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Twisted Convex Hulls in the Heisenberg Group

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We define and investigate the notion of twisted convex hull for a subset of the Heisenberg group $I\!H$. We show that, while the twisted convex hull of two points is always a bounded set, the twisted convex hull of $n > 2$ points is, in general, an unbounded prism, with edges parallel to the t axis. These results shed some light on the twisted convex subsets of $I\!H$; in particular, we give a characterization of their interior.

Keywords: Heisenberg group, twisted convex combination, twisted convex hull, horizontal planes

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1. Introduction

Motivated by the role played by convex functions in the study of fully nonlinear subelliptic PDE's, recently many authors introduced and studied several notions of convexity in sub–Riemannian geometries as Carnot groups, and, in particular, the Heisenberg group (see [1], [4], [5]).

In [6] a definition of convex set of geometric type is considered and discussed; a subset A of the geodesic metric space $(I\!H, d)$ – where d denotes the Carnot–Carathéodory metric on $I\!H$ – is said to be geodetically convex if the image of any geodesic connecting two elements in A is contained in A. The unexpected result is that, by contrast to the Euclidean case, the geodetic convex envelope of three points is, in general, the whole group.

In this paper we consider a definition of convexity that is based on the group and dilation structure of the Heisenberg group. This convexity, that was called strong H–convexity in [4], relies on the notion of twisted convex (tw-convex) combination of two points g and g' given by $\gamma_{g,g'}(\lambda) = g\delta_\lambda(g^{-1}g')$, $\lambda \in [0, 1]$. We say that a set A is tw–convex if the curve $\gamma_{g,g'}$ is contained in A, for every $g, g' \in A$. A weaker notion of convexity can be given if g' is required to belong to the horizontal plane H_g; this is called weak H–convexity in [4]. In [3], we investigated an optimization problem within this framework.

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Our goal in this paper is to describe the tw–convex hull of subsets of $\mathbb{H}$. In particular, we show that the tw–convex hull of two points is always a bounded set, contained in a plane. In contrast, the tw–convex hull of more than two points is an unbounded subset of $\mathbb{H}$, unless all points belong to the same left translate of a 2–dimensional subgroup. This is reminiscent of the result of Monti and Rickly ([6]), but in our setting, the tw–convex hull is not the whole group in general. Actually, we prove that, in general, the tw–convex hull of n points is a prism with edges parallel to the t axis. Our results show that the tw–convex sets are scarce. In particular, a bounded tw–convex set is always contained in a left translate of some 2–dimensional subgroup; obviously, the gauge balls are not tw–convex.

2. The Heisenberg group

The Heisenberg group $\mathbb{H} = \mathbb{H}^1$ is the Lie group given by the underlying manifold $\mathbb{R}^3$ with the non commutative group law

$$gg' = (x, y, t)(x', y', t') := (x + x', y + y', t + t' + 2(x'y - xy')) .$$

The unit element is $e = (0, 0, 0)$ and the inverse of $g = (x, y, t)$ is $g^{-1} = (-x, -y, -t)$. Left translations and anisotropic dilations are, in this setup, $L_{g_0}(g) = g_0 g$ and $\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$.

The differentiable structure on $\mathbb{H}$ is determined by the left invariant vector fields

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \qquad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}, \qquad T = \frac{\partial}{\partial t}, \quad \text{with } [X, Y] = -4T;$$

the vector field T commutes with the vector fields X and Y.

Let $\mathfrak{h}$ be the Lie algebra of $\mathbb{H}$; then $\mathfrak{h} = \mathbb{R}^3 = V_1 \oplus V_2$, where $V_1 = \text{span}\{X, Y\}$, $V_2 = \text{span}\{T\}$. Via the exponential map exp we identify the vector $\alpha X + \beta Y + \gamma T$ in $\mathfrak{h}$ with the point (α, β, γ) in $\mathbb{H}$. For the inverse $\xi : \mathbb{H} \to \mathfrak{h}$ of the exponential mapping, we have the unique decomposition $\xi = (\xi_1, \xi_2)$ with $\xi_i : \mathbb{H} \to V_i$.

Definition 2.1. Given a point $g_0 \in \mathbb{H}$, the horizontal plane H_{g_0} associated to g_0 is the plane in $\mathbb{H}$ defined by $H_{g_0} = L_{g_0}(exp(V_1)) = \{g = (x, y, t) \in \mathbb{H} : t = t_0 + 2y_0 x - 2x_0 y\}$.

It can be easily seen that: $g' \in H_g$, $g \in H_{g'}$, and $g^{-1}g' \in H_e = \{(x, y, 0),\ x, y \in \mathbb{R}\}$ are three equivalent statements. Moreover, it is worth noticing that every horizontal plane is associated to a unique point g_0, and that two horizontal planes are parallel, in the Euclidean sense, if and only if the associated points have the same projection on the (x, y)–plane. Therefore their intersection is, in general, a straight line. Every horizontal plane splits $\mathbb{H}$ in two parts. Hence, given $g \notin H_{g_0}$, we say that $g = (x, y, t)$ is *above* H_{g_0} when $t > t_0 + 2y_0 x - 2x_0 y$, and g is *below* H_{g_0} in the other case.

We recall that a curve γ, where $\gamma(s) = (x(s), y(s), t(s)) : [0, 1] \to \mathbb{H}$, is said to be *horizontal* if $\gamma'(s) \in \text{span}_{\mathbb{R}}\{X(\gamma(s)), Y(\gamma(s))\}$ for almost every $s \in [0, 1]$ and hence $t'(s) - 2y(s)x'(s) + 2x(s)y'(s) = 0$.

Given two points g and g', we consider the set $\Gamma_{g,g'}$ of all possible horizontal curves joining these points. The Carnot–Carathéodory distance d is then defined as

$$d(g,g') = \inf_{\gamma \in \Gamma_{g,g'}} \int_0^1 |\gamma'(s)|\, ds.$$

The infimum is actually a minimum, and the curve with minimum length is a geodesic. In the case that $g' \in H_g$, we call *horizontal segment* the horizontal geodesic that realizes $d(g,g')$.

Definition 2.2. Given $a, b, c \in I\!R$, $a^2 + b^2 \neq 0$, we define a vertical plane in $I\!H$ as the following subset π

$$\pi = \{g = (x,y,t) \in I\!H : \ ax + by = c\}.$$

It is easy to prove that every vertical plane is left translate of a vertical plane through the origin. A curve γ is said to be *vertical* if $\gamma'(s) \in \mathrm{span}_{I\!R}\{T(\gamma(s))\}$ for almost every $s \in [0,1]$, and hence x and y are constant. A *vertical segment* is the image of a vertical curve. More in general, we call *segment* any convex closure, in the Euclidean sense, of the set of two points of $I\!H$; clearly, horizontal and vertical segments are segments.

It is worthwhile noticing that left translations preserve horizontal planes, horizontal segments, vertical planes and vertical segments.

3. Twisted convex combination of points of $I\!H$

The main purpose of this paper is the investigation of the convex hull of a finite number of points in $I\!H$, where the underlying convex combination is the tw–convex combination already introduced in [4], [5]. Let us recall the following definition:

Definition 3.1. Given two points $g, g' \in I\!H$, the twisted convex (tw–convex) combination of g and g' based on g is the point g_λ defined by

$$g_\lambda = g\delta_\lambda(g^{-1}g'), \quad \lambda \in [0,1].$$

Explicit computations give us that, if $g = (x,y,t)$ and $g' = (x', y', t')$, then

$$g_\lambda = ((1-\lambda)x + \lambda x', (1-\lambda)y + \lambda y', t + 2\lambda(x'y - xy') + \lambda^2(-t + t' - 2x'y + 2xy')).$$

From now on, we denote by $\gamma_{g,g'}$ the curve defined by

$$\gamma_{g,g'}(\lambda) = g_\lambda, \quad \lambda \in [0,1].$$

Notice that, for every $g, g' \in I\!H$,

$$U(\gamma_{g,g'}) = \gamma_{(Ug),(Ug')}, \tag{1}$$

where U is a group isomorphism which commutes with dilations. In particular, this is true for $U = L_{g_0}$, for every $g_0 \in I\!H$.

In the following propositions, we list some properties of $\gamma_{g,g'}$ that can be derived with easy computations starting from the explicit expression of g_λ.

Proposition 3.2. *Let $g, g' \in I\!H$. Then $\gamma_{g,g'}$ is a segment if and only if $g' \in H_g$, or $(x,y) = (x',y')$. In particular, it is horizontal in the first case, and vertical in the second.*

Proposition 3.3. *Let $g, g' \in I\!H$, such that $\gamma_{g,g'}$ is not a segment, and denote by $\pi_{g,g'}$ the vertical plane through g and g'. We have that*

i) $\gamma_{g,g'}$ *and* $\gamma_{g',g}$ *are contained in* $\pi_{g,g'}$*;*

ii) *the curves* $\gamma_{g,g'}$ *and* $\gamma_{g',g}$ *are symmetric with respect to the point* $g_m = \left(\frac{x+x'}{2}, \frac{y+y'}{2}, \frac{t+t'}{2}\right) \in \pi_{g,g'}$*, i.e., for every* $\lambda \in [0,1]$*, the midpoint of the segment with end points* $\gamma_{g,g'}(\lambda)$ *and* $\gamma_{g',g}(1-\lambda)$ *is* g_m*;*

iii) *the line* r_g *tangent to the curve* $\gamma_{g,g'}$*, at the point* g*, is the intersection of the planes* H_g *and* $\pi_{g,g'}$*; its expression is* $r_g(s) = (x + s(-x + x'), y + s(-y + y'), t + 2s(x'y - xy'))$, $\quad s \in I\!R$*; the line* $r_{g'}$ *tangent to the curve* $\gamma_{g',g}$ *at the point* g' *is parallel to the line* r_g*;*

iv) *if* g_λ *is a point of* $\gamma_{g,g'}$ *such that* $r_{g_\lambda} \subset H_{g_\lambda}$*, then* $g_\lambda = g$*.*

The line r_g, tangent to the curve $\gamma_{g,g'}$ at the point g, will play a role in the characterization of the tw–convex hull of two points (see Theorem 4.2).

Now, we give the definition of tw–convex set in $I\!H$.

Definition 3.4. A set $C \subseteq I\!H$ is twisted convex (tw–convex) if and only if

$$\gamma_{g,g'} \subset C, \quad \forall g, g' \in C.$$

A weaker notion of convexity, the so called *weak H–convexity*, requires that $\gamma_{g,g'} \subset C$ for every $g \in C$ and $g' \in H_g \cap C$ (see, for instance, [4]).

The notion of tw–convexity coincides with the strong H–convexity (see [4, Remark 7.5]). It is known that the Carnot–Carathéodory ball $\{g \in I\!H : d(e,g) \leq 1\}$ is not weakly H–convex, and therefore not even tw–convex. In [4] it is proved that the ball $\{g \in I\!H : \|g\| \leq 1\}$, defined via the gauge

$$\|(x,y,t)\| = ((x^2+y^2)^2 + t^2)^{1/4},$$

is weakly H–convex, and it is claimed, without proof, that it is not tw–convex. In the following, we set ourselves the aim of investigating this kind of convex sets.

A similar investigation was carried out in [6], where geodetically convex sets were characterized. The authors considered the Carnot–Carathéodory metric in $I\!H$; within this framework, a subset $A \subset I\!H$ is said to be geodetically convex if any geodesic, i.e., length minimizing horizontal curve, connecting two elements of A, lies in A. They showed that, for any $A \subset I\!H$ containing three points not lying on the same geodesic, the geodetic envelope is the whole group $I\!H$.

Definition 3.5. Given $A \subseteq I\!H$, the tw–convex hull of A is the set

$$CH_A = \bigcap \{C \subseteq I\!H : \ C \text{ tw–convex}, \ A \subseteq C\}.$$

If $A = \{g_1, \ldots, g_n\}$, we will denote CH_A by $CH(g_1, \ldots, g_n)$.

Remark 3.6. i) From (1), it follows that, for every group isomorphism U which commutes with dilations,

$$C \text{ is tw–convex} \ \Leftrightarrow \ U(C) \text{ is tw–convex}.$$

In particular, the tw–convexity is invariant by left translation; therefore, for every $A \subset I\!\!H$ and $g \in I\!\!H$

$$CH_A = L_g \left(CH_{L_{g^{-1}}A} \right).$$

ii) For every $A \subset I\!\!H$ and $g \in CH_A$,

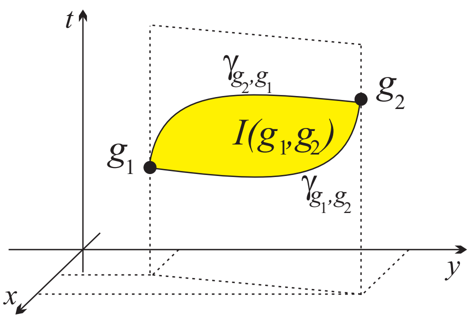
$$CH_A = CH_{A \cup \{g\}}.$$

In the sequel, when dealing with tw–convex hulls of subsets A of $I\!\!H$, we will assume, without loss of generality, that e belongs to A.

4. The twisted convex hull of two points

In this section, we analyse the tw–convex hull of two points $g_1, g_2 \in I\!\!H$.

If γ_{g_1,g_2} is a segment, then $CH(g_1, g_2) = \gamma_{g_1,g_2}$, and $CH(g_1, g_2)$ is geodetically convex, with respect to the Heisenberg notion of geodesics (see [6]); we will not consider this trivial case.



If γ_{g_1,g_2} is not a segment, then we know that γ_{g_1,g_2} is an arc of parabola; moreover, the arcs γ_{g_1,g_2} and γ_{g_2,g_1} draw on their vertical plane the border of the symmetric leaf $I(g_1, g_2)$. This closed set is exactly the convex hull of γ_{g_1,g_2} and γ_{g_2,g_1} according to the Euclidean idea of convexity. Using Propositions 3.2 and 3.3 *i*), we can see that any point inside this leaf is a tw–convex combination of a couple of points on the border; in particular, the set $I(g_1, g_2)$ is included in $CH(g_1, g_2)$. We notice that the Euclidean convex hull of g_1 and g_2 is always contained in the $CH(g_1, g_2)$. One can wonder whether the leaf, i.e., the Euclidean convex hull of $\gamma_{g_1,g_2} \cup \gamma_{g_2,g_1}$, is closed under the tw–convex combination of any couple of points in $I(g_1, g_2)$. We will show in the next example that the answer is negative; we will provide a result that characterizes geometrically $CH(g_1, g_2)$.

Example 4.1. Let $g_1 = e$ and $g_2 = (1, 0, 1)$. We prove that $I(g_1, g_2)$ is not tw–convex. Indeed, easy computations give that $\gamma_{g_1,g_2}(\lambda) = (\lambda, 0, \lambda^2)$, $\gamma_{g_2,g_1}(\lambda) = (1 - \lambda, 0, 1 - \lambda^2)$, $\lambda \in [0, 1]$. Taking λ equal to $\frac{1}{3}$ and $\frac{2}{3}$ in $\gamma_{g_1,g_2}(\lambda)$, we get the points $g_3 = \left(\frac{1}{3}, 0, \frac{1}{9}\right)$ and $g_4 = \left(\frac{2}{3}, 0, \frac{4}{9}\right)$, respectively. Then $\gamma_{g_3,g_4}(\lambda) = \left(\frac{1+\lambda}{3}, 0, \frac{1+3\lambda^2}{9}\right)$, i.e., γ_{g_3,g_4} is the curve $t = \frac{1+3(3x-1)^2}{9}$, where $x \in [\frac{1}{3}, \frac{2}{3}]$, and it lies on the (x, t)–plane. Since $\frac{1+3(3x-1)^2}{9} < x^2$ if and only if $\frac{1}{3} < x < \frac{2}{3}$, it follows that $\gamma_{g_3,g_4} \not\subset I(g_1, g_2)$, showing that $I(g_1, g_2)$ is not tw–convex. $\square$

Let us state and prove our result that characterizes geometrically $CH(g_1, g_2)$.

Theorem 4.2. *Let g_1 and g_2 be two points in $\mathbb{H}$, with $g_2 \notin H_{g_1}$ and having distinct projections on the (x, y)–plane. Then*

$$CH(g_1, g_2) = R(g_1, g_2) \cup \{g_1\} \cup \{g_2\},$$

where $R(g_1, g_2)$ denotes the parallelogram lying in the vertical plane π_{g_1,g_2}, with two edges that are vertical segments through g_1 and g_2, and the other two that are horizontal segments on the lines r_{g_1} and r_{g_2} respectively.

In particular, if g_2 is above H_{g_1}, then

$$\begin{aligned} R(g_1, g_2) =\Big\{&(x, y, t) \in \mathbb{H} : \ x = (1-\lambda)x_1 + \lambda x_2, \ y = (1-\lambda)y_1 + \lambda y_2, \\ &t_1 - 2(x_1y_2 - x_2y_1)\lambda < t < t_2 + 2(x_1y_2 - x_2y_1)(1-\lambda), \quad \lambda \in (0,1)\Big\}. \end{aligned} \tag{2}$$

Proof. Let us assume that g_2 is above g_1. Since $CH(g_1, g_2) = L_{g_1}(CH(e, g_0))$, where $g_0 = g_1^{-1}g_2 = (x_0, y_0, t_0)$, with $t_0 > 0$ and $(x_0, y_0) \neq (0,0)$, from Remark 3.6 and Proposition 3.3 it suffices to prove that

$$CH(e, g_0) = R(e, g_0) \cup \{e\} \cup \{g_0\},$$

where

$$R(e, g_0) = \{g \in \mathbb{H} : \ g = (\lambda x_0, \lambda y_0, \mu t_0), \, 0 < \lambda < 1, \, 0 < \mu < 1\}.$$

For the sake of clearness, we split the proof into three parts.

i) First, we show that one can define in $\mathbb{H}$ two sequences $\{v_n\}$ and $\{z_n\}$ satisfying the properties

$$\begin{array}{ll} v_0 = e & z_0 = g_0 \\ v_{n+1} \in \gamma_{v_n,z_n} & z_{n+1} \in \gamma_{v_n,z_n} \end{array}$$

in such a way that the limit of v_n is as close as possible to $(x_0, y_0, 0)$.

Indeed, fix two real numbers a and b, with $0 < a < b < 1$, and consider the two points in γ_{e,g_0}

$$v_1 = (ax_0, ay_0, a^2t_0), \qquad z_1 = (bx_0, by_0, b^2t_0).$$

The idea is the following: on the curve γ_{v_1,z_1} we take the two points v_2 and z_2 corresponding to the choice of the parameters $\lambda = a$ and $\lambda = b$. Then, starting from γ_{v_2,z_2}, we define v_3 and z_3 according to the same rule, and so on. Let us denote by A_n and B_n the following sums

$$A_n = \sum_0^{n-1} (b-a)^j, \qquad B_n = \sum_0^{n-1} (b^2 - a^2)^j.$$

We show by induction that v_n and z_n can be written as:

$$v_n = (ax_0A_n, ay_0A_n, a^2t_0B_n)$$

and

$$z_n = (ax_0A_n + x_0(b-a)^n, ay_0A_n + y_0(b-a)^n, a^2t_0B_n + (b^2-a^2)^nt_0).$$

Indeed, it can be easily verified that this is true for $n = 1$. Assume that v_n and z_n have the expressions above, and compute v_{n+1} and z_{n+1} according to the same rule. From the definition of tw–convex combination we get that the three components of $v_n\delta_a(v_n^{-1}z_n)$ are

$$\begin{gathered}(1-a)ax_0A_n + a(ax_0A_n + x_0(b-a)^n),\\ (1-a)ay_0A_n + a(ay_0A_n + y_0(b-a)^n),\\ a^2t_0B_n + a^2(-a^2t_0B_n + a^2t_0B_n + (b^2-a^2)^n t_0),\end{gathered}$$

i.e., after reductions,

$$ax_0A_{n+1}, \qquad ay_0A_{n+1}, \qquad a^2t_0B_{n+1},$$

that are actually the components of v_{n+1}. Similar computations show that $z_{n+1} = v_n\delta_b(v_n^{-1}z_n)$.

Let us now consider the limit of v_n as $n \to \infty$; we get that

$$\lim_{n\to\infty} v_n = \left(\frac{ax_0}{1-(b-a)}, \frac{ay_0}{1-(b-a)}, \frac{a^2t_0}{1-(b^2-a^2)}\right).$$

This limit point can be chosen as close as we like to the point $(x_0, y_0, 0)$ provided we make a suitable choice of the numbers a and b; indeed, if we take $a_k = 1/k$ and $b_k = 1-(1/k^{3/2})$, we have that

$$\left(\frac{a_kx_0}{1-(b_k-a_k)}, \frac{a_ky_0}{1-(b_k-a_k)}, \frac{a_k^2t_0}{1-(b_k^2-a_k^2)}\right) \to (x_0, y_0, 0),$$

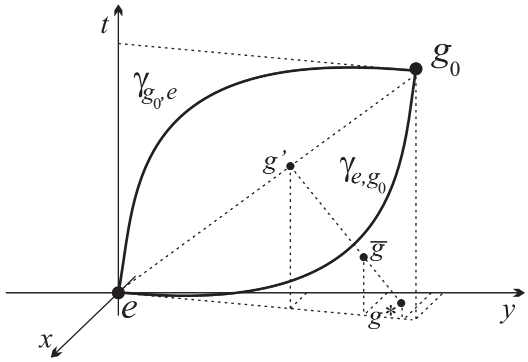
if $k \to \infty$, thereby showing our first claim.

Notice that the proof above can be adapted to show that also the point $(0, 0, t_0)$ is the limit of a suitable sequence $\{v'_n\}$ defined in an analogous way starting from two points of $\gamma_{g_0,e}$.

ii) We show that any point $\overline{g} \in R(e, g_0)$ belongs to $CH(e, g_0)$, i.e., $CH(e, g_0)$ fills the open rectangle $R(e, g_0)$. Since we already know that $I(e, g_0)$ is contained in $CH(e, g_0)$, we need to prove that the inclusion $\overline{g} \in CH(e, g_0)$ holds for any $\overline{g} \in R(e, g_0) \setminus I(e, g_0)$. Indeed, fix such a point $\overline{g} = (\overline{x}, \overline{y}, \overline{t})$. We can assume that $\overline{g}$ is below γ_{e,g_0}; a similar reasoning holds if $\overline{g}$ is above $\gamma_{g_0,e}$. Consider the set

$$\begin{aligned}A_{\overline{g}} = \Bigg\{(x, y, t) \in R(e, g_0) : \min\left\{x_0, \frac{\overline{x}+x_0}{2}\right\} < x < \max\left\{x_0, \frac{\overline{x}+x_0}{2}\right\},\\ \min\left\{y_0, \frac{\overline{y}+y_0}{2}\right\} < y < \max\left\{y_0, \frac{\overline{y}+y_0}{2}\right\},\ 0 < t < \frac{\overline{t}}{2}\Bigg\}.\end{aligned}$$

From the first part of the proof, there exists at least one point $g^* = (x^*, y^*, t^*) \in A_{\overline{g}}$ that can be reached by a finite number of tw–convex combinations starting from points in γ_{g_1,g_2}. Denote by g' the intersection between the lines joining $\overline{g}$ with g^*, and e with g_0. Since $g' \in I(e, g_0)$ and $g^*, \overline{g}, g'$ are aligned, we argue that $\overline{g} \in I(g^*, g')$, thereby showing that $\overline{g} \in CH(e, g_0)$.



iii) Finally, we show that the set

$$R(e, g_0) \cup \{e\} \cup \{g_0\}$$

is tw–convex. Indeed, take any two points $g' = (x', y', t')$, $g'' = (x'', y'', t'')$ in $R(e, g_0) \cup \{e\} \cup \{g_0\}$, and consider $g'\delta_\lambda((g')^{-1}g'')$. Since the two points belong to the vertical plane through e and g_0, we get that

$$\gamma_{g',g''}(\lambda) = ((1-\lambda)x' + \lambda x'', (1-\lambda)y' + \lambda y'', t' + \lambda^2(-t' + t'')).$$

In particular, if g' or g'' belongs to $R(e, g_0)$, and $\lambda \in (0, 1)$, it is easy to see that $\gamma_{g',g''}(\lambda)$ is in $R(e, g_0)$. The cases where both points belong to $\{e\} \cup \{g_0\}$ are trivial. $\square$

It is interesting to compare formula (2) with the following one, that provides the tw–convex hull of g_1 and g_2 in the case $g_2 \in H_{g_1}$:

$$CH(g_1, g_2) = \Big\{(x, y, t) \in I\!H : x = (1-\lambda)x_1 + \lambda x_2,\ y = (1-\lambda)y_1 + \lambda y_2,$$
$$t = t_1 - 2(x_1 y_2 - x_2 y_1)\lambda, \quad \lambda \in [0, 1]\Big\}.$$

Remark 4.3. From the results of this section, it is not hard to infer that the Euclidean convex hull of any subset A of $I\!H$ is contained in CH_A.

5. The twisted convex hull of 3 points

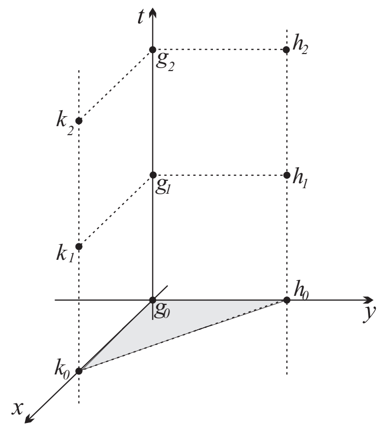
In this section, we study the tw–convex hull $CH(g_1, g_2, g_3)$. We will see that, in general, such set is not closed and not open, as in the case $n = 2$; moreover, it is unbounded.

Let us first show an example that sheds some light on this situation. Starting from three points of the plane $t = 0$, we show that it is possible to climb in the vertical direction and to "reach" ∞. We are not interested in the study of the boundary of the tw–convex hull, but we will give an intuitive idea of the climb in order to obtain an unbounded set.

Example 5.1. Let us consider the sequences $\{k_n\}_{n\geq 0}$, $\{h_n\}_{n\geq 0}$ and $\{g_n\}_{n\geq 0}$ of points of $I\!H$ defined by

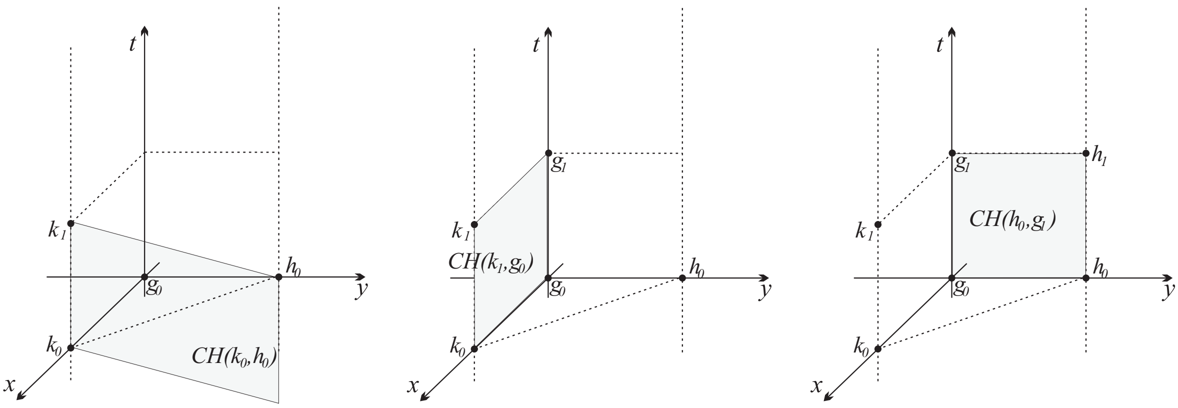
$$k_n = (1, 0, 2n), \qquad h_n = (0, 1, 2n), \qquad g_n = (0, 0, 2n),$$

for every integer n. We will prove that the closure of $CH(k_0, h_0, g_0)$ contains all the points of the previous three sequences. We show that, for every n, starting from the points



k_n, h_n and g_n, it is possible to obtain, through three suitable convex hull of two points, the points k_{n+1}, h_{n+1} and g_{n+1}. In particular, let n be fixed; from previous Theorem 4.2, it is easy to see that

$$k_{n+1} \in \overline{CH(k_n, h_n)}, \qquad g_{n+1} \in \overline{CH(k_{n+1}, g_n)}, \qquad h_{n+1} \in \overline{CH(h_n, g_{n+1})},$$



Hence, $\{k_{n+1}, h_{n+1}, g_{n+1}\} \subset \overline{CH(k_n, h_n, g_n)}$. We remark that $\{k_{n+1}, h_{n+1}, g_{n+1}\}$ is not a point of $CH(k_n, h_n, g_n)$, so our thinking is not rigorous, but an approximation argument gives us that

$$\{(x, y, t) : 0 < x < 1, 0 < y < 1 - x\} \subset CH(k_0, h_0, g_0).$$

In the next theorem, we prove a result that characterizes geometrically $CH(g_1, g_2, g_3)$ in the case that the projections of such points on the (x, y)–plane are not aligned. In this case, the interior of $CH(g_1, g_2, g_3)$ is the interior of an unbounded prism with vertical edges; in particular, its section with the (x, y)–plane is an open triangle whose vertices are the projections of g_1, g_2 and g_3 on this plane. The boundary of $CH(g_1, g_2, g_3)$ is the union of the three sets $CH(g_1, g_2)$, $CH(g_1, g_3)$ and $CH(g_2, g_3)$.

Theorem 5.2. *Let* $g_1 = (x_1, y_1, t_1)$, $g_2 = (x_2, y_2, t_2)$, $g_3 = (x_3, y_3, t_3)$ *be points in* $\mathbb{H}$, *with non aligned projections on the* (x, y)*–plane. Then the tw–convex hull* $CH(g_1, g_2, g_3)$ *is the set*

$$CH(g_1, g_2, g_3) = R(g_1, g_2, g_3) \cup CH(g_1, g_2) \cup CH(g_1, g_3) \cup CH(g_2, g_3),$$

where

$$R(g_1, g_2, g_3) = \left\{ g \in I\!\!H : g = \left(\sum_{i=1}^{3} \lambda_i x_i, \sum_{i=1}^{3} \lambda_i y_i, t \right), 0 < \lambda_1, \lambda_2, \lambda_3 < 1, \sum_{i=1}^{3} \lambda_i = 1 \right\}$$

Proof. For the sake of clearness, we split the proof into four parts. The first three parts show that the interior of $CH(g_1, g_2, g_3)$ is $R(g_1, g_2, g_3)$.

i) Let τ be the function $\tau : T \to (-\infty, \infty]$, defined by

$$\tau(x, y) = \sup\{t : (x, y, t) \in CH(g_1, g_2, g_3)\}, \quad (x, y) \in T,$$

where T is the open triangle in $I\!\!R^2$ defined by

$$T = \left\{ (x, y) : x = \sum_{i=1}^{3} \lambda_i x_i,\ y = \sum_{i=1}^{3} \lambda_i y_i,\ 0 < \lambda_1, \lambda_2, \lambda_3 < 1, \sum_{i=1}^{3} \lambda_i = 1 \right\}.$$

We prove that τ is continuous on T.

Let us denote by $t(x, y)$ the set $\{t : (x, y, t) \in CH(g_1, g_2, g_3)\}$. Since the Euclidean convex hull of g_1, g_2 and g_3 is contained in $CH(g_1, g_2, g_3)$, $t(x, y)$ is not empty. Moreover, the tw–convex hull of (x, y, t^0), (x, y, t^*), where $t^0, t^* \in t(x, y)$, is a vertical segment, implying that $t(x, y)$ is an interval whenever $\operatorname{card}(t(x, y)) \geq 2$.

Let $(x', y'), (x'', y'') \in T$, $t' \in t(x', y')$ and $t'' \in t(x'', y'')$. Since the segment joining the points (x', y', t') and (x'', y'', t'') belongs to $CH((x', y', t'), (x'', y'', t''))$, we have that

$$\tau((\lambda x' + (1 - \lambda)x'', \lambda y' + (1 - \lambda)y'') \geq \lambda t' + (1 - \lambda)t'',$$

for every $t' \in t(x', y')$ and $t'' \in t(x'', y'')$. Hence τ is concave and consequently continuous on T.

ii) We prove that τ is unbounded from above.

Let us suppose that there exists $k > 0$ such that $\tau(x, y) < k$, for every $(x, y) \in T$. Fix (x', y'), $(x'', y'') \in T$, distinct. First, we prove that

$$\tau(x'', y'') - \tau(x', y') + 2(x'y'' - x''y') = 0.$$

Assume, by contradiction, that

$$\tau(x'', y'') - \tau(x', y') + 2(x'y'' - x''y') = \alpha > 0. \tag{3}$$

For every ϵ, $0 < \epsilon < \alpha/2$, there exist $t' \in t(x', y')$ and $t'' \in t(x'', y'')$ such that

$$0 \leq \tau' - t' < \epsilon/2, \qquad 0 \leq \tau'' - t'' < \epsilon/2. \tag{4}$$

Since (x'', y'', t'') is above $H_{(x',y',t')}$, an easy computation using (2) gives that

$$\big((1-\lambda)x' + \lambda x'', (1-\lambda)y' + \lambda y'', t'' + 2(x'y'' - x''y')(1-\lambda) - \epsilon/2\big) \in CH\big((x', y', t'), (x'', y'', t'')\big), \tag{5}$$

for every $\lambda \in (0,1)$. Clearly, $CH\big((x',y',t'),(x'',y'',t'')\big) \subset CH(g_1,g_2,g_3)$ and by (3)–(5) we get

$$\begin{aligned}\tau((1-\lambda)x'+\lambda x'',(1-\lambda)y'+\lambda y'') &\geq t''+2(x'y''-x''y')(1-\lambda)-\epsilon/2\\ &> \tau(x'',y'')-\epsilon+2(x'y''-x''y')(1-\lambda)\\ &= \tau(x',y')+\alpha-\epsilon-2(x'y''-x''y')\lambda\\ &> \tau(x',y')+\alpha/2-2(x'y''-x''y')\lambda.\end{aligned} \tag{6}$$

From (6) and the continuity of τ we obtain that

$$\begin{aligned}\tau(x',y') &= \lim_{\lambda\to 0^+} \tau((1-\lambda)x'+\lambda x'',(1-\lambda)y'+\lambda y'')\\ &> \alpha/2+\tau(x',y'),\end{aligned}$$

a contradiction, since $\alpha > 0$. Hence, for every $(x',y'),\ (x'',y'') \in T$ fixed and distinct we have

$$\tau(x'',y'')-\tau(x',y')+2(x'y''-x''y') = 0.$$

This equation gives that, for every $(x',y'),\ (x'',y''),(x''',y''') \in T$,

$$(x'y''-x''y')+(x''y'''-x'''y'')+(x'''y'-x'y''') = 0,$$

i.e., these points are aligned. We conclude that the projections on $t=0$ of the points g_1,g_2,g_3 are aligned, contradicting our assumptions. So, τ is unbounded.

iii) Our purpose in this third part is to prove that τ is identically ∞.

If there exists at least one point $(x,y) \in T$ such that $\tau(x,y)=\infty$, the concavity of τ implies our assertion. Let us suppose that $\tau(T) \subset (-\infty,\infty)$. Then we can find a sequence of points $\big\{(x'_n,y'_n)\big\}_n$ of T such that $\tau(x'_n,y'_n) > n$. For every $\epsilon > 0$ there exists a sequence $\{t'_n\}_n$ such that

$$\tau(x'_n,y'_n)-t'_n < \epsilon, \qquad (x'_n,y'_n,t'_n) \in CH(g_1,g_2,g_3),$$

for every n. Let us fix a point $(x',y',t') \in CH(g_1,g_2,g_3)$; for n large, the point (x'_n,y'_n,t'_n) is above $H_{(x',y',t')}$. Using (2), we have that, for n large enough,

$$\big((1-\lambda)x'+\lambda x'_n,(1-\lambda)y'+\lambda y'_n,t'_n+2(x'y'_n-x'_ny')(1-\lambda)-\epsilon\big) \in CH\big((x',y',t'),(x'_n,y'_n,t'_n)\big),$$

for every $\lambda \in (0,1)$. The continuity of τ gives that, for every n,

$$\begin{aligned}\tau(x',y') &= \lim_{\lambda\to 0^+} \tau((1-\lambda)x'+\lambda x'_n,(1-\lambda)y'+\lambda y'_n)\\ &\geq \lim_{\lambda\to 0^+} (t'_n+2(x'y'_n-x'_ny')(1-\lambda)-\epsilon)\\ &> n+2(x'y'_n-x'_ny')-\epsilon.\end{aligned}$$

Hence $\tau(x',y')=\infty$.

In a similar way it is possible to prove that $\inf\{t : (x,y,t) \in CH(g_1,g_2,g_3)\} = -\infty$, for every $(x,y) \in T$.

iv) From iii), it follows that

$$CH(g_1, g_2, g_3) \supseteq R(g_1, g_2, g_3) \cup CH(g_1, g_2) \cup CH(g_1, g_3) \cup CH(g_2, g_3).$$

We need to show that the right hand side of the inclusion is tw–convex. Of course, $CH(g_i, g_j)$ is tw–convex, and $R(g_1, g_2, g_3)$ is tw–convex by iii). Suppose that $g \in CH(g_i, g_j)$ and $g' \in R(g_1, g_2, g_3)$ or $g' \in CH(g_i, g_k)$ where the indexes i, j, k are all different; then, $R(g, g') \subset R(g_1, g_2, g_3)$. This concludes the proof. $\square$

Finally, let us consider the set $CH(g_1, g_2, g_3)$ where the projections of g_1, g_2, g_3 onto the (x, y)–plane are aligned. We assume, without loss of generality, that one of the three points is e.

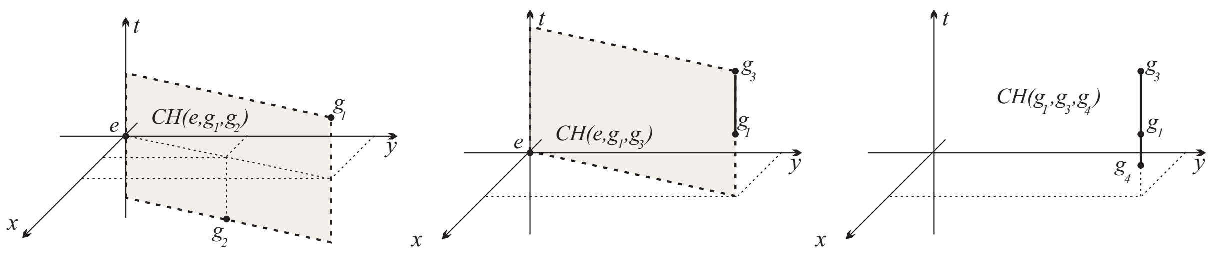
Remark 5.3. Let e, g_1, g_2 be distinct points in $I\!H$ that belong to the same 2–dimensional subgroup. Let $g_1 = (\alpha s_1, \beta s_1, t_1)$, $g_2 = (\alpha s_2, \beta s_2, t_2)$. Denote by s_m and s_M the minimum and the maximum, respectively, of $\{s_1, s_2, 0\}$, and by t_m and t_M the minimum and the maximum of $\{t_1, t_2, 0\}$. Then the tw–convex hull $CH(e, g_1, g_2)$ is the set

$$CH(e, g_1, g_2) = R'(e, g_1, g_2) \cup CH(e, g_1) \cup CH(g_1, g_2) \cup CH(g_2, e),$$

where $R'(e, g_1, g_2) = \{g \in I\!H : (\alpha s, \beta s, t),\ s_m < s < s_M,\ t_m < t < t_M\}$.

The proof is left to the reader. We only note that the previous remark includes the different situations where three, two or none of the projections are equal. The following example gives an idea of the situation.

Example 5.4. Let $g_1 = (2, 8, 2)$, $g_2 = (1, 4, -2)$, $g_3 = (2, 8, 4)$ and $g_4 = (2, 8, 1)$. In the next picture some tw–convex hulls are sketched, where the points of the broken lines are not included and the points of the continuous lines are included.



6. The twisted convex hull of a set A

Let us first consider the tw–convex hull of n points of $I\!H$ in the case $n \geq 4$. At this point there should be no surprises to the reader; indeed, passing from the case of three points to the case of n points, with $n \geq 4$, and with non aligned projections on $t = 0$, the situation is not a novelty.

Theorem 6.1. *Let $A = \{g_1, g_2, \dots, g_n\} \subset I\!H$, where $g_i = (x_i, y_i, t_i)$. Suppose that the projection onto the (x, y)–plane of any $B \subseteq A$, with card $(B) = 3$ is not contained in a straight line. Then the tw–convex hull $CH(g_1, g_2, \dots, g_n)$ is the set*

$$CH(g_1, g_2, \dots, g_n) = R(g_1, g_2, \dots, g_n) \cup \Big(\bigcup_{1 \leq i < j \leq n} CH(g_i, g_j) \Big),$$

where

$$R(g_1, g_2, \ldots, g_n) = \left\{ g \in I\!H : g = \left(\sum_{i=1}^{n} \lambda_i x_i, \sum_{i=1}^{n} \lambda_i y_i, t \right), 0 < \lambda_1, \ldots, \lambda_n < 1, \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

The proof may be omitted since it is very similar to the case $n = 3$. If there are three points g_i, g_j, g_k that belong to the same 2–dimensional subgroup, we can argue as in Example 5.4.

Finally, assume that $A \subseteq I\!H$. From previous arguments, we easily deduce the following result:

Theorem 6.2. *Let A be a subset of $I\!H$. Denote by A_0 the interior of the Euclidean convex hull in $I\!R^2$ of the set $\{(x, y) : \exists t \in I\!R, (x, y, t) \in A\}$.*

If A is an open subset of $I\!H$, then the tw–convex hull CH_A of A is given by

$$CH_A = \{g = (x, y, t) \in I\!H : (x, y) \in A_0, t \in I\!R\}.$$

If A is not open, then the interior of the tw–convex hull CH_A of A is

$$\operatorname{int}(CH_A) = \{g = (x, y, t) \in I\!H : (x, y) \in A_0, t \in I\!R\}.$$

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