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Uniqueness and Continuous Dependence on Boundary Data for Integro-Extremal Minimizers of the Functional of the Gradient

Sandro Zagatti
S.I.S.S.A., Via Beirut 2/4, 34014 Trieste, Italy

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We study some qualitative properties of the integro-extremal minimizers of the functional of the gradient defined on Sobolev spaces with Dirichlet boundary conditions. We discuss their use in the non-convex case via viscosity methods and give conditions under which they are unique and depend continuously on boundary data.

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1. Introduction

Consider the functional of the gradient

$$\mathcal{F}(u) = \int_\Omega f(Du)\,dx, \quad u \in \varphi + W_0^{1,p}(\Omega), \tag{1}$$

where $p \geq 1$, $\varphi \in W^{1,p}(\Omega)$ is a given datum and $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function satisfying natural growth condition ensuring the coercivity of $\mathcal{F}$ on the Sobolev space $W^{1,p}(\Omega)$. If the lagrangian f is convex the functional turns out to be sequentially weakly lower semicontinuous and, consequently, by the direct method of the Calculus of Variations, we know that it admits at least one minimizer. Dropping convexity the existence of minimum points is no more guaranteed but in recent years several improvements in the study of non-convex case have been obtained; we refer in particular to [4], [5], [12], [15] [8], [11], [16], [3], [2] and to references quoted there. The leading idea is the following: introduce the bipolar function f^{**} of f (which is convex and pointwise less or equal than f) and the relaxed functional

$$\overline{\mathcal{F}}(u) = \int_\Omega f^{**}(Du)\,dx, \quad u \in \varphi + W_0^{1,p}(\Omega).$$

Clearly the set S of minimizers of $\overline{\mathcal{F}}$ is nonempty and the solution of the minimum problem for the non-convex functional $\mathcal{F}$ may be obtained by finding an element $\tilde{u} \in S$ such that

$$f^{**}(D\tilde{u}) - f(D\tilde{u}) = 0 \tag{2}$$

almost everywhere in Ω. This result may be obtained under the following assumptions:

(i) all the elements of S are (classically) differentiable at almost every point of Ω;

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(ii) the map $\xi \mapsto f^{**}(\xi)$ is locally affine on the set

$$X \doteq \{\xi \in \mathbb{R}^n : f^{**}(\xi) < f(\xi)\}.$$

In paper [16] it is shown that, under these conditions, equation (2) is solved by the *integro-extremal* elements of the set S of minimizers of $\overline{\mathcal{F}}$, that is to say by any map $\overline{u}$, $\underline{u}$ in S such that

$$\int_\Omega \overline{u}\,dx \geq \int_\Omega u\,dx \quad \forall u \in S, \qquad \int_\Omega \underline{u}\,dx \leq \int_\Omega u\,dx \quad \forall u \in S,$$

whose existence is ensured by the coercivity of the functional and by Weierstrass theorem.

In this paper we devote our attention to these special minimizers showing that $\overline{u}$ is a viscosity solution of equation (2), that $\underline{u}$ is a viscosity solution of the equation $f(Du) - f^{**}(Du) = 0$, (Theorem 3.4 and Remark 3.7) and investigating the well posedness of their definition. In particular we exhibit conditions under which the maps $\overline{u}$ and $\underline{u}$ are unique (Theorem 4.2) and depend continuously on boundary data (Theorem 4.3).

2. Preliminaries and notations.

We denote respectively by $\langle\cdot,\cdot\rangle$ and by $|\cdot|$ the inner product and the euclidean norm in $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $r > 0$, $B(x,r)$ is the open ball of center x and radius r. Given $E \subseteq \mathbb{R}^n$, $\mathrm{meas}(E)$ denotes the Lebesgue measure of a (Lebesgue measurable) subset E of $\mathbb{R}^n$; E^c, ∂E and $\mathrm{int}(E)$ are, respectively, the complement, the boundary and the interior of E, while by $\dim(E)$ we mean the algebraic dimension of the smallest subspace of $\mathbb{R}^n$ containing E. The letter $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, \dots\}$ while $\mathbb{N}_0$ is the set $\mathbb{N} \cup \{0\}$. We adopt the notion of extremal face and of extremal point of a convex set, and refer to [13] for a complete exposition. Given a map $f : \mathbb{R}^n \to \mathbb{R}$ we define its epigraph as the set

$$\mathrm{epi}(f) \doteq \{(\xi, t) \in \mathbb{R}^n \times \mathbb{R} : t \geq f(\xi)\},$$

remarking that, whenever f is a convex function, $\mathrm{epi}(f)$ is a convex subset of $\mathbb{R}^n \times \mathbb{R}$. We denote by f^{**} the bipolar function of f and refer to [7] for its definition as well as for other basic arguments of the Calculus of Variations widely used in the paper like, for example, sequential weak lower semicontinuity of convex functionals defined on Sobolev spaces. We denote the conjugate exponent of $p \in [1, \infty]$ by $p' \doteq \frac{p}{p-1}$.

Throughout the paper Ω is an open bounded subset of $\mathbb{R}^n$ and we consider real valued functions $u : \Omega \to \mathbb{R}$. We use the spaces $C^k(\Omega)$, $C^k_c(\Omega)$ for $k = 0, 1, \dots$, $L^p(\Omega)$, $W^{1,p}(\Omega)$ and $W^{1,p}_0(\Omega)$, for $1 \leq p \leq \infty$, with their usual (strong and weak) topologies and identify a Sobolev function with its precise representative as defined, for example, in [10].

We need the following notion and refer to the monograph [1] (Chapter II) for proofs, general setting and for the definition of viscosity solution of Hamilton-Jacobi equations that we will consider in Remark 3.7 below.

Definition 2.1. Let $U \subseteq \mathbb{R}^n$ be open, $v \in C^0(U)$ and $x_0 \in U$. We set

$$D^- v(x_0) \doteq \left\{\xi \in \mathbb{R}^n : \liminf_{x \to x_0} \frac{v(x) - v(x_0) - \langle \xi, x - x_0\rangle}{|x - x_0|} \geq 0\right\}, \tag{3}$$

$$D^+ v(x_0) \doteq \left\{\xi \in \mathbb{R}^n : \limsup_{x \to x_0} \frac{v(x) - v(x_0) - \langle \xi, x - x_0\rangle}{|x - x_0|} \leq 0\right\}. \tag{4}$$

We call these sets, respectively, *super* and *sub* differentials (or semidifferentials) of u at the point x_0 and set also

$$A^-(v) \doteq \left\{x \in U : D^-v(x) \neq \emptyset\right\}. \tag{5}$$

$$A^+(v) \doteq \left\{x \in U : D^+v(x) \neq \emptyset\right\}, \tag{6}$$

We recall the following

Lemma 2.2. *Let $U \subseteq \mathbb{R}^n$ be open, $v \in C^0(U)$ and $x_0 \in U$.*

(*i*) *$\xi \in D^-v(x_0)$ if and only if there exists a function $\phi \in C^1(U)$ such that $D\phi(x_0) = \xi$ and and the function $x \mapsto v(x) - \phi(x)$ has local minimum at the point x_0.*

(*ii*) *$\xi \in D^+v(x_0)$ if and only if there exists a function $\phi \in C^1(U)$ such that $D\phi(x_0) = \xi$ and and the function $x \mapsto v(x) - \phi(x)$ has local maximum at the point x_0.*

(*iii*) *$D^+v(x_0)$ and $D^-v(x_0)$ are closed convex possibly empty subsets of $\mathbb{R}^n$.*

(*iv*) *If v is differentiable at the point x_0 then $D^+v(x_0) = D^-v(x_0) = \{Dv(x_0)\}$.*

(*v*) *If both $D^+v(x_0)$ and $D^-v(x_0)$ are nonempty then u is differentiable at x_0 and $D^+v(x_0) = D^-v(x_0) = \{Dv(x_0)\}$.*

(*vi*) *The sets $A^-(v)$ and $A^+(v)$ are dense in U.*

Remark 2.3. The proof can be found in [1] and we take from such textbook the following argument (Lemma 1.7, p. 29) which is a direct consequence of definitions (4), (3) and of Lemma 2.2.

(i) Let $\xi \in D^-v(x_0)$. Then there exist $\rho > 0$ and a continuous increasing function $\sigma : [0, +\infty[\to [0, +\infty[$ such that $\sigma(0) = 0$ and

$$v(x) \geq v(x_0) + \langle \xi, x - x_0\rangle - \sigma(|x - x_0|)|x - x_0| \quad \forall x \in B(x_0, \rho). \tag{7}$$

(ii) Let $\xi \in D^+v(x_0)$. Then there exist $\rho > 0$ and a continuous increasing function $\sigma : [0, +\infty[\to [0, +\infty[$ such that $\sigma(0) = 0$ and

$$v(x) \leq v(x_0) + \langle \xi, x - x_0\rangle + \sigma(|x - x_0|)|x - x_0| \quad \forall x \in B(x_0, \rho). \tag{8}$$

In the proof of our main result we shall need the following refined version of a well known argument (see [9], [15], [16]).

Lemma 2.4. *Let U be an open subset of $\mathbb{R}^n$, $p \in [1, \infty]$, $v \in W^{1,p}(U) \cap C^0(U)$, $x_0 \in A^-(v)$, $\xi \in D^-v(x_0)$, $r > 0$, and $\rho > 0$ such that $B(x_0, \rho) \subseteq U$. Let $\alpha \in L^\infty(U)$ such that $\alpha(y) > 0$ for a.e. $y \in B(x_0, \rho)$. Then there exists a map $\hat{v} \in W^{1,p}(U) \cap C^0(U)$ with the following properties*

$$\hat{v} - v \in W_0^{1,p}(U); \tag{9}$$

$$v(x) \leq \hat{v}(x) \textit{ for a.e. } x \in U; \tag{10}$$

$$\hat{\Lambda} \doteq \{x \in U \colon \hat{v}(x) > v(x)\} \textit{ is nonempty and } \hat{\Lambda} \subseteq B(x_0, \rho); \tag{11}$$

$$\begin{cases} |D\hat{v}(x) - \xi| = r, & \textit{for a.e. } x \in \hat{\Lambda} \\ D\hat{v}(x) = Dv(x), & \textit{for a.e. } x \in U \setminus \hat{\Lambda}; \end{cases} \tag{12}$$

$$\int_U \alpha\hat{v}\, dx > \int_U \alpha v\, dx; \tag{13}$$

$$\int_{\hat{\Lambda}} D\hat{v}\, dx = \int_{\hat{\Lambda}} Dv\, dx. \tag{14}$$

Proof. Recall definition (5) of $A^-(v)$ and take $\xi \in D^-v(x_0)$. By (7) there exists $s>0$ such that

$$\frac{s}{r} \le \rho \tag{15}$$

and

$$\frac{v(x)-v(x_0)-\langle \xi, x-x_0\rangle}{|x-x_0|} \ge -\frac{r}{2} \quad \forall x \in B\left(x_0, \frac{s}{r}\right). \tag{16}$$

Inserting (15) in (16) we have, in particular,

$$v(x)-v(x_0)-\langle \xi, x-x_0\rangle \ge -\frac{s}{2} \quad \forall x \in B\left(x_0, \frac{s}{r}\right). \tag{17}$$

Define a map w on $B\left(x_0, \frac{s}{r}\right)$ by setting

$$w(x) \doteq \max\left\{v(x), v(x_0)+\langle \xi, x-x_0\rangle + \frac{s}{4} - r|x-x_0|\right\} \tag{18}$$

and introduce the set

$$\hat{\Lambda} \doteq \left\{x \in B\left(x_0, \frac{s}{r}\right) : w(x) > v(x)\right\}. \tag{19}$$

Since both the functions v and

$$B(x_0,\rho) \ni x \mapsto v(x_0)+\langle \xi, x-x_0\rangle + \frac{s}{4} - r|x-x_0| \tag{20}$$

are continuous and the value of the second one in x_0 is strictly greater than $v(x_0)$, it follows that the set $\hat{\Lambda}$ is nonempty. In addition the map defined in (20) is Lipschitz continuous and then belongs to $W^{1,\infty}\left(B\left(x_0, \frac{s}{r}\right)\right)$ which is contained in $W^{1,p}\left(B\left(x_0, \frac{s}{r}\right)\right)$; hence, by Stampacchia's theorem and by the continuity of v and of the map in (20), w belongs to $W^{1,p}\left(B\left(x_0, \frac{s}{r}\right)\right) \cap C^0\left(B\left(x_0, \frac{s}{r}\right)\right)$ and we have:

$$Dw(x) = \begin{cases} \xi - rD|x-x_0|, & \text{for a.e. } x \in \hat{\Lambda}, \\ Dv(x), & \text{for a.e. } x \in B\left(x_0, \frac{s}{r}\right) \setminus \hat{\Lambda}. \end{cases} \tag{21}$$

We observe that, for any $x \in B\left(x_0, \frac{s}{r}\right)$ such that $|x-x_0| > \frac{3}{4}\frac{s}{r}$, we have, by (17),

$$\begin{aligned} v(x_0)+\langle \xi, x-x_0\rangle + \frac{s}{4} - r|x-x_0| &< v(x_0)+\langle \xi, x-x_0\rangle + \frac{s}{4} - \frac{3}{4}s \\ &= v(x_0)+\langle \xi, x-x_0\rangle - \frac{s}{2} \le v(x). \end{aligned}$$

Hence

$$\hat{\Lambda} \subseteq B\left(x_0, \frac{s}{r}\right) \tag{22}$$

and the map w coincides with v on $B\left(x_0, \frac{s}{r}\right) \setminus \overline{B\left(x_0, \frac{3}{4}\frac{s}{r}\right)}$. By definition (18) and by standard notions on Sobolev functions (see for example [10]), this implies that

$$(w-v)\,|_{B\left(x_0, \frac{s}{r}\right)} \in W_0^{1,p}\left(B\left(x_0, \frac{s}{r}\right)\right). \tag{23}$$

Now we set

$$\hat{v}(x) \doteq \begin{cases} w(x) & \text{for } x \in B\left(x_0, \frac{s}{r}\right), \\ v(x) & \text{for } x \in U \setminus B\left(x_0, \frac{s}{r}\right). \end{cases} \tag{24}$$

From (18), (22) (and the consideration which follows), (23) and (24) we obtain that $\hat{v}$ lies in $W^{1,p}(U) \cap C^0(U)$. We see now that conditions (9)-(14) hold. Property (9) is a trivial consequence of (18), (23) and (24); (10) follows trivially from definitions (18) and (24). Property (11) follows directly from definition (19) and, in order to show (12), we observe that, by (21) and by (24), we have that $|D\hat{v}(x) - \xi| = rD|x - x_0| = r$ for almost every $x \in \hat{\Lambda}$ and that $D\hat{v}(x) = Dv(x)$ for almost every $x \in U \setminus \hat{\Lambda}$. Inequality (13) is a consequence of (10), of the positivity of α on the ball $B(x_0, \rho)$ and of the fact that the open set $\hat{\Lambda}$, being nonempty, has positive measure. In order to prove (14) observe that by (23) and by divergence theorem we have

$$\int_{B\left(x_0, \frac{s}{r}\right)} D(w - v)dx = 0.$$

Then (14) follows from (24) and (22).

With obvious modification to the above arguments, and taking into account (6) and (8) in place of (5) and (7), we have the following

Lemma 2.5. *Let U, u, r and ρ as in Lemma 2.4, $x_0 \in A^+(v)$, $\xi \in D^+v(x_0)$. Let $\alpha \in L^\infty(U)$ such that $\alpha(y) < 0$ for a.e. $y \in B(x_0, \rho)$. Then there exists a map $\check{v} \in W^{1,p}(U) \cap C^0(U)$ satisfying the following properties.*

$$\check{v} - v \in W_0^{1,p}(U); \tag{25}$$

$$v(x) \geq \check{v}(x) \text{ for a.e. } x \in U; \tag{26}$$

$$\check{\Lambda} \doteq \{x \in U : \check{v}(x) < v(x)\} \text{ is nonempty and } \check{\Lambda} \subseteq B(x_0, \rho); \tag{27}$$

$$\begin{cases} |D\check{v}(x) - \xi| = r, & \text{for a.e. } x \in \check{\Lambda} \\ D\check{v}(x) = Du(x), & \text{for a.e. } x \in U \setminus \check{\Lambda}; \end{cases} \tag{28}$$

$$\int_U \alpha\check{v}\, dx > \int_U \alpha v\, dx; \tag{29}$$

$$\int_{\check{\Lambda}} D\check{v}\, dx = \int_{\check{\Lambda}} Dv\, dx. \tag{30}$$

The following result is rather elementary and we give the proof for convenience of the reader.

Lemma 2.6. *Let Ω be on open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$ and $p \in [1, +\infty[$. Let $t \in \mathbb{R}^+$ be small and $u, v \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ be such that*

$$\|u - v\|_{C(\partial\Omega)} \leq t. \tag{31}$$

Then there exist an open subset $\Omega_t \subseteq \Omega$ and a map $w_t \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ such that

(i) $\quad \operatorname{meas}(\Omega \setminus \Omega_t) \to 0$ *as* $t \to 0+$;

(ii) $\quad \|w_t\|_{W^{1,p}(\Omega)} \leq M$, *where M is a positive constant independent on t;*

(*iii*) $w_t = u$ *on* Ω_t;
(*iv*) $w_t = v$ *on* $\partial\Omega$.

Proof. Step 1. For $p = 1$ the statement is trivial. Indeed set

$$\Omega_t \doteq \{x \in \Omega : \operatorname{dist}(x, \Omega^c) > t\}$$

and take a map $\theta_t \in C^1_c(\Omega)$ such that $0 \le \theta_t \le 1$ on Ω, $\theta_t \equiv 1$ on Ω_t and $\|D\theta_t\|_{L^1(\Omega)} \le C$, where C is a positive constant depending on the $(n-1)$-dimensional measure of $\partial\Omega$. The required map w_t may be given by the formula

$$w_t \doteq \theta_t u + (1 - \theta_t)v.$$

To prove (*ii*) we observe that, by explicit differentiation and by Hölder inequality, we have

$$\|Dw_t\|_{L^1(\Omega)} \le \|Du\|_{L^1(\Omega)} + \|Dv\|_{L^1(\Omega)} + C\left(\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)}\right).$$

The other properties can be checked by direct inspection.

Step 2. Let $1 < p < \infty$ and assume $u \equiv 0$. Condition (31) takes the form

$$\|v\|_{C(\partial\Omega)} \le t. \tag{32}$$

We localize the problem by considering a point $x \in \partial\Omega$ and $r > 0$ such that, up to orthogonal change of variables and for a suitable Lipschitz continuous function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$, we have:

$$\Omega \cap Q(x, r) = \{y = (y', y_n) : y_n > \gamma(y')\} \cap Q(x, r), \tag{33}$$

where we adopt the notation $y = (y_1, \ldots, y_{n-1}, y_n) = (y', y_n)$ and $Q(x, r)$ denotes the cube

$$Q = Q(x, r) \doteq \{y = (y_1, \ldots, y_n) : |y_i - x_i| < r, i = 1, \ldots, n\}.$$

We set also

$$L \doteq \|D\gamma\|_{L^\infty(\Delta(x,r))}, \tag{34}$$

where

$$\Delta(x, r) \doteq \{y = (y_1, \ldots, y_{n-1}) : |y_i - x_i| \le r, i = 1, \ldots, n-1\}. \tag{35}$$

Since we are interested in small values of the parameter t, we may assume that

$$\left\{y = (y', y_n) : y' \in \Delta(x, r), \gamma(y') \le y_n \le \gamma(y') + t^{p'}\right\} \subseteq Q(x, r) \cap \overline{\Omega}. \tag{36}$$

Define the maps $I^\pm : Q(x, r) \cap \overline{\Omega} \to \mathbb{R}$ by setting

$$\begin{array}{ll} I^+(y', y_n) = t - t^{1-p'}(y_n - \gamma(y')) & \text{for } \gamma(y') \le y_n \le \gamma(y') + t^{p'}; \\ I^-(y', y_n) = -t + t^{1-p'}(y_n - \gamma(y')) & \text{for } \gamma(y') \le y_n \le \gamma(y') + t^{p'}; \\ I^\pm(y', y_n) = 0 & \text{for } y_n > \gamma(y') + t^{p'}. \end{array} \tag{37}$$

We observe that the functions $I^\pm$ are Lipschitz continuous on $Q(x, r) \cap \overline{\Omega}$, hence they belong to $W^{1,\infty}(\Omega)$ and we have:

$$|D_n I^\pm| = \begin{cases} 0, & \text{a.e. on } y_n > \gamma(y') + t^{p'} \\ t^{1-p'}, & \text{a.e. on } \gamma(y') < y_n < \gamma(y') + t^{p'}, \end{cases} \tag{38}$$

while, for $j = 1, \dots, n-1$, we have:

$$|D_j I^{\pm}| = \begin{cases} 0, & \text{a.e. on } y_n > \gamma(y') + t^{p'} \\ t^{1-p'} |D_j \gamma(y')|, & \text{a.e. on } \gamma(y') < y_n < \gamma(y') + t^{p'}. \end{cases} \tag{39}$$

Formulas (34), (35), (37), (38) and (39) imply that for every $j = 1, \dots, n$, we have:

$$\begin{aligned} &\int_{Q(x,r)\cap\Omega} \left|D_j I^{\pm}(y)\right|^p dy \\ \le\ &\max\{1, L^p\} \int_{\Delta(x,r)} \left(\int_{\gamma(y')}^{\gamma(y')+t^{p'}} t^{(1-p')p}\, dy_n \right) dy' \le \max\{1, L^p\}\, (2r)^{n-1}. \end{aligned} \tag{40}$$

Then (33), (36), (37) and (40) imply that $I^{\pm} \in W^{1,p}(\Omega \cap Q) \cap C^0(\overline{\Omega \cap Q})$ and that there exists $M_1 > 0$ independent on t such that

$$\|I^{\pm}\|_{W^{1,p}(\Omega)} \le M_1. \tag{41}$$

In addition, setting

$$\Omega_t(x) \doteq \left\{ y = (y', y_n) : y_n \ge \gamma(y') + t^{p'} \right\} \cap Q(x,r), \tag{42}$$

we have

$$\operatorname{meas}\left(Q(x,r) \cap (\Omega \setminus \Omega_t(x))\right) \to 0 \quad \text{as } t \to 0. \tag{43}$$

Now define a map $z : Q(x,r) \cap \overline{\Omega} \to \mathbb{R}$ by setting

$$z(y) \doteq \min\{v(y), I^+(y)\}. \tag{44}$$

Recall (36) and (37): since $I^+(y) = 0$ for $y \in \Omega_t(x)$, we have

$$z(y) \le 0 \quad \forall y \in \Omega_t(x). \tag{45}$$

Since $I^+(y) = t$ and, recalling (32), $v(y) \le t$ for $y \in \partial\Omega$, we have

$$z(y) = v(y) \quad \forall y \in \partial\Omega \cap Q. \tag{46}$$

Define the map $w : Q(x,r) \cap \overline{\Omega} \to \mathbb{R}$ by setting

$$w(y) \doteq \max\{z(y), I^-(y)\}. \tag{47}$$

By (45) and since and $I^-(y) = 0$ for $y \in \Omega_t(x)$, we have

$$w(y) = 0 \quad \forall y \in \Omega_t(x). \tag{48}$$

Since $I^-(y) = -t$ and, by (32) and (46), $z(y) = v(y) \ge -t$ for $y \in \partial\Omega \cap Q$, we have

$$w(y) = v(y) \quad \forall y \in \partial\Omega \cap Q. \tag{49}$$

Putting together (48) and (49), we obtain that

$$\begin{cases} w(y) = 0 & \text{for } y \in \Omega_t(x), \\ w(y) = v(y) & \text{for } y \in \partial\Omega \cap Q. \end{cases} \tag{50}$$

In addition, by (40), (44), (47) and by Stampacchia's theorem, we have that w lies in $W^{1,p}(\Omega \cap Q) \cap C^0(\overline{\Omega} \cap Q)$ and

$$\|Dw\|_{L^p(\Omega\cap Q)} \leq \max\left\{\|DI^-\|_{L^p(\Omega\cap Q)}, \|DI^+\|_{L^p(\Omega\cap Q)}, \|Dv\|_{L^p(\Omega)}\right\}.$$

Recalling (41) this implies that there exists $M_2 > 0$ independent on the parameter t such that

$$\|w\|_{W^{1,p}(\Omega\cap Q)()} \leq M_2. \tag{51}$$

Step 3. In order to globalize the construction let $\{x^j, j = 1, \dots, m\}$ be a family of points in $\partial\Omega$ and let $\{Q_j = Q(x^j, r_j),\ j = 1, \dots, m\}$ be a corresponding covering of $\partial\Omega$ of cubes for which previous conditions hold. Take a an open subset Ω_0 compactly contained in Ω such that

$$\Omega \subseteq \left(\bigcup_{j=1}^{m} Q_j\right) \bigcup \Omega_0,$$

and choose a partition of unity $\{\phi_j, j = 1, \dots, m\}$ of $\partial\Omega$ associated to the family $\{Q_j\}$. For each $j \in \{1 \dots, m\}$ set $v_j \doteq \phi_j v$ and, assuming that the equivalent of condition (36) is satisfied for every $j \in \{1 \dots, m\}$, define, as in previous step, sets $\Omega_t(x^j)$ and maps $w_j \in W^{1,p}(Q_j \cap \Omega) \cap C^0(Q_j \cap \overline{\Omega})$ with the following properties (see (50) and (51)):

$$\begin{aligned} w_j &= 0 \quad \text{on } \Omega_t(x^j), \\ w_j &= v_j \quad \text{on } \partial\Omega \cap Q_j, \\ \|w_j\|_{W^{1,p}(Q_j\cap\Omega)} &\leq M_3 \end{aligned}$$

for a suitable $M_3 > 0$ independent on t and for every $j \in \{1 \dots, m\}$, with the condition (see (42) and (43)):

$$\text{meas}\left(Q_j \cap (\Omega \setminus \Omega_t(x^j))\right) \to 0 \quad \text{as } t \to 0 \quad \forall j \in \{1 \dots, m\}.$$

Now define

$$w_t(y) = \begin{cases} 0, & y \in \Omega_0, \\ \sum_{j=1}^{m} \phi_j(y) w_j(y), & y \in \left(\bigcup_{j=1}^{m} Q_j\right) \cap \Omega. \end{cases}$$

It is easy to see that w_t fulfills (i)-(iv).

Step 4. To end the proof we are left to remove the assumption $u \equiv 0$. Set $\tilde{v} \doteq v - u$, $\tilde{u} \equiv 0$ and define a map $\tilde{w}_t$ as in Step 3 relatively to functions $\tilde{u}$ and $\tilde{v}$; then set

$$w_t \doteq u + \tilde{w}_t.$$

Properties (i)-(iv) can be easily verified.

We conclude this section recalling from [14] (Theorem 3, p. 449) the following classical result.

Theorem 2.7. *Let Ω be an open and bounded subset of $\mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that*

$$g(\xi) \geq a|\xi|^p - b$$

for some $p \in]1, \infty[$ and $a, b > 0$. Let $w_k, w : \Omega \to \mathbb{R}^n$ $(k \in \mathbb{N})$ be measurable functions such that

(i) $\displaystyle\int_\Omega g(w_k)\,dx \to \int_\Omega g(w)\,dx;$

(ii) $(w(x), g(w(x)))$ *is an extremal point of* $\mathrm{epi}(g)$ *for a.e.* $x \in \Omega$;

(iii) $w_k \rightharpoonup w$ *weakly in* $L^1_{\mathrm{loc}}(\Omega)$.

Then

$$w_k \to w \quad \textit{strongly in } L^p(\Omega).$$

3. Minimization of non-convex functionals.

In this section we recall the main result of [16] concerning non-convex variational problems. We consider a continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$, its bipolar f^{**} (which is convex) and define the open set

$$X \doteq \{\xi \in \mathbb{R}^n \colon f(\xi) > f^{**}(\xi)\}\,. \tag{52}$$

For $p \in [1, \infty[$ and given a boundary datum $\varphi \in W^{1,p}(\Omega)$, we introduce the functionals

$$\mathcal{F}(u) = \int_\Omega f(Du)\,dx, \qquad u \in \varphi + W_0^{1,p}(\Omega);$$

$$\overline{\mathcal{F}}(u) = \int_\Omega f^{**}(Du)\,dx, \qquad u \in \varphi + W_0^{1,p}(\Omega).$$

Remark 3.1. The functional $\overline{\mathcal{F}}$ is sequentially lower semicontinuous with respect to weak convergence in $W^{1,p}$ and, if standard growth conditions (see hypothesis H2 below) are satisfied, it is also coercive; consequently the set S of its minimizers is nonempty. Suppose that there exists $\tilde{u} \in S$ such that $D\tilde{u}(x) \in \mathbb{R}^n \setminus X$ for almost every $x \in \Omega$. Then $\tilde{u}$ is a minimizer of $\mathcal{F}$. Indeed $f(\xi) \geq f^{**}(\xi)$ for every $\xi \in \mathbb{R}^n$ and then $\inf \mathcal{F} \leq \min \overline{\mathcal{F}}$. Since

$$f(D\tilde{u}(x)) = f^{**}(D\tilde{u}(x)) \quad \text{for a.e. } x \in \Omega,$$

it turns out that

$$\mathcal{F}(\tilde{u}) = \int_\Omega f(D\tilde{u})\,dx = \int_\Omega f^{**}(D\tilde{u})\,dx = \min \overline{\mathcal{F}}.$$

We recall from [16] the conditions (hypotheses H1, H2, H3) under which the argument of Remark 3.1 may be performed.

Hypothesis H1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous. The function f^{**} is locally affine on the set X. More precisely, for every $\eta \in X$ there exist $r = r(\eta) > 0$, $m = m(\eta) \in \mathbb{R}^n$ and $q = q(\eta) \in \mathbb{R}$ such that $B(\eta, r) \subseteq X$ and

$$f^{**}(\xi) = \langle m, \xi\rangle + q \quad \forall \xi \in B(\eta, r); \tag{53}$$

$$f^{**}(\xi) \geq \langle m, \xi \rangle + q \quad \forall \xi \in \mathbb{R}^n. \tag{54}$$

Hypothesis H2. The set S of minimizers of $\overline{\mathcal{F}}$ is nonempty and sequentially compact in $L^1(\Omega)$.

It is well known (by Rellich and De La Vallée Poussin theorems) that such condition is ensured by the following growth conditions, to which we will refer in the paper by saying that the functional $\mathcal{F}$ is coercive on $W^{1,p}(\Omega)$.

$$\text{Case } p = 1: \qquad f(\xi) \geq \psi(|\xi|) \quad \forall \xi \in \mathbb{R}^n,$$

$$\text{Case } 1 < p < \infty: \qquad f(\xi) \geq a|\xi|^p - b \quad \forall \xi \in \mathbb{R}^n,$$

where $a, b > 0$ and $\psi : [0, +\infty[\to \mathbb{R}$ is superlinear at infinity, i.e.

$$\lim_{t \to +\infty} \frac{\psi(t)}{t} = +\infty.$$

Hypothesis H3. All the elements of S are continuous on Ω and (classically) differentiable almost everywhere in Ω.

It is known (see for example [3]) that such condition is verified if the following growth conditions are satisfied.

$$\text{Case } 1 < p \leq n. \quad \exists\, a, b, c, d > 0 : a|\xi|^p - b \leq f(\xi) \leq c|\xi|^p + d \quad \forall \xi \in \mathbb{R}^n.$$

$$\text{Case } n < p < \infty. \quad \exists\, a, b > 0 : f(\xi) \geq a|\xi|^p - b \quad \forall \xi \in \mathbb{R}^n.$$

We give in Theorem 3.4 and Corollary 3.5 a refined version of Theorem 1 in [16] for which we need a preliminary notion.

Definition 3.2. Let $\alpha \in L^\infty(\Omega)$. We say that α has essentially locally definite sign if for almost every $x \in \Omega$ there exists $\rho = \rho(x) > 0$ such that the ball $B(x, \rho)$ is contained in Ω and

$$\text{either} \quad \alpha(y) > 0 \quad \text{for a.e. } y \in B(x, \rho) \tag{55}$$

$$\text{or} \quad \alpha(y) < 0 \quad \text{for a.e. } y \in B(x, \rho). \tag{56}$$

We set

$$Z_\alpha^+ \doteq \{x \in \Omega : (55) \text{ holds}\} \tag{57}$$

$$Z_\alpha^- \doteq \{x \in \Omega : (56) \text{ holds}\} \tag{58}$$

Lemma 3.3. *Let $\alpha \in L^\infty(\Omega)$ and let S be a (sequentially) compact subset of $L^1(\Omega)$. Then there exist $\overline{u}_\alpha, \underline{u}_\alpha \in S$ such that*

$$\int_\Omega \alpha \overline{u}_\alpha \, dx \geq \int_\Omega \alpha u \, dx \quad \forall u \in S; \tag{59}$$

$$\int_\Omega \alpha \underline{u}_\alpha \, dx \leq \int_\Omega \alpha u \, dx \quad \forall u \in S. \tag{60}$$

Proof. The functional

$$L^1(\Omega) \ni u \mapsto \int_\Omega \alpha u \, dx$$

is (sequentially) continuous. Then the thesis follows by the compactness of S and by Weierstrass theorem.

Theorem 3.4. *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, $p \in [1, \infty[$, $\varphi \in W^{1,p}(\Omega)$ and let $\alpha \in L^\infty(\Omega)$ have essentially locally definite sign. Assume hypotheses* H1 *and* H2 *and that all the elements of the set S of minimizers of $\overline{\mathcal{F}}$ are continuous on Ω. Define the maps $\overline{u}_\alpha, \underline{u}_\alpha \in S$ as in Lemma* 3.3*. Then*

(*i*) *For every $x \in Z_\alpha^+ \cap A^-(\overline{u}_\alpha)$ we have*

$$f(\xi) = f^{**}(\xi) \quad \forall \xi \in D^-\overline{u}_\alpha(x); \tag{61}$$

(*ii*) *For every $x \in Z_\alpha^- \cap A^+(\overline{u}_\alpha)$ we have*

$$f(\xi) = f^{**}(\xi) \quad \forall \xi \in D^+\overline{u}_\alpha(x); \tag{62}$$

(*iii*) *For every $x \in Z_\alpha^+ \cap A^+(\underline{u}_\alpha)$ we have*

$$f(\xi) = f^{**}(\xi) \quad \forall \xi \in D^+\underline{u}_\alpha(x); \tag{63}$$

(*iv*) *For every $x \in Z_\alpha^- \cap A^-(\underline{u}_\alpha)$ we have*

$$f(\xi) = f^{**}(\xi) \quad \forall \xi \in D^-\underline{u}_\alpha(x). \tag{64}$$

Proof. We consider the map $\overline{u}_\alpha$ and prove (61) and (62); the proof of (63) and (64) for $\underline{u}_\alpha$ is analogous and makes use of (60) in place of (59) (see below).

Recall that $f^{**} \le f$ and definitions (52) and (57). We prove that for every $x \in A^-(\overline{u}_\alpha) \cap Z_\alpha^+$ and for every $\xi \in D^-(\overline{u}_\alpha)$ we have $\xi \in X^c$. We argue by contradiction: assume that there exist a point $x_0 \in Z_\alpha^+$ such that $D^-\overline{u}_\alpha(x_0) \neq \emptyset$ and a vector $\xi \in D^-\overline{u}_\alpha(x_0) \cap X$. Recalling hypothesis H1 we select $\overline{r}$ in such a way that

$$0 < \overline{r} < r(\xi) \tag{65}$$

and choose $\rho > 0$ such that (55) holds. Apply Lemma 2.4 with Ω, $\overline{u}_\alpha$, $\overline{r}$ in place of U, u, r respectively and construct a map u^+ satisfying the following conditions:

$$u^+ - \overline{u}_\alpha \in W_0^{1,p}(\Omega) \cap C^0(\Omega); \tag{66}$$

$$\Lambda \doteq \{x \in \Omega \colon u^+(x) > \overline{u}_\alpha(x)\} \textit{ is nonempty and } \Lambda \subseteq B(x_0, \rho); \tag{67}$$

$$\begin{cases} |Du^+(x) - \xi| = \overline{r}, & \textit{for a.e. } x \in \Lambda, \\ Du^+(x) = D\overline{u}_\alpha(x), & \textit{for a.e. } x \in \Omega \setminus \Lambda; \end{cases} \tag{68}$$

$$\int_\Omega \alpha u^+ \, dx > \int_\Omega \alpha \overline{u}_\alpha \, dx; \tag{69}$$

$$\int_\Lambda Du^+ \, dx = \int_\Lambda D\overline{u}_\alpha \, dx. \tag{70}$$

First we prove that u^+ lies in S. By (66) it belongs to $\overline{u}_\alpha + W_0^{1,p}(\Omega) = \varphi + W_0^{1,p}(\Omega)$ and we can compute, using (67) and (68),

$$\begin{aligned} \overline{\mathcal{F}}(u^+) &= \int_{\Omega\setminus\Lambda} f^{**}(Du^+)\,dx + \int_\Lambda f^{**}(Du^+)\,dx \\ &= \int_{\Omega\setminus\Lambda} f^{**}(D\overline{u}_\alpha)\,dx + \int_\Lambda f^{**}(Du^+)\,dx. \end{aligned} \tag{71}$$

Again by (68) and by the choice (65) of $\overline{r}$, we have that

$$Du^+(x) \in B(\xi, r(\xi)) \quad \text{a.e. in } \Lambda. \tag{72}$$

Using (72), (53), (54) and (70) we have

$$\begin{aligned} \int_\Lambda f^{**}(Du^+)dx &= \int_\Lambda \big(\langle m, Du^+\rangle + q\big)\,dx \\ &= \left\langle m, \int_\Lambda Du^+\,dx \right\rangle + q\,\mathrm{meas}(\Lambda) \\ &= \left\langle m, \int_\Lambda D\overline{u}_\alpha\,dx \right\rangle + q\,\mathrm{meas}(\Lambda) \\ &= \int_\Lambda (\langle m, D\overline{u}_\alpha\rangle + q)\,dx \;\leq\; \int_\Lambda f^{**}(D\overline{u}_\alpha)\,dx. \end{aligned} \tag{73}$$

Inserting (73) in (71), we have

$$\overline{\mathcal{F}}(u^+) = \int_\Omega f^{**}(Du^+)\,dx \leq \int_\Omega f^{**}(D\overline{u}_\alpha)\,dx = \min \overline{\mathcal{F}}.$$

Hence u^+ belongs to S and (69) contradicts (59). This proves (61).

Assume now that there exist $x_0 \in Z_\alpha^- \cap A^+(\overline{u}_\alpha)$ and a vector $\xi \in D^+\overline{u}_\alpha(x_0) \cap X$. Apply Lemma 2.5 and construct a function u^- with properties (25)-(30). By identical computations and taking into account (29) and (58) we prove (62).

Theorem 3.4 has the following consequence.

Corollary 3.5. *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, $p \in [1,\infty[$, $\varphi \in W^{1,p}(\Omega)$ and let $\alpha \in L^\infty(\Omega)$ have essentially locally definite sign. Assume hypotheses* H1, H2, H3*. Then the functional $\mathcal{F}$ attains its minimum at any function $\overline{u}_\alpha$ and $\underline{u}_\alpha$ as defined in Lemma 3.3.*

Proof. Take any map $\overline{u}_\alpha$ (or $\underline{u}_\alpha$) as defined in Lemma 3.3. Recalling point *(iv)* in lemma 2.1 we have that,

$$D^-\overline{u}_\alpha(x) = D^+\overline{u}_\alpha(x) = \{D\overline{u}_\alpha(x)\} \quad \text{for a.e. } x \in \Omega,$$

where the last set is the singleton of the differential of $\overline{u}_\alpha$ at x. Hence by previous lemma we have

$$f(D\overline{u}_\alpha(x)) = f^{**}(D\overline{u}_\alpha(x)) \quad \text{for a.e. } x \in \Omega.$$

Since $\overline{u}_\alpha \in S$, and recalling Remark 3.1, this ends the proof.

This result can be reformulated as follows.

Corollary 3.6. *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, $p \in [1,\infty[$, $\varphi \in W^{1,p}(\Omega)$ and let $\alpha \in L^\infty(\Omega)$ have essentially locally definite sign. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let X be the (possibly empty) open set on which f is locally affine in the sense of* (53) *and* (54) *of hypothesis* H1*. Assume that the set S of minimizers of $\mathcal{F}$ is nonempty and sequentially compact with respect to the strong topology of $L^1(\Omega)$ and that all the elements of S are continuous on Ω and differentiable almost everywhere in Ω. Then, for every element $\overline{u}_\alpha$ and $\underline{u}_\alpha$ in S defined as in Lemma* 3.3*, the sets $\{x \in \Omega : D\overline{u}_\alpha(x) \in X\}$ and $\{x \in \Omega : D\underline{u}_\alpha(x) \in X\}$ have measure zero.*

Remark 3.7. Theorem 3.4 and Corollary 3.5 says that, in a certain sense, the minimizers $\overline{u}_\alpha$ and $\underline{u}_\alpha$ are sorts of "viscosity semisolutions" of the equation (2).

Consider the special case $\alpha \equiv 1$ and call $\overline{u}$ and $\underline{u}$ the corresponding elements of S defined in Lemma 3.3. Since $Z_\alpha^+ = \Omega$ and $f^{**}(\xi) - f(\xi) \le 0$ for every $\xi \in \mathbb{R}^n$, it follows from (61) that $\overline{u}$ is a viscosity solution of the Hamilton-Jacobi equation

$$f^{**}(Du) - f(Du) = 0,$$

while it follows from (63) that $\underline{u}$ is a viscosity solution of the Hamilton-Jacobi equation

$$f(Du) - f^{**}(Du) = 0.$$

4. Uniqueness and continuous dependence on boundary data.

Given an index p with $1 < p < \infty$, a continuous function f and a boundary datum $\varphi \in W^{1,p}(\Omega)$, we consider the functional defined in (1) and concentrate our attention on some properties of the special minimizers of $\mathcal{F}$ introduced in previous section. In view of Remark 3.7 we restrict ourselves to the case $\alpha \equiv 1$, corresponding to elements $\overline{u}$ and $\underline{u}$ of the set S of minimizers of $\mathcal{F}$ such that

$$\int_\Omega \overline{u}\, dx \ge \int_\Omega u\, dx \quad \forall u \in S; \qquad \int_\Omega \underline{u}\, dx \le \int_\Omega u\, dx \quad \forall u \in S. \tag{74}$$

We call $\overline{u}$ and $\underline{u}$, respectively, *integro-maximal* and *integro-minimal* minimizers (in general *integro-extremal* minimizers) and look for conditions under which they are unique and depend continuously on boundary data in the natural Sobolev strong topology in which the variational problem is formulated. We assume that the lagrangian f is convex since previous section clarifies the relation with the non-convex case and impose suitable conditions (hypotheses H4, H5, H6).

Hypothesis H4. The function $f : \mathbb{R}^n \to \mathbb{R}$ is convex and there exist $a, b, d > 0$ such that

$$a|\xi|^p - b \le f(\xi) \le d(1 + |\xi|^p) \quad \forall \xi \in \mathbb{R}^n. \tag{75}$$

It is well known that, under this condition, there exists $c > 0$ such that

$$|f(\xi) - f(\eta)| \le c\left(1 + |\xi|^{p-1} + |\eta|^{p-1}\right)|\xi - \eta| \quad \forall \xi, \eta \in \mathbb{R}^n. \tag{76}$$

Hypothesis H5. The epigraph epi(f) of the function f has at most one proper face of dimension greater than zero and all the other proper faces are extremal points.

Remark 4.1. 1. First of all notice that hypothesis H4 subsumes the growth conditions mentioned in hypothesis H3 of previous section and then, in particular, the functional $\mathcal{F}$ is coercive on $W^{1,p}(\Omega)$ and all its minimizers are continuous on Ω and differentiable almost everywhere in Ω. Assuming that the datum φ is in $W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ – as it will be done in Theorem 4.3 – we have that all the elements of S lie in $C^0(\overline{\Omega})$.

Hypothesis H5 has the following consequences.

2. There exists at most one closed convex set $F_0 \subseteq \mathbb{R}^n$ of dimension greater than zero and $m \in \mathbb{R}^n$, $q \in \mathbb{R}$ such that

$$f(\xi) = \langle m, \xi \rangle + q \quad \forall \xi \in F_0; \tag{77}$$

$$f(\xi) \geq \langle m, \xi \rangle + q \quad \forall \xi \in \mathbb{R}^n. \tag{78}$$

3. If the set F_0 of item 2 is empty all proper faces of epi(f) are extremal points; if it is nonempty the unique proper face of epi(f) of nonzero dimension is the set

$$F \doteq \{(\xi, f(\xi)) \in \mathbb{R}^n \times \mathbb{R} : \xi \in F_0\} \tag{79}$$

and, in particular, for every $\xi \in (\mathrm{int}(F_0))^c$, the point $(\xi, f(\xi))$ is an extremal point of epi(f).

4. Given two distinct points $\xi, \eta \in \mathbb{R}^n$ such that at least one of them lies in the complement F_0^c of F_0 we have

$$f\left(\frac{1}{2}\xi + \frac{1}{2}\eta\right) < \frac{1}{2}f(\xi) + \frac{1}{2}f(\eta). \tag{80}$$

We start by giving a uniqueness result.

Theorem 4.2. *Let Ω be an open and bounded subset of $\mathbb{R}^n$ and let $\varphi \in W^{1,p}(\Omega)$. Assume hypothesis H5 and that the set S of minimizers of the functional $\mathcal{F}$ is nonempty and sequentially compact in $L^1(\Omega)$. Then the integro-extremal elements of S $\overline{u}$ and $\underline{u}$ defined in* (74) *are unique, in the sense that*

(i) *if $u, v \in S$ and $\int_\Omega v\,dx = \int_\Omega u\,dx = \max\left\{\int_\Omega z\,dx, z \in S\right\}$, then $v = u$;*
(ii) *if $u, v \in S$ and $\int_\Omega v\,dx = \int_\Omega u\,dx = \min\left\{\int_\Omega z\,dx, z \in S\right\}$, then $v = u$.*

Proof. Step 1. First of all consider the case in which there are no proper faces of epi(f) of positive dimension, i.e. the boundary of such set is the union of its extremal points. This means that f is strictly convex and then, by a standard argument, the set S of minimizers of $\mathcal{F}$ is a singleton and there is nothing to prove.

Step 2. Let now F as in (79) be the unique proper face of epi(f) of nonzero dimension. For every $u \in S$ define the set

$$\Omega_F^u \doteq \{x \in \Omega : Du(x) \in F_0\}. \tag{81}$$

Claim. Take any pair (u, v) of elements of S. We have

$$\mathrm{meas}\,(\Omega_F^u \triangle \Omega_F^v) = 0, \tag{82}$$

where the symbol $\triangle$ stands for the symmetric difference. If both Ω_F^u and Ω_F^v have measure zero there is nothing to prove. Assume, by contradiction, that the set

$$G \doteq \Omega_F^u \setminus \Omega_F^v$$

has positive measure. Recalling item 4 in Remark 4.1, (80) and (81) imply that

$$f\left(\frac{1}{2}Du(x) + \frac{1}{2}Dv(x)\right) < \frac{1}{2}f(Du(x)) + \frac{1}{2}f(Dv(x)) \quad \text{for a.e. } x \in G. \tag{83}$$

Define the map

$$w \doteq \frac{1}{2}u + \frac{1}{2}v.$$

Clearly w lies in the set $\varphi + W^{1,p}(\Omega)$ and, by (83), we have

$$\begin{aligned}
\mathcal{F}(w) &= \int_G f(Dw)\,dx + \int_{\Omega\setminus G} f(Dw)\,dx \\
&= \int_G f\left(\frac{1}{2}Du + \frac{1}{2}Dv\right)\,dx + \int_{\Omega\setminus G} f\left(\frac{1}{2}Du + \frac{1}{2}Dv\right)\,dx \\
&< \frac{1}{2}\int_G f\left(Du\right)\,dx + \frac{1}{2}\int_G f\left(Dv\right)\,dx + \int_{\Omega\setminus G} f\left(\frac{1}{2}Du + \frac{1}{2}Dv\right)\,dx \\
&\leq \frac{1}{2}\int_\Omega f\left(Du\right)\,dx + \frac{1}{2}\int_\Omega f\left(Dv\right)\,dx \\
&= \frac{1}{2}\mathcal{F}(u) + \frac{1}{2}\mathcal{F}(v) = \min \mathcal{F}
\end{aligned} \tag{84}$$

Inequality (84) is a contradiction and this proves that $\text{meas}(G) = 0$. Interchanging the role of u and v we prove (82).

Choose any $v \in S$ and set

$$\Omega_F \doteq \Omega_F^v. \tag{85}$$

As a consequence of (82) and of definition (85) the following properties hold:

$$\text{for a.e. } x \in \Omega_F \quad Du(x) \in F_0 \quad \forall u \in S; \tag{86}$$

$$\text{for a.e. } x \in \Omega \setminus \Omega_F \quad Du(x) \in (F_0)^c \quad \forall u \in S. \tag{87}$$

Claim. Take any pair (u, v) of elements of S. We have

$$Du(x) = Dv(x) \quad \text{for a.e. } x \in \Omega \setminus \Omega_F. \tag{88}$$

Assume, by contradiction, that there exists a set $G \subseteq \Omega \setminus \Omega_F$ with $\text{meas}(G) > 0$ such that

$$Dv(x) \neq Du(x) \quad \text{for a.e. } x \in G. \tag{89}$$

Recalling item 3 in Remark 4.1, (86), (87) and (89) imply that, for a.e. $x \in G$, the points

$$(Du(x), f(Du(x))) \quad \text{and} \quad (Dv(x), f(Dv(x)))$$

are distinct extremal points of the epigraph of f and then, by (80), we have

$$f\left(\frac{1}{2}Du(x)+\frac{1}{2}Dv(x)\right)<\frac{1}{2}f(Du(x))+\frac{1}{2}f(Dv(x))\quad \text{for a.e. } x\in G.$$

Introduce as above the map $w \doteq \frac{1}{2}u+\frac{1}{2}v$: by the same computations of (84) we obtain a contradiction. Hence (88) is proved.

Step 3. Assume now that

$$\dim(F)=\dim(F_0)\in\{1,\dots,n-1\}.$$

Take any pair (u,v) of elements of S; by (88) we have that, up to null sets, the gradients Du and Dv differ only on the set Ω_F on which they take values inside the convex set F_0 whose dimension is strictly less than n. This implies that there exists a direction $\mathbf{n}$ in $\mathbb{R}^n$ such that

$$\frac{\partial}{\partial \mathbf{n}}(u-v)=0\quad \text{a.e. on } \Omega. \tag{90}$$

Since the difference $u-v$ lies in $W_0^{1,p}(\Omega)$, (90) and Poincaré inequality imply that $u=v$ a.e. in Ω. Hence, also in this case, the set of minimizers of $\mathcal{F}$ is a singleton and there is nothing more to be proved.

Step 4. Assume

$$\dim(F)=\dim(F_0)=n.$$

Recalling (77) and (78) we set

$$\tilde f(\xi)\doteq f(\xi)-\langle m,\xi\rangle-q\quad \forall \xi\in\mathbb{R}^n$$

and introduce the functional

$$\tilde{\mathcal{F}}(u)=\int_\Omega \tilde f(Du(x))dx,\quad u\in\varphi+W_0^{1,p}(\Omega).$$

By divergence theorem, for every $u\in\varphi+W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}\tilde{\mathcal{F}}(u) &= \int_\Omega f(Du(x))dx-\int_\Omega (\langle m,Du\rangle+q\,)\,dx\\ &= \int_\Omega f(Du(x))dx-\left\langle m,\int_\Omega Du\,dx\right\rangle-q\,\mathrm{meas}(\Omega)\\ &= \mathcal{F}(u)-\left\langle m,\int_\Omega D\varphi\,dx\right\rangle-q\,\mathrm{meas}(\Omega).\end{aligned}$$

Hence the set of minimizers of $\tilde{\mathcal{F}}$ coincides with the set S of minimizers of $\mathcal{F}$ and for this reason we may assume, replacing f with $\tilde f$,

$$f(\xi)=0\quad \forall\xi\in F_0. \tag{91}$$

Step 5. Take now any pair (u,v) of elements of S and define the map

$$w(x)=(u\wedge v)(x)\doteq\max\{u(x),v(x)\}\quad \text{for a.e. } x\in\Omega. \tag{92}$$

Claim. $w \in S$.

Clearly w lies in $\varphi + W^{1,p}(\Omega)$. By (88) we have

$$Dw(x) = Du(x) = Dv(x) \quad \text{for a.e. } x \in \Omega \setminus \Omega_F.$$

Consequently, by Stampacchia's theorem, we have

$$Dw(x) = \begin{cases} Du(x) & \text{for a.e. } x \in \Omega \setminus (G \cap \Omega_F) \\ Dv(x) & \text{for a.e. } x \in G \cap \Omega_F, \end{cases} \tag{93}$$

where we have set

$$G \doteq \{x \in \Omega : v(x) > u(x)\}.$$

Formula (93) implies that

$$\mathcal{F}(w) = \int_{\Omega \setminus (G \cap \Omega_F)} f(Du)\, dx + \int_{G \cap \Omega_F} f(Dv)\, dx. \tag{94}$$

But, by (86) and (91), we have

$$f(Du(x)) = f(Dv(x)) = 0 \quad \text{for a.e. } x \in G \cap \Omega_F \tag{95}$$

and inserting (95) in (94) we obtain that

$$\mathcal{F}(w) = \int_\Omega f(Du)\, dx = \mathcal{F}(u) = \min \mathcal{F}.$$

This proves the claim.

Step 6. Let now (u, v) be a pair of integro-maximal element of S, that is to say:

$$\int_\Omega v\, dx = \int_\Omega u\, dx = \max\left\{ \int_\Omega z\, dx, z \in S \right\}. \tag{96}$$

Assume, by contradiction, that there exists a set G of positive measure such that $v(x) > u(x)$ for a.e. $x \in G$. Define the map $w \doteq u \wedge v \in S$ as in (92) and observe that necessarily we have

$$\int_\Omega w\, dx > \int_\Omega u\, dx. \tag{97}$$

By Step 5 the map w lies in S and then (97) contradicts (96). Hence we have

$$v(x) \le u(x) \quad \text{for a.e. } x \in \Omega. \tag{98}$$

Formulas (96) and (98) imply that $v = u$ almost everywhere on Ω and this ends the proof of item (i) of the theorem.

The proof of item (ii) is analogous.

We turn now our attention to the dependence on the boundary data and formulate the following hypotheses. Condition H5' strengthen H5. The use of extremality to get strong convergence made in Theorem 4.3 below recalls the main result contained in [6].

Hypothesis H5'. The epigraph epi(f) of the function f has at most one proper face of dimension n and all the other proper faces are extremal points.

Hypothesis H6. Let $\{\varphi_k, k = 0, 1, \dots\}$ be a sequence in $W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$, bounded in $W^{1,p}(\Omega)$. We assume that

$$\varphi_k \lfloor_{\partial\Omega} \to \varphi_0 \lfloor_{\partial\Omega} \quad \text{as } k \to \infty \text{ in } C^0(\partial\Omega). \tag{99}$$

For every $k \in \mathbb{N}_0$ introduce the set

$$\mathcal{W}_k \doteq \varphi_k + W^{1,p}(\Omega),$$

the variational problem

$$\text{Minimize } \mathcal{F}(u) = \int_\Omega f(Du)\, dx; \quad u \in \mathcal{W}_k,$$

the minimum

$$m_k \doteq \min\{\mathcal{F}(u); u \in \mathcal{W}_k\},$$

and the corresponding set of minimizers

$$S_k \doteq \{u \in \mathcal{W}_k : \mathcal{F}(u) = m_k\}.$$

Assuming that the hypotheses of Theorem 4.2 hold, for every $k \in \mathbb{N}_0$ we denote by $\overline{u}_k$ and $\underline{u}_k$ the unique integro-extremal elements of S_k as defined in (74). We have the following

Theorem 4.3. *Let Ω be an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary. Assume hypotheses* H4, H5', H6. *Then*

$$\begin{aligned} \overline{u}_k &\to \overline{u}_0 \quad \text{strongly in } W^{1,p}(\Omega); \\ \underline{u}_k &\to \underline{u}_0 \quad \text{strongly in } W^{1,p}(\Omega). \end{aligned}$$

Proof. We give the proof for the integro-maximal minimizers $\overline{u}_k$. The one for the integro-minimal minimizers $\underline{u}_k$ is identical. We reason assuming that there exists a face of dimension n of the epigrapf of f since the alternative case corresponds to a strictly convex function f and the result follows easily from Theorem 2.7.

Step 1. For every $\epsilon > 0$ and for every $k \in \mathbb{N}_0$ we define the set

$$S_k^\epsilon \doteq \{u \in \mathcal{W}_k : \mathcal{F}(u) \le m_k + \epsilon\}. \tag{100}$$

Let (u_j) be a sequence in S_k^ϵ; by the coercivity of $\mathcal{F}$ (see (75)) we have that there exists $M > 0$ such that

$$\|Du_j\|_{W^{1,p}(\Omega)} \le M \quad \forall j \in \mathbb{N}.$$

It follows that (u_j) admits a subsequence, still denoted by (u_j), weakly converging in $W^{1,p}(\Omega)$ to $u \in \mathcal{W}_k$. By sequential weak lower semicontinuity of $\mathcal{F}$, we have

$$\mathcal{F}(u) \le \liminf \mathcal{F}(u_j) \le m_k + \epsilon;$$

hence u lies in S_k^ϵ and, in particular, by Rellich theorem, such set turns out to be sequentially compact in $L^1(\Omega)$. Invoking Lemma 3.3, for every $k \in \mathbb{N}_0$ and for every $\epsilon > 0$, we may select an element $\overline{u}_k^\epsilon \in S_k^\epsilon$ such that

$$\int_\Omega \overline{u}_k^\epsilon \, dx \geq \int_\Omega u \, dx \quad \forall u \in S_k^\epsilon. \tag{101}$$

Claim. For every $k \in \mathbb{N}_0$ we have

$$\overline{u}_k^\epsilon \rightharpoonup \overline{u}_k \quad \text{weakly in } W^{1,p}(\Omega) \text{ as } \epsilon \to 0+\,. \tag{102}$$

Fix $k \in \mathbb{N}_0$. By the coercivity of $\mathcal{F}$, and with the same argument used above, we deduce that there exists a positive constant M such that

$$\|\overline{u}_k^\epsilon\|_{W^{1,p}(\Omega)} \leq M, \quad \forall \epsilon > 0.$$

Take any sequences (ϵ_j) and $(\overline{u}_k^{\epsilon_j})$ such that $\epsilon_j \to 0+$ and

$$\overline{u}_k^{\epsilon_j} \rightharpoonup v \quad \text{weakly in } W^{1,p}(\Omega) \text{ as } j \to \infty \tag{103}$$

for some $v \in \mathcal{W}_k$. By sequential weak lower semicontinuity of $\mathcal{F}$ we have

$$\mathcal{F}(v) \leq \liminf_{j\to\infty} \mathcal{F}(\overline{u}_k^{\epsilon_j})$$

and, since $\mathcal{F}(\overline{u}_k^{\epsilon_j}) \leq m_k + \epsilon_j \; \forall j \in \mathbb{N}$ and $\epsilon_j \to 0+$, we obtain that $\mathcal{F}(v) \leq m_k + \epsilon_j \; \forall j \in \mathbb{N}$. It follows that $\mathcal{F}(v) \leq m_k$ and then

$$v \in S_k. \tag{104}$$

On the other hand, since $S_k \subseteq S_k^\epsilon$ for every $\epsilon > 0$, we have

$$\int_\Omega \overline{u}_k^\epsilon \, dx \geq \int_\Omega \overline{u}_k \, dx \quad \forall \epsilon > 0 \tag{105}$$

and consequently, by the continuity of the integral operator with respect to weak convergence in $W^{1,p}$, (103) and (105) imply that

$$\int_\Omega v \, dx \geq \int_\Omega \overline{u}_k \, dx. \tag{106}$$

Inequality (106), inclusion (121) and the uniqueness result of Theorem 4.2 imply that $v = \overline{u}_k$. Then the arbitrariness of (ϵ_j) and $(\overline{u}_k^{\epsilon_j})$ proves (102) and, in particular, by Rellich theorem, we have

$$\int_\Omega \overline{u}_k^\epsilon \, dx \to \int_\Omega \overline{u}_k \, dx \quad \text{as } \epsilon \to 0+\,. \tag{107}$$

Step 2. Consider the sequence $(m_k)_{k\in\mathbb{N}}$.

Claim.

$$m_k \to m_0 \quad \text{as } k \to \infty. \tag{108}$$

First of all remark that hypotheses H4 and H6 imply that (m_k) is bounded in $\mathbb{R}$. Take any converging subsequence $(\tilde{m}_k)$ and call $\tilde{m}$ its limit. Assume first $\tilde{m} < m_0$: there exist some positive δ, an index $k_\delta \in \mathbb{N}$ and elements $v_k \in S_k$ such that

$$\mathcal{F}(v_k) = \tilde{m}_k \leq m_0 - \delta, \quad \forall k > k_\delta. \tag{109}$$

By coercivity of $\mathcal{F}$ we may extract from (v_k) a subsequence, still denoted (v_k), weakly converging in $W^{1,p}(\Omega)$ to v. By convergence (99) of hypothesis H6 we have that v lies in $\mathcal{W}_0$ and, in addition, by weak lower semicontinuity of $\mathcal{F}$ and by (109),

$$\mathcal{F}(v) \leq \liminf \mathcal{F}(v_k) \leq m_0 - \delta < \min \{\mathcal{F}(u); u \in \mathcal{W}_0\}. \tag{110}$$

Inequality (110) is absurd and then

$$\tilde{m} \geq m_0. \tag{111}$$

Recalling item 1 in Remark 4.1 and convergence (99) of hypothesis H6 we apply Lemma 2.6, defining a family (Ω_k) of open subsets of Ω and a sequence $(u_k)_{k\in\mathbb{N}}$ in $W^{1,p}(\Omega)$ such that for every $k \in \mathbb{N}$ and for a suitable positive M independent on k, the following properties hold:

$$\|u_k\|_{W^{1,p}(\Omega)} \leq M, \tag{112}$$
$$u_k = \overline{u}_0 \quad \text{on } \Omega_k, \tag{113}$$
$$u_k = \varphi_k \quad \text{on } \partial\Omega \tag{114}$$

and, in addition,

$$\text{meas}(\Omega \setminus \Omega_k) \to 0 \quad \text{as } k \to \infty. \tag{115}$$

We stress that $\overline{u}_0$ is the integro-maximal minimizer of the variational problem with boundary datum φ_0. Conditions (112)-(115) imply that $u_k \in \mathcal{W}_k$ for every k and that

$$u_k \to \overline{u}_0 \quad \text{strongly in } W^{1,p}(\Omega) \text{ as } k \to \infty. \tag{116}$$

Then, using (76) of hypothesis H4, we estimate

$$\begin{aligned}
|\mathcal{F}(u_k) - \mathcal{F}(\overline{u}_0)| &= \left| \int_\Omega (f(Du_k) - f(D\overline{u}_0))\, dx \right| \\
&\leq c \int_\Omega (1 + |Du_k|^{p-1} + |D\overline{u}_0|^{p-1}) |Du_k - D\overline{u}_0|\, dx \\
&\leq c \left(1 + M^{p-1} + \|D\overline{u}_0\|^{p-1}_{L^p(\Omega)}\right) \|Du_k - D\overline{u}_0\|_{L^p(\Omega)} \overset{k\to\infty}{\longrightarrow} 0,
\end{aligned}$$

where we have used Hölder inequality and (116). Hence we have

$$\mathcal{F}(u_k) \to \mathcal{F}(\overline{u}_0) \quad \text{as } k \to \infty \tag{117}$$

and, as a consequence of (117), $\forall \delta > 0$ there exists $k_\delta \in \mathbb{N}$ such that

$$\tilde{m}_k = \mathcal{F}(\overline{u}_k) \leq \mathcal{F}(u_k) \leq m_0 + \delta, \quad \forall k > k_\delta. \tag{118}$$

By the arbitrariness of δ we deduce from (118) that

$$\tilde{m} \leq m_0. \tag{119}$$

Inequality (111) and (119) imply that $\tilde{m} = m_0$ and we conclude that any converging subsequence of (m_k) tends to m_0. This proves (108).

Step 3. Consider now the sequence $(\overline{u}_k)_{k\in\mathbb{N}}$.

Claim.

$$\overline{u}_k \rightharpoonup \overline{u}_0 \quad \text{weakly in } W^{1,p}(\Omega) \text{ as } k \to \infty. \tag{120}$$

First of all remark that, by the boundedness of the sequence (φ_k) in $W^{1,p}(\Omega)$ (hypothesis H6) and by the coercivity of $\mathcal{F}$, $(\overline{u}_k)$ is bounded in $W^{1,p}(\Omega)$. Take any weakly converging (in $W^{1,p}(\Omega)$) subsequence, still denoted $(\overline{u}_k)$, and call v its limit. Convergence (99) in Hypothesis H6 implies that $v \in \mathcal{W}_0$ and, by weak lower semicontinuity of $\mathcal{F}$ and by (108), we have

$$\mathcal{F}(v) \leq \liminf \mathcal{F}(\overline{u}_k) = m_0.$$

Hence

$$v \in S_0. \tag{121}$$

Consider the sequence (u_k) defined in Step 2 with properties (112)-(115). By (108) and (117) we obtain that $[\mathcal{F}(\overline{u}_k) - \mathcal{F}(u_k)] \to 0$ as $k \to \infty$; recalling definition (100) it follows that for every $\epsilon > 0$ there exists $k_\epsilon \in \mathbb{N}$ such that $u_k \in S_k^\epsilon$ for all $k > k_\epsilon$. Hence, by (101), we have

$$\int_\Omega u_k \, dx \leq \int_\Omega \overline{u}_k^\epsilon \, dx \quad \forall k > k_\epsilon. \tag{122}$$

Recalling (107) and (116) we deduce from (122) and from $\overline{u}_k \rightharpoonup v$ in $W^{1,p}(\Omega)$ that

$$\int_\Omega \overline{u}_0 \, dx \leq \int_\Omega v \, dx;$$

then the definition of $\overline{u}_0$ and (121) imply that

$$\int_\Omega \overline{u}_0 \, dx = \int_\Omega v \, dx. \tag{123}$$

By Theorem 4.2 we deduce from (123) that $v = \overline{u}_0$. Hence any weakly converging subsequence of $(\overline{u}_k)$ tends weakly to $\overline{u}_0$ in $W^{1,p}(\Omega)$ and this proves (120).

Step 4. If there are no proper faces of the epigraph of f of positive dimension, by hypothesis H5' we have that the point $(D\overline{u}_0(x), f(D\overline{u}_0(x)))$ is an extremal point of epi(f) for a.e. $x \in \Omega$.

Hypothesis H4 implies in particular that the conditions of hypothesis H3 which guarantee differentiability almost everywhere of the minimizers are satisfied. If there exists a proper face F of the epigraph of f of dimension n we invoke Corollary 3.6 of previous section and, recalling (77) and (78), we obtain that

$$D\overline{u}_0(x) \in (\mathrm{int}(F_0))^c \quad \text{for a.e. } x \in \Omega. \tag{124}$$

Item 3 in Remark 4.1 and (124) imply that, also in this case, the point

$$(D\overline{u}_0(x), f(D\overline{u}_0(x)))$$

is an extremal point of epi(f) for a.e. $x \in \Omega$. In addition, recalling (108), we have that

$$\mathcal{F}(\overline{u}_k) = m_k \to m_0 = \mathcal{F}(\overline{u}_0) \quad \text{as } k \to \infty. \tag{125}$$

Then (120), (125) and Theorem 2.7 of Section 2, imply that

$$\overline{u}_k \to \overline{u}_0 \quad \text{strongly in } W^{1,p}(\Omega) \text{ as } k \to \infty.$$

This concludes the proof.

Remark 4.4. In Theorem 4.3 the replacement of hypothesis H5 with H5' is necessary. If only H5 holds, in general, we have uniqueness of the minimizer but not necessarily strong convergence in the sense of Theorem 4.3. This is clear from the following example.

Assume that the epigraph of the convex integrand f has a face F of dimension $d \in \{1, \ldots, n-1\}$. Let (ϕ_k) be a sequence as in hypothesis H6 such that $D\phi_k$ lies in F_0 a.e. in Ω for every $k \in \mathbb{N}_0$; assume that ϕ_k converges weakly but not strongly to ϕ_0 in $W^{1,p}(\Omega, \mathbb{R})$. By the convexity of f, ϕ_k is a minimizer of $\mathcal{F}$ on $\mathcal{W}_k$ for every $k \in \mathbb{N}_0$; by Step 3 in Theorem 4.2 the set of minimizers of the functional is a singleton and, in particular, $\overline{u}_k = \phi_k$ for all $k \in \mathbb{N}_0$. The choice of the sequence (ϕ_k) implies that continuous dependence in the sense of Theorem 4.3 fails.

Example. We give here a simple example which briefly illustrates our results. Take any open bounded subset Ω of $\mathbb{R}^n$ and set

$$f(\xi) \doteq \left(|\xi|^2 - 1\right)^2, \quad \xi \in \mathbb{R}^n.$$

By direct inspection we see that

$$f^{**}(\xi) = \begin{cases} \left(|\xi|^2 - 1\right)^2, & |\xi| \geq 1 \\ 0, & |\xi| < 1 \end{cases}$$

and that the set X defined at the beginning of Section 3 is the open unit ball in $\mathbb{R}^n$, while it is easy to show that hypotheses H1, H3, H4 and H5' are satisfied (with $p = 4$). Consider the functionals $\mathcal{F}$ and $\overline{\mathcal{F}}$ defined as in previous sections and a boundary datum $\varphi \in W^{1,\infty}(\Omega)$ such that $|D\varphi| < 1$ a.e. in Ω. Clearly any map $u \in \varphi + W_0^{1,\infty}(\Omega)$ such that $|Du| \leq 1$ a.e. in Ω is a minimizer of $\overline{\mathcal{F}}$ while any map $u \in \varphi + W_0^{1,\infty}(\Omega)$ solving the eikonal equation $|Du| = 1$ a.e. in Ω is a minimizer of $\mathcal{F}$. Then for the solutions of this minimum problems we cannot expect uniqueness and continuous dependence on boundary data in classical sense. Our results say that, by selecting the integro-extremal minimizers $\overline{u}$ and $\underline{u}$ of the relaxed functional, we solve the minimum problem for $\mathcal{F}$ with the viscosity properties discussed in Section 3 and that well posedness conditions specified in Section 4 hold true.

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