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Centrally Symmetric Convex Sets

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There are some conditions under which a compact, convex set is centrally symmetric. The conditions were established by W. Blaschke and G. Hessenberg [1], G. D. Chakerian and M. S. Klamkin [3], L. Montejano [7]. In this article we give some new conditions.

Keywords: Convex set, symmetry, sections and projections of sets

1. Introduction

First we formulate some notions connected with central symmetry of sets in the space $\mathbb{R}^n$.

Definition 1.1. Let $M \subset \mathbb{R}^n$ be a set and $q \in \mathbb{R}^n$ be a point. The set $M' = 2q - M$ is said to be *centrally symmetric to M with respect to the center q*. In other words, M' is centrally symmetric to M with respect to the center q if and only if for every point $x \in M$ there is a point $x' \in M'$ such that $\frac{1}{2}(x + x') = q$ and for every point $y' \in M'$ there is a point $y \in M$ such that $\frac{1}{2}(y + y') = q$.

We note that if M' is centrally symmetric to M with respect to the center q and M'' is centrally symmetric to M with respect to the center r, then M'' is obtained from M' under the translation with the vector $2(r-q)$, i.e., the equality $M'' = 2(r-q) + M'$ holds.

Definition 1.2. A set $M \subset \mathbb{R}^n$ is *centrally symmetric* if there exists a point $q \in \mathbb{R}^n$ such that M is centrally symmetric to itself with respect to q; the point q is said to be the *center of symmetry for M*.

We note that if a *compact* set $M \subset \mathbb{R}^n$ is centrally symmetric, then the center of symmetry for M is determined uniquely.

In [1], [3], [6] some conditions for central symmetry of compact, convex sets are given. G. D. Chakerian and M. S. Klamkin established [3] the following necessary and sufficient condition under which a compact set $M \subset \mathbb{R}^n$ (not necessarily convex) is centrally symmetric:

Theorem 1.3. *Let $M \subset \mathbb{R}^n$ be a compact set. The set M is centrally symmetric if and only if for every three-point set $Q = \{a, b, c\}$ contained in M there exists a point $q \in \mathbb{R}^n$ such that the set Q' symmetric to Q with respect to q also is contained in M.*

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In particular, Theorem 1.3 holds for compact, convex sets. Another necessary and sufficient condition under which a compact, convex set $M \subset \mathbb{R}^n$ is centrally symmetric was established by W. Blaschke and G. Hessenberg [1]:

Theorem 1.4. *Let $M \subset \mathbb{R}^n$ be a compact, convex set. The set M is centrally symmetric if and only if for every two-dimensional plane $L \subset \mathbb{R}^n$ the orthogonal projection of M into L is a centrally symmetric set.*

Corollary 1.5. *Let $M \subset \mathbb{R}^n$ be a compact, convex set and k be a fixed integer, $2 \le k \le n-1$. The set M is centrally symmetric if and only if for every k-dimensional plane $L \subset \mathbb{R}^n$ the orthogonal projection of M into L is a centrally symmetric set.*

Corollary 1.6. *Let $M \subset \mathbb{R}^n$ be a compact, convex set and ε be a positive number. The set M is centrally symmetric if and only if its ε-neighborhood $U_\varepsilon(M)$ is centrally symmetric.*

The following necessary and sufficient condition under which a compact, convex set $M \subset \mathbb{R}^n$ is centrally symmetric was proved by L. Montejano [6]:

Theorem 1.7. *Let $M \subset \mathbb{R}^n$ be a compact, strictly convex body, k be a fixed integer, $1 \le k \le n-2$, and Γ be a fixed k-dimensional subspace of $\mathbb{R}^n$. The body M is centrally symmetric if and only if for every $(k+1)$-dimensional subspace $L \supset \Gamma$ the orthogonal projection of M into L is a centrally symmetric set.*

In this article we give some new necessary and sufficient conditions under which a compact, convex set $M \subset \mathbb{R}^n$ is centrally symmetric.

2. The results

In Section 3 we give a proof of Theorem 1.4. That proof allows us to obtain the following generalization of Montejano's theorem in which instead of the requirement of strict convexity we use a weaker condition:

Theorem 2.1. *Let $M \subset \mathbb{R}^n$ be a compact, convex body, k be a fixed integer, $1 \le k \le n-2$, and Γ be a fixed k-dimensional subspace of $\mathbb{R}^n$. Assume that there exists a hyperplane H such that H is orthogonal to a line contained in Γ and each support hyperplane of M parallel to H has only one common point with M. The body M is centrally symmetric if and only if for every $(k+1)$-dimensional subspace $L \supset \Gamma$ the orthogonal projection of M into L is a centrally symmetric set.*

Example 2.2. Let $M \subset \mathbb{R}^n$ be an n-dimensional compact, convex body and $I = [a, b]$ be an its diameter. If M is not strictly convex, then Theorem 1.7 is not applicable. But from Theorem 2.1 we deduce (even in the case when M is not strictly convex) that if for every 2-dimensional plane $L \supset I$ the orthogonal projection of M into L is a centrally symmetric set, then M is centrally symmetric. This is a more strong result than Theorem 1.4.

We note that H. Groemer [4] had obtained a result that is similar to our Theorem 2.1.

Theorem 2.3. *Let $M \subset \mathbb{R}^n$ be a compact, convex set. The set M is centrally symmetric if and only if for every n-dimensional simplex $T \subset \mathbb{R}^n$ that contains M there exists a point $q \in \mathbb{R}^n$ such that the simplex T' symmetric to T with respect to q also contains M.*

Definition 2.4. Let $T \subset \mathbb{R}^n$ be a k-dimensional simplex, $2 \le k \le n-1$, and $N \subset \mathbb{R}^n$ be the orthogonal complement of its affine hull aff$T$. The vector sum $B = T \oplus N$ we name a *k-beam* with the base T in $\mathbb{R}^n$.

Theorem 2.5. *Let $M \subset \mathbb{R}^n$ be a compact, convex set. The set M is centrally symmetric if and only if for every 2-beam $B \subset \mathbb{R}^n$ that contains M there exists a point $q \in \mathbb{R}^n$ such that the 2-beam B' symmetric to B with respect to q also contains M.*

We note that, in a sense, Theorem 2.5 is *dual* to Theorem 1.3, since instead of three points we consider three support hyperplanes.

Corollary 2.6. *Let $M \subset \mathbb{R}^n$ be a compact, convex set and k be an integer, $2 \le k \le n-1$. The set M is centrally symmetric if and only if for every k-beam $B \subset \mathbb{R}^n$ that contains M there exists a point $q \in \mathbb{R}^n$ such that the k-beam B' symmetric to B with respect to q also contains M.*

Theorem 2.7. *Let $M \subset \mathbb{R}^n$ be a compact, convex body and Γ be a fixed k-dimensional subspace, $1 \le k \le n-2$. Assume that there exists a hyperplane H orthogonal to a line contained in Γ such that each support hyperplane of M parallel to H has only one common point with M. The body M is centrally symmetric if and only if for every $(k+1)$-dimensional subspace $L \supset \Gamma$ and every $(k+1)$-beam $B \supset M$ with a base $T \subset L$ there exists a point $q \in \mathbb{R}^n$ such that the k-beam B' symmetric to B with respect to q also contains M.*

Corollary 2.8. *Let $M \subset \mathbb{R}^n$ be a compact, strictly convex body and Γ be a fixed k-dimensional subspace of $\mathbb{R}^n$, $1 \le k \le n-2$. The body M is centrally symmetric if and only if for every $(k+1)$-dimensional subspace $L \supset \Gamma$ and every $(k+1)$-beam B with a base $T \subset L$ such that $B \supset M$ there exists a point $q \in \mathbb{R}^n$ such that the k-beam B' symmetric to B with respect to q also contains M.*

Example 2.9. Let $L \subset \mathbb{R}^n$ be a two-dimensional subspace and P_1, P_2, P_3 be closed half-planes of L. The intersection $P_1 \cap P_2 \cap P_3$ (if it is two-dimensional) can be either a triangle or an unbounded set. It is easily shown that if in the definition of 2-beam we replace the triangle T by an unbounded set $P_1 \cap P_2 \cap P_3$, then Theorem 2.5 (also Theorem 2.7 and Corollaries 2.6, 2.8) fails. Indeed, assume that bd$P_1 \cap$ bdP_3 is not contained in P_2 and $M \subset P_1 \cap P_2 \cap P_3$. Denoting by b the symmetry axes of $P_1 \cap P_3$, we conclude that for any point $q \in b \cap P_1 \cap P_2 \cap P_3$ far enough from bdP_2 the set symmetric to $P_1 \cap P_2 \cap P_3$ with respect to q contains M, independently on central symmetry of M.

Theorem 1.3 directly implies the following two theorems.

Theorem 2.10. *Let $M \subset \mathbb{R}^n$ be a compact set and k be a fixed integer, $2 \le k \le n-2$. The set M is centrally symmetric if and only if there exists a point $x \in \mathbb{R}^n$ such that for every $(k+1)$-dimensional plane $L \subset \mathbb{R}^n$ passing through x the intersection $L \cap M$ is centrally symmetric.*

Theorem 2.11. *Let $M \subset \mathbb{R}^n$ be a compact set, k be a fixed integer with $1 \le k \le n-4$, and $\Gamma \subset \mathbb{R}^n$ be a fixed k-dimensional subspace. The set M is centrally symmetric if and only if there exists a point $x \in \mathbb{R}^n$ such that for every $(k+3)$-dimensional plane $L \subset \mathbb{R}^n$ containing $x + \Gamma$ the intersection $L \cap M$ is centrally symmetric.*

We note that for the case of convex sets Theorems 2.10 and 2.11 follow immediately from the classical Roger's theorem and its proof [8]; moreover, Theorems 2.10 and 2.11 are related with the classical false center theorem (see [2]).

3. Proofs

Theorems 1.3, 1.4 and 1.7 are known. But for completeness of the article we give here their proofs.

Proof of Theorem 1.3. The part "only if" is evident. We prove the part "if". Thus for every three-point set $Q = \{a, b, c\} \subset M$ there exists a point $q \in \mathbb{R}^n$ such that the set Q' symmetric to Q with respect to q is contained in M.

Let $[a, b]$ be a diameter of M, i.e., $\| x - y \| \leq \| a - b \|$ for arbitrary points $x, y \in M$. Let, furthermore, m be the midpoint of the segment $[a, b]$ and $d = \| a - b \|$. Choose an arbitrary point $c \in M$. Then there is a point $q \in \mathbb{R}^n$ such that the set Q' symmetric to $Q = \{a, b, c\}$ with respect to q is contained in M. If q is distinct from m, then at least one of the distances $\| q - a \|$, $\| q - b \|$ is greater than $\frac{1}{2}d$, say $\| q - a \| > \frac{1}{2}d$. The point $a' = 2q - a$ (that is symmetric to a with respect to q) belongs to M. Since q is the midpoint of the segment $[a, a']$, we have $\| a' - a \| = 2 \| q - a \| > d$, contradicting that d is the diameter of the set M. This contradiction shows that q coincides with m. Thus for every point $c \in M$ the point $c' = 2m - c$ belongs to M, i.e., M is centrally symmetric with respect to the center m. $\square$

Proof of Theorem 1.4. The part "only if" is evident. We prove the part "if". It is sufficient to consider the case when M is a compact, convex *body*, i.e., M has a nonempty interior in $\mathbb{R}^n$ (in the opposite case it is possible to replace $\mathbb{R}^n$ by the affine hull of M). Thus we assume that M is a compact, convex body and its orthogonal projection into any two-dimensional plane is a centrally symmetric set.

For every unit vector $v \in \mathbb{S}^{n-1}$ denote by $\Pi(v)$ the support half-space of M with the outward normal v, i.e., $\Pi(v)$ is a closed half-space with the outward normal v such that $\Pi(v) \supset M$ and $\mathrm{bd}\Pi(v)$ is a support hyperplane of M. The set of all vectors $v \in \mathbb{S}^{n-1}$ for which the intersection $M \cap \mathrm{bd}\Pi(v)$ contains more than one point is a set of the first category in $\mathbb{S}^{n-1}$. Hence there exists a unit vector $v_0 \in \mathbb{S}^{n-1}$ such that each intersection $M \cap \mathrm{bd}\Pi(v_0)$, $M \cap \mathrm{bd}\Pi(-v_0)$ consists of only one point. Denote by $p(v_0)$ and $p(-v_0)$ the corresponding intersection points, i.e. $M \cap \mathrm{bd}\Pi(v_0) = \{p(v_0)\}$, $M \cap \mathrm{bd}\Pi(-v_0) = \{p(-v_0)\}$. The midpoint of the segment $[p(v_0), p(-v_0)]$ denote by q.

Let now $v_1 \in \mathbb{S}^{n-1}$ be an arbitrary unit vector distinct from $\pm v_0$. Denote by N the $(n-2)$-dimensional plane $\mathrm{bd}\Pi(v_0) \cap \mathrm{bd}\Pi(v_1)$ and by L the 2-dimensional subspace that is the orthogonal complement of N. Let $\pi_L : \mathbb{R}^n \to L$ be the orthogonal projection. Then the set $\pi_L(N)$ consists of only one point and each of the images $\pi_L(\mathrm{bd}\Pi(v_0))$, $\pi_L(\mathrm{bd}\Pi(v_1))$ is a line in L. Consequently each of the sets $S_0 = \pi_L(\Pi(v_0) \cap \Pi(-v_0))$ and $S_1 = \pi_L(\Pi(v_1) \cap \Pi(-v_1))$ is a strip in the plane L. The intersection $S_0 \cap S_1$ it a circumscribed parallelogram of the figure $\pi_L(M)$. By the hypothesis, $\pi_L(M)$ is a centrally symmetric figure, and hence the center of the parallelogram $S_0 \cap S_1$ coincides with the center of symmetry of $\pi_L(M)$.

The intersection $\pi_L(M) \cap \pi_L(\mathrm{bd}\Pi(v_0))$ coincides with the point $\pi_L(p(v_0))$, and analogously the intersection $\pi_L(M) \cap \pi_L(\mathrm{bd}\Pi(-v_0))$ coincides with the point $\pi_L(p(-v_0))$. Moreover,

$r = \pi_L(p(v_0))$ and $r' = \pi_L(p(-v_0))$ are the single intersection points of two opposite sides of the parallelogram $S_0 \cap S_1$ with the boundary of the figure $\pi_L(M)$. Consequently the midpoint $m = \pi_L(q)$ of the segment $[r, r']$ coincides with the center of the parallelogram $S_0 \cap S_1$. This means that the boundary lines of the strip S_1 are situated in the *equal distances* from the point m. Hence, returning to the space $\mathbb{R}^n$, we conclude that the hyperplanes $\mathrm{bd}\Pi(v_1)$ and $\mathrm{bd}\Pi(-v_1)$ are situated in the *equal distances* from the midpoint q of the segment $[p(v_0), p(-v_0)]$. In other words, *every two parallel hyperplanes of the body M are situated in the equal distances from the midpoint q of the segment* $[p(v_0), p(-v_0)]$. From this we conclude that the body M is centrally symmetric and q is its center of symmetry.

Indeed, assume that q is not the center of symmetry of the body M, i.e., there is a point $x \in M$ such that the point $x' = 2q - x$ does not belong to M. Let Γ be a hyperplane through x' with $\Gamma \cap M = \emptyset$. Denote by $v \in \mathbb{S}^{n-1}$ the unit vector orthogonal to Γ such that the half-space with the boundary Γ and the outward normal v contains M in its interior. Let Γ' be the hyperplane that is parallel to Γ and passes through x. Furthermore, denote by Γ_1 and Γ_2 the support hyperplanes of M which are parallel to Γ and have the outward normals v and $-v$, respectively. Then the hyperplanes Γ and Γ' are situated in the same distance d from q, whereas the distance of Γ_1 from q is *lesser* than d and the distance of Γ_2 from q is *greater or equal* than d. Thus two parallel hyperplanes Γ_1 and Γ_2 of M are situated in different distances from q, contradicting what was proved above. This contradiction shows that q is the center of symmetry of the body M. □

Proof of Corollary 1.5. As in the proof of Theorem 1.4, we have to establish only the part "if" and we may assume that M is a compact, convex *body*. Let $L \subset \mathbb{R}^n$ be an arbitrary two-dimensional plane and $K \supset L$ be a k-dimensional plane. For the orthogonal projections $\pi_K : \mathbb{R}^n \to K$ and $\pi_L : \mathbb{R}^n \to L$ we have $\pi_L(M) = \pi_L(\pi_K(M))$. By the hypothesis, the set $N = \pi_K(M)$ is centrally symmetric. Hence the set $\pi_L(M) = \pi_L(N)$ is centrally symmetric, too. Thus for every two-dimensional plane L the projection $\pi_L(M)$ is centrally symmetric. Hence, by Theorem 1.4, M is centrally symmetric. □

Proof of Corollary 1.6. If M is centrally symmetric, then evidently $U_\varepsilon(M)$ is centrally symmetric. We prove the opposite affirmation: if M is not centrally symmetric, then $U_\varepsilon(M)$ is not centrally symmetric, too.

First we consider the case when M is a compact, convex *body*. Let us conserve the notations v_0, q introduced in the proof of Theorem 1.4. If every two parallel hyperplanes of the body M are situated in equal distances from q, then (by the proof of Theorem 1.4) M is centrally symmetric. Since M is not centrally symmetric, there is a vector $v_1 \in \mathbb{S}^{n-1}$ such that the support hyperplanes of M with outward normals $\pm v_1$ are situated in *different* distances from q. Now it is clear that for the plane L as in the proof of Theorem 1.4 the projection $\pi_L(U_\varepsilon(M))$ is not centrally symmetric. Hence $U_\varepsilon(M)$ is not centrally symmetric.

If the affine hull of M is distinct from $\mathbb{R}^n$, then the above reasoning shows that $U_\varepsilon(M) \cap (\mathrm{aff}M)$ is not centrally symmetric. Consequently $U_\varepsilon(M)$ is not centrally symmetric, too. □

Proof of Theorem 1.7. Theorem 1.7 is a particular case of Theorem 2.1 (obtained when we replace the existence of the hyperplane H in Theorem 2.1 by the more strong requirement that M is strictly convex). Therefore we can restrict ourselves by a proof of Theorem 2.1.

Proof of Theorem 2.1. First we consider the case $k = 1$. The part "only if" is evident. We prove the part "if". Thus $M \subset \mathbb{R}^n$ is a compact, convex body and $\Gamma \subset \mathbb{R}^n$ is a line. Denote by H the orthogonal complement of the line Γ. By the hypothesis, each support hyperplane of M parallel to H has only one common point with M and for every 2-dimensional subspace $L \supset \Gamma$ the orthogonal projection of M into L is a centrally symmetric set. Denote by v_0 the unit vector orthogonal to H. Now, using word by word the proof of Theorem 1.4, we obtain that M is centrally symmetric.

Assume now that $k > 1$. By the hypothesis, there exists a hyperplane H orthogonal to a line l contained in Γ such that each support hyperplane of M parallel to H has only one common point with M and, moreover, for every $(k+1)$-dimensional subspace $L \supset \Gamma$ the set $\pi_L(M)$ is centrally symmetric. Let P be an arbitrary two-dimensional subspace containing l. Then either $L = P \oplus \Gamma$ is a $(k+1)$-dimensional subspace and $L \supset \Gamma$, or $L = \Gamma$. By the hypothesis, the projection $\pi_L(M)$ is centrally symmetric. Consequently, as in the proof of Corollary 1.5, we conclude that $\pi_P(M)$ is centrally symmetric, and hence the above considered case $k = 1$ shows that M is centrally symmetric. □

Proof of Theorem 2.3. For the case when M is a *compact, convex body* this theorem follows directly from the following result established by E. Lutwak [5]: *Let K and L be convex bodies in $\mathbb{R}^n$. If every n-dimensional simplex that contains K also contains a translate of L, then K contains a translate of L.*

We note that no simple inductive argument allows to obtain Theorem 5 for a compact convex set $M \subset \mathbb{R}^n$ with empty interior if even this theorem holds for smaller dimensions. Indeed, if T is an n-dimensional simplex, containing M, then the intersection $T \cap \mathrm{aff}M$, in general, is not a *simplex*, and therefore the case of smaller dimension is useless.

Nevertheless, it is possible to prove Theorem 2.3 in general case, using Lutwak's result. Let $T \subset \mathbb{R}^n$ be an n-dimensional simplex, r be the radius of its inscribed ball, and $\varepsilon < r$ be a positive number. We denote by T_ε^- the maximal simplex T' satisfying the inclusion $U_\varepsilon(T') \subset T$. The simplex T_ε^- may be defined by the following way. Let $\Gamma_0, \Gamma_1, ..., \Gamma_n$ be the hyperplanes which contain the facets of the simplex T. For every $i = 0, 1, ..., n$ denote by Γ'_i the hyperplane parallel to Γ_i such that $\Gamma'_i \cap \mathrm{int}\, T \neq \emptyset$ and the distance between Γ'_i and Γ_i is equal to ε. Furthermore, by Π'_i denote the closed half-space with the boundary Γ'_i that does not contain Γ_i. Then $T_\varepsilon^- = \Pi'_0 \cap \Pi'_1 \cap ... \cap \Pi'_n$.

Let now $M \subset \mathbb{R}^n$ be a compact, convex set as in the statement of Theorem 2.3 and $N = \mathrm{cl}\, U_\varepsilon(M)$, where ε is a positive number. We may suppose that M is distinct from a point. Let T be an n-dimensional simplex that contains N. Then ε is less than the radius of the inscribed ball of the simplex T. Since $T \supset U_\varepsilon(M)$, the simplex T_ε^- contains M. Consequently there exists a point $q \in \mathbb{R}^n$ such that the simplex T' symmetric to T_ε^- with respect to q contains M. Hence $\mathrm{cl}\, U_\varepsilon(T') \supset N$. This implies that the simplex symmetric to T with respect to q contains N. Thus N satisfies the condition of Theorem 2.3. Now the result of E. Lutwak implies that N is centrally symmetric (since N is a compact, convex *body*). Hence, by Corollary 1.6, the set M also is centrally symmetric. □

Proof of Theorem 2.5. Let $L \subset \mathbb{R}^n$ be a two-dimensional subspace, $T \subset L$ be a triangle and $B = T \oplus N$ be the corresponding 2-beam, where N is the orthogonal complement of L. Assume that $B \supset M$. By the hypothesis, there is a point $q \in \mathbb{R}^n$ such that the beam B' symmetric to B with respect to q also contains M. Without loss of generality, we may assume that $q \in L$. Thus $B' = T' \oplus N$ where $T' \subset L$ is the triangle symmetric to T with respect to q. The inclusion $M \subset B'$ is equivalent to the inclusion $\pi_L(M) \subset T'$. Thus for every triangle T with $\pi_L(M) \subset T$ there exists a point $q \in L$ such that $\pi_L(M) \subset T'$. Theorem 2.3 implies now that the set $\pi_L(M) \subset L$ is centrally symmetric. Since that holds for every two-dimensional subspace $L \subset \mathbb{R}^n$, we conclude from Theorem 1.4 that M is centrally symmetric. □

Proof of Corollary 2.6 is quite analogous to the proof of Corollary 1.5. □

Proof of Theorem 2.7 is an evident combination of the proofs of Theorems 2.1 and 2.3 (taking into account Corollary 2.6). □

Proof of Corollary 2.8. This is a direct consequence of Theorem 2.7, since the condition (1) follows immediately from strict convexity of M. □

Proof of Theorem 2.10. Let $Q = \{a, b, c\} \subset M$ be a three-point set. Consider a k-dimensional plane $L \subset \mathbb{R}^n$ that contains the points x, a, b, c. By the hypothesis, $L \cap M$ is centrally symmetric. Let q be its center of symmetry. Then the set Q' symmetric to Q with respect to q is contained in $L \cap M$, i.e., $Q' \subset M$. By Theorem 1.3, the set M is centrally symmetric. □

Proof of Theorem 2.11. Let $Q = \{a, b, c\} \subset M$ be a three-point set. Choose a $(k+3)$-dimensional subspace $K \subset \mathbb{R}^n$ containing $\Gamma \cup Q$. Since $K \supset \Gamma$, the set $K \cap M$ is centrally symmetric. Let q be its center of symmetry. Then the set Q' symmetric to Q with respect to q is contained in $K \cap M$, i.e., $Q' \subset M$. By Theorem 1.3, M is centrally symmetric. □

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