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A Conjecture of De Giorgi on the Square Distance Function

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We prove a conjecture of De Giorgi on the regularity of the square distance function from manifolds of arbitrary codimension in $\mathbb{R}^N$.

1. Introduction

In [7] De Giorgi made the following conjecture.

Let $E \subseteq \mathbb{R}^N$ be a set and Ω be an open subset of $\mathbb{R}^N$, such that $E \cap \Omega = \overline{E} \cap \Omega$. Define

$$\eta_E(x) = \frac{1}{2}\left[\operatorname{dist}(x, E)^2\right] = \frac{1}{2}\inf_{y \in E}|y - x|^2, \quad x \in \mathbb{R}^N, \tag{1}$$

and assume that $\eta_E \in \mathcal{C}^\omega(\Omega)$ (respectively $\mathcal{C}^\infty(\Omega)$). Then

$$E = \bigcup_{h=0}^{N} E_h,$$

where $E_h \in V_h\mathcal{C}^\omega(\Omega)$ (respectively $E_h \in V_h\mathcal{C}^\infty(\Omega)$) and $E_i \cap E_j = \emptyset$ for i and $j \in \{0, \dots, N\}$, $i \neq j$.

The symbol $\mathcal{C}^\omega(\Omega)$ denotes the class of real analytic functions in Ω, and $V_h\mathcal{C}^\omega(\Omega)$ stands for the class of h-dimensional real analytic manifolds without boundary in Ω; an element of $V_0\mathcal{C}^\omega(\Omega)$ consists of a discrete set of points which is locally finite in Ω, and by definition $\emptyset \in V_h\mathcal{C}^\omega(\Omega)$ for any $h \in \{0, \dots, N\}$.

To understand the meaning of the conjecture some comments are in order.

(i) If we have given a compact h-dimensional manifold M of class $\mathcal{C}^\infty$ without boundary embedded in $\mathbb{R}^N$, then the square distance function η_M from M is of class $\mathcal{C}^\infty$ in a tubular

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neighbourhood of M (see [1]). The conjecture is concerned with the opposite conclusion, namely to infer the regularity properties of M once we know that η_M is sufficiently smooth.

(ii) If ∂A is a compact $(N-1)$-dimensional manifold of class $\mathcal{C}^k$, $k \geq 2$, then the signed distance function from ∂A (and the square distance function from ∂A) is of class $\mathcal{C}^k$ in a tubular neighbourhood of ∂A, see for instance [13]. Conversely, if the signed distance function from the compact ∂A is of class $\mathcal{C}^k$ in a suitable tubular neighbourhood of ∂A, then ∂A is of class $\mathcal{C}^k$. Hence the conjecture becomes interesting for manifolds of codimension *higher* than one.

(iii) As a consequence of the conjecture, there are no N-dimensional connected components of E strictly contained in Ω; in other words, if E_N is nonempty then it is composed of a union of connected components of Ω.

The aim of this short note is to prove the conjecture, and some slightly stronger versions of it. The interest in the square distance function and in its connections with the second fundamental form of a manifold of arbitrary codimension was originated, as far as we know, from the above mentioned paper of De Giorgi, and the more recent papers [8], [9]. Those ideas were developed in [15], [3], [4] and [5] in connection with mean curvature flow in arbitrary codimension. We also recall the papers [2], [14], [10] where it is possible to find a rather complete description of the geometric meaning of the higher derivatives of the square distance function in terms of the second fundamental form and of its derivatives.

Finally, one of our motivations for proving such a kind of regularity result is that it can be considered as a preliminary step toward an alternative proof of the local existence of a unique smooth mean curvature flow in arbitrary codimension besides the one given in [12]. We recall that the short time existence result for mean curvature flow in the one-codimensional case using the signed-distance function was proved in [11].

2. Proof.

Given $q \in \{1, \dots, N\}$ we let $\mathrm{Id}_q := \mathrm{diag}(1, \cdots, 1, 0, \cdots, 0)$ be the matrix having 1 (resp. 0) repeated q (resp. $N-q$) times. Given two n-dimensional symmetric matrices T_1, T_2, when we write $T_1 \geq T_2$ we mean that $T_1 - T_2$ is positive semidefinite. We set $\mathbb{S}^{N-1} = \{v \in \mathbb{R}^N : |v| = 1\}$, and for $x \in \mathbb{R}^N$ and $\rho > 0$ we let $B_\rho(x) := \{y \in \mathbb{R}^N : |x-y| < \rho\}$; $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^N$, ∇ (resp. ∇^2) denotes the gradient (resp. the Hessian). Given a closed set C and $x \in \mathbb{R}^N$, we let

$$\Pi^C(x) := \{y \in C : |y - x| = \mathrm{dist}(x, C)\}.$$

We begin with the following preliminary result (see [1] and [6] for related results).

Lemma 2.1. *Let E be a closed subset of $\mathbb{R}^N$ and let η_E be defined as in* (1). *Then η_E is differentiable on $E = \{\eta_E = 0\}$. In addition, at any point $x \in \mathbb{R}^N$ where η_E is differentiable, we have*

$$|\nabla \eta_E(x)|^2 = 2\eta_E(x). \tag{2}$$

In particular

$$E = \{\nabla \eta_E = 0\}. \tag{3}$$

Proof. Fix $y \in E$. For any $x \in \mathbb{R}^N$ we have

$$0 \leq \eta_E(x) = \eta_E(x) - \eta_E(y) \leq \frac{|x-y|^2}{2},$$

which implies that η_E is differentiable at y with $\nabla \eta_E(y) = 0$.

Let now $x \in \mathbb{R}^N$ be a differentiability point of η_E. If $\eta_E(x) > 0$ then also $d_E = \sqrt{2\eta_E}$ is differentiable at x, with $\nabla \eta_E(x) = d_E(x)\nabla d_E(x)$, hence (2) follows from the eikonal equation $|\nabla d_E(x)| = 1$ satisfied by the distance function. If $\eta_E(x) = 0$, from the previous arguments it follows $\nabla \eta_E(x) = 0$, hence (2) is satisfied. □

Remark 2.2. In general η_E is not of class $\mathcal{C}^1$ on $\{\eta_E = 0\}$, in the sense that there are no open sets containing $\{\eta_E = 0\}$ where η_E is $\mathcal{C}^1$. Indeed if $E := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 = 0\}$ then η_E is not differentiable on $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2\}$.

Theorem 2.3. *Let $M \subset R^N$ be an embedded h-dimensional manifold without boundary of class $\mathcal{C}^k$, $k \geq 2$. Then for any $y \in M$ there exists an open set U of $\mathbb{R}^N$ containing y such that $\eta_M \in \mathcal{C}^{k-1}(U)$ and $\eta_M(y+p) = \frac{1}{2}|p|^2$ for any p in the normal space $N_M(y)$ to M at y, with $y + p \in U$. In particular the matrix $\nabla^2 \eta_M(y)$ represents the orthogonal projection on $N_M(y)$.*

Proof. The proof is the same as the one in [1], which is given under the additional assumptions that M is compact and $k = +\infty$. □

As proved for instance in [13], if $h = N - 1$, then η_M in Theorem 2.3 if of class $\mathcal{C}^k(U)$.

The next theorem is the main result of the present paper.

Theorem 2.4. *Let $E \subset \mathbb{R}^N$ be a closed set, η_E be defined as in (1), let $k \geq 3$ be an integer and let $\Omega \subseteq \mathbb{R}^N$ be an open set. Assume that*

$$\eta_E \in \mathcal{C}^k(\Omega). \tag{4}$$

Then any connected component of $E \cap \Omega$ is a manifold of class $\mathcal{C}^{k-1}(\Omega)$ without boundary.

Proof. From Lemma 2.1 we have

$$E \cap \Omega = \{y \in \Omega : \eta_E(y) = 0\} = \{y \in \Omega : \nabla \eta_E(y) = 0\}. \tag{5}$$

For all $y \in E \cap \Omega$ we define the tangent cone $C_E(y)$ to $E \cap \Omega$ at y as

$$C_E(y) := \left\{ \alpha\nu : \ \alpha \geq 0, \ \nu = \lim_{E\cap\Omega \ni z_n \to y, z_n \neq y} \frac{z_n - y}{|z_n - y|} \right\}.$$

Observe that $C_E(y) = \{0\}$ if y is an isolated point of $E \cap \Omega$. We indicate with $T_E(y)$ the smallest vector subspace of $\mathbb{R}^N$ containing $C_E(y)$ and with $N_E(y)$ the orthogonal complement of $T_E(y)$.

Since η_E is of class $\mathcal{C}^k(\Omega)$ and $k \geq 3$, we can consider the Hessian $\nabla^2 \eta_E$ on $E \cap \Omega$. In particular, given $y \in E \cap \Omega$ and $\nu \in \mathbb{S}^{n-1}$, we can define

$$\lambda_y(\nu) := \langle \nu, \nabla^2 \eta_E(y)\nu \rangle.$$

We divide the proof into seven steps.

Step 1. We have

$$\nabla^2 \eta_E(y) \leq \mathrm{Id}_N \quad \forall y \in E \cap \Omega. \tag{6}$$

Let $y \in E \cap \Omega$ and $\nu \in \mathbb{S}^{N-1}$; we expand η_E at the point $y + t\nu$ (for $|t|$ positive and small enough so that the segment which connects y with $y + t\nu$ is contained in Ω). From (4) and (5) we obtain

$$\frac{1}{2}t^2 \geq \eta_E(y + t\nu) = \frac{1}{2}\langle \nu, \nabla^2\eta_E(y)\nu\rangle t^2 + o(t^2) = \frac{1}{2}\lambda_y(\nu)t^2 + o(t^2). \tag{7}$$

Dividing by t and letting $t \to 0$ we conclude that

$$\lambda_y(\nu) \leq 1. \tag{8}$$

Since $\nu \in \mathbb{S}^{N-1}$ is arbitrary, (6) follows.

Step 2. We have

$$\nabla^2\eta_E \;\Big|_{T_E(y)} = 0 \quad \forall y \in E \cap \Omega. \tag{9}$$

Let $y \in E \cap \Omega$ and $\nu \in \mathbb{S}^{N-1}$. Let $|t|$ be small enough in such a way that $y + t\nu \in \Omega$. Let $y_t \in E \cap \Omega$ be a point such that $\eta_E(y + t\nu) = \frac{1}{2}|y + t\nu - y_t|^2$. From (7) we have

$$\frac{2}{t^2}\eta_E(y + t\nu) = \left|\frac{y - y_t}{t} + \nu\right|^2 = \lambda_y(\nu) + o(1), \tag{10}$$

hence

$$\lim_{t\to 0} \frac{2}{t^2}\eta_E(y + t\nu) = \lambda_y(\nu). \tag{11}$$

Take now $\nu \in C_E(y) \cap \mathbb{S}^{N-1}$ and choose a sequence $\{y_n\} \subset E \cap \Omega$ converging to y and with $\frac{y_n - y}{|y_n - y|} \to \nu$ as $n \to +\infty$. Setting $t_n := |y_n - y|$, from (10) and (11) it follows that

$$0 \leq \lambda_y(\nu) = \lim_{n\to\infty}\left|\frac{y - y_{t_n}}{t_n} + \nu\right|^2 \leq \lim_{n\to\infty}\left|\frac{y - y_n}{t_n} + \nu\right|^2 = 0.$$

Therefore $\lambda_y(\nu) = 0$ for all $\nu \in C_E(y) \cap \mathbb{S}^{N-1}$ and (9) follows.

Step 3. We have

$$\lambda_y(\nu) = 1 \quad \forall y \in E \cap \Omega, \quad \forall \nu \in N_E(y) \cap \mathbb{S}^{N-1}. \tag{12}$$

Let $y \in E \cap \Omega$ and $\nu \in N_E(y) \cap \mathbb{S}^{N-1}$. By (10) it follows that $|\frac{y - y_t}{t}|^2$ is bounded for $|t|$ small enough. Therefore, possibly passing to a (not relabelled) subsequence and recalling the definition of $T_E(y)$, we can suppose that $\frac{y - y_t}{t}$ converges to a vector $w \in T_E(y)$ as $t \to 0$.

It follows from (8), (10) and the orthogonality between ν and w that

$$1 \geq \lambda_y(\nu) = \lim_{t\to 0}\left|\frac{y - y_t}{t} + \nu\right|^2 = |\nu|^2 + |w|^2 = 1 + |w|^2 \geq 1,$$

and (12) follows.

Step 4. Assume that $\Omega \setminus E \neq \emptyset$ and let $x \in \Omega \setminus E$. Then

$$\nabla\eta_E(x) \in N_E(y) \setminus \{0\} \quad \forall y \in \Pi^E(x) \cap \Omega. \tag{13}$$

First we note that $\nabla\eta_E(x) \neq 0$ is a consequence of (3). Therefore, by *Steps 2* and *3*, to show (13) it is enough to prove that $\lambda_y(\nabla\eta_E(x)/|\nabla\eta_E(x)|) = 1$. Set $z_t := (1-t)y+tx$ with $t \in [0,1]$. Note that there exists $\delta \in (0,1/2)$ such that $z_t \in \Omega$ for any $t \in [0,\delta] \cup [1-\delta,1]$. Moreover $y \in \Pi^E(z_t)$ and $\eta_E(z_t) = \frac{1}{2}|z_t - y|^2$ for any $t \in [0,1]$. Hence

$$\lambda_y\Big(\frac{\nabla\eta_E(x)}{|\nabla\eta_E(x)|}\Big) = \frac{d^2}{dt^2}\frac{1}{2}|z_t - y|^2_{|t=0} = \frac{d^2}{dt^2}\frac{t^2}{2}_{|t=0} = 1.$$

Step 5. For all $\overline{y} \in E\cap\Omega$ such that $\dim T_E(\overline{y}) = n < N$ there exist $\sigma > 0$ and an embedded manifold $\Gamma(\overline{y})$ of dimension n and of class $\mathcal{C}^{k-1}(B_\sigma(\overline{y}))$ such that $B_\sigma(\overline{y}) \subset \Omega$ and

$$E \cap \Omega \cap B_\sigma(\overline{y}) = \Gamma(\overline{y}).$$

Let $\overline{y}$ be a point of $E \cap \Omega$ and $\{\overline{\nu}^1, \dots, \overline{\nu}^{N-n}\}$ be an orthonormal basis of $N_E(\overline{y})$. Define

$$F_i(y) := \langle \nabla\eta_E(y), \overline{\nu}^i\rangle, \quad i = 1, \dots, N-n,\ y \in \Omega,$$

$$F := \big(F_1, \dots, F_{N-n}\big) : \Omega \to \mathbb{R}^{N-n}.$$

We observe that $F \in \mathcal{C}^{k-1}(\Omega;\mathbb{R}^{N-n})$ and the Jacobian of F at $\overline{y}$ coincides with the orthogonal projection of $\nabla^2\eta_E(\overline{y})$ on $N_E(\overline{y})$. In addition $\{y \in \Omega : \nabla\eta_E(y) = 0\} \subseteq \{y \in \Omega : F(y) = 0\}$. Therefore from (5) we have

$$\{y \in \Omega : F(y) = 0\} \supseteq E \cap \Omega. \tag{14}$$

We choose $\sigma > 0$ so that the Jacobian of F has constant rank on $B_\sigma(\overline{y})$. This is possible since, from *Steps 2* and *3*, the rank of the $(N \times N)$-matrix $\nabla^2\eta_E(\overline{y})$ is $N - n$. Let us define

$$\Gamma(\overline{y}) := B_\sigma(\overline{y}) \cap \{y \in \Omega : F(y) = 0\}.$$

The Implicit Function Theorem ensures that $\Gamma(\overline{y})$ is a submanifold of class $\mathcal{C}^{k-1}(B_\sigma(\overline{y}))$ of dimension n. Note that the (14) implies

$$\Gamma(\overline{y}) \supseteq E \cap \Omega \cap B_\sigma(\overline{y}) \quad \text{and} \quad T_{\Gamma(\overline{y})}(\overline{y}) = T_E(\overline{y}). \tag{15}$$

We now prove that

$$\Gamma(\overline{y}) \subseteq E \cap \Omega. \tag{16}$$

Let us consider the distance function $d_E = \sqrt{2\eta_E}$; then $d_E \in \mathcal{C}^k(\Omega \setminus E)$. Define, for $\varepsilon > 0$, the sets

$$\Sigma_\varepsilon := \left\{y \in \Omega : \eta_E(y) = \varepsilon^2/2\right\}.$$

Since $d_E = \varepsilon > 0$ and $|\nabla d_E| = 1$ in Σ_ϵ, by the Implicit Function Theorem we have that, if $\Sigma_\varepsilon \neq \emptyset$, then Σ_ε is a $(N-1)$-dimensional manifold of class $\mathcal{C}^k$.

By the regularity properties of $\eta_{\Gamma(\overline{y})}$ (see Theorem 2.3), we have that $\eta_{\Gamma(\overline{y})}$ is of class at least $\mathcal{C}^1$ in a neighbourhood of $\overline{y}$; moreover for any $\overline{v} \in N_{\Gamma(\overline{y})}(\overline{y}) \cap \mathbb{S}^{N-1}$ and $\varepsilon > 0$ small enough, setting $x_\varepsilon(\overline{y}) := \overline{y} + \varepsilon\overline{v}$, we have $\eta_{\Gamma(\overline{y})}(x_\varepsilon(\overline{y})) = \varepsilon^2/2$. From (15) we have $\eta_{\Gamma(\overline{y})}(x_\varepsilon(\overline{y})) \leq \eta_E(x_\varepsilon(\overline{y}))$, and since $\eta_E(x_\varepsilon(\overline{y})) \leq \frac{1}{2}|x_\varepsilon(\overline{y}) - \overline{y}|^2 = \varepsilon^2/2$, we conclude that $\eta_E(x_\varepsilon(\overline{y})) = \varepsilon^2/2$. Therefore $x_\varepsilon(\overline{y}) \in \Sigma_\varepsilon$.

From *Step 4* we deduce that $\overline{v}$ and $\nabla\eta_E(x_\varepsilon(\overline{y}))$ are parallel, more precisely $\overline{v} = \nabla d_E(x_\varepsilon(\overline{y}))$, and $\overline{y} = x_\varepsilon(\overline{y}) - \nabla d_E(x_\varepsilon(\overline{y}))$. From (13) it also follows that $T_E(\overline{y}) \subseteq T_{\Sigma_\epsilon}(x_\varepsilon(\overline{y}))$.

Take an orthonormal basis $\{\overline{\tau}^1, \dots, \overline{\tau}^n\}$ of $T_E(\overline{y})$ and let

$$L := \mathrm{span}\left\{\overline{\tau}^1, \dots, \overline{\tau}^n, \nabla d_E(x_\varepsilon(\overline{y}))\right\}$$

be the vector space generated by $T_E(\overline{y})$ and by the normal to Σ_ϵ at $x_\varepsilon(\overline{y})$. Define

$$\Lambda := \Sigma_\epsilon \cap (L + x_\varepsilon(\overline{y})).$$

Then Λ is a n-dimensional manifold of class $\mathcal{C}^k$ in a neighbourhood of $x_\varepsilon(\overline{y})$.

Let us consider the map Π^E between two n-dimensional manifolds of class $\mathcal{C}^{k-1}$ defined as

$$\Pi^E : V \cap \Lambda \to \Sigma \cap B_\sigma(\overline{y}) \subseteq \Gamma(\overline{y}), \qquad \Pi^E(x) := x - \nabla\eta_E(x),$$

where V is a suitable neighbourhood of $x_\varepsilon(\overline{y})$. Observe that

$$\Pi^E(\Sigma_\varepsilon) \subseteq E \cap \Omega. \tag{17}$$

From *Step 2* we can assume that $|\langle \nabla^2\eta_E(x_\varepsilon(\overline{y}))\overline{\tau}_i, \overline{\tau}_i\rangle| \leq 1/2$ for $\varepsilon > 0$ sufficiently small and for any $i = 1, \dots, n$. It follows that the (tangential) Jacobian of Π^E is nonvanishing at $x_\varepsilon(\overline{y})$, hence by the Implicit Function Theorem we have that Π^E is a local diffeomorphism around $x_\varepsilon(\overline{y})$. This, together with (17), concludes the proof of *Step 5.*

Step 6. Let Σ be a connected component of $E \cap \Omega$. Then there exists an integer $q \in \{0, \dots, N\}$ such that

$$\nabla^2\eta_E \Big|_{N_E(y)} = \mathrm{Id}_q \quad \forall y \in \Sigma.$$

Since the trace of $\nabla^2\eta_E(y)$ is a continuous function with respect to y and Σ is connected, from *Steps 2* and *3* we have that the dimensions of $N_E(y)$ and $T_E(y)$ are independent of the choice of $y \in \Sigma$.

The following step, together with *Step 5*, concludes the proof of the theorem.

Step 7. Let $\overline{y} \in E \cap \Omega$ be such that $\dim(T_E(\overline{y})) = N$. Then $\overline{y}$ belongs to the interior part of $E \cap \Omega$.

Assume by contradiction that $\overline{y}$ belongs to the boundary of $E\cap\Omega$. Let $\rho > 0$ be sufficiently small, $x \in B_\rho(\overline{y}) \setminus \overline{y}$, and write $x_t := t\overline{y} + (1-t)x$. Define $t^* := \sup\{t \in [0,1] : x_s \in E \cap \Omega \text{ for all } s \in [0,t]\}$. Note that $t^* < 1$. Moreover $x_{t^*} \in E \cap \Omega$, since $E \cap \Omega$ is closed. Denoting by Σ the connected component of $E \cap \Omega$ containing $\overline{y}$, we point out that $x_s \in \Sigma$ for any $s \in [0, t^*]$. Let us pick a decresing sequence $\{s_n\} \subset (0,1)$ converging to t^* as $n \to +\infty$ and such that $x_{s_n} \in B_\rho(\overline{y}) \setminus (E \cap \Omega)$ for any $n \in \mathbb{N}$. By our assumption, *Step 2* and *Step 6* we have $\nabla^2\eta_E(t^*) = 0$. On the other hand, from *Step 4* it follows that 1 is an eigenvalue of $\nabla^2\eta_E(s_n)$ for any $n \in \mathbb{N}$. We conclude that η_E is not of class $\mathcal{C}^2$ at x_{t^*}, a contradiction. $\square$

Remark 2.5. From the proof of Theorem 2.3 it follows that:

(i) given a connected component Σ of $E \cap \Omega$ there is an open set containing Σ which does not intersect any other connected component of $E \cap \Omega$;

(ii) any connected component of $E \cap \Omega$ of dimension N is a connected component of Ω;

(iii) the assumption $k \geq 3$ in Theorem 2.4 is used only in the proof of *Step 5*, in order to show that $\eta_{\Gamma(\overline{y})}$ is of class $\mathcal{C}^1$ in a neighbourhood of $\overline{y}$ and $x_\varepsilon(\overline{y}) \in \Sigma_\varepsilon$;

(iv) if in the statement we substitute k with ∞ (resp. ω), then the thesis holds with $k-1$ substituted by ∞ (resp. ω); hence the conjecture of De Giorgi follows.

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