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Minmax via Differential Inclusion

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The asymptotic behavior of the solution of a differential inclusion provides a simple proof of a minmax theorem.

1. Introduction

Consider a finite zero-sum game (I, J, A) where $I = \{1, ..., \ell\}$ (resp. $J = \{1, ..., m\}$) is the set of pure strategies of player 1 (the maximizer), (resp. of player 2 (the minimizer)) and A is the $\ell \times m$ payoff matrix. Denote by X and Y the sets of mixed strategies of player 1 and 2 (i.e., X and Y are the unit simplices in $\mathbb{R}^\ell$ and in $\mathbb{R}^m$, respectively). Hence, if $x \in X$ and $y \in Y$ are the players' mixed actions, the payoff is $xAy = \sum_{ij} x_i A_{ij} y_j$. Assume that $\max_X \min_Y xAy = 0$. We prove that $\min_Y \max_X xAy \leq 0$.

The proof is based on an approximation of a discrete dynamics[1] related to Blackwell's approachability procedure (Blackwell [3]; Sorin [9]), by a continuous one, in the spirit of Benaim, Hofbauer and Sorin [2]. The goal of player 2 is to ensure that the payoff corresponding to any pure strategy of player 1 is less than or equal to zero, or in other words, that the state (in the vector payoff space $\mathbb{R}^\ell$ – a payoff for each pure strategy of player 1) will reach the negative orthant. At any time the current error, from player 2's point of view, is the difference between the time-average state and its projection on the negative orthant. As in approachability theory, his strategy is a function of this error and in particular, the dynamics depends only on the behavior of player 2.

Other dynamical approaches have been used to prove the minmax theorem. Brown and von Neumann [4] introduced a differential equation for two person symmetric zero-sum games. The variable is a symmetric mixed strategy that converges to the set of optimal strategies. Other procedures (e.g., fictitious play, Robinson [7]) exhibit similar properties for non-symmetric games while the state variable is in the product space of mixed strategy. In these procedures the dynamics of each player's strategy depends on *both* players' strategies. In contrast, our dynamics is autonomous: it is defined over the vector payoff space of player 2 and does not involve player's 1 behavior.

[1]Similar discrete dynamics have been used by Lehrer and Sorin [6] and Lehrer [5].

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2. A discrete dynamics

Consider a sequence $\{y_n\}$ of player 2's mixed strategies and let $\bar{y}_n$ be the average of its n first elements. Thus,

$$\bar{y}_{n+1} - \bar{y}_n = \frac{1}{n+1}(y_{n+1} - \bar{y}_n). \tag{1}$$

Let $g_n = Ay_n \in I\!R^\ell$ be the corresponding sequence of vector payoffs, hence its average $\bar{g}_n$ satisfies:

$$\bar{g}_{n+1} - \bar{g}_n = \frac{1}{n+1}(g_{n+1} - \bar{g}_n). \tag{2}$$

For any $g \in I\!R^\ell$ define $g^+ = (g^{+1}, ..., g^{+n})$ by $g^{+k} = \max(g^k, 0)$. If $g^+ \neq 0$, then g^+ is proportional to some $x \in X$, and thus, by assumption, there exists $y \in Y$ that satisfies $xAy \leq 0$, which implies $\langle g^+, Ay \rangle \leq 0$.

Define a correspondence N from $I\!R^\ell$ to Y by:

$$N(g) = \{y \in Y; \langle g^+, Ay \rangle \leq 0\}.$$

The values of N are non-empty. Furthermore, N is an upper semi-continuous correspondence with compact convex values.

The discrete dynamics, respectively in the space of player 2's mixed strategies and in the space of vector payoffs are defined by

$$y_{n+1} \in N(A\bar{y}_n) \quad \text{and} \quad g_{n+1} \in AN(\bar{g}_n). \tag{3}$$

3. The theorem and its proof

We turn now to the formal presentation of the theorem and its proof.

Theorem 3.1. *If $\max_X \min_Y xAy = 0$, then there exists a point $y^* \in Y$ such that all the coordinates of the vector Ay^* are less than or equal to 0.*

Proof. Let D denote the negative orthant, $I\!R^\ell_-$, and let $C = AY$, which is a compact convex image of Y.

Consider the continuous dynamics in the vector payoffs space defined by the following differential inclusion (note the similarity to equation (3)):

$$\dot{g} \in AN(g) - g. \tag{4}$$

Since N is an upper semi-continuous correspondence with non-empty compact convex values, there exists a solution $g(t)$ of (4) (see e.g., Theorem 3, p. 98 in Aubin and Cellina [1]). When $g(t)$ is on the boundary of C, $\dot{g}(t)$ points inside C. Thus, if $g(0) \in C$, then $g(t) \in C$ for all $t \geq 0$ (see e.g., Theorem 5.7, p. 129 in Smirnov [8]).

Let $Z(g) = \|g^+\|^2$ be the square of the distance from a point $g \in I\!R^\ell$ to D. Hence, $\nabla Z(g) = 2g^+$. Let $z(t) = Z(g(t))$, then for almost all $t \geq 0$:

$$\begin{aligned}\dot{z}(t) &= \langle \nabla Z(g(t)), \dot{g}(t) \rangle = 2\langle g^+(t), \dot{g}(t) \rangle \leq -2\langle g^+(t), g(t) \rangle \\ &= -2\langle g^+(t), g^+(t) \rangle = -2Z(g(t)) = -2z(t),\end{aligned}$$

where the inequality is due to $\dot{g}(t) + g(t) \in AN(g)$ and $\langle g^+, Ay \rangle \leq 0$ for any $y \in N(g)$. Thus, $z(t)$ (hence the distance between $g(t)$ and D) decreases exponentially to 0 and if it reaches 0 it keeps this value.[2]

Any accumulation point g^* of $g(t)$ (as $t \to \infty$) satisfies $Z(g^*) = 0$, which means that $g^* \in D$. Since $g^* \in C$, there exists y^* with $Ay^* \in D$, as desired. □

Remark. The proof immediately extends to the case where X is a finite dimensional simplex, Y is a convex compact subset of a topological vector space and the payoff is a real bilinear function on $X \times Y$, continuous on Y for each $x \in X$.

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[2] This is in opposition with the discrete dynamics whose distance to D does not converge monotonically to D.