|_\omega \\ \leq \quad & \|u\|_\omega + \|v_{m_k}\chi_{A_k} + (1 - r)v_{m_k}\chi_{D_k} + v_{m_k}(1 - w_{m_k})\chi_{N\setminus(A_k\cup D_k)}\|_\omega \leq 2 - rb/16. \end{aligned}$$

This contradiction with (9) for suffitiently large k finishes the proof of sufficiency.

Necessity. We observe that if $\lambda_{\varphi,\omega} \in (\mathbf{LUR})$, then, by Theorem 3.2, $\varphi \in \Delta_2^{\lambda_\omega}$, $\varphi(b_\varphi)\omega(1) \geq 1$ and $\lambda_\omega \in \mathbf{LLUM}$, whence, by Propositon 4.2, $\sum_{i=1}^{\infty} \omega(i) = \infty$. Therefore $\lambda_\omega \hookrightarrow c_o$ and condition $\Delta_2^{\lambda_\omega}$ means condition $\Delta_2(0)$. Since $\mathbf{LUR} \Rightarrow \mathbf{R}$, from Corollary 3.3 in [29], we get that φ is strictly convex on $[0, \gamma_2]$. Thus, in order to finish the proof we need only to show that φ is strictly convex on $[\gamma_2, \gamma_1]$ whenever $\varphi^* \notin \Delta_2(0)$.

Suppose that φ fails to be strictly convex on $[\gamma_2, \gamma_1]$ and $\varphi^* \notin \Delta_2(0)$. Then there are $a, b \in [\gamma_2, \gamma_1)$, $a < b$, and sequences $d(i) \downarrow 0$, $p(i) \downarrow 0$ such that

$$\varphi\left(\frac{a+b}{2}\right) = \frac{\varphi(a) + \varphi(b)}{2}, \quad \text{and} \quad \varphi\left(\frac{d(i)}{2}\right) \geq (1 - p(i))\frac{\varphi(d(i))}{2}$$

for any $i \in N$. Let $c \in (0, \gamma_2)$ be such that $\varphi(b)\omega(1) + \varphi(c)\omega(2) = 1$. For any $n \in N$ we find $d(i_n)$ and m_n such that $d(i_1) < a$, $d(i_{n+1}) < d(i_n)$, $m_{n+1} > m_n$ and

$$\begin{aligned} 1 - \frac{1}{n} \quad & \leq \quad \varphi(a)\omega(1) + \varphi(c)\omega(2) + \varphi(d(i_n)) \sum_{k=3}^{m_n} \omega(k) \leq 1 \\ & < \quad \varphi(a)\omega(1) + \varphi(c)\omega(2) + \varphi(d(i_n)) \sum_{k=3}^{m_n+1} \omega(k). \end{aligned}$$

Define

$$u = \varphi(b)e_1 + \varphi(c)e_2, \quad v_n = \varphi(a)e_1 + \varphi(c)e_2 + \varphi(d(i_n)) \sum_{k=3}^{m_n} e_k + f_n e_{m_n+1},$$

where $f_n < \varphi(d(i_n))$ is choosen in such a way that $\|v_n\|_\omega = 1$. We have

$$\|u - v_n\|_\omega \geq (\varphi(b) - \varphi(a))\omega(1) > 0, \quad \|uw_n\|_\omega = 0$$

for any $n \in N$ and

$$\|u + v_n(1 - w_n)\|_\omega \geq (\varphi(b) + \varphi(a))\omega(1) + 2\varphi(c)\omega(2) + (1 - p(i_n))\varphi(d(i_n)) \sum_{k=3}^{m_n} \omega(k) \to 2$$

as $n \to \infty$. So, by Theorem 3.2, $\lambda_{\varphi,\omega} \notin$ (**LUR**).

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