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Hausdorff Dimension of Cut Loci of Generic Subspaces of Euclidean Spaces

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Let F be a closed set of the Euclidean space $\mathbb{E}^d$, with $\emptyset \neq F \neq \mathbb{E}^d$ and $d \geq 2$. Let $\mathcal{N}$ be the set of centers of all open balls contained in $\mathbb{E}^d \setminus F$ which are maximal with respect to inclusion. We prove that the Hausdorff dimension $\dim_{\mathrm{H}}(\mathcal{N})$ of $\mathcal{N}$ equals d when F is, in the sense of Baire categories, a generic compact subset of $\mathbb{E}^d$, or when $\mathbb{E}^d \setminus F$ is the interior of a generic convex body of $\mathbb{E}^d$.

If C is a generic convex body, we deduce that the set of all points of ∂C where the "upper curvature" of ∂C is positive and finite, is of Hausdorff dimension $d-1$. Let CurvCt be the set of centers of upper curvature of ∂C, and ω be any non empty open subset of $\mathbb{E}^d$. We also prove that $\dim_{\mathrm{H}}(\omega \cap \mathrm{CurvCt}) = d$.

Let B be a generic compact subset of $\mathbb{E}^d$, or a generic convex body of $\mathbb{E}^d$. Let $\mathrm{a}\mathcal{N}$ be the set of centers of all closed balls containing B which are minimal with respect to inclusion. We also prove that $\dim_{\mathrm{H}}(\mathrm{a}\mathcal{N}) = d$.

The proofs employ some of the ideas used in [19] to construct large cut loci in $\mathbb{E}^d$.

Keywords: Cut locus, skeleton, medial axis, Hausdorff dimension, critical value, curvature, convex body, farthest distance

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1. Notation, Introduction

The aim of this section is to introduce the definitions and to present in a simplified form our main results.

Throughout this paper, d is an integer ≥ 2. $\mathbb{R}^d$ is endowed with its usual Euclidean structure, with inner product $(. \mid .)$ and with induced norm $||.||$.

First we recall some definitions and introduce some notations, concerning Hausdorff measures and generic properties.

1.1. Hausdorff measures

We denote by $[a, b]$, $]a, b[$, $[a, b[$, and $]a, b]$ the real intervals, respectively closed, open, left-closed-right-open and left-open-right-closed. We use the Landau notation

$$f = \mathrm{o}(g) \quad \text{and} \quad f \sim g$$

denoting that $f = \theta g$ with θ of limit respectively 0 and 1.

By a *dimension function*, we mean a homeomorphism $h : [0, +\infty[\to [0, +\infty[$.

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Definition 1.1. Let A be a subset of a metric space (E, δ) and h a dimension function.

- $\operatorname{diam} A$ denotes the diameter $\sup\{\delta(a,b) \mid a \in A \text{ and } b \in B\}$ of the set A.
- $\mathcal{H}^h_\varepsilon(A) = \inf\{\sum h(\operatorname{diam} A_n) \mid (A_n)$ is a countable covering of A by sets satisfying $\operatorname{diam} A_n \leq \varepsilon\}$, for $\varepsilon > 0$.
- $\mathcal{H}^h(A) = \sup_{\varepsilon>0} \mathcal{H}^h_\varepsilon(A)$ is the *Hausdorff measure* of A with respect to the dimension function h.

When $h(t) = t^s$, we also write

$$\mathcal{H}^s \quad \text{and} \quad \mathcal{H}^s_\varepsilon$$

instead of $\mathcal{H}^h$ and $\mathcal{H}^h_\varepsilon$. Here $\mathcal{H}^s$ is the *s-dimensional measure.*

The *Hausdorff dimension*

$$\dim_{\mathrm{H}}(A)$$

of A is defined by the fact that $\mathcal{H}^s(A) = 0$ if $s > \dim_{\mathrm{H}}(A)$ and that $\mathcal{H}^s(A) = \infty$ if $0 < s < \dim_{\mathrm{H}}(A)$.

The Lebesgue measure of a subset A of $\mathbb{R}^d$ is given by

$$\operatorname{vol}(A) = \frac{\mathcal{H}^d(A)}{\mathcal{H}^d([0,1]^d)}.$$

Using dimension functions, we can precise the information given by the Hausdorff dimension of a set. For instance, let us suppose that $h(t) \sim t^d |\ln t|$ (in zero) and that $\mathcal{H}^h(A) > 0$ for some metric space A; then we have $\dim_{\mathrm{H}} A \geq d$, and moreover A has no countable covering (A_n) by sets of Hausdorff dimensions $< d$. Similarly, if we suppose that $h(t) \sim 1/|\ln t|$ and that $\mathcal{H}^h(A) = 0$, then we have $\dim_{\mathrm{H}} A = 0$.

1.2. Generic properties

Let X be a topological space and $A \subset X$.

The subset A is a G_δ of X if it is the intersection of a countable family of open subsets of X. The subset A is a F_σ of X if it is the union of a countable family of closed subsets of X. The subset A is *meager* if it is included in the union of a countable family of closed subsets of X of empty interiors.

A property P is said to be "generic" in X (in the sense of Baire categories), or shared by "most elements" of X, or by "generic" elements of X, or by "typical" elements of X, if the exceptional set of all the $x \in X$ not satisfying P is meager.

We will consider more specifically the generic properties of two topological spaces.

Definition 1.2 (hyperspaces).

- $\operatorname{Cpct}(\mathbb{R}^d)$ denotes the space of all non empty compact subsets of $\mathbb{R}^d$. It is a metric space, boundedly compact, with respect to the Pompeiu-Hausdorff metric defined by

$$\operatorname{dist}_{\mathrm{H}}(A,B) = \max\left(\sup_{a\in A}\inf_{b\in B} ||a-b||, \sup_{b\in B}\inf_{a\in A} ||b-a||\right).$$

- $\operatorname{Cvb}(\mathbb{R}^d)$ denotes the subspace of $\operatorname{Cpct}(\mathbb{R}^d)$, of all *convex bodies* of $\mathbb{R}^d$, i.e. of all convex compact subsets of non empty interior.

- conv B denotes the convex hull of the set $B \subset \mathbb{R}^d$.

Observe that the mapping $B \mapsto \operatorname{conv} B$ is a retraction of $\mathrm{Cpct}(\mathbb{R}^d)$ to the closure A of its subspace $\mathrm{Cvb}(\mathbb{R}^d)$ (A is the set of all non empty compact convex subsets of $\mathbb{R}^d$).

See the papers by Schneider [25] for a survey of differential properties of convex surfaces, and by Gruber [9] and Zamfirescu [33] for the properties of generic convex surfaces.

1.3. Cut locus

Let us define the set $\mathcal{N}_F$ in which we are mainly interested, and also some other sets or notions more or less obviously related to it.

Definition 1.3 (metric projection). Let $F \subset \mathbb{R}^d$, $a \in \mathbb{R}^d$ and $A \subset \mathbb{R}^d$. We define

- $\operatorname{dist}(a, F) = \inf\{||a - p|| \mid p \in F\}$,
- $\operatorname{Proj}_F(a) = \{p \in F \mid ||a - p|| = \operatorname{dist}(a, F)\}$,
- $\operatorname{Proj}_F(A) = \bigcup_{b \in A} \operatorname{Proj}_F(b)$.

When there is only one projection of a to F, we denote it by $\operatorname{proj}_F(a)$, we have then $\operatorname{Proj}_F(a) = \{\operatorname{proj}_F(a)\}$.

When $a, b \in \mathbb{R}^d$ we write $[a, b] = \{(1 - \lambda)a + \lambda b \mid 0 \leq \lambda \leq 1\}$.

Definition 1.4 (cut locus). Let F be a non empty closed subset of $\mathbb{R}^d$. We define

- $\mathcal{M}_F = \{a \in \mathbb{R}^d \mid \operatorname{card} \operatorname{Proj}_F(a) \geq 2\}$,
- $\mathcal{N}_F = \{a \in \mathbb{R}^d \setminus F \mid \forall b \in \mathbb{R}^d \setminus \{a\} \text{ and } p \in \operatorname{Proj}_F(b),\ a \notin [b, p]\}$.

In other words, $\mathcal{M}_F$ is the set of points of $\mathbb{R}^d$ which have at least two projections to F, and $\mathcal{N}_F$ is the set of points of $\mathbb{R}^d$ which are never "crossed" by a projection ray to F.

Remark. If F is the complement $\mathbb{R}^d \setminus \operatorname{int} C$ of the interior $\operatorname{int} C$ of a convex body C of $\mathbb{R}^d$, then the boundary ∂C of C satisfies

$$\mathcal{M}_F = \mathcal{M}_{\partial C} \quad \text{and} \quad \mathcal{N}_F = \mathcal{N}_{\partial C}.$$

Example 1.5 Let $d = 2$ and F be a non circular ellipse. Consider the two points p and q of F where the curvature of F is maximal, and a, b the two curvature centers of F at these points. Then

$$\mathcal{N}_F = [a, b] \quad \text{and} \quad \mathcal{M}_F = [a, b] \setminus \{a, b\}.$$

Moreover we have $p = \operatorname{proj}_F(a)$, $q = \operatorname{proj}_F(b)$, and $\operatorname{card} \operatorname{Proj}_F(c) = 2$ for all $c \in \mathcal{M}_F$.

Remarks. We always have $\mathcal{M}_F \subset \mathcal{N}_F$. The set $\mathcal{M}_F$ is considered more frequently than $\mathcal{N}_F$, whose definition is less easy to understand (other equivalent characterizations of elements of $\mathcal{N}_F$ could be given). $\mathcal{M}_F$ is often called the *ambiguous locus* of the metric projection. The word *medial axis* is also used, sometimes for $\mathcal{M}_F$ as in [3], and sometimes for $\mathcal{N}_F$ as in [4]. When the ambient space is a Riemannian manifold, it is better to consider the set of points having at least two projection ray to F, to define the ambiguous locus.

Still the set $\mathcal{N}_F$ has been considered in many situations and with various names: the "rib" in [17] and [18], the "skeleton" in mathematical morphology ([15] and [23]), the *cut locus* in Riemannian Geometry and its generalizations (first and mainly in the case where F is reduced to a point of a Riemannian manifold E) see [26], [11], [12], [34] and the recent panorama [2].

Because we are concerned by the size of the cut locus, we first precise some known results about it.

For most non empty compact subset F of $\mathbb{R}^d$, $\mathcal{N}_F$ is dense in $\mathbb{R}^d$, as proved by Zamfirescu [32] and also by De Blasi and Myjak in [5] (in separable Hilbert spaces E of dimension ≥ 2 instead of $E = \mathbb{R}^d$). $\mathcal{N}_F$ is also dense in C when $F = \mathbb{R}^d \setminus \operatorname{int} C$ for a generic convex body C of $\mathbb{R}^d$ (this follows for instance from Theorem 4.6, but a direct simple proof can be given).

For every non empty proper closed subset F of $\mathbb{R}^d$, $\mathcal{N}_F$ is a G_δ subset of $\mathbb{R}^d$ while $\mathcal{M}_F$ is an F_σ subset of $\mathbb{R}^d$, and $\mathcal{M}_F$ is dense in $\mathcal{N}_F$. Thus, when $\mathcal{N}_F$ is dense in the open set $\Omega = \mathbb{R}^d \setminus F$, most points of Ω are in $\mathcal{N} \setminus \mathcal{M}$.

For any $0 \leq s \leq d$, there exists a convex body C of $\mathbb{R}^d$ such that $\dim_{\mathrm{H}}(\mathcal{N}_F \setminus \mathcal{M}_F) = s$, if we set $F = \mathbb{R}^d \setminus \operatorname{int} C$ [19]. The same result is proved in [20] when $d = 2$ for convex bodies of class C^2, by more analytical methods. If F is a non empty proper closed subset of $\mathbb{R}^d$ such that the closure of $\mathcal{N}_F$ has a non empty interior, then we have $\dim_{\mathrm{H}}(\mathcal{N}_F \setminus \mathcal{M}_F) \geq 1$ [19].

Theorem 3.9, p. 841 and Theorem 4.3, p. 843 imply the following result.

Theorem 1.6. *We have* $\dim_{\mathrm{H}} \mathcal{N}_F = d$ *when* F *is a generic non empty compact subset of* $\mathbb{R}^d$*, and also when* $F = \mathbb{R}^d \setminus \operatorname{int} C$*, with* C *a generic convex body of* $\mathbb{R}^d$.

It was only known that $\dim_{\mathrm{H}} \mathcal{N}_F \setminus \mathcal{M}_F \geq 1$ and $\dim_{\mathrm{H}} \mathcal{M}_F = d - 1$.

A similar cut locus $\mathrm{a}\mathcal{N}_B$ can also be associated to a bounded subset B of $\mathbb{R}^d$ when we consider, instead of the distance from F, the "antidistance" from B defined by $\operatorname{adist}(a, B) = \sup\{||a - p||, \text{ for } p \in B\}$, related to the farthest projection problem. We will describe it in Section 5.

Theorem 5.11, p. 852 is analogous to Theorems 3.9, 4.3 and 4.6, for the cut locus generated by the farther distance. It implies the following result.

Theorem 1.7. *Let* B *be a generic compact subset of* $\mathbb{E}^d$*, or a generic convex body of* $\mathbb{E}^d$*. Let* $\mathrm{a}\mathcal{N}_B$ *be the set of centers of all closed balls containing* B *which are minimal for inclusion. Then* $\dim_{\mathrm{H}}(\mathrm{a}\mathcal{N}_B) = d$.

1.4. Curvature

Remark. Let us assume that $d = 2$ and that the boundary ∂F of F is a C^3-submanifold of $\mathbb{R}^2$. If $a \in \mathcal{N} \setminus \mathcal{M}$, then a is a center of curvature of ∂F at $p = \operatorname{proj}_F(a)$, moreover the curvature of ∂F has a zero derivative at p, and one may expect that p is a local maximum of this curvature. Conversely, if $a \in \mathbb{R}^d \setminus F$, $p \in \operatorname{Proj}_F(a)$ and a is the curvature center of ∂F at p, then $a \in \mathcal{N}_F$.

These elementary observations and the example p. 825 should motivate the following definitions.

Definition 1.8 (local metric projection). For $F \subset \mathbb{R}^d$ and $a \in \mathbb{R}^d$, we define

- $\mathrm{Projloc}_F(a) = \{p \in F \mid \exists r > 0, \forall q \in F, ||q-p|| < r \Rightarrow ||q-a|| \geq ||p-a||\}$.

Definition 1.9 (upper curvature). For a closed subset F of $\mathbb{R}^d$ and $q \in \partial F$, we define

- $\mathrm{curv}_q(F) = \inf\{1/||a-q|| \mid a \in \mathbb{R}^d \setminus \{q\} \text{ and } q \in \mathrm{Projloc}_F(a)\} \in [0, +\infty]$,
- $\mathrm{GrCurv}_F = \{(p,a) \in F \times \mathbb{R}^d$ such that for any real $t \geq 0$ we have $t < 1 \Rightarrow p \in \mathrm{Projloc}_F(p + t(a-p))$ and $t > 1 \Rightarrow p \notin \mathrm{Projloc}_F(p + t(a-p))\}$.
- $\mathrm{CurvCt}_F = \mathrm{pr}_2(\mathrm{GrCurv}_F) = \{a \mid (p,a) \in \mathrm{GrCurv}_F\}$.
- $X_F = \mathrm{pr}_1(\mathrm{GrCurv}_F) = \{p \mid (p,a) \in \mathrm{GrCurv}_F\}$.

$\mathrm{curv}_p(F)$ is the "upper curvature" at p of ∂F when $p \in \partial F$ and $\mathbb{R}^d \setminus F$ is convex.

When $(p,a) \in \mathrm{GrCurv}_F$, then a is a kind of *curvature center* of F at p. CurvCt_F is the set of these curvature centers of F. X_F is the set of points of ∂F such that $0 < \mathrm{curv}_p(F) < \infty$.

Question 1.10. Have we $\mathcal{H}^d(\mathrm{CurvCt}_F) = 0$ for every closed subset of $\mathbb{R}^d$?

When $\mathbb{R}^d \setminus F$ is convex, we will prove (Property 4.9, p. 847) that

$$\dim_{\mathrm{H}} \mathcal{N}_F = d \Rightarrow \dim_{\mathrm{H}} X_F = d-1.$$

Thus we have the following result, wich is refined in Theorem 4.10, p. 847.

Theorem 1.11. *For most convex bodies C of $\mathbb{R}^d$ and $F = \mathbb{R}^d \setminus \mathrm{int}\, C$ we have* $\dim_{\mathrm{H}} X_F = d-1$.

Nothing was known about $\dim_{\mathrm{H}} X_F$. However Zamfirescu [29], [30], has also proved, using [1], that (for most convex bodies C of $\mathbb{R}^d$ and $F = \mathbb{R}^d \setminus \mathrm{int}\, C$)

$$\mathcal{H}^{d-1}(X_F) = 0.$$

Sectional curvatures.

We will recall here some classical definitions. They will not be used in other parts of the paper, but we need them here to precise the mentioned result of Zamfirescu, and also to precise the meaning of Theorem 1.11.

Definition 1.12 (Sectional curvatures). Let C be a convex body of $\mathbb{R}^d$, $p \in \partial C$, $x \in \mathbb{R}^d$ with $||x|| = 1$, $H = x^\perp = \{y \in \mathbb{R}^d \mid (x \mid y) = 0\}$ and $\varphi : H \to \mathbb{R}$ a nonnegative convex function with $\varphi(0) = 0$ and such that, for some $r > 0$, we have $\forall y \in H, ||y|| < r \Rightarrow p + y + \varphi(y)x \in \partial C$.

We suppose φ differentiable at 0 (thus ∂C is smooth at p, one also says that p is a regular point, or a smooth point, of ∂C). Let $y \in H$ with $||y|| = 1$. We define

- $\mathrm{curvsup}_{p,y}(\partial C) = \limsup_{0<t\to 0} 2\frac{\varphi(p+ty)}{t^2}$,
- $\mathrm{curvinf}_{p,y}(\partial C) = \liminf_{0<t\to 0} 2\frac{\varphi(p+ty)}{t^2}$,
- $\mathrm{curv}_{p,y}(\partial C) = \mathrm{curvinf}_{p,y}(\partial C)$ when $\mathrm{curvsup}_{p,y}(\partial C) = \mathrm{curvinf}_{p,y}(\partial C)$.
- We say that ∂C *has sectional curvatures at* p if for all $z \in H$ with $||z|| = 1$, we have $\mathrm{curvsup}_{p,z}(\partial C) = \mathrm{curvinf}_{p,z}(\partial C) < \infty$.

- $\mathrm{SC}(\partial C)$ denotes the set of points q of ∂C at which ∂C is smooth and has sectional curvatures.

From these definitions we have

$$0 \leq \mathrm{curvinf}_{p,y}(\partial C) \leq \mathrm{curvsup}_{p,y}(\partial C) \leq \mathrm{curv}_F(p) \leq \infty$$

if $F = \mathbb{R}^d \setminus \operatorname{int} C$. Moreover, using the convexity of φ, one easily gets

$$\mathrm{curv}_F(p) = \sup\{\mathrm{curvsup}_{p,z}(\partial C) \mid z \in H \text{ and } ||z|| = 1\}.$$

The point p is called a *Normal point*, or an *Euler point*, of C if $p \in \mathrm{SC}(\partial C)$ and if $\mathrm{curv}_{p,z}(F)$ is a quadratic function of z. For example if φ has a second derivative in 0, then p is an Euler point of ∂C.

Theorem 1.13 (Klee, [13]). *Most convex bodies C of $\mathbb{R}^d$ are smooth (∂C is a C^1 submanifold of $\mathbb{R}^d$) and strictly convex.*

Theorem 1.14 (Zamfirescu, [29], [30]). *For most convex bodies C of $\mathbb{R}^d$, for each $p \in \partial C$ and for each normal tangent vector y to ∂C at p we have*

$$\mathrm{curvinf}_{p,y}(\partial C) = 0 \quad \text{or} \quad \mathrm{curvsup}_{p,y}(\partial C) = \infty.$$

So if in Definition 1.4 we suppose C generic and $p \in \mathrm{SC}(\partial C)$, then for all $z \in H$ with $||z|| = 1$ we have $\mathrm{curvsup}_{p,z}(\partial C) = 0$. Thus p is an Euler point of C. Moreover $curv_F(p) = 0$ if $F = \mathbb{R}^d \setminus \operatorname{int} C$. In other words,

$$\varphi(z) = \mathrm{o}(||z||^2)$$

for $z \in H$ and in zero.

Thus, combining Theorem 1.11 and Theorem 1.14, we get:

Theorem 1.15 (Euler points). *For most convex bodies C of $\mathbb{R}^d$,*

$$\dim_{\mathrm{H}} \partial C \setminus \mathrm{SC}(\partial C) = d - 1$$

and for each $p \in \partial C \setminus \mathrm{SC}(\partial C)$, p is not a Euler point of C.

1.5. Local cut locus

There is not a canonical way to localise the notion of $\mathcal{M}_F$ or of $\mathcal{N}_F$, but we will need the following ones.

Definition 1.16. For a non empty proper closed subset F of $\mathbb{R}^d$, we define

- $\mathcal{N}_{\mathrm{LOC},F} = \{a \in \mathbb{R}^d \mid \forall b \in \mathbb{R}^d \setminus \{a\} \text{ and } p \in \mathrm{Projloc}(b) \setminus \{a\},\ a \notin [b, p]\}$,
- $\mathcal{N}_{\mathrm{loc},F} = \{a \in \mathbb{R}^d$ such that for some non empty closed subset F' of F we have $a \in \mathcal{N}_{F'}$ and $\mathrm{Proj}_{F'}(a) \subset \mathrm{int}_F(F')$ (the interior in F of F')$\}$,
- $\mathcal{M}_{\mathrm{loc},F} = \{a \in \mathbb{R}^d \mid \exists p, q \in \mathrm{Projloc}_F(a) \text{ with } p \neq q \text{ and } ||a - p|| = ||a - q||\}$.

In other words, the point a is in $\mathcal{N}_{\mathrm{LOC},F}$, if it is never "crossed" by a local projection ray to F. We also have

$$\mathcal{M}_{\mathrm{loc},F} = \{a \in \mathbb{R}^d \mid \exists F' \subset F \text{ with } \operatorname{card} \operatorname{Proj}_{F'}(a) \cap \operatorname{int}_F(F') \geq 2\}.$$

We have the following elementary relations, about "local cut locus" and curvature centers:

$$\mathcal{N}_{\mathrm{LOC},F} \setminus F \subset \mathcal{N}_F \subset \mathcal{N}_{\mathrm{loc},F}$$

and

$$\mathcal{N} \setminus \mathcal{M} \subset \mathrm{CurvCt}_F \supset \mathcal{N}_{\mathrm{loc},F} \setminus \mathcal{M}_{\mathrm{loc},F}.$$

Remarks. We always have $\mathcal{H}^d(\mathcal{N}_F) = 0$, as proved in [18], and thus $\mathcal{H}^d(\mathcal{N}_{\mathrm{loc},F}) = 0$. The set $\mathcal{M}_F$ is always of $\mathcal{H}^{d-1}$-measure σ-finite, as proved in [7], so $\mathcal{M}_{\mathrm{loc},F}$ is also of $\mathcal{H}^{d-1}$-measure σ-finite.

Theorem 4.6, p. 846 is a local version of Theorem 4.3, and a stronger result. It implies the following.

Theorem 1.17. *For most convex bodies C of $\mathbb{R}^d$, for $F = \mathbb{R}^d \setminus \operatorname{int} C$ and for every non empty open subset ω of $\mathbb{R}^d$, we have*

$$\dim_{\mathrm{H}}(\omega \cap \mathcal{N}_{\mathrm{loc},F} \cap \mathcal{N}_{\mathrm{LOC},F} \setminus \mathcal{M}_{\mathrm{loc},F}) = d$$

and hence $\dim_{\mathrm{H}} \omega \cap \mathrm{CurvCt}_F = d$.

Curvature of boundaries of convex bodies. Let C be a convex body of $\mathbb{R}^d$. In this paper we are concerned by two kinds of curvature of ∂C, the upper curvature and the lower curvature. We are also concerned by two specific cases, namely

(1) C is a generic convex body of $\mathbb{R}^d$, and
(2) C is the convex hull of a generic non empty compact subset of $\mathbb{R}^d$.

Concerning the upper curvature, we will see in Section 4 that the cases (1) and (2) are very different. In the case (1), ∂C has an everywhere large set of curvature centers (Theorem 1.17), and the set of corresponding points of ∂C is a large subset of ∂C (Theorem 1.11). In the case (2), ∂C has no curvature centers (Property 4.12, p. 848).

Concerning the lower curvature, we will see in Section 5 that the cases (1) and (2) are similar. In both of them, ∂C has an everywhere large ($\dim_{\mathrm{H}} = d$) set of curvature centers, but the set of corresponding points of ∂C is a small ($\dim_{\mathrm{H}} = 0$) subset of ∂C (Theorem 5.11, p. 852).

1.6. Porosity

Section 2 is devoted to a preliminary result that we want to describe here. Roughly speaking, it says that there are large compact subsets K of $\mathbb{R}^d$ which are small, with respect to some orthogonal projections.

Definition 1.18. For $a \in \mathbb{R}^d$ and $r > 0$,

- $\mathrm{S}(a, r) = \{b \in \mathbb{R}^d \mid ||b - a|| = r\}$,

- $\mathrm{B}(a,r) = \{b \in \mathbb{R}^d \mid ||b-a|| < r\}$ and
- $\overline{\mathrm{B}}(a,r) = \{b \in \mathbb{R}^d \mid ||b-a|| \leq r\}$

are the sphere, the open ball and the closed ball of center a and of radius r. We will also denote by

$\mathcal{D}_{\mathrm{H}}$ the set of all affine hyperplanes of $\mathbb{R}^d$, affinely generated by subsets of $\mathbb{Q}^d$ (the "rational hyperplanes"); and by

$\mathcal{D}_{\mathrm{S}}$ the set of all spheres $\mathrm{S}(c,r)$ with $c \in \mathbb{Q}^d$ and $r \in \mathbb{Q}_+^*$ (the "rational spheres").

Definition 1.19 (orthogonal projection). Let p be a point of a submanifold F of $\mathbb{R}^d$ of class C^1, $a \in \mathbb{R}^d$ and $A \subset \mathbb{R}^d$.

- $\mathrm{T}_p(F)$ denotes the linear subspace of $\mathbb{R}^d$, of all vectors tangent to F at p.
- $\mathrm{N}_p(F)$ denotes the affine subspace of $\mathbb{R}^d$ normal to F at p, that is $\{x \in \mathbb{R}^d$ such that $x-p$ is orthogonal to $\mathrm{T}_p(F)\}$.
- $\pi_F(a)$ denotes the orthogonal projection of a to F, that is $\{q \in F \mid a \in \mathrm{N}_q(F)\}$.
- $\pi_F(A)$ denotes $\bigcup_{b \in A} \pi_F(b)$.

Remark. We obviously have $\mathrm{Projloc}_F(a) \subset \pi_F(a)$.

We will be concerned by the following notions of smallness of a set A.

Definition 1.20 (porosity). Let $a \in A \subset F \subset \mathbb{R}^d$.

- The set A is *strongly porous* in F at the point a if there exists a sequence (x_n) in $F \setminus \{a\}$ converging to a such that $\mathrm{dist}(x_n, A)/||x_n - a||$ converges to 1.
- The set A is *strongly porous* in F if it is strongly porous in F at every point of A.
- The set A is *radially strongly porous* at the point a if there exists two sequences (r_n) and (ε_n) of positive numbers, converging to zero, and satisfying $A \cap \mathrm{B}(a, r_n) \setminus \mathrm{B}(a, \varepsilon_n r_n) = \emptyset$.
- The set A is *radially strongly porous* if it is strongly porous at every point of A.

Remark. If F is a C^1-submanifold of $\mathbb{R}^d$ of dimension ≥ 1, and $A \subset F$ is radially strongly porous at the point a, then A is also strongly porous in F at the point a.

Proposition 2.1 implies the following.

Proposition 1.21. *Let A be an arbitrary subset of $\mathbb{R}^d$ with $\mathcal{H}^d(A) = 0$.*

There exists a compact subset K of $\mathbb{R}^d \setminus A$ such that $\dim_{\mathrm{H}} K = d$ and satisfying, for all $F \in \mathcal{D}_{\mathrm{H}} \cup \mathcal{D}_{\mathrm{S}}$, the following properties.

- *The orthogonal projection $\pi_F(K)$ of K to F is radially strongly porous.*
- *$\forall p, q \in K$, $p \neq q \Rightarrow \pi_F(p) \cap \pi_F(q) = \emptyset$.*

1.7. Out line of some proofs of the paper

Section 2 is used in each of the three others, while Sections 3, 4 and 5 are mainly independant.

In Section 2, we will prove a preliminary result, Proposition 2.1, that we will use in each of the other sections (for the proofs of Theorems 3.9, 4.3, 4.6 and 5.11) in a similar way. To give an idea of these proofs, we outline now the proof of the first statement of Theorem 1.6.

If we want to prove, for instance, that for most $F \in \mathrm{Cpct}(\mathbb{R}^d)$ we have $\dim_{\mathrm{H}}(\mathcal{N}_F) = d$, then we just have to prove the existence of some compact subset K of $\mathbb{R}^d$ such that $\dim_{\mathrm{H}} K = d$ and satisfying

$$(A)_K: \text{ for most } F \in \mathrm{Cpct}(\mathbb{R}^d), \text{ we have } K \subset \mathcal{N}_F.$$

We now choose K as in Proposition 1.21 with A being the union of all the $\mathcal{M}_{H \cup H'}$ for $H, H' \in \mathcal{D}_{\mathrm{H}}$. To prove $(A)_K$ we observe that, because of the compacity of K, the set $G_K = \{F \in \mathrm{Cpct}(\mathbb{R}^d) \text{ such that } K \subset \mathcal{N}_F\}$ is a G_δ subset of $\mathrm{Cpct}(\mathbb{R}^d)$. Thus we only have to prove the density of G_K. Here we use an approximation result, Lemma 3.4, p. 838 (in other cases Lemma 4.1, p. 842 or 5.9, p. 850). It allows us to choose $F' \in G_K$ near F when for each $a \in K$ there exists $p \in \mathrm{Proj}_F(a)$ such that ∂F equals a rational hyperplane H in a neighborhood of p.

In that case, we also need another approximation result like Lemma 3.7, p. 839 to prove the density of the set of such sets F in $\mathrm{Cpct}(\mathbb{R}^d)$. At this point, we need that orthogonal projections of K to rational spheres are radially strongly porous, and not only strongly porous.

2. A large compact set with small projections

In this section we use the definitions given in Sections 1.1 and 1.6., but we need nothing about cut locus.

2.1. Statement of Proposition 2.1

Proposition 2.1. *Let h be a dimension function such that $t^d = \mathrm{o}(h(t))$ and let $\mathcal{D}$ be a countable family of non empty submanifolds of $\mathbb{R}^d$ of class C^2 and of dimensions $\in \{1, \ldots, d-1\}$. Let A be an arbitrary subset of $\mathbb{R}^d$ with $\mathcal{H}^d(A) = 0$.*

Then there exists a compact subset K of $\mathbb{R}^d \setminus A$ such that $\mathcal{H}^h(K) > 0$ and such that for each $F \in \mathcal{D}$, we have the following properties.

I) $\forall p \in F$, $\mathrm{card}\, K \cap \mathrm{N}_p(F) \leq 1$.
Hence we have a continuous mapping π_F^{-1} from $\pi_F(K)$ to K (which is onto K when F is closed).

II) *$\pi_F(K)$ is radially strongly porous.*

Moreover we will see in the proof that we can also ask K to be radially strongly porous.

Remarks. It follows that $\pi_F(K)$ is strongly porous in F.

It is not difficult to prove that most compact subset K of $\mathbb{R}^d$ satisfy *I*) and *II*), but they are also small: $\dim_{\mathrm{H}}(K) = 0$, and we need a large one.

Let us consider some particular cases of this proposition:

Case 1. All $F \in \mathcal{D}$ are lines.
Case 2. All $F \in \mathcal{D}$ are hyperplanes.
Case 3. All $F \in \mathcal{D}$ are circles.
Case 4. All $F \in \mathcal{D}$ are spheres (homeomorphic with S^{d-1}).

The cases 2 and 4 will be enough for our main aim, but we find interesting to give and prove a more general statement (we do not know how much it can be generalized).

If we replace in Proposition 2.1 "radially strongly porous" by "strongly porous in F", then the case 2 of the new proposition becomes an easy consequence of the case 1.

Lemma 2 of [19] is essentially the case 1 (or 2) when $d = 2$ and $\operatorname{card} \mathcal{D} = 1$, but our argumentation here seems to us more simple.

If we want to use the case 2 of the proposition to find a compact set K such that $\dim_H(K) > d-1$, we choose h such that $h(t) = \mathrm{o}(t^s)$ for some $s > d-1$. It is known then that we have $\mathcal{H}^{d-1}(\mathrm{Proj}_F(K)) > 0$ for $\mathcal{H}^{d-1}$-almost all x of the sphere $S^{d-1} = \mathrm{S}(0,1)$ of $\mathbb{R}^d$ and for the orthogonal hyperplane $F = x^\perp$ (see [8], p. 85). Hence for such an F, the orthogonal projection $\mathrm{Proj}_F(K)$ is not strongly radially porous. However, if we have choosed $\mathcal{D} = \mathcal{D}_{\mathrm{H}}$ then it is not difficult to check that for most $x \in S^{d-1}$, $F = x^\perp$ satisfies *I*) and *II*) of the proposition.

2.2. Dyadic net measures

We will use net measure ideas in a rather standard way (see [22], §7 for a general description of such measures).

Definition 2.2 (dyadic cubes). Let $n \in \mathbb{N}$.
$\mathcal{F}_n$ denotes the set of dyadic cubes $2^{-n}\big(x + [0,1]^d\big)$, where $x \in \mathbb{Z}^d$. We also set $\mathcal{F} = \bigcup_{k\geq 0} \mathcal{F}_k$, $\mathcal{F}_{\leq n} = \bigcup_{0\leq k\leq n} \mathcal{F}_k$, $\mathcal{F}_{\geq n} = \bigcup_{k\geq n} \mathcal{F}_k$, ... etc.

Definition 2.3 (Net measures). Let h be a dimension function and $A \subset \mathbb{R}^d$. We define

- $\widetilde{h}(C) = h(2^{-n})$ if $n \in \mathbb{N}$ and $C \in \mathcal{F}_n$,
- $\widetilde{h}(\mathcal{G}) = \sum_{A\in\mathcal{G}} \widetilde{h}(A)$ if $\mathcal{G} \subset \mathcal{F}$,
- $\widetilde{\mathcal{H}}^h_N(A) = \inf\{\widetilde{h}(\mathcal{G})$, where $\mathcal{G} \subset \mathcal{F}_{\geq N}$ is a covering of $A\}$ if $N \in \mathbb{N}$,
- $\widetilde{\mathcal{H}}^h(A) = \sup_{N\in\mathbb{N}} \widetilde{\mathcal{H}}^h_N(A)$,
- $\widehat{\mathcal{H}}^h_N(A) = \inf\{\widetilde{h}(\mathcal{G})$, where $\mathcal{G} \subset \mathcal{F}_{\geq N}$ is a *finite* covering of $A\}$ if $N \in \mathbb{N}$,
- $\widehat{\mathcal{H}}^h(A) = \sup_{N\in\mathbb{N}} \widehat{\mathcal{H}}^h_N(A)$.

We have for every $A \subset \mathbb{R}^d$,

$$\widetilde{\mathcal{H}}^h(A) \leq 3^d \mathcal{H}^h(A). \tag{1}$$

Indeed a bounded subset B of $\mathbb{R}^d$ is included in the union of at most 3^d elements of $\mathcal{F}_n$ for the first integer n such that B is not included in any cube $x + [0, 2^{-n}]^d$. We also have for every compact subset A of $\mathbb{R}^d$,

$$\widehat{\mathcal{H}}^h(A) \leq 3^d \widetilde{\mathcal{H}}^h(A), \tag{2}$$

as one sees when replacing the covering $\mathcal{G} \subset \mathcal{F}$ of A by $\mathcal{G}' = \{x + ry + r[0,1]^d$ where $x + r[0,1]^d \in \mathcal{G}$ and $y \in \{-1,0,1\}^d\}$. We have then $\widetilde{h}(\mathcal{G}') \leq 3^d \widetilde{h}(\mathcal{G})$, and moreover every $a \in A$ has a neighborhood which is the union of a finite subfamily of $\mathcal{G}'$, so $\mathcal{G}'$ contains a finite subcovering of A.

2.3. Construction of large compact subsets of $\mathbb{R}^d$

We will use the following lemma to construct the large compact set K mentioned in Proposition 2.1. It is a standard construction, but we do not know a precise reference for it, so we prove it. One may found similar ideas related with porosity in [16], p. 64, and also in [14], p. 302 and p. 305.

Lemma 2.4. *Let h be a dimension function such that $t^d = \mathrm{o}(h(t))$ and $\theta(t) := t^{-d}h(t)$ is nonincreasing on $]0, +\infty[$.*

Let $(K_n)_{n\geq 0}$ be a decreasing sequence of non empty compact subsets of $\mathbb{R}^d$, such that each K_n is the union of a finite family of elements of $\mathcal{F}_{M_n}$, for some increasing sequence (M_n) in $\mathbb{N}$.

Suppose that for a sequence (N_n) of integers and a sequence (λ_n) in $]0, 1[$ we have for each $n \geq 0$

i) $1 + M_n < N_n < -1 + M_{n+1}$,

ii) $\lambda_n \theta(2^{-N_n}) > \theta(2^{-M_n})$ *and*

iii) *each $C \in \mathcal{F}_{N_n}$ such that $C \subset K_n$ satisfies $\mathcal{H}^d(C \cap K_{n+1}) \geq \lambda_n \mathcal{H}^d(C)$; moreover, there exists $C' \in \mathcal{F}_{M_{n+1}}$ such that $C' \subset K_{n+1} \cap \operatorname{int} C$.*

Then the compact set $K = \bigcap_{n\geq 0} K_n$ satisfies $\mathcal{H}^h(K) > 0$.

Proof. We can suppose that $M_0 = 0$ and that $K_0 = [0, 1]^d \in \mathcal{F}_0$. Because of (1) and (2), it will be enough to prove that $\widehat{\mathcal{H}}^h(K) > 0$.

For $\mathcal{G} \subset \mathcal{F}$ and $C \in \mathcal{F}$, we define (in this proof)

$$\mathcal{G}^C = \{A \in \mathcal{G} \mid A \subset C\}.$$

Let $\mathcal{G} \subset \mathcal{F}$ be a finite covering of K, it will be enough to prove that $\widetilde{h}(\mathcal{G}) \geq h(1)$. For this, we can suppose that $\mathcal{G}$ is minimal (i.e. that no proper subset of $\mathcal{G}$ is a covering of K). We use then an induction over the first integer n such that $\mathcal{G} \subset \mathcal{F}_{\leq M_{n+1}}$. Since the case $n = -1$ is obvious, we suppose that $n \geq 0$.

Observe that $\mathcal{G}$ is also a covering of K_{n+1}, because for every $C \in \mathcal{F}_{M_{n+1}}$ such that $C \subset K_{n+1}$ we have $K \cap \operatorname{int} C \neq \emptyset$, from *i*) and *iii*).

Now let $C \in \mathcal{F}_{M_n} \setminus \mathcal{G}$ such that $\mathcal{G}^C \neq \emptyset$, we only have to prove that

$$\widetilde{h}(\mathcal{G}^C) \geq \widetilde{h}(C).$$

Indeed, this allows to replace all $A \in \mathcal{G}^C$ by C in $\mathcal{G}$ and doing that for all such elements of $\mathcal{F}_{M_n}$ enables to reduce our problem to the induction hypothesis.

The idea is to decompose C in two parts $C_\mathcal{A}$ and $C_\mathcal{B}$, such that in each of them the "wanted inequality is true in mean".

More precisely, we consider $\mathcal{A} = \mathcal{G}^C \cap \mathcal{F}_{\leq N_n}$, $C_\mathcal{A} = \bigcup_{A\in\mathcal{A}} A$, $\mathcal{B} = \{B \in \mathcal{F}_{N_n}$ such that $B \subset C$ and $B \not\subset C_\mathcal{A}\}$ and $C_\mathcal{B} = \bigcup_{B\in\mathcal{B}} B$.

Then we have $C = C_\mathcal{A} \cup C_\mathcal{B}$ and $\operatorname{vol} C = \operatorname{vol} C_\mathcal{A} + \operatorname{vol} C_\mathcal{B}$.

Let us observe that if $A \in \mathcal{F}_N$ then $\operatorname{vol}(A) = 2^{-Nd}$.

Because θ is not increasing, we have for every $A \in \mathcal{A}$,

$$\widetilde{h}(A) \geq \theta(2^{-M_n}) \operatorname{vol}(A).$$

Adding these inequalities gives

$$\widetilde{h}(\mathcal{A}) \geq \theta(2^{-M_n}) \operatorname{vol}(C_{\mathcal{A}}). \tag{3}$$

As $\mathcal{G}$ is a covering of K_{n+1} and using *iii*), then *ii*), we have for every $B \in \mathcal{B}$: $\widetilde{h}(\mathcal{G}^B) \geq \sum_{A\in\mathcal{G}^B} \theta(2^{-N_n}) \operatorname{vol}(A) \geq \theta(2^{-N_n}) \sum_{A\in\mathcal{G}^B} \operatorname{vol}(A \cap K_{n+1}) = \theta(2^{-N_n}) \operatorname{vol}(B \cap K_{n+1}) \geq \theta(2^{-N_n})\lambda_n \operatorname{vol}(B) \geq \theta(2^{-M_n}) \operatorname{vol}(B)$.

Adding these inequalities gives us

$$\widetilde{h}(\mathcal{G}^C \setminus \mathcal{A}) \geq \theta(2^{-M_n}) \operatorname{vol}(C_{\mathcal{B}}),$$

that we add to (3) to get $\widetilde{h}(\mathcal{G}^C) \geq \theta(2^{-M_n}) \operatorname{vol}(C) = \widetilde{h}(C)$ as wanted.

2.4. Proof of a particular case of Proposition 2.1

Here we want to prove Proposition 2.1 when $\mathcal{D}$ is reduced to one line $F = \mathbb{R} \times \{0\}$, where we identify $\mathbb{R}^d$ with $\mathbb{R} \times \mathbb{R}^{d-1}$ (when also $A = \emptyset$ and $t \mapsto t^{-d}h(t)$ is nonincreasing).

This proof will not be formally used for the general case, but it is convenient to present the main ideas.

We use Lemma 2.4 and take $M_0 = 1$ and $K_0 = [0,1]^d$.

Suppose $n \geq 0$, we must explain how to choose λ_n, N_n, M_{n+1} and K_{n+1}. Take $\lambda_n = 2^{-n-d+1}$, choose N_n large enough to get *ii*) and take $M_{n+1} = N_n + n + d - 1$.

We will use an enumeration $\{0,1\}^{d-1} = \{y_1, y_2, \ldots, y_{2^{d-1}}\}$ to describe K_{n+1}. Let X be the finite set of all $m \in \mathbb{N}$ such that $K_n \cap \left(\frac{m+[0,1]}{2^{N_n}} \times \mathbb{R}^{d-1}\right)$ is of the type $\left(\frac{m+[0,1]}{2^{N_n}}\right) \times \left(\frac{z_m+[0,1]^{d-1}}{2^n}\right)$.

We then define

$$K_{n+1} = \bigcup_{m\in X, 1\leq k\leq 2^{d-1}} \left(\frac{m2^{n+d-1} + (k-1)2^n + [0,1]}{2^{M_{n+1}}}\right) \times \left(\frac{2z_m + y_k + [0,1]^{d-1}}{2^{n+1}}\right).$$

Then it is easy to check that this defines a sequence like in Lemma 2.4, and that $K = \bigcap_{n\geq 0} K_n$ is as wanted. Indeed, at the n^{th} step we have guaranteed that the subset K of K_{n+1} satisfies

- $\forall x \in F$, $\operatorname{diam} K \cap \mathrm{N}_x(F) \leq 2^{-n-1}\sqrt{d-1}$ (*I* follows), and
- $\pi_F(K)$ has a covering by intervals I_k with $\operatorname{diam} I_k = 2^{-M_{n+1}}$ and with $k < l \Rightarrow \inf I_l - \sup I_k \geq (2^n - 1)2^{-M_{n+1}}$ (*II* follows).

Remarks. If we want to prove the case 1 of Proposition 2.1, we have to consider now successively the lines of $\mathcal{D}$ (coming back infinitely to each of them). Moreover, we have an adaptation to make because the canonical axes of $\mathbb{R}^d$ are no more associated to lines of $\mathcal{D}$.

For the general case, we have other (more or less standard) adaptations to make.

We will use the following property.

Lemma 2.5. *If $A \subset \mathbb{R}^d$ satisfies $\mathcal{H}^d(A) = 0$, then there exists a dimension function h such that $t^d = \mathrm{o}(h(t))$ and $\mathcal{H}^h(A) = 0$.*

According to [22], p. 81, A. S. Besicovitch (1956) proved more generally that given a metric space A and a dimension function g such that $\mathcal{H}^g(A) = 0$, there always exists another dimension function h with $g = \mathrm{o}(h)$ and such that $\mathcal{H}^h(A) = 0$.

2.5. Proof of Proposition 2.1

1) Let $F \in \mathcal{D}$, $\alpha = \dim F$ and $\beta = d - \alpha$.

We will consider "local normal" mappings $\Phi : \mathbb{R}^\alpha \times \mathbb{R}^\beta \to \mathbb{R}^d$ which are of class C^1, such that $\Phi(.,0)$ is a C^2-imbedding of $\mathbb{R}^\alpha$ into F, and such that for each $x \in \mathbb{R}^\alpha$, $\Phi(x,.)$ is an affine isometry from $\mathbb{R}^\beta$ to the affine normal space $\mathrm{N}_{\Phi(x)}F$ of F at $\Phi(x)$ (so we have the orthogonality relation $\Phi(x,y) - \Phi(x,0) \perp \mathrm{T}_{\phi(x,0)}F$). We then write

- $\mathrm{Crit}(\Phi) = \{\Phi(x,y)$ such the differential $\mathrm{d}_{\Phi(x,y)}$ of Φ at (x,y) is not invertible$\}$ the set of all critical values of Φ,
- $\omega_\Phi = \Phi(\mathbb{R}^\alpha, 0) \subset F$,
- $\Omega_\Phi = \{a \in \mathbb{R}^d$ such that $\mathrm{card}\, \Phi^{-1}(a) = 1\}$, and
- $\Omega_\Phi^\varepsilon = \{a \in \mathbb{R}^d$ such that $\mathrm{dist}(a, \mathbb{R}^d \setminus \Omega_\Phi) \geq \varepsilon\}$, for $\varepsilon > 0$.

Choose a countable set X_F of such mappings Φ in such a way that $\{\omega_\Phi, \Phi \in X\}$ is a basis of open sets of F, and denote by $\mathrm{Crit}(F)$ the union of the $\mathrm{Crit}(\Phi)$, for $\Phi \in X_F$. In other words, $\mathrm{Crit}(F)$ is the set of the critical values of the canonical mapping from the normal fiber $\mathrm{N}(F)$ to $\mathbb{R}^d$. By the Sard theorem we have

$$\mathrm{vol}\, \mathrm{Crit}(F) = 0.$$

Lemma 2.5 allows us (possibly after replacing h by some dimension function nearer from $t \mapsto t^d$) to suppose that for each $F \in \mathcal{D}$, $\mathrm{Crit}(F)$ and the cut locus of F are of null $\mathcal{H}^h$-measure, and that $\mathcal{H}^h(A) = 0$. In other words

$$\mathcal{H}^h(A \cup \mathrm{Crit}(\mathcal{D})) = 0,$$

if we denote by $\mathrm{Crit}(\mathcal{D})$ the union of all the $\mathrm{Crit}(F)$ for $F \in \mathcal{D}$.

We can also suppose that $t \mapsto t^{-d}h(t)$ is non increasing (for instance, change h on $]0,+\infty[$ with $t \mapsto t^d[1 + \inf_{0<s\leq t} s^{-d}h(s)]$, as checked in [19], p. 222).

Let X be the countable set of all $(F, \Phi, 1/n)$ for $F \in \mathcal{D}$, $\Phi \in X_F$ and $n \in \mathbb{N}^*$. We choose a sequence (ξ_n) in X such that for any $\xi \in X$, the set of the integers n with $\xi_n = \xi$ is infinite.

2) We use now Lemma 2.4 to build a compact set K_∞, and we take $M_0 = 0$ and $K_0 = [0,1]^d$.

We suppose $n \geq 0$ and explain how to choose λ_n, N_n, M_{n+1} and K_{n+1}.

First we set $\xi_n = (F, \Phi, \varepsilon)$, $\dim F = \alpha$, $\beta = d - \alpha$. Let N be the first integer $N \geq 0$ such that $2^{-N}\sqrt{d} < \varepsilon$, and denote by K_n' the union of all the sets $C \cap K_n$, where $C \in \mathcal{F}_N$ is such that $C \cap K_n \cap \Omega_\Phi^\varepsilon \neq \emptyset \neq K_n \cap \mathrm{int}\, C$. Thus K_n' is a compact set such that

$$K_n \cap \Omega_\Phi^\varepsilon \subset K_n' \subset K_n \cap \mathrm{int}\, \Omega_\Phi.$$

Observe that Φ is a homeomorphism from $\Phi^{-1}(\operatorname{int}\Omega_\Phi)$ to $\operatorname{int}\Omega_\Phi$, so $\Phi^{-1}(K'_n)$ is compact and we can choose $y_0 \in \mathbb{R}^\beta$ and $r > 0$ such that $\Phi^{-1}(K'_n) \subset \mathbb{R}^\alpha \times (y_0 + [0,r]^\beta)$. We precise that we make the choice of y_0 and r depending only of ξ_n and not precisely of n (we make the same choice at the step m of the construction if $\xi_m = \xi_n$), this is possible because we are constructing a non increasing sequence (K_n).

Take $\lambda_n = 2^{-n\beta-nd-1}$ and choose N_n large enough to get *ii*).

We will use an enumeration $\{0,1,2,\ldots,2^n-1\}^\beta = \{y_1, y_2, \ldots, y_{2^{n\beta}}\}$ to describe K_{n+1}. We will also consider the mappings $f_1 : \mathbb{R}^\alpha \times \mathbb{R}^\beta \to \mathbb{R}$ and $g : \mathbb{R}^\alpha \times \mathbb{R}^\beta \to \mathbb{R}^\beta$ defined by $f_1(x,y) = x_1$ and by $g(x,y) = y$.

$C^{(n)}$ will denote $z + [0, \rho 2^{-n}]^d$, for a cube $c = z + [0,\rho]^d$, with $\rho > 0$. K''_n denotes the closure of $K_n \setminus K'_n$. Then for each $P, M \in \mathbb{N}$ we define A_P and $K_{P,M}$ by the following three conditions.

- A_P is the union of the $C^{(n)}$, for $C \in \mathcal{F}_P$ such that for some integers $m \in \mathbb{Z}$ and $k \in \{1,2,\ldots,2^{n\beta}\}$ we have $f_1(C) = 2^{-P}(m2^{n\beta} + [k-1,k])$ and $g(C) \subset y_0 + 2^{-n}\left(y_k + [0,r]^\beta\right)$.
- $K_{P,M} \cap K''_n = K''_n$.
- $K_{P,M} \setminus K''_n$ is the union of all the $C \setminus K''_n$, for $C \in \mathcal{F}_M$ such that $C \subset \Phi(A_P) \cap K'_n$.

For each $C \in \mathcal{F}_{N_n}$ such that $C \subset K'_n$, one sees that we have then

$$\lim_{P\to\infty} \frac{\mathcal{H}^d(C \cap \Phi(A_P))}{\mathcal{H}^d(C)} = 2\lambda_n,$$

and when $\mathcal{H}^d(C \cap \Phi(A_P)) > 0$, we also have

$$\lim_{M\to\infty} \frac{\mathcal{H}^d(C \cap K_{P,M})}{\mathcal{H}^d(C \cap \Phi(A_P))} = 1.$$

We can thus choose P large enough so that for each $C \in \mathcal{F}_{N_n}$ such that $C \subset K'_n$, we have $\frac{\mathcal{H}^d(C\cap\Phi(A_P))}{\mathcal{H}^d(C)} > \lambda_n$. Then we can choose $M_{n+1} = M$ large enough so that the condition *iii*) will be true when we set $K_{n+1} = K_{P,M}$.

It is now easy to check that we obtain thus a sequence like in Lemma 2.4, so $K_\infty = \bigcap_{n\geq 0} K_n$ satisfies $\mathcal{H}^h(K_\infty) > 0$.

Since $A \cup Y \cup \operatorname{Crit}(\mathcal{D})$ is included in some G_δ-subset G of $\mathbb{R}^d$ such that $\mathcal{H}^h(G) = 0$, we can now choose in the Borel set $K_\infty \setminus G$ a compact subset K such that $\mathcal{H}^h(K) > 0$, (see Th. 57 of [22] for a general result about the existence of compact subspaces of positive and finite $\mathcal{H}^h$-measure in a given space X of positive $\mathcal{H}^h$-measure).

3) We explain now why K is as wanted.

Let $F \in \mathcal{D}$, $\alpha = \dim F$, $\beta = d - \alpha$, $p \in F$ and $a, b \in K \cap \mathrm{N}_p(F)$. Because a and b are not in $\operatorname{Crit}(F)$, p, a and b are all in the same Ω^ε_Φ for some $\xi = (F, \Phi, \varepsilon) \in X$. Consider one of the infinitely many integers n such that $\xi_n = \xi$.

We have then $d\left(\Phi^{-1}(a), \Phi^{-1}(b)\right) \leq r2^{-n}\sqrt{\beta}$, hence $d\left(\Phi^{-1}(a), \Phi^{-1}(b)\right) = 0$, hence $a = b$, hence I) is satisfied.

Observe that $\Phi^{-1}(K \cap K'_n)$ is radially strongly porous at $\Phi^{-1}(a)$. From this we deduce that K is strongly radially porous at a. This justifies the affirmation after the proposition if at least one of the manifolds in $\mathcal{D}$ is closed, and we can always assume that $\mathcal{D}$ contains for instance a sphere.

The first projection $\mathrm{pr}_1\left(\Phi^{-1}(K \setminus K''_n)\right) \subset \mathbb{R}^\alpha$ of $\Phi^{-1}(K \setminus K''_n)$ is also radially strongly porous at $\Phi^{-1}(p)$. From this we deduce that its image $\Phi\left(\mathrm{pr}_1\left(\Phi^{-1}(K \setminus K''_n)\right)\right)$ is radially strongly porous at p.

On the other hand we have $\Phi\left(\mathrm{pr}_1\left(\Phi^{-1}(K)\right)\right) = \pi_F(K) \cap \omega_\Phi$ and also $\Phi\left(\mathrm{pr}_1\left(\Phi^{-1}(K \setminus K''_n)\right)\right) = \pi_F(K \setminus K''_n) \cap \omega_\Phi$. But π_F^{-1} is continuous on $\pi_F(K)$ and $K \setminus K''_n$ is a neighborhood of $a = \pi_F^{-1}(p)$ in K, hence $\pi_F(K \setminus K''_n) \cap \omega_\Phi$ is a neighborhood of p in $\pi_F(K)$. So $\pi_F(K)$ is radially strongly porous at p, thus $\pi_F(K)$ is radially strongly porous. This ends the proof.

3. Cut loci of generic compact subsets of $\mathbb{R}^d$

3.1. Class properties

We will need some easy class properties.

Lemma 3.1. *For a given compact subset K of $\mathbb{R}^d$, the set of all non empty compact subsets F of $\mathbb{R}^d$ such that $K \setminus F \subset \mathcal{N}_F$ is a G_δ of* $\mathrm{Cpct}(\mathbb{R}^d)$.

Indeed, for a given $\varepsilon > 0$ the following condition defines a closed subspace of $\mathrm{Cpct}(\mathbb{R}^d)$: $\exists a \in K$, $b \in \mathbb{R}^d$ and $p \in \mathrm{Proj}_F(b)$ such that $a \in [b, p]$, $||a - p|| \geq \varepsilon$ and $||a - b]] \geq \varepsilon$.

Lemma 3.2. *For a given compact subspace K of $\mathbb{R}^d$, the set of all non empty compact subsets F of $\mathbb{R}^d$ such that $K \cap \mathcal{M}_{\mathrm{loc},F} \neq \emptyset$ is a F_σ of* $\mathrm{Cpct}(\mathbb{R}^d)$.

Indeed, for a given $\varepsilon > 0$ the following condition defines a closed subspace of $\mathrm{Cpct}(\mathbb{R}^d)$: $\exists a \in K$, $r \geq \varepsilon$, $p, q \in \mathrm{S}(a, r) \cap F$ such that $||p - q|| \geq \varepsilon$ and $F \cap \mathrm{B}(a, r) \cap (\mathrm{B}(p, \varepsilon) \cup \mathrm{B}(q, \varepsilon)) = \emptyset$.

Remark. For a given compact subspace K of $\mathbb{R}^d$, it is easy to check that the set of all non empty compact subsets F of $\mathbb{R}^d$ such that $K \subset \mathcal{N}_{\mathrm{LOC},F}$ is a G_δ of $\mathrm{Cpct}(\mathbb{R}^d)$ (this seems untrue for $\mathcal{N}_{\mathrm{loc},F}$). However we will not use this fact because the only way we know to prove the density of this set (in Propositions 3.8 and 4.5) is to found in it a dense G_δ subset of $\mathrm{Cpct}(\mathbb{R}^d)$.

Lemma 3.3. *Let K, A be compact subsets of $\mathbb{R}^d$. Then $G_{K,A} := \{F \in \mathrm{Cpct}(\mathbb{R}^d)$ such that $F \cap A \neq \emptyset$, $F \cap \partial A = \emptyset$ and $K \subset \mathcal{N}_{F \cap A}\}$ is a G_δ of $\mathrm{Cpct}(\mathbb{R}^d)$. Moreover, $F \in G_{K,A} \Rightarrow K \subset \mathcal{N}_{\mathrm{loc},F}$.*

Indeed $U = \{F \in \mathrm{Cpct}(\mathbb{R}^d)$ such that $F \cap \mathrm{int}\, A \neq \emptyset\}$ and $V = \{F \in \mathrm{Cpct}(\mathbb{R}^d)$ such that $F \cap \partial A = \emptyset = F \cap K \cap A\}$ are open in $\mathrm{Cpct}(\mathbb{R}^d)$. So $U \cap V$ is an open subset of $\mathrm{Cpct}(\mathbb{R}^d)$ containing $G_{K,A}$. As the mapping $F \mapsto A \cap F$ is continuous on $U \cap V$, $G_{K,A} = \{F \in U \cap V$ such that $K \setminus (F \cap A) \subset \mathcal{N}_{F \cap A}\}$ is G_δ of $U \cap V$ (and then of $\mathrm{Cpct}(\mathbb{R}^d)$), by Lemma 3.1. The last statement is obvious.

3.2. Approximation of F at a compact subset

The following lemma is essentially a simple generalization of the first point of Lemma 1 of [19], which is used there to prove the existence of a closed subset F of $\mathbb{R}^2$ such that

$\dim_{\mathrm{H}} \mathcal{N}_F = 2$.

Lemma 3.4. *Let $F \in \mathrm{Cpct}(\mathbb{R}^d)$, K be a compact subset of $\mathbb{R}^d \setminus F$, $a \in K$ and $p = \mathrm{proj}_F(a)$. We suppose that*

- *$\forall b \in K \setminus \{a\}$, $p \notin \mathrm{Proj}_F\, b$,*
- *for some $r > 0$, $(p + (a-p)^{\perp}) \cap \mathrm{B}(p,r) = \partial F \cap \mathrm{B}(p,r)$, and*
- *$\mathrm{Proj}_F\, K$ is strongly porous at p in ∂F.*

We consider

$$F' = \mathbb{R}^d \setminus \bigcup_{b \in K} \mathrm{B}(b, \mathrm{dist}(b, F)).$$

Then $(p,a) \in \mathrm{GrCurv}_{F'}$, hence $a \in \mathcal{N}_{F'}$.

Remarks. The set F' is a kinf of "approximation of F at K". It is a usefull tool for the study of the cut locus.

For instance, for any $K \subset \mathbb{R}^d$ we have $\mathcal{N}_{F'} \cap \mathrm{int}\, K = \mathcal{N}_F \cap \mathrm{int}\, K$. From this, one deduces that, for the study of the local properties of $\mathcal{N}_F$, one can always suppose that the boundary ∂F has some Lipschitz regularity.

We also have $K \subset \mathcal{N}_F \Rightarrow K \subset \mathcal{N}_{F'}$. This is used in [19] to get closed sets F with a prescribed dimension $\dim_{\mathrm{H}} \mathcal{N}_F$ of the cut locus.

In this paper we will also use two adaptations of Lemma 3.4: Lemma 4.1, p. 842 and Lemma 5.9, p. 850.

Proof of Lemma 3.4. We can suppose that $||a - p|| = 1$.

We choose a sequence (p_n) in $\mathrm{B}(p,r) \cap \partial F \setminus \{p\}$ converging to p, satisfying $||p_n - p|| < 1$, and such that $\delta_n := \mathrm{dist}(p_n, \mathrm{Proj}_F\, K) \sim ||p_n - p||$. Denote by p'_n the point of $\partial F'$ "above" p_n: $p'n = p_n + \lambda_n(a-p)$ for a $\lambda_n \geq 0$ as small as possible. It will then be enough to prove that

$$\limsup_{n\to\infty} \frac{\lambda_n}{||p - p_n||^2} \geq 1/2. \tag{4}$$

We can choose a_n in K such that $p'_n \in \mathrm{Proj}_{F'}(a_n)$, and then (a_n) converges to a, so $R_n := ||a_n - p'_n||$ converges to 1. Denote by a'_n the projection of a_n to the hyperplane $p + (a-p)^{\perp}$.

For n large enough, we have $a'_n = \mathrm{proj}_F\, a_n$, we also have then $||a_n - a'_n|| = ||a_n - p'_n||$ and $||a'_n - p_n|| \geq \delta_n$. So

$$(R_n - \lambda_n)^2 = R_n^2 - ||a'_n - p_n||^2 \leq R_n^2 - \delta_n^2,$$

so

$$2\lambda_n \sim 2\lambda_n R_n - \lambda_n^2 \geq \delta_n^2$$

and (4) follows (and actually $2\lambda_n \sim ||p - p'_n||^2$).

3.3. Existence of a flat local projection

Property 3.5. Let A be a compact and strongly porous subset of $\mathbb{R}^d$. Then for every $\lambda \geq 1$ and $\varepsilon > 0$, there exists a finite covering of A by balls $\mathrm{B}(a_i, r_i)$ satisfying $a_i \in A$,

$0 < r_i < \varepsilon$ and

$$i \neq j \Rightarrow \emptyset = \overline{\mathrm{B}}(a_i, \lambda r_i) \cap \overline{\mathrm{B}}(a_j, \lambda r_j).$$

Proof. Indeed, we can first choose a finite covering $(B_i)_{i \in I}$ of A by balls $B_i = \mathrm{B}(a_i, r_i)$ such that $a_i \in A$, $0 < r_i < \varepsilon$ and $A \cap \overline{\mathrm{B}}(a_i, 2\lambda r_i) \setminus \mathrm{B}(a_i, r_i) = \emptyset$. Because it is finite, we can also assume that $(B_i)_{i \in I}$ is minimal; that is, no proper subfamily is a covering of A. Then $(B_i)_{i \in I}$ is as wanted.

It is in the use of the following lemma, and of Lemma 4.4, that the radial strong porosity is useful for our purpose.

Lemma 3.6. *let $c \in \mathbb{R}^d$, $r > 0$ and K be a compact subset of $\mathbb{R}^d \setminus \mathrm{B}(c, r)$ such that $\mathrm{Proj}_{\mathrm{S}(c,r)}(K)$ is radially strongly porous.*

Then there exists a finite subset A of $\mathbb{Q}^d \cap \mathrm{B}(c, r)$ such that the rational polytope $F = \mathrm{conv}(A)$ satisfies: $\forall a \in K$, ∂F is flat at $p = \mathrm{proj}_F(a)$ (in other words, ∂F contains a neighborhood in $p + (a - p)^\perp$ of p).

Proof of Lemma 3.6. Denote by S the sphere $\mathrm{S}(c, r)$ and take $\lambda > \frac{\pi}{2} \max\{||c - a||/r,$ for $a \in K\}$. By Property 3.5, we can now find a finite covering of $\mathrm{Proj}_S(K)$ by balls $\mathrm{B}(p_i, \rho_i)$ with $p_i \in \mathrm{Proj}_S(K)$, $2\lambda\rho_i < r\pi$ and such that

$$i \neq j \Rightarrow \emptyset = \overline{\mathrm{B}}(p_i, \lambda\rho_i) \cap \overline{\mathrm{B}}(p_j, \lambda\rho_j).$$

Observe now that the convex body

$$Q = \mathrm{conv}\left(S \setminus \bigcup_i \mathrm{B}(p_i, \lambda\rho_i)\right)$$

is flat at each point of $\mathrm{Proj}_Q(K)$. So we can choose a polytope $F \subset Q$ which is flat at each point of $\mathrm{Proj}_F(K)$. Finally, by a small move of the vertexes of ∂F, we can also assume that they are all in $\mathrm{B}(a, r) \cap \mathbb{Q}^d$.

The following result is an immediate consequence of the lemma above.

Lemma 3.7. *Let K be a compact subset of $\mathbb{R}^d$ such that for all $S \in \mathcal{D}_{\mathrm{S}}$ (defined in p. 830), $\mathrm{Proj}_S(K)$ is radially strongly porous. Then a dense subset of $\mathrm{Cpct}(\mathbb{R}^d)$ is constituted by all $F \in \mathrm{Cpct}(\mathbb{R}^d)$ satisfying the following properties.*

- *$\emptyset = F \cap K$ and $F = \bigcup_{k=1}^N P_k$, where the P_k are convex polytopes of non empty interiors, with $k \neq l \Rightarrow P_k \cap P_l = \emptyset$.*
- *$\forall a \in K$, ∂F is flat at each $p \in \mathrm{Projloc}_F(a)$ (in other words ∂F contains a neighborhood in $p + (a - p)^\perp$ of p).*
- *the P_k are rational (i.e. of the form $\mathrm{conv}\, A$ for A finite subset of $\mathbb{Q}^d$).*

Remarks. Any subset K of $\mathbb{R}^d$ satisfying the conclusion of this lemma is of null Lebesgue measure. To check this, one can obviously suppose K bounded and then observe that $\mathrm{vol}_d(K) \leq (\mathrm{diam}\, K)\, \mathrm{vol}_{d-1}(\partial P_1)$.

It is easy to check that most (non empty) compact subsets K of $\mathbb{R}^d$ satisfy the conclusion of the lemma.

3.4. Cut loci of most compact subsets of $\mathbb{R}^d$

Proposition 3.8. *Let h be a dimension function such that $t^d = \mathrm{o}(h(t))$. Then there exists, by Proposition 2.1, a compact subset K of $\mathbb{R}^d$ such that $\mathcal{H}^h(K) > 0$ and satisfying the following three properties.*

(a) $\forall F \in \mathcal{D}_{\mathrm{H}}$, *$K$ satisfies I) and II) of Proposition 2.1.*
(b) $\forall S \in \mathcal{D}_{\mathrm{S}}$, $\mathrm{Proj}_S(K)$ *is radially strongly porous.*
(c) $\forall H, H' \in \mathcal{D}_{\mathrm{H}}$, $K \cap \mathcal{M}_{H \cup H'} = \emptyset$.

Moreover, these three properties imply that, for most $F \in \mathrm{Cpct}(\mathbb{R}^d)$,

$$K \subset \mathcal{N}_F \cap \mathcal{N}_{\mathrm{LOC},F} \setminus \mathcal{M}_{\mathrm{loc},F}.$$

Remarks. 1) We always have $\mathcal{N}_F \cap \mathcal{N}_{\mathrm{LOC},F} = \mathcal{N}_{\mathrm{LOC},F} \setminus F$.

2) It is easily seen that we also have, for K as in the proposition and for most F,

$$\forall a \in K, \mathrm{card}\,\mathrm{Projloc}_F(a) = \mathrm{card}\,\mathbb{N},$$

because $K \cap \mathcal{M}_{\mathrm{loc},F} = \emptyset$.

3) The set $\mathrm{AbsN}(\mathbb{R}^d)$ of all the subsets K of $\mathbb{R}^d$ satisfying $K \subset \mathcal{N}_F$ for most $F \in \mathrm{Cpct}(\mathbb{R}^d)$, is a σ-ideal of $\mathcal{H}^d$-null sets (many similar σ-ideals are involved with this work). For instance $S^{d-1} \notin \mathrm{AbsN}(\mathbb{R}^d)$. More precisely, if ω is a non empty open subset of $\mathbb{R}^d$, if $p \in \partial\omega$, and if A contains a neighborhood of p in $\partial\omega$, $A \notin \mathrm{AbsN}(\mathbb{R}^d)$.

Most compact subsets K of $\mathbb{R}^d$ satisfy the properties in this proposition, except for $\mathcal{H}^h(K) > 0$. So they are in $\mathrm{AbsN}(\mathbb{R}^d)$.

Proof of Proposition 3.8. 1) The existence of K is an immediate consequence of Proposition 2.1.

2) Now we prove that most F satisfy $K \subset \mathcal{N}_F \cap \mathcal{N}_{\mathrm{LOC},F}$. It will be enough to prove that for all rational closed ball B (thus $\partial B \in \mathcal{D}_{\mathrm{S}}$), the set $G_B := \{F \in \mathrm{Cpct}(\mathbb{R}^d)$ such that $F \cap \partial B = \emptyset$ and $F \cap B \neq \emptyset \Rightarrow K \subset \mathcal{N}_{F\cap B}\}$, which is a G_δ of $\mathrm{Cpct}(\mathbb{R}^d)$ by Lemma 3.3, is everywhere dense in $\mathrm{Cpct}(\mathbb{R}^d)$.

Consider such a rational closed ball B, and the open subset $U = \{F \in \mathrm{Cpct}(\mathbb{R}^d)$ such that $F \cap \partial B = \emptyset$ and $F \cap B \cap K = \emptyset\}$ of $\mathrm{Cpct}(\mathbb{R}^d)$, which is everywhere dense because $\mathrm{int}(K \cup \partial B) = \emptyset$. Also consider the open subset $V = \{F \in U$ such that $F \cap B \neq \emptyset\}$ of U. As $U \setminus V \subset G_B$, we only have to prove the density in V of G_B.

So let $F_0 \in V$ and W be some open neighborhood of F_0 in V. By Lemma 3.7, we can choose $F_1 \in W$ as in this lemma. Observe that for all $p \in \partial F_1$ such that ∂F_1 is flat at p, we have $\mathrm{card}\{a \in K$ such that $p \in \mathrm{Projloc}_{F_1}(a)\} \leq 1$, because of I) of (a). Moreover (c) implies that $K \cap \mathcal{M}_{F_1} = \emptyset$. For $F = B \cap F_1$ and

$$F' = \mathbb{R}^d \setminus \bigcup_{b \in K} \mathrm{B}(b, \mathrm{dist}(b, F)),$$

we have $K \subset \mathcal{N}_{F'}$ by Lemma 3.4.

For $0 < \varepsilon < \mathrm{dist}(K, \partial B)$, Lemma 3.4 also gives that $F_2 := (\overline{\mathrm{B}}(F, \varepsilon) \cap F') \cup (F_1 \setminus B) \in G_B$. We can choose ε small enough so that $F_2 \in W$. So G_B is dense in V.

3) Observe now that $G'_K = \{F \in \mathrm{Cpct}(\mathbb{R}^d)$ such that $K \cap \mathcal{M}_{\mathrm{loc},F} = \emptyset\}$ is a G_δ by Lemma 3.2, which is dense because of (*c*) and by Lemma 3.7. Thus most F satisfy $K \cap \mathcal{M}_{\mathrm{loc},F} = \emptyset$, and this ends the proof.

Theorem 3.9. *Let h be a dimension function such that $t^d = \mathrm{o}(h(t))$. Then for most $F \in \mathrm{Cpct}(\mathbb{R}^d)$, we have $\mathcal{H}^h(\mathcal{N}_F) > 0$ and consequently $\dim_{\mathrm{H}} \mathcal{N}_F = d$.*

More precisely, if ω is any non empty open subset of $\mathbb{R}^d$, then

$$\mathcal{H}^h(\omega \cap \mathcal{N}_F \cap \mathcal{N}_{\mathrm{LOC},F} \setminus \mathcal{M}_{\mathrm{loc},F}) > 0 \quad \textit{and} \quad \dim_{\mathrm{H}}(\omega \cap \mathcal{N}_F \cap \mathcal{N}_{\mathrm{LOC},F} \setminus \mathcal{M}_{\mathrm{loc},F}) = d.$$

Proof. Consider one K as in Proposition 3.8. All the elements of the countable set $\mathcal{A} = \{x + rK$, where $x \in \mathbb{Q}^d$ and $r \in \mathbb{Q}_+^*\}$ share the properties of K. Thus we have for most $F \in \mathrm{Cpct}(\mathbb{R}^d)$ and all $A \in \mathcal{A}$: $A \subset \mathcal{N}_F \cap \mathcal{N}_{\mathrm{LOC},F} \setminus \mathcal{M}_{\mathrm{loc},F}$. The theorem follows by taking as usual $h(t) \sim t^d |\ln t|$ to get the Hausdorff dimension.

Remarks. 1) We have also $\mathcal{H}^h(\mathcal{N} \setminus \mathcal{M}) > 0$ and $\dim_{\mathrm{H}} \mathrm{Proj}_F(\mathbb{R}^d \setminus F) = \dim_{\mathrm{H}} F = 0$; moreover, proj_F is (continuous and) one to one between $\mathcal{N} \setminus \mathcal{M}$ and a subset of $\mathrm{Proj}_F(\mathbb{R}^d \setminus F)$.

2) Recall that $\mathcal{M}_{\mathrm{loc},F}$ is always of $\mathcal{H}^{d-1}$-σ-finite measure and that for most F, we have $\dim_{\mathrm{H}} F = 0$. So in the theorem above, the main information is that $\mathcal{H}^h(\omega \cap \mathcal{N}_{\mathrm{LOC},F}) > 0$.

3.5. Cut loci of most closed subsets of $\mathbb{R}^d$

Let us observe that generic properties of $\mathrm{Cpct}(\mathbb{R}^d)$ can also be viewed as generic properties of non empty closed subsets of $\mathbb{R}^d$.

1) Indeed, at least for the problems considered in this paper, pertinent topologies on the set $\mathrm{Cl}(\mathbb{R}^d)$ of these closed subsets are obtained by an imbedding $F \mapsto \widehat{F}$ into $\mathrm{Cpct}(\widehat{\mathbb{R}^d})$ (endowed with the Vietoris topology; concerning hyperspace topologies, one can see Section 1.1 of [10], or the fourth chapter of [21]), where $\widehat{\mathbb{R}^d}$ is a compactification of $\mathbb{R}^d$ (that is a compact topological space having $\mathbb{R}^d$ as an everywhere dense open subset) and where $\widehat{F}$ denotes the closure of F in $\widehat{\mathbb{R}^d}$. Then $\mathrm{Cpct}(\mathbb{R}^d)$ is an everywhere dense open subspace of the compact space $\mathrm{Cpct}(\widehat{\mathbb{R}^d})$, and hence of $\mathrm{Cl}(\mathbb{R}^d)$.

The simplest case is when $\widehat{\mathbb{R}^d} = \mathbb{R}^d \cup \{\infty\}$ is the one point compactification of $\mathbb{R}^d$. A more pertinent case, for the study at infinity of the cut locus (this is done in [17]), is $\widehat{\mathbb{R}^d} = \mathbb{R}^d \cup \mathbb{S}(\mathbb{R}^d)$, where $\mathbb{S}(\mathbb{R}^d)$ is the sphere at infinity of $\mathbb{R}^d$; it is named the cosmic compactification in [21], and is associated to the "total" convergence in $\mathrm{Cl}(\mathbb{R}^d)$.

2) It is also natural to endow $\mathrm{Cl}(\mathbb{R}^d)$ with the usual Painlevé-Kuratowski convergence, which is associated here with the Wijsman topology, and also with the Fell topology. It is also the topology obtained by the imbedding $F \mapsto F \cup \{\infty\}$ of $\mathrm{Cl}(\mathbb{R}^d)$ into $\mathrm{Cpct}(\widehat{\mathbb{R}^d})$, where $\widehat{\mathbb{R}^d} = \mathbb{R}^d \cup \{\infty\}$ is the one point compactification of $\mathbb{R}^d$. $\mathrm{Cl}(\mathbb{R}^d)$ is then a metrisable separable locally compact space (an interesting distance on it is described in [21]).

Then $\mathrm{Cpct}(\mathbb{R}^d)$ is an everywhere dense, but meager, subspace of $\mathrm{Cl}(\mathbb{R}^d)$. However, generic elements of these two spaces behave very similarly for our purpose. More precisely, Proposition 3.8 and Theorem 3.9 remain true if we substitute in them $\mathrm{Cl}(\mathbb{R}^d)$ for $\mathrm{Cpct}(\mathbb{R}^d)$. To check this, one has to make similar adaptation for Lemma 3.1, 3.2 and 3.3 (adding

$||a-p|| \leq 1/\varepsilon$ in the proof of the the first, and $r \leq 1/\varepsilon$ in the proof of the second). The proof of the new proposition is then almost the same, the proof of the new theorem is the same.

4. Cut loci of the boundaries of most convex bodies of $\mathbb{R}^d$

4.1. Approximation of a convex body at a compact subset

The following adaptation of Lemma 3.4 is also essentially a simple generalization of the second point of Lemma 1 of [19], but our proof here is simpler.

Lemma 4.1. *Let $c \in \mathbb{R}^d$, $R > 0$ and F be a non empty closed subset of $\mathbb{R}^d$ such that $F \supset \mathbb{R}^d \setminus \mathrm{B}(c,R)$. Let K be a compact subset of $\mathbb{R}^d \setminus F$, $a \in K$ and $p = \mathrm{proj}_F(a)$. We suppose that $\forall b \in K \setminus \{a\}$, $p \notin \mathrm{Proj}_F(b)$, that for some $r > 0$,*

$$\mathrm{S}(c,R) \cap \mathrm{B}(p,r) = \partial F \cap \mathrm{B}(p,r),$$

and that $\mathrm{Proj}_F K$ is strongly porous at p in ∂F.

If $F' = \mathbb{R}^d \setminus \mathrm{conv} \bigcup_{b\in K} \mathrm{B}(b, \mathrm{dist}(b,F))$, then $(p,a) \in \mathrm{GrCurv}_{F'}$ and so we have $a \in \mathcal{N}_{F'}$.

Proof. We can suppose that $a \neq c$ and that $R = 1$.

Choose a sequence (p_n) in ∂F converging to p, satisfying $0 < ||p_n - p|| < ||a-p||/2$, $||p_n - p|| < r/2$, and such that $\delta_n := \mathrm{dist}(p_n, \mathrm{Proj}_F K) \sim ||p_n - p||$.

We will focus on the part $\widetilde{A_n}$ of $\partial F'$ above $A_n = \overline{\mathrm{B}}(p_n, \delta_n) \cap \mathrm{S}(c,1)$. More precisely, for $q \in A_n$ we denote by $\widetilde{q}$ the point of $\partial F'$ above q: $\widetilde{q} = q + \lambda(c-q)$ for a $\lambda \geq 0$ as small as possible and $\widetilde{A_n} := \{\widetilde{q}$ for $q \in A_n\}$. Put $h(x) := \mathrm{dist}(x,F)$ and $\mathrm{S}(x) := \mathrm{S}(x, h(x))$.

Let $X_n = \{q_1 \mid \mu_1 q_1 + \cdots + \mu_d q_d \in \widetilde{A_n}$, where $a_1, \ldots, a_d \in K$, $\mu_k > 0$, $\mu_1 + \cdots + \mu_d = 1$ and $q_k \in S(a_k)\}$, $K_n = \{b \in K \mid S(b) \cap X_n \neq \emptyset\}$ and $r_n = \sup\{h(x) \mid x \in K_n\}$.

We observe that $p \in \mathrm{Proj}_{F'}(a)$ and that $\partial F' \subset \overline{\mathrm{B}}(c,1)$. Hence $\partial F'$ contains no segment line $[p,q]$ with $q \neq p$. From this and because $\sup\{||p-q||, q \in \widetilde{A_n}\} \to 0$, we deduce that $\sup\{||p-q||, q \in X_n\} \to 0$, then that $\sup\{||a-b||, b \in K_n\} \to 0$. Hence (r_n) converges to $r_\infty = ||a-p||$.

For n large enough, we also have $\forall b \in K_n$, $\mathrm{proj}_F(b) \in \mathrm{S}(c,1)$.

We consider for such n the set

$$C_n = \mathrm{conv} \bigcup_{q \in \mathrm{S}(c,1) \setminus \mathrm{B}(p_n,\delta_n)} \overline{\mathrm{B}}(q + r_n(c-q), r_n).$$

Then $\widetilde{A_n} \subset C_n$ and F' contains the flat part $\mathrm{conv}\, S_n$ of ∂C_n, which is delimited by a $d-2$ dimensional sphere S_n of radius $\delta'_n(1-r_n)$, where $\delta'_n \sim \delta_n$ is the radius of the sphere $\mathrm{S}(p_n, \delta_n) \cap \mathrm{S}(c,1)$. With $q_n = \mathrm{proj}_{S_n}(p) \in F'$, we have $||q_n - p|| \sim ||p_n - p|| r_\infty$ and then

$$\limsup_{n\to\infty} \frac{(c-p \mid q_n - p)}{||p - q_n||^2} \geq \frac{r_\infty}{2}.$$

The conclusion follows.

Remark. In the proof above we have $||q_n - p||/||p_n - p|| \sim ||a-p||/R$. The case $||a-p|| = R$ is obvious and in the case $R = +\infty$, meaning that ∂F is flat at p, the lemma becomes false.

4.2. Generic cut locus

The following proposition will immediately yield Theorem 4.3, which we consider the main result of our paper, and which could also be deduced from Theorem 4.6. However we find better to give a direct proof.

Proposition 4.2. *Let K be a compact subset of $\mathbb{R}^d$ satisfying, for all $F \in \mathcal{D}_{\mathrm{S}}$, I) and II) of Proposition 2.1, $K \cap F = \emptyset$, and such that for every $S, S' \in \mathcal{D}_{\mathrm{S}}$ we have $K \cap \mathcal{M}_{S \cup S'} = \emptyset$.*

Then for most convex bodies C of $\mathbb{R}^d$ and $F = \mathbb{R}^d \setminus \operatorname{int} C$, we have $K \setminus F \subset \mathcal{N}_F$.

Proof. By Lemma 3.1, we only have to prove that the set G of such C is dense in $\mathrm{Cvb}(\mathbb{R}^d)$. So take $C_0 \in \mathrm{Cvb}(\mathbb{R}^d)$ and $\varepsilon > 0$, we can choose $C_1 \in \mathrm{Cvb}(\mathbb{R}^d)$ such that $\mathrm{dist}_{\mathrm{H}}(C_0, C_1) < \varepsilon$ and such that C_1 is the intersection of a finite family of closed balls of boundaries in $\mathcal{D}_{\mathrm{S}}$. Then $\emptyset = K \cap \partial C_1$. Now we can choose a finite subset A of $\operatorname{int} C_1$, such that $\mathrm{proj}_{\partial C_1}(A) \cap \mathrm{proj}_{\partial C_1}(K) = \emptyset$ and large enough so that $\mathrm{dist}_{\mathrm{H}}(C_0, C_2) < \varepsilon$, where

$$C_2 = \mathrm{conv} \bigcup_{a \in A \cup K \cap \operatorname{int} C_1} \overline{\mathrm{B}}(a, \mathrm{dist}(a, \partial C_1)).$$

We get then $C_2 \in G$ by Lemma 4.1. The proposition is proved.

Theorem 4.3. *Let h be a dimension function such that $t^d = \mathrm{o}(h(t))$.*

Then for most convex bodies C of $\mathbb{R}^d$ and $F = \mathbb{R}^d \setminus \operatorname{int} C$, we have $\mathcal{H}^h(\mathcal{N}_F) > 0$. Consequently we also have $\dim_{\mathrm{H}} \mathcal{N}_F = d$. More precisely, for every non empty open subset ω of $\operatorname{int} C$, we have $\mathcal{H}^h(\mathcal{N}_F \cap \omega) > 0$.

Proof. Using Proposition 2.1, we choose K as in Proposition 4.2 and such that $\mathcal{H}^h(K) > 0$ and we consider the countable set D of all compact subsets of the form $x + rK$, where $x \in \mathbb{Q}^d$ and $0 < r \in \mathbb{Q}$. It is obvious that these compact sets satisfy the same property as K. So for $C \in \mathrm{Cvb}(\mathbb{R}^d)$ generic, $F = \mathbb{R}^d \setminus \operatorname{int} C$ and for all $K' \in D$, we have $K' \subset \operatorname{int} C \Rightarrow K' \subset \mathcal{N}_F$. The theorem follows by taking $h(t) = t^d |\ln t|$ for t small enough to get $\dim_{\mathrm{H}} \mathcal{N}_F = d$.

4.3. Existence of non trivial local projection

In view of Lemma 4.4 we define

- $\mathrm{Projloc}^*_F(a) = \{p \in F$ such that for some neighborhood F' of p in F, we have $||a - p|| = \mathrm{dist}(a, F') < \mathrm{dist}(a, F'')$, where F'' is the boundary $\partial_F F'$ of F' in F (we consider $\mathrm{dist}(a, \emptyset) = +\infty)\}$.

Lemma 4.4. *Let K be a compact subset of $\mathbb{R}^d$ and U_K the set of all convex bodies C of $\mathbb{R}^d$ such that for every $a \in K$, $\mathrm{Projloc}^*_F(a) \not\subset \{a\}$, where $F = \mathbb{R}^d \setminus \operatorname{int} C$. Then U_K is an open subset of $\mathrm{Cvb}(\mathbb{R}^d)$.*

Moreover if for all $S \in \mathcal{D}_{\mathrm{S}}$ (defined in p. 830), $\mathrm{Proj}_S(K)$ is radially strongly porous, then U_K is everywhere dense in $\mathrm{Cvb}(\mathbb{R}^d)$

Similarly to our observation after Lemma 3.7, a subset K of $\mathbb{R}^d$ satisfying the last conclusion of Lemma 4.4 must be a $\mathcal{H}^d$-null set.

Proof. U_K is open because the set of all (a, F) such that $\mathrm{Projloc}^*_F(a) \not\subset \{a\}$ is an open subset of $\mathbb{R}^d \times \mathrm{Cpct}(\mathbb{R}^d)$. So we only have to prove the density of U_K.

Consider N rational closed balls $B_k = \overline{\mathrm{B}}(c_k, r_k) \subset \mathbb{R}^d \setminus K$, for $1 \leq k \leq N$ (thus $c_k \in \mathbb{Q}^d$ and $0 < r_k \in \mathbb{Q}$). We only have to prove that one can choose for each k a subset C_k of B_k in such a way that $C := \mathrm{conv}(C_1 \cup \cdots \cup C_N) \in U_K$.

Define $Q := \mathrm{conv}(B_1 \cup \cdots \cup B_N)$ and $\omega = \{p \in \partial Q$ such that a neighborhood in ∂Q of p is included in one of the rational spheres $\partial B_k\}$. Then, using Definition 5.1, p. 848 of the antiprojection Aproj, we have $\mathrm{Aproj}_Q(K) \subset \omega$ and $\varepsilon := \mathrm{dist}(\mathrm{Aproj}_Q(K), \partial Q \setminus \omega) > 0$.

Take $k \in \{1, \ldots, N\}$. We choose C_k in a way similar to that of the choice of Q in the proof of Lemma 3.6. With $\lambda > \frac{\pi}{2} \max\{||c_k - a||/r_k$, for $a \in K\}$. Because of the radial strong porosity of $\mathrm{Proj}_S(K)$, and thus of $A := \mathrm{Aproj}_Q(K) \cap \partial B_k$, we can now by Property 3.5, p. 838, find a finite covering of A by open balls $\mathrm{B}(p_i, \rho_i)$ with $p_i \in A$, $\lambda\rho_i \leq \varepsilon$, $2\lambda\rho_i \leq r_k\pi$ and

$$i \neq j \Rightarrow \emptyset = \mathrm{B}(p_i, \lambda\rho_i) \cap \mathrm{B}(p_j, \lambda\rho_j).$$

Observe now that for each $a \in K$ such that $B_k \cap \mathrm{Aproj}_Q(a) \neq \emptyset$,

$$C_k := \mathrm{conv}\left(B_k \setminus \bigcup_i \mathrm{B}(p_i, \lambda\rho_i)\right)$$

is flat at some point p of $\mathrm{Projloc}_{F_k}(a) \setminus \{a\}$ if $F_k = \mathbb{R}^d \setminus \mathrm{int}\, C_k$. We have then $p \in \mathrm{Projloc}^*_F(a) \setminus \{a\}$ if $F = \mathbb{R}^d \setminus \mathrm{int}\, C$, and thus $C \in U_K$.

4.4. Generic local cut locus

Proposition 4.5. *Let K be a compact subset of $\mathbb{R}^d$ satisfying I) and II) of Proposition 2.1 for all $F \in \mathcal{D}_\mathrm{S}$, and such that $K \cap \mathcal{M}_{S\cup S'} = \emptyset$ for every $S, S' \in \mathcal{D}_\mathrm{S}$. Then for most convex bodies C of $\mathbb{R}^d$ and $F = \mathbb{R}^d \setminus \mathrm{int}\, C$ we have*

$$K \subset \mathcal{N}_{\mathrm{loc},F} \cap \mathcal{N}_{\mathrm{LOC},F} \setminus \mathcal{M}_{\mathrm{loc},F}.$$

Remark. If K is a compact subset of

$$\mathbb{R}^d \setminus \bigcup_{S,S' \in \mathcal{D}_\mathrm{S}} \mathcal{M}_{S\cup S'}, \quad \text{or of} \quad \mathbb{R}^d \setminus \bigcup_{H,H' \in \mathcal{D}_\mathrm{H}} \mathcal{M}_{H\cup H'},$$

then we have, for most convex bodies C of $\mathbb{R}^d$ and $F = \mathbb{R}^d \setminus \mathrm{int}\, C$,

$$K \cap \mathcal{M}_{\mathrm{loc},F} = \emptyset.$$

Proof. 1) Observe that $G' = \{C \in \mathrm{Cvb}(\mathbb{R}^d)$ such that $F = \mathbb{R}^d \setminus \mathrm{int}\, C \Rightarrow K \cap \mathcal{M}_{\mathrm{loc},F} = \emptyset\}$ is a G_δ by Lemma 3.2, which is dense because it contains every finite intersection of rational closed balls (of not empty interior), because of the assumption $\emptyset = K \cap \mathcal{M}_{S\cup S'}$. So it remains to prove that for most $C \in \mathrm{Cvb}(\mathbb{R}^d)$ and $F = \mathbb{R}^d \setminus \mathrm{int}\, C$ we have $K \subset \mathcal{N}_{\mathrm{loc},F} \cap \mathcal{N}_{\mathrm{LOC},F}$.

2) Let H be a closed rational half space of $\mathbb{R}^d$ (then $\partial H \in \mathcal{D}_\mathrm{H}$). We denote by U_H the set of the convex bodies of $\mathbb{R}^d$ such that $\mathrm{int}\, C \setminus H \neq \emptyset \neq H \cap \mathrm{int}\, C$. Put $G_H = \{C \in U_H$ such

that if $F_{H,C} := \mathbb{R}^d \setminus \operatorname{int}(H \cup C)$ and $K_{H,C} := \{a \in K \cap \operatorname{int} H$ such that $\operatorname{Proj}_{F_{H,C}}(a) \setminus H \neq \emptyset\} \subset \mathcal{N}_{F_{H,C}}\}$. U_H is obviously an open subset of $\operatorname{Cvb}(\mathbb{R}^d)$, and G_H is a G_δ of U_H because for every $\varepsilon > 0$, $\{C \in U_H$ such that there exist $b \in \mathbb{R}^d$, $p \in \operatorname{Proj}_{F_{H,C}}(b)$ and $a \in K \cap [p,b]$ such that $||a-b|| \geq \varepsilon$, $\operatorname{dist}(p,H) \geq \varepsilon$ and $\operatorname{dist}(a, \mathbb{R}^d \setminus H) \geq \varepsilon\}$ is closed in U_H.

We prove now that G_H is dense in U_H. Consider $C_0 \in U_H$ and V an open neighborhood in U_H of C_0. We choose $C_1 \in V$ which is the intersection of a finite number of rational closed balls.

For $\varepsilon > 0$ we define $C_1^{\geq\varepsilon} = \{a \in C_1$ such that $\operatorname{dist}(a, \mathbb{R}^d \setminus C_1) \geq \varepsilon\}$,

$$C_2 = C_1^{\geq\varepsilon} \cup \left(\bigcup_{a \in K \cap H} \overline{\mathrm{B}}(a, \operatorname{dist}(a, F_{H,C_1})) \setminus \operatorname{int} H \right)$$

and $C_3 = \operatorname{conv} C_2$. We choose ε small enough so that $C_3 \in V$. Then we claim that $C_3 \in G_H$. To check this, observe first that $C_3 \subset C_1$ and $K_{C_3,H} \setminus K_{C_1,H} \subset \mathcal{M}_{F_{H,C_3}} \subset \mathcal{N}_{F_{H,C_3}}$. We choose thus $a \in K_{C_1,H} \setminus \mathcal{M}_{F_{H,C_3}}$ and we must check that $a \in \mathcal{N}_{F_{H,C_3}}$. Observe that $p := \operatorname{proj}_{F_{H,C_3}}(a) = \operatorname{proj}_{F_{H,C_1}}(a) \in \partial C_1 \setminus H$ has a neighborhood in ∂C_1 which is included in a rational sphere $\mathrm{S}(c,r)$ such that $C_1 \subset \overline{\mathrm{B}}(c,r)$.

We also have, by Lemma 4.1, $(p,a) \in \operatorname{GrCurv}_F$, where

$$F = \mathbb{R}^p \setminus \operatorname{conv} \bigcup_{b \in L} \mathrm{B}(b, \operatorname{dist}(b, F_{H,C_1}))$$

and where $L := \{b \in K \cap H$ such that $\mathrm{B}(b, \operatorname{dist}(b, F_{C_1,H})) \subset \mathrm{B}(c,r)\}$ is a closed neighborhood of a in K.

So we just have to check that ∂F contains some neighborhood of p in $\partial F_{H,C_3}$, i.e. in ∂C_3. We can choose now a closed half space H' of $\mathbb{R}^d$ not containing p but containing the following subset A of C_1, and hence of $\overline{B}(c,r)$,

$$A = C_1^{\geq\varepsilon} \cup \left(\bigcup_{b \in K \cap H \setminus L} \overline{\mathrm{B}}(b, \operatorname{dist}(b, F_{H,C_1})) \setminus \operatorname{int} H \right)$$

Observe then that any point of $\partial C_3 \setminus \partial F$ must lie on some tangent line Δ such that $\Delta \cap \mathrm{B}(a, ||a-p||) \setminus H = \emptyset$ and $\Delta \cap \overline{\mathrm{B}}(c,r) \cap H' \neq \emptyset$. Since the union X of all such lines is a closed subset of $\mathbb{R}^d$ and $p \notin X$, we have proved that G_H is everywhere dense in U_H.

Thus G_H is a G_δ dense subset of the open U_H of $\operatorname{Cvb}(\mathbb{R}^d)$, so $G''_H = G_H \cup (\operatorname{Cvb}(\mathbb{R}^d) \setminus U_H)$ is a G_δ dense subset of $\operatorname{Cvb}(\mathbb{R}^d)$.

3) Thus most $C \in \operatorname{Cvb}(\mathbb{R}^d)$ are in the intersection G'', over all $H \in \mathcal{D}_{\mathrm{S}}$, of the sets G''_H; hence they are also in $G = G' \cap G'' \cap G'''$, where $G''' = \{C \in \operatorname{Cvb}(\mathbb{R}^d)$ such that if $F = \mathbb{R}^d \setminus \operatorname{int} C$, then $\forall a \in K$, $\operatorname{Projloc}^*_F(a) \not\subset \{a\}\}$ is a dense open subset of $\mathbb{R}^d$ by Lemma 4.4.

Let then $C \in G$, $F = \mathbb{R}^d \setminus \operatorname{int} C$ and $a \in K$. As $C \in G'''$, we can choose $p \in \operatorname{Projloc}_F(a) \setminus \{a\}$. Because $C \in G'$, we can also choose H a rational closed half space such that $C \in U_H$, $a \in \operatorname{int} H$ and $p = \operatorname{proj}_{F_{H,C}}(a) \notin H$. As $C \in G \subset G'' \subset G''_H$, we also have $C \in G_H$. We also have $a \in K_{H,C}$, hence $a \in \mathcal{N}_{F_C}$ and $a \in \mathcal{N}_{\mathrm{loc},F}$.

Because $a \in \mathcal{N}_{F_{H,C}} \setminus \mathcal{M}_{F_{H,C}}$ we also have $(p, a) \in \mathrm{GrCurv}_{F_{H,C}}$, hence $(p, a) \in \mathrm{GrCurv}_F$. As this is true for every $p \in \mathrm{Projloc}_F(a) \setminus \{a\}$, we also have $a \in \mathcal{N}_{\mathrm{LOC},F}$. This ends the proof.

As Theorem 4.3 was deduced from Proposition 4.2, we deduce now the following result from Proposition 4.5.

Theorem 4.6. *Let h be a dimension function such that $t^d = \mathrm{o}(h(t))$. Then for most convex bodies C of $\mathbb{R}^d$ and $F = \mathbb{R}^d \setminus \mathrm{int}\, C$, we have for every non empty open subset ω of $\mathbb{R}^d$*

$$\mathcal{H}^h(\omega \cap \mathcal{N}_{\mathrm{loc},F} \cap \mathcal{N}_{\mathrm{LOC},F} \setminus \mathcal{M}_{\mathrm{loc},F}) > 0.$$

Consequently we also have $\dim_{\mathrm{H}}(\omega \cap \mathcal{N}_{\mathrm{loc},F} \cap \mathcal{N}_{\mathrm{LOC},F} \setminus \mathcal{M}_{\mathrm{loc},F}) = d$.

For $a \in \mathbb{R}^d \setminus \mathcal{M}_{\mathrm{loc},F}$ we have $\mathrm{card}\,\mathrm{Projloc}_F(a) \leq \mathrm{card}\,\mathbb{N}$. For $a \in \mathbb{R}^d$ and for most $C \in \mathrm{Cvb}(\mathbb{R}^d)$, and $F = \mathbb{R}^d \setminus \mathrm{int}\, C$, we have $\mathrm{card}\,\mathrm{Projloc}_F(a) = \mathrm{card}\,\mathbb{N}$. This is also true for most $C \in \mathrm{Cvb}(\mathbb{R}^d)$ and for most $a \in \mathbb{R}^d$.

Question 4.7. What can be said for most $C \in \mathrm{Cvb}(\mathbb{R}^d)$, and $F = \mathbb{R}^d \setminus \mathrm{int}\, C$, of $\dim_{\mathrm{H}}\{a \in \mathbb{R}^d$ such that $\mathrm{Projloc}_F(a)$ is infinite$\}$?

4.5. Curvature of a generic convex surface

Let C be a closed convex subset of $\mathbb{R}^d$ of non empty interior, with $C \neq \mathbb{R}^d$. For $p \in \partial C$, let $\varphi(p)$ be the least upper bound of the numbers $||a - p||$, where $a \in C$ and $p \in \mathrm{Proj}_{\partial C}(a)$. Let $\widetilde{C}$ be the subspace of the Euclidean space $\mathbb{R}^d \times \mathbb{R} \simeq \mathbb{R}^{d+1}$ of all the $(p, t) \in \partial C \times \mathbb{R}_+$ such that $t \leq \varphi(p)$. We have then a continuous mapping $\Phi_C : \widetilde{C} \to C$ which associates to (p, t) the unique $a \in C$ such that $p \in \mathrm{Proj}_{\partial C}(a)$ and $||a - p|| = t$. Moreover, φ is upper semi continuous, so $\widetilde{C}$ is closed in $\mathbb{R}^{d+1}$ (and compact when C is compact).

Lemma 4.8. *The mapping Φ_C defined above is 1-Lipschitzian on $\widetilde{C}$.*

Proof. Let us suppose that $a, b \in C$, $p \in \mathrm{Proj}_{\partial C}(a)$ and $q \in \mathrm{Proj}_{\partial C}(b)$. Considering the tangent hyperplanes at p and q of ∂C, we must have then $(a - p \mid q - p) \geq 0$ and $(b - q \mid p - q) \geq 0$. Let $r = ||a - p||$, $\rho = ||b - q||$ and $s = ||p - q||$. We must then prove that

$$||a - b|| \leq \sqrt{s^2 + (r - \rho)^2}. \tag{5}$$

We can suppose that $a \neq b$ and $r > 0$. Let $C' = \mathrm{conv}(\mathrm{S}(a, r) \cup \mathrm{S}(b, \rho))$, then $p, q \in \partial C'$, so we can suppose that $C = C'$.

Then ∂C is of revolution around the line $\Delta = a + \mathbb{R}(b - a)$, we have $p \in A := \partial C \cap \mathrm{S}(a, r)$ and $q \in B := \partial C \cap \mathrm{S}(b, \rho)$. Moreover, there are two points a' and b' of Δ such that $A = \{x \in \mathrm{S}(a, r)$ such that $(x - a \mid b - a) \leq (a' - a \mid b - a)\}$ and $B = \{x \in \mathrm{S}(b, \rho)$ such that $(x - a \mid b - a) \geq (b' - a \mid b - a)\}$ (we also have $(a' - a \mid b - a) \leq (b' - a \mid b - a)$).

(In what follows a, b, r and ρ are known and we look for the lower value of s.) We can suppose that a, b, p and q are in a same plane H, and even in the same half plane H_+ delimited by Δ. Now it is easy to see that the lower value of s is obtain when $p - a \perp b - a$ and $q - b \perp b - a$. Then (5) is true as an equality.

Lemma 4.8 immediately implies the following result.

Property 4.9. If C is a closed convex subset of $\mathbb{R}^d$ of non empty interior, $C \neq \mathbb{R}^d$ and $A \subset C$, then we have the inequality

$$\dim_{\mathrm{H}}(A) \leq 1 + \dim_{\mathrm{H}}(\mathrm{Proj}_{\partial C} A).$$

Indeed, we have $\Phi^{-1}(A) \subset \mathrm{Proj}_{\partial C}(A) \times \mathbb{R}$, and thus

$$\dim_{\mathrm{H}}(A) \leq \dim_{\mathrm{H}}(\mathrm{Proj}_{\partial C}(A) \times \mathbb{R}) = 1 + \dim_{\mathrm{H}} \mathrm{Proj}_{\partial C} A$$

(see for instance [8], p. 95 for the last equality). From Property 4.9 and Theorem 4.6 we get the next result.

Theorem 4.10. *For most convex bodies C of $\mathbb{R}^d$ and $F = \mathbb{R}^d \setminus \mathrm{int}\, C$,*

$$\dim_{\mathrm{H}} \mathrm{Proj}_{\partial C}(\mathcal{N}_F \cap \mathcal{N}_{\mathrm{LOC},F} \setminus \mathcal{M}_{\mathrm{loc},F}) = d - 1.$$

More precisely, for every non empty open interval I of $\mathbb{R}_+^*$ and for every non empty open subset J of ∂C, we also have

$$\dim_{\mathrm{H}} P_{I,J} = d - 1,$$

where $P_{I,J} := \mathrm{pr}_1(G_{I,J})$, and $G_{I,J} = \{(p,a)$ such that $a \in \mathcal{N}_{\mathrm{loc},F} \cap \mathcal{N}_{\mathrm{LOC},F} \setminus \mathcal{M}_{\mathrm{loc},F}$, $||a-p|| \in I$ and $p \in \mathrm{Projloc}_F(a) \cap J\} \subset \mathrm{GrCurv}_F$.

Thus we have a large set $X = \{p$ where $(p,a) \in \mathrm{GrCurv}_F\}$ of points in ∂C where the upper curvature of ∂C is finite and positive:

$$\dim_{\mathrm{H}}(X) = d - 1$$

and an analogous property restricted to the cases where the curvature radius belongs to I and where the local projection belongs to J.

Proof. The first equality is an immediate consequence of Property 4.9 and Theorem 4.3. For the second equality, we suppose C "generic", $F = \mathbb{R}^d \setminus \mathrm{int}\, C$ and I, J as above. We can also suppose that $\lambda := \inf I > 0$. Consider $p \in J$ and $x \in \mathbb{R}^d$ such that $||x|| = 1$ and for each $a \in C$, $(a - p \mid x) \geq 0$. Because C is strictly convex (Theorem 1.13), if we choose $r > 1 + \mathrm{diam}\, J + \lambda$ large enough then we will have for all $a \in \mathrm{B}(p + rx, 1)$, $\mathrm{Projloc}_F(a) \setminus \{a\} \subset J$.

We use Theorem 4.6 to choose $a \in \mathrm{B}(p + rx, 1) \cap \mathcal{N}_{\mathrm{loc},F}$. We can now choose $q \in \mathrm{Projloc}_F(a) \setminus \{a\}$, so we have $q \in J$ and $||a - q|| > \lambda$. Thus we can choose $b \in [a, q]$ and some closed neighborhood A in F of q such that $||b - q|| \in I$, $A \cap \partial C \subset J$ and $q = \mathrm{proj}_A(b)$. As C is strictly convex, we can more precisely suppose that $A = H_A \setminus \mathrm{int}\, C$ for some closed half-space H_A such that $b \notin H_A$.

The set $\omega := \{c \in \mathbb{R}^d$ such that $\mathrm{Proj}_A(c) \subset \mathrm{int}_F(A)$ and $\mathrm{dist}(c, A) \in I\}$ is an open neighborhood of b. Then $\mathrm{Proj}_A(\omega \cap \mathcal{N}_{\mathrm{loc},F} \cap \mathcal{N}_{\mathrm{LOC},F} \setminus \mathcal{M}_{\mathrm{loc},F}) \subset P_{I,J}$. By Theorem 4.6 we have $\dim_{\mathrm{H}}(\omega' \cap \mathcal{N}_{\mathrm{loc},F} \cap \mathcal{N}_{\mathrm{LOC},F} \setminus \mathcal{M}_{\mathrm{loc},F}) = d$ for every non empty open subset ω' of ω.

With $\varepsilon > 0$ small enough so that $\mathrm{B}(b, \varepsilon) \subset \omega$, we associate to ε the intersection C_ε of all closed half spaces H containing C and such that $(\partial H) \cap \overline{\mathrm{B}}(q, \varepsilon) \cap A \cap \partial C \neq \emptyset$. We also consider the union B_ε of all closed balls $\overline{\mathrm{B}}(c, \mathrm{dist}(c, A))$ for $c \in \overline{\mathrm{B}}(b, \varepsilon)$. Observe that $H_A \cap B_\varepsilon \subset C_\varepsilon$, so we can choose ε small enough so that $B_\varepsilon \subset C_\varepsilon$.

Now we observe that for all $c \in B(b,\varepsilon) \setminus \mathcal{M}_{\mathrm{loc},F}$ we have $\mathrm{proj}_A(c) \in \mathrm{Proj}_{\partial C_\varepsilon}(c)$, thus Property 4.9 applied to the convex body C_ε gives

$$\dim_{\mathrm{H}} \mathrm{Proj}_A(B(b,\varepsilon) \cap \mathcal{N}_{\mathrm{loc},F} \cap \mathcal{N}_{\mathrm{LOC},F} \setminus \mathcal{M}_{\mathrm{loc},F}) = d-1,$$

hence $\dim_{\mathrm{H}} P_{I,J} = d-1$. This ends the proof.

Question 4.11. For $\lambda > 0$ and for a "generic convex body" C (and $F = \mathbb{R}^d \setminus \mathrm{int}\, C$), what can be said about $\dim_{\mathrm{H}} X_{C,\lambda}$, where $X_{C,\lambda} := \{p \in \partial C \text{ such that } \mathrm{curv}_p(F) = \lambda\}$?

It can be proved (mainly by a Baire class argumentation) that $X_{C,\lambda}$ contains a Cantor set (so $\mathrm{card}\, X_{C,\lambda} = \mathrm{card}\, \mathbb{R}$) but we know nothing about its Hausdorff dimension.

4.6. Convex hull of a generic compact subset of $\mathbb{R}^d$

We end this Section 4 by an interesting, but much easier property (whose the last point is also proved in [28], with a different argumentation).

Property 4.12. Let h be a dimension function. For most non empty compact subsets K of $\mathbb{R}^d$, if we set $C = \mathrm{conv}\, K$ and $F = \mathbb{R}^d \setminus \mathrm{int}\, C$, then

- $\mathrm{CurvCt}_F = \emptyset$,
- $\mathrm{Proj}_F(\mathrm{int}\, C) = \{\lambda_1 p_1 + \cdots + \lambda_d p_d \in \partial C$, where $p_1, \ldots, p_d$ are affinely independant points of $K \cap \partial C$, $\lambda_k > 0$ and $\lambda_1 + \cdots + \lambda_d = 1\}$ (thus ∂C is precisely flat on $\mathrm{Proj}_F(\mathrm{int}\, C)$ and $\mathrm{curv}_F = +\infty$ on $\partial C \setminus \mathrm{Proj}_F(\mathrm{int}\, C)$),
- for every integer n, $\mathcal{H}^h(K^n) = 0$. Thus $\dim_{\mathrm{H}} \partial C \setminus \mathrm{Proj}_F(\mathrm{int}\, C) = d-2$.

(Roughly speaking, the property says that C behaves like a convex polytope.)

So we have then $\mathcal{N}_F = \mathcal{M}_F$, and it can be proved that $\overline{\mathcal{N}_F} = \mathcal{N}_F \cup \partial C \setminus \mathrm{Proj}_F(\mathrm{int}\, C)$, that $\overline{\mathcal{N}_F}$ is contractible and locally arcwise connected.

Proof of Property 4.12. For the second point, observe that for every $\varepsilon > 0$, the following set is closed in $\mathrm{Cpct}(\mathbb{R}^d)$: $\{K \in \mathrm{Cpct}(\mathbb{R}^d)$ for which there exists $p_1, \ldots, p_{d-1} \in K$ and $a \in \mathrm{int}\, C$ with $\mathrm{dist}(a,F) \geq \varepsilon$ and $\mathrm{Proj}_F(a) \cap \mathrm{conv}(p_1, \ldots, p_{d-1}) \neq \emptyset\}$. The first point follows from the second. For the third point, observe that for every $\varepsilon > 0$, the set $\{K \in \mathrm{Cpct}(\mathbb{R}^d)$ such that there exists a finite covering $(A_k)_{1 \leq k \leq N}$ with $N^n \sum h(\mathrm{diam}\, A_k) < \varepsilon\}$ is open.

5. Adaptation to the antiprojection problem

5.1. Definitions, introduction

We describe now the cut locus $\mathrm{a}\mathcal{N}_B$ generated by the farthest projection to B. The notations will emphasize the analogy with the cut locus $\mathcal{N}_F$ generated by the nearest projection to F.

Definition 5.1 (metric farthest projection). Let $B \subset \mathbb{R}^d$, $a \in \mathbb{R}^d$ and $A \subset \mathbb{R}^d$. We define

- $\mathrm{adist}(a,B) = \sup\{||a-p|| \mid p \in B\}$,
- $\mathrm{Aproj}_B(a) = \{p \in B \mid ||a-p|| = \mathrm{adist}(a,B)\}$,

- $\mathrm{Aproj}_B(A) = \bigcup_{b\in A} \mathrm{Aproj}_B(b)$.

When $\mathrm{card}\,\mathrm{Aproj}_B(a) = 1$ we set $\mathrm{Aproj}_B(a) = \{\mathrm{aproj}_B(a)\}$.

Definition 5.2 (cut locus). Let B be a non empty compact subset of $\mathbb{R}^d$. We define

- $\mathrm{a}\mathcal{M}_B = \{a \in \mathbb{R}^d \mid \mathrm{card}\,\mathrm{Aproj}_F(a) \geq 2\}$,
- $\mathrm{a}\mathcal{N}_F = \{a \in \mathbb{R}^d \mid \forall b \in \mathbb{R}^d \setminus \{a\}$ and $p \in \mathrm{Aproj}_B(b)$, $b \notin [a,p]\}$.

Example 5.3 Let $d = 2$ and B be a non circular ellipse. Consider the two points p and q of B where the curvature of B is minimal, and the two curvature centers a, b of B at these points. Then

$$\mathrm{a}\mathcal{N}_B = [a,b] \quad \text{and} \quad \mathrm{a}\mathcal{M}_B = [a,b] \setminus \{a,b\}.$$

Moreover, we have $p = \mathrm{aproj}_B(a)$, $q = \mathrm{aproj}_B(b)$, and $\mathrm{card}\,\mathrm{Aproj}_B(c) = 2$ for all $c \in \mathrm{a}\mathcal{M}_B$.

The properties of $\mathrm{a}\mathcal{N}$ are very similar to those of $\mathcal{N}$. For example we always have $\mathrm{a}\mathcal{M}_B \subset \mathrm{a}\mathcal{N}_B$. It can be checked that $\mathrm{a}\mathcal{N}_B$ is always of null $\mathcal{H}^d$-measure (by the same way as for $\mathcal{N}_F$ in [18]). Despite the fact that $\mathrm{adist}(.,B)$ is less often considered than $\mathrm{dist}(.,B)$, the properties of $\mathrm{a}\mathcal{N}$ are rather simpler than those of $\mathcal{N}$. One explanation for this is that the multivalued operator $A : a \mapsto a - (\mathrm{conv}\,\mathrm{Aproj}_B(a))$ is more regular ($-I + A$ is monotone) than $B : a \mapsto \mathrm{conv}\,\mathrm{Proj}_B(a) - a$ ($I + B$ is monotone).

Strong connectedness properties of $\mathrm{a}\mathcal{M}$ have been proved by Westphal and Schwartz [27]. De Blasi and Myjak [5] have shown the density of $\mathrm{a}\mathcal{M}$ for most non empty compact subsets B of a strictly convex separable Banach space E of dimension ≥ 2.

Definition 5.4 (local farthest pojection). Let $B \subset \mathbb{R}^d$, $a \in \mathbb{R}^d$ and $A \subset \mathbb{R}^d$. We define

- $\mathrm{Aprojloc}_B(a) = \{p \in B \mid \exists r > 0, \forall q \in B, ||q-p|| < r \Rightarrow ||q-a|| \leq ||p-a||\}$,
- $\mathrm{Aprojloc}_B(A) = \bigcup_{b\in A} \mathrm{Aprojloc}_B(b)$,

Definition 5.5 (curvature). Let B be a non empty compact subset of $\mathbb{R}^d$, we define

- $\mathrm{GrAcurv}_B = \{(p,a) \in B \times \mathbb{R}^d \mid t > 1 \Rightarrow p \in \mathrm{Aprojloc}_B(p + t(a-p))$ and $0 \leq t < 1 \Rightarrow p \notin \mathrm{Aprojloc}_B(p + t(a-p))\}$.
- $\mathrm{AcurvCt}_B = \{a \mid (p,a) \in \mathrm{GrAcurv}_B\}$.

Thus if C is a convex body of $\mathbb{R}^d$, the relation $(p,a) \in \mathrm{GrAcurv}_C$ means that a is a curvature center of ∂C at p, relatively to the “lower curvature” $\mathrm{acurv}_p(C)$, that is the supremum of $1/||p-a||$ for $a \in \mathbb{R}^d$ such that $p \in \mathrm{Aprojloc}_C(a)$. $\mathrm{AcurvCt}_C$ is the set of all such centers of curvature.

Definition 5.6 (local cut locus). Let B be a compact subset of $\mathbb{R}^d$, we define

- $\mathrm{a}\mathcal{M}_{\mathrm{loc},B} = \{a \in \mathbb{R}^d, \exists p,q \in \mathrm{Aprojloc}_B(a), ||a-p|| = ||a-q||$ and $p \neq q\}$,
- $\mathrm{a}\mathcal{N}_{\mathrm{LOC},B} = \{a \in \mathbb{R}^d \mid \forall b \in \mathbb{R}^d \setminus \{a\}$ and $p \in \mathrm{Aprojloc}_B(b), b \notin [a,p]\}$.

We note some elementary relations.

Property 5.7. For any non empty compact subset B of $\mathbb{R}^d$ and $C = \mathrm{conv}\,B$, we have

$$\mathrm{a}\mathcal{N}_{\mathrm{LOC},B} \subset \mathrm{a}\mathcal{N}_B,$$

$$\mathrm{adist}(.,B) = \mathrm{adist}(.,C),\ \mathrm{Aproj}_B = \mathrm{Aproj}_C,\ \mathrm{a}\mathcal{M}_B = \mathrm{a}\mathcal{M}_C,\ \mathrm{a}\mathcal{N}_B = \mathrm{a}\mathcal{N}_C,$$
$$\mathrm{Aprojloc}_B \cap \partial C \supset \mathrm{Aprojloc}_C,\ \mathrm{a}\mathcal{M}_{\mathrm{loc},B} \supset \mathrm{a}\mathcal{M}_{\mathrm{loc},C},\ \mathrm{a}\mathcal{N}_{\mathrm{LOC},B} \subset \mathrm{a}\mathcal{N}_{\mathrm{LOC},C}$$
$$\text{and} \quad \mathrm{GrAcurv}_B \supset \mathrm{GrAcurv}_C.$$

Proof. We prove only the last inclusion, which is least simple. We assume that $(p,a) \in \mathrm{GrAcurv}_C \backslash \mathrm{GrAcurv}_B$, and find a contradiction. We can assume that $||p-a|| = 1$. Because $(p,a) \in \mathrm{GrAcurv}_C$, we can choose $R > 1$ large enough so that $C \subset \overline{\mathrm{B}}(p + R(a-p), R)$. Because $(p,a) \notin \mathrm{GrAcurv}_B$, we can choose $0 < \rho < 1$ and $r > 0$ such that $B \cap \mathrm{B}(p,r) \subset \overline{\mathrm{B}}(p + \rho(a-p), \rho)$. Consider a half space $H = \{x \in \mathbb{R}^d \mid (x-p \mid a-p) \geq \varepsilon\}$, for some $\varepsilon > 0$ small enough so that $\overline{\mathrm{B}}(p + R(a-p), R) \setminus H \subset \mathrm{B}(p,r)$. Then C is included in the convex body

$$C' = \mathrm{conv}\left((\overline{\mathrm{B}}(p + \rho(a-p), \rho) \setminus H) \cup (\overline{\mathrm{B}}(p + R(a-p), R) \cap H)\right).$$

But we have, for $r' > 0$ small enough, $\mathrm{B}(p,r') \cap C' = \mathrm{B}(p,r') \cap \overline{\mathrm{B}}(p + \rho(a-p), \rho)$, hence $\mathrm{B}(p,r') \cap C \subset \overline{\mathrm{B}}(p + \rho(a-p), \rho)$, in contradiction with $(p,a) \in \mathrm{GrAcurv}_C$.

Lemma 5.8. *For a given compact subspace K of $\mathbb{R}^d$, the set of all non empty compact subsets B of $\mathbb{R}^d$ such that $K \cap \mathrm{a}\mathcal{M}_{\mathrm{loc}} \neq \emptyset$ is a F_σ of $\mathrm{Cpct}(\mathbb{R}^d)$.*

Indeed for a given $\varepsilon > 0$ the following condition define a closed subspace of $\mathrm{Cpct}(\mathbb{R}^d)$: $\exists a \in K$, $r \geq \varepsilon$, $p, q \in \mathrm{S}(a,r) \cap B$ such that $||p-q|| \geq \varepsilon$ and $B \cap (\mathrm{B}(p,\varepsilon) \cup \mathrm{B}(q,\varepsilon)) \setminus \overline{\mathrm{B}}(a,r) = \emptyset$.

5.2. Approximation of B at a compact subset

Lemma 5.9. *Let $c \in \mathbb{R}^d$, $r > 0$, $R > 0$ and C be a convex body of $\mathbb{R}^d$ containing the closed ball $\overline{\mathrm{B}}(c,R)$. Let K be a compact subset of $\mathbb{R}^d$, $a \in K$ and $p = \mathrm{aproj}_C(a)$. Assume that $\forall b \in K \setminus \{a\}$, $p \notin \mathrm{Aproj}_C\, b$, that*

$$\mathrm{S}(c,R) \cap \mathrm{B}(p,r) = \partial C \cap \mathrm{B}(p,r),$$

and that $\mathrm{Aproj}_C\, K$ is strongly porous at p in ∂C. Define

$$C' = \bigcap_{b \in K} B(b),$$

where $B(b) := \overline{\mathrm{B}}(b, \mathrm{adist}(b,C))$.

Then $(p,a) \in \mathrm{GrAcurv}_{C'}$ and therefore $a \in \mathrm{a}\mathcal{N}_{C'}$.

Proof. It is similar to the proof of Lemma 4.1. We can suppose that $a \neq c$ and that $R = 1$.

Choose a sequence (p_n) in $\mathrm{B}(p,r) \cap \partial C \setminus \{p\}$ converging to p, satisfying $||p_n - p|| < r/2$, and such that $\delta_n := \mathrm{dist}(p_n, \mathrm{Aproj}_C\, K) \sim ||p_n - p||$. We have thus $||p - p_n|| = \delta_n + \varepsilon_n \delta_n$, where (ε_n) converges to zero.

Denote by Δ_n the half line $p_n + \mathbb{R}_+(p_n - c)$ and we set $\Delta_n \cap \partial C' = \{p'_n\}$. We can choose $a_n \in K$ such that $p'_n \in \mathrm{Aproj}_{C'}(a_n)$. Then (a_n) converges to a and $r_n := ||a_n - p'_n||$ converges to $r_\infty = ||a - p||$; moreover, for n large enough, $\mathrm{aproj}_C(a_n) \in \mathrm{S}(c,1)$.

For such n we consider now the convex body

$$C_n = \bigcap_{q \in \mathrm{S}(c,1) \setminus \mathrm{B}(p_n,\delta_n)} B_n(q),$$

where $B_n(q) := \overline{\mathrm{B}}(q + r_n(c-q), r_n)$. Set $\Delta_n \cap \partial C_n = \{p_n''\}$ and $S_n = \mathrm{S}(c,1) \cap \mathrm{S}(p_n,\delta_n)$. We observe that $p_n'' \in C'$ and that $p_n'' \in \partial B_n(q)$ for all $q \in S_n$. We thus consider $q_n = \mathrm{proj}_{S_n}(p)$, $H_n = q_n + (c - q_n)^\perp$ and $\{q_n'\} = H_n \cap \Delta_n$.

Use the decomposition $p_n'' - p = (p_n'' - q_n') + (q_n' - q_n) + (q_n - p)$ and observe that $(q_n - p \mid c - p) \sim (\varepsilon_n \delta_n)^2/2 = \varepsilon_n^2 \delta_n^2/2$, $(q_n' - q_n \mid c - p) \sim (\varepsilon_n \delta_n)\delta_n = \varepsilon_n \delta_n^2$ and $(p_n'' - q_n' \mid c - p) \sim \frac{\delta_n^2}{2r_n} \sim \frac{\delta_n^2}{2r_\infty}$.

Thus $(p_n'' - p \mid c - p) \sim \frac{\delta_n^2}{2r_\infty} \sim \frac{||p-p_n||^2}{2r_\infty}$, hence $(p,a) \in \mathrm{GrAcurv}_{C'}$. The conclusion follows.

5.3. Generic cut loci for the farthest projection

Proposition 5.10. *Let K be a compact subset of $\mathbb{R}^d$ satisfying for all $F, F' \in \mathcal{D}_\mathrm{S}$, I) and II) of Proposition 2.1, and $K \cap \mathrm{a}\mathcal{M}_{F \cup F'} = \emptyset$. Then*

$$K \subset \mathrm{a}\mathcal{N}_{\mathrm{LOC},B} \setminus \mathrm{a}\mathcal{M}_{\mathrm{loc},B} \qquad (\subset \mathrm{a}\mathcal{N}_B \setminus \mathrm{a}\mathcal{M}_B)$$

for most non empty compact subset B of $\mathbb{R}^d$, and also for most convex bodies B of $\mathbb{R}^d$.

Proof. 1) $G_K' = \{B \in \mathrm{Cpct}(\mathbb{R}^d)$ such that $K \cap \mathrm{a}\mathcal{M}_{\mathrm{loc},B} = \emptyset\}$ is a G_δ by Lemma 5.8, which is dense because it contains all finite union of disjointed closed rational balls, from the condition $K \cap \mathrm{a}\mathcal{M}_{F \cup F'} = \emptyset$ for $F, F' \in \mathcal{D}_\mathrm{S}$. Thus for most non empty compact subset B of $\mathbb{R}^d$ and for $C = \mathrm{conv}\, B$, we have $K \cap \mathrm{a}\mathcal{M}_{\mathrm{loc},C} = \emptyset$. This also holds for most convex bodies C of $\mathbb{R}^d$ because $B \mapsto \mathrm{conv}\, B$ is a retraction of $\mathrm{Cpct}(\mathbb{R}^d)$ to the closure of its subspace $\mathrm{Cvb}(\mathbb{R}^d)$, and because $\mathrm{a}\mathcal{M}_{\mathrm{loc},B} \supset \mathrm{a}\mathcal{M}_{\mathrm{loc},\mathrm{conv}\, B}$.

2) We prove now that $K \subset \mathrm{a}\mathcal{N}_{\mathrm{LOC},B}$ for most B. We associate to $S \in \mathcal{D}_\mathrm{S}$ the set $G_S = \{B \in \mathrm{Cpct}(\mathbb{R}^d)$ such that $B \cap \mathrm{conv}\, S \neq \emptyset \Rightarrow K_{S,B} \subset \mathrm{a}\mathcal{N}_{B \cap \mathrm{conv}\, S}\}$, where $K_{S,B} = \{a \in K$ such that $\mathrm{Aproj}_{B \cap \mathrm{conv}\, S}(a) \setminus S \neq \emptyset\}$. We only have to prove that most B are in all G_S, for $S \in \mathcal{D}_\mathrm{S}$.

So take $S = \mathrm{S}(c,r) \in \mathcal{D}_\mathrm{S}$. G_S is a G_δ subset of $\mathrm{Cpct}(\mathbb{R}^d)$ because for every $\varepsilon > 0$ the set $\{B \in \mathrm{Cpct}(\mathbb{R}^d)$ such that there exist $b \in \mathbb{R}^d$, $p \in \mathrm{Aproj}_{B \cap \mathrm{conv}\, S}(b)$, $a \in K$, $\lambda > 1$ such that $a = p + \lambda(b - p)$, $||a - b|| \geq \varepsilon$ and $\mathrm{dist}(p, S) \geq \varepsilon\}$ is closed.

It remains to prove that G_S is everywhere dense in $\mathrm{Cpct}(\mathbb{R}^d)$. Take $B_0 \in \mathrm{Cpct}(\mathbb{R}^d)$ such that $B_0 \cap \overline{\mathrm{B}}(c,r) \neq \emptyset$ and V an open neighborhood of C_0 in $\mathrm{Cpct}(\mathbb{R}^d)$. Choose the union $B \subset \mathbb{R}^d \setminus S$ of a finite family of mutually disjoint rational closed balls, such that $B \in V$ and $B \cap B(c,r) \neq \emptyset$. Then Lemma 5.9 gives, for $\varepsilon > 0$, $B_1 \in G_S$, where

$$B_1 := (B \setminus \mathrm{conv}\, S) \cup \left(\overline{\mathrm{B}}\,(B \cap \mathrm{conv}\, S, \varepsilon) \cap \bigcap_{a \in K} \overline{\mathrm{B}}\,(a, \mathrm{adist}(a, B \cap \mathrm{conv}\, S)) \right)$$

(we have here $K = K_{S,B} = K_{S,B_1}$). Moreover $B_1 \in V$ if ε is small enough, and thus G_S is everywhere dense.

3) Finally let C_0 be a convex body of $\mathbb{R}^d$ such that $C_0 \cap \overline{\mathrm{B}}(c,r) \neq \emptyset$. We apply the construction above to $B_0 = C_0$, we can ask also that $\operatorname{conv} B_0 \in V$. Then we have also $C_1 := \operatorname{conv} B_1 \in G_S$. Indeed, if $a \in K_{S,C_1}$ and $p \in \mathrm{B}(a,r) \cap \mathrm{Aproj}_{C_1 \cap \operatorname{conv} S}(a)$, then $p \in \partial B_1$, hence $p \in \mathrm{Aproj}_{B_1 \cap \operatorname{conv} S}(a)$. But $B_1 \cap \operatorname{conv} S \subset C_1 \cap \operatorname{conv} S$ and $a \in \mathrm{a}\mathcal{N}_{B_1 \cap \operatorname{conv} S}$, thus $a \in \mathrm{a}\mathcal{N}_{C_1 \cap \operatorname{conv} S}$.

We also have $C_1 \in V$ if ε is small enough. Thus we have $K \subset \mathrm{a}\mathcal{N}_{\mathrm{LOC},C}$ for most convex bodies C, this ends the proof.

We obtain now by Proposition 2.1 the next result.

Theorem 5.11. *Let h be a dimension function such that $t^d = \mathrm{o}(h(t))$. Then for most non empty compact subset B of $\mathbb{R}^d$ and for every non empty open subset ω of $\mathbb{R}^d$, we have*

$$\mathcal{H}^h(\omega \cap \mathrm{a}\mathcal{N}_{\mathrm{LOC},B} \setminus \mathrm{a}\mathcal{M}_{\mathrm{loc},B}) > 0.$$

Thus $\dim_{\mathrm{H}}(\omega \cap \mathrm{AcurvCt}_{\operatorname{conv} B}) = d$, *hence* $\dim_{\mathrm{H}}(\omega \cap \mathrm{AcurvCt}_B) = d$. *We also have*

$$\mathcal{H}^h\left(\mathrm{Aprojloc}_B(\mathbb{R}^d)\right) = 0.$$

Similarly for most convex bodies C of $\mathbb{R}^d$ and for every non empty open subset ω of $\mathbb{R}^d$, we have

$$\mathcal{H}^h(\omega \cap \mathrm{a}\mathcal{N}_{\mathrm{LOC},C} \setminus \mathrm{a}\mathcal{M}_{\mathrm{loc},C}) > 0 \quad \textit{and} \quad \mathcal{H}^h\left(\mathrm{Aprojloc}_C(\mathbb{R}^d)\right) = 0.$$

Thus $\dim_{\mathrm{H}}(\omega \cap \mathrm{AcurvCt}_C) = d$.

Results of Wieacker [28] immediately imply that $0 = \dim_{\mathrm{H}} \mathrm{Aproj}_B(\mathbb{R}^d)$ (for most B); he also proved that for most B and $C = \operatorname{conv} B$, $\mathrm{Aproj}_C(\mathbb{R}^d)$ is homeomorphic to the topological product of $\mathbb{Q} \times \{0,1\}^{\mathbb{N}}$ of $\mathbb{Q}$ and the Cantor set. According to Gruber [9], p. 1332, $0 = \dim_{\mathrm{H}} \mathrm{Aproj}_C(\mathbb{R}^d)$ (for most convex bodies C) is contained in the results of Schneider and Wieacker [24], and Zamfirescu [31].

Proof. The assertions about $\mathrm{a}\mathcal{N}_{\mathrm{LOC},B} \setminus \mathrm{a}\mathcal{M}_{\mathrm{loc},B}$ and $\mathrm{a}\mathcal{N}_{\mathrm{LOC},C} \setminus \mathrm{a}\mathcal{M}_{\mathrm{loc},C}$ are obtained from Proposition 5.10 in the same way as Theorem 3.9 was deduced from Proposition 3.8. Moreover, from Property 5.7 we have $\mathrm{a}\mathcal{N}_{\mathrm{LOC},B} \setminus \mathrm{a}\mathcal{M}_{\mathrm{loc},B} \subset \mathrm{a}\mathcal{N}_{\mathrm{LOC},\operatorname{conv} B} \setminus \mathrm{a}\mathcal{M}_{\mathrm{loc},\operatorname{conv} B} \subset \mathrm{AcurvCt}_{\operatorname{conv} B} \subset \mathrm{AcurvCt}_B$. We also have for most B, $\mathcal{H}^h(\mathrm{Aprojloc}_B(\mathbb{R}^d)) \leq \mathcal{H}^h(B) = 0$.

It remains to prove that $\mathcal{H}^h\left(\mathrm{Aprojloc}_C(\mathbb{R}^d)\right) = 0$ for most convex bodies C of $\mathbb{R}^d$. For this we only have to prove that $\mathcal{H}^h(L_{R,r,C}) = 0$, where $R, r > 0$ and $L_{R,r,C} = \{p \in \partial C \mid \exists a \in \mathbb{R}^d$, with $||a - p|| \leq R$ and $p \in \mathrm{Aproj}_{C \cap \overline{\mathrm{B}}(p,r)}(a)\}$. But $L_{R,r,C}$ is upper semi continuous in C, and the dense case where C is a polytope lets us conclude. More precisely, for every $\varepsilon > 0$, the set of all convex bodies C such that $L_{R,r,C}$ admits a finite covering (A_n) with $\sum h(\operatorname{diam} A_n) < \varepsilon$ is a dense open subset of $\mathrm{Cvb}(\mathbb{R}^d)$.

5.4. Combination of Propositions 2.1 to 5.10

We can use Proposition 2.1 to get large compact subsets K satisfying the properties of Propositions 3.8, 4.5 and 5.10. We can thus obtain more precise results, as for example we have the following theorem.

Theorem 5.12. *Let h be a dimension function such that $t^d = \mathrm{o}(h(t))$. Then for most non empty compact subset B of $\mathbb{R}^d$, for most convex bodies C of $\mathbb{R}^d$ and for every non empty open subset ω of $\mathbb{R}^d$, we have $\mathcal{H}^h(X) > 0$, where $X = \omega \cap \mathrm{a}\mathcal{N}_{\mathrm{LOC},B} \cap \mathcal{N}_{\mathrm{LOC},B} \cap \mathrm{a}\mathcal{N}_{\mathrm{LOC},C} \cap \mathcal{N}_{\mathrm{loc},F} \cap \mathcal{N}_{\mathrm{LOC},F} \setminus (B \cup \mathrm{a}\mathcal{M}_{\mathrm{loc},B} \cup \mathcal{M}_{\mathrm{loc},B} \cup \mathrm{a}\mathcal{M}_{\mathrm{loc},C} \cup \mathcal{M}_{\mathrm{loc},F})$ and where $F = \mathbb{R}^d \setminus \mathrm{int}\, C$.*

As observed by Zamfirescu [31], p. 202, for most convex bodies C of $\mathbb{R}^d$ we have $\emptyset = (\mathrm{Proj}_{\partial C}\, \mathrm{int}\, C) \cap \mathrm{Aproj}_C(\mathbb{R}^d)$, because of Theorem 1.14. This also holds for most non empty compact subset B of $\mathbb{R}^d$ and for $C = \mathrm{conv}\, B$. De Blasi and Zhivkov [6] have proved the density in E, for generic non empty compact subsets B of a separable Hilbert space E of dimension ≥ 2, of the set $C^{n,m}(B)$ of all $a \in E$ such that $n = \mathrm{card}\, \mathrm{Proj}_B(a)$ and $m = \mathrm{card}\, \mathrm{Aproj}_B(a)$, when $m + n \leq 2 + \dim E$.

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