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On Uniform Rotundity in Every Direction in Calderón-Lozanovskiĭ Sequence Spaces

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We find a criterion for uniform rotundity in every direction ($URED$) of Calderón-Lozanovskiĭ sequence spaces solving Problem XII from [7]. In order to do it, we study properties of the directed modulus of convexity of Banach spaces. Next we introduce and study new notions such as uniform rotundity in every interval ($UREI$) and uniform monotonicity in every interval ($UMEI$). They are crucial to get the main criterion, that is a Köthe sequence space is $UREI$ iff it is $URED$ and order continuous. Then we show the important (for further investigations) characterization of the property $UREI$ on the positive cone of Köthe sequence space. Applying that we prove the characterization mentioned at the beginning of the abstract. As a corollary, we obtain the criterion for $URED$ of Orlicz-Lorentz sequence spaces, which has not been proved until now.

Keywords: Köthe space, Calderón-Lozanovskiĭ spaces, Orlicz-Lorentz space, uniform rotundity in every direction

2000 Mathematics Subject Classification: 46E30, 46B20, 46B42, 46A45

1. Introduction

The fundamental geometric property called uniform rotundity (introduced by J. A. Clarkson in 1936, [8]) is applied in many subjects of mathematics (approximation theory, operator theory, fixed point theory and others). There is a lot of generalizations of this property. The uniform rotundity in every direction ($URED$) is one of them. It was introduced by A. L. Garkavi in [15] to characterize those Banach spaces in which every bounded subset has at most one Chebyshev center. On the other hand, $URED$ is an important notion in the metric fixed point theory (see [6]). Namely, V. Zizler in [36] showed that a Banach space that is $URED$ possesses normal structure, a property that guarantees the weak fixed point property ($WFPP$) (see [23]). Consequently, if a Banach space is $URED$, then it has $WFPP$. Note also that an $URED$ Banach space need not have the fixed point property (FPP) in general. Really, it is enough to take the l^∞ equipped with the norm defined by M. A. Smith in [33]. Such a renormed l^∞ is $URED$. On the other hand, P. N. Dowling, C. J. Lennard and B. Turett proved that l^∞ cannot be renormed

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to have FPP (see [11]). Note also that some generalizations of $URED$ were considered by M. A. Smith in [32] and [34]. It is worth to mention that the class of $URED$ Banach spaces is quite large. For example, each separable Banach space has an equivalent norm that is $URED$ (see [36]).

Our paper deals with the property $URED$ in Calderón-Lozanovskiĭ sequence spaces. The Calderón-Lozanovskiĭ spaces are widely investigated because of their role in the interpolation theory (see [28], [29]). The study of geometric properties of these spaces became interesting for many mathematicians (see e.g. [5], [12], [13], [14], [17], [25]). Sufficient conditions for various properties of Calderón-Lozanovskiĭ spaces have been presented in [5], [13] and [17], but the necessity of some among those conditions was only proved and it was concluded that some of sufficient conditions are not necessary. It has been shown in [5, Remark 3] that geometry of Calderón-Lozanovskiĭ space E_φ can be "good" even if geometry neither of E nor of φ is "good". Namely, a couple of E and φ can be found such that φ is not strictly convex and E is not strictly convex but E_φ is uniformly convex. On the base of this phenomena there was posed a natural open problem in [7] to find criteria for geometric properties such as rotundity, uniform rotundity, local uniform rotundity, $URED$ and others in Calderón-Lozanovskiĭ spaces. We solve this problem for $URED$ in the case of Calderón-Lozanovskiĭ sequence spaces. The general approach comes from [25]. However, the case of $URED$ required a lot of new ideas. First we introduced a new notion of uniform rotundity in every interval ($UREI$) that is equivalent in Köthe sequence spaces to both $URED$ and order continuity. This made possible studying of $URED$ in Calderón-Lozanovskiĭ sequence spaces e_φ with e being order continuous. Further, it is known that the crucial point in the investigations of geometric properties of spaces E_φ is the appropriate monotonicity property (see [14], [18], [25]). We introduce the notion of uniform monotonicity in every interval ($UMEI$) which is suitable for $UREI$. Moreover, to find criteria for a given geometric property A in Calderón-Lozanovskiĭ spaces E_φ, it is very important to prove that any Köthe space E has the property A if and only if E_+ has the property A. We showed the result of this type for $UREI$. Finally, we applied our general theorems to characterize $URED$ in Orlicz-Lorentz sequence spaces.

2. Preliminaries

Let $S(X)$ (resp. $B(X)$) be the unit sphere (resp. the closed unit ball) of a real Banach space $(X, \|\cdot\|_X)$.

We say that a Banach space X is *rotund* ($X \in (R)$ for short) if for every $x, y \in S(X)$ with $x \neq y$ we have $\|x+y\|_X < 2$. X is called *uniformly rotund in every direction* ($X \in (URED)$ for short) if for any $\varepsilon \in (0,1)$ and $z \in S(X)$ there exists $\delta(\varepsilon, z) \in (0,1)$ such that

$$\left\|\frac{x+y}{2}\right\|_X \leq 1 - \delta(\varepsilon, z)$$

for any $x, y \in S(X)$ with $x - y = \varepsilon z$. This form of the definition of $URED$ comes from [36, Proposition 1, (10)]. A function $\delta_X^{\rightarrow} : [0,2] \times X \setminus \{0\} \to [0,1]$ defined by the formula

$$\delta_X^{\rightarrow}(\varepsilon, z) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\|_E : x, y \in B(X),\ x - y = \lambda z \text{ for some } \lambda > 0,\ \|x-y\|_X \geq \varepsilon\right\}$$

is said to be the *directed modulus of convexity*. Clearly, $X \in (URED)$ if and only if for any $\varepsilon \in (0,2]$ and $z \in X \setminus \{0\}$ we have $\delta_X^{\rightarrow}(\varepsilon, z) > 0$ (see also [21]).

Let (T, Σ, μ) be a $\sigma-$finite and complete measure space. By $L^0 = L^0(T)$ we denote the set of all μ-equivalence classes of real valued measurable functions defined on T.

A Banach space $E = (E, \|\cdot\|_E)$ is said to be a *Köthe space* if E is a linear subspace of L^0 and

(i) if $x \in E, y \in L^0$ and $|y| \le |x|$ μ-a.e., then $y \in E$ and $\|y\|_E \le \|x\|_E$;
(ii) there exists a function x in E that is positive on the whole T (see [22] and [27]).

Every *Köthe space* is a Banach lattice under the obvious partial order ($x \ge 0$ if $x(t) \ge 0$ for μ-a.e. $t \in T$). In particular, if we consider the space E over the non-atomic measure μ, then we shall say that E is a *Köthe function space.* If we replace the measure space (T, Σ, μ) by the counting measure space $\left(\mathbb{N}, 2^{\mathbb{N}}, m\right)$, then we will say that E is a *Köthe sequence space* and we denote it by e. In the last case the symbol $e_i = (0, ..., 0, 1, 0, ...)$ stands for the i-th unit vector.

The set $E_+ = \{x \in E : x \ge 0\}$ is called the *positive cone of* E. For any subset $A \subset E$ define $A_+ = A \cap E_+$.

A point $x \in E$ is said to have *order continuous norm* if for any sequence (x_m) in E such that $0 \le x_m \le |x|$ and $x_m \to 0$ μ-a.e. we have $\|x_m\|_E \to 0$. A Köthe space E is called *order continuous* ($E \in (OC)$) if every element of E has an order continuous norm (see [22] and [27]).

E is said to be *strictly monotone* ($E \in (SM)$) if for each $0 \le y \le x$ with $y \ne x$ we have $\|y\|_E < \|x\|_E$. We say that E is *uniformly monotone* ($E \in (UM)$) provided for every $q \in (0, 1)$ there exists $p \in (0, 1)$ such that for all $0 \le y \le x$ satisfying $\|x\|_E = 1$ and $\|y\|_E \ge q$ we have $\|x - y\|_E \le 1 - p$ (see [3], [18]).

In the whole paper φ denotes an *Orlicz function*, i.e. $\varphi : \mathbb{R} \to [0, \infty]$, it is convex, even, vanishing and continuous at zero, left continuous on $(0, \infty)$ and not identically equal to zero. Denote

$$a_\varphi = \sup\{u \ge 0 : \varphi(u) = 0\} \quad \text{and} \quad b_\varphi = \sup\{u \ge 0 : \varphi(u) < \infty\}.$$

We write $\varphi > 0$ when $a_\varphi = 0$ and $\varphi < \infty$ if $b_\varphi = \infty$. Let $\varphi_r = \varphi\chi_{G_\varphi}$, where

$$G_\varphi = \begin{cases} \{0\} \cup (a_\varphi, b_\varphi] & \text{if } \varphi(b_\varphi) < \infty \\ \{0\} \cup (a_\varphi, b_\varphi) & \text{otherwise.} \end{cases} \tag{1}$$

The function φ is *strictly convex* if $\varphi((u+v)/2) < (\varphi(u) + \varphi(v))/2$ for all $u, v \in \mathbb{R}$ with $u \ne v$.

Define on L^0 a convex semimodular I_φ by

$$I_\varphi(x) = \begin{cases} \|\varphi \circ x\|_E & \text{if } \varphi \circ x \in E \\ \infty & \text{otherwise,} \end{cases}$$

where $(\varphi \circ x)(t) = \varphi(x(t))$, $t \in T$. By the *Calderón-Lozanovskiĭ space* E_φ we mean

$$E_\varphi = \{x \in L^0 : I_\varphi(cx) < \infty \text{ for some } c > 0\}$$

equipped with so called *Luxemburg norm* defined by

$$\|x\|_{\varphi} = \inf \{\lambda > 0 : I_{\varphi}(x/\lambda) \leq 1\}.$$

We will assume in the whole paper that E has the *Fatou property*, that is, if $0 \leq x_n \uparrow x \in L^0$ with $(x_n)_{n=1}^{\infty}$ in E and $\sup_n \|x_n\|_E < \infty$, then $x \in E$ and $\|x\|_E = \lim_n \|x_n\|_E$. Since E has the Fatou property, E_{φ} has also this property (see [12]), whence E_{φ} is a Banach space.

We say an Orlicz function φ satisfies *condition* $\Delta_2(0)$ (resp. $\Delta_2(\infty)$) if there exist $K > 0$ and $u_0 > 0$ such that $\varphi(u_0) > 0$ (resp. $\varphi(u_0) < \infty$) and the inequality $\varphi(2u) \leqslant K\varphi(u)$ holds for all $u \in [0, u_0]$ (resp. $u \in [u_0, \infty)$). If there exists $K > 0$ such that $\varphi(2u) \leqslant K\varphi(u)$ for all $u \geqslant 0$, then we say that φ satisfies *condition* $\Delta_2(\mathbb{R}_+)$. We write for short $\varphi \in \Delta_2(0)$, $\varphi \in \Delta_2(\infty)$, $\varphi \in \Delta_2(\mathbb{R}_+)$, respectively.

For a Köthe space E and an Orlicz function φ we say that φ satisfies *condition* Δ_2^E ($\varphi \in \Delta_2^E$ for short) if:

1) $\varphi \in \Delta_2(0)$ whenever $E \hookrightarrow L^{\infty}$;
2) $\varphi \in \Delta_2(\infty)$ whenever $L^{\infty} \hookrightarrow E$;
3) $\varphi \in \Delta_2(\mathbb{R}_+)$ whenever neither $L^{\infty} \hookrightarrow E$ nor $E \hookrightarrow L^{\infty}$ (see [17]),

where the symbol $E \hookrightarrow F$ stands for the continuous embedding of the space E into the space F.

Relationships between the modular I_{φ} and the norm $\|\cdot\|_{\varphi}$ are collected in [25].

If $E = L^1$ ($e = l^1$), then E_{φ} (e_{φ}) is the Orlicz function (sequence) space equipped with the Luxemburg norm. If $E = \Lambda_{\omega}$ ($e = \lambda_{\omega}$), then E_{φ} (e_{φ}) is the corresponding Orlicz-Lorentz function (sequence) space denoted by $(\Lambda_{\omega})_{\varphi}$ ($(\lambda_{\omega})_{\varphi}$) and equipped with the Luxemburg norm (see [4], [17], [19] and [20]).

3. Results

For any $\varepsilon \in [0, 1]$ and $z \in E \setminus \{0\}$ define

$$\begin{aligned} &\delta_{E_+}^{\rightarrow}(\varepsilon, z) \\ &= \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\|_E : x, y \in B(E)_+,\ x - y = \lambda z \text{ for some } \lambda > 0,\ \|x - y\|_E \geq \varepsilon \right\}. \end{aligned}$$

The following lemma was proved by A. Kamińska and B. Turett in [21] for the directed modulus of convexity $\delta_X^{\rightarrow}(\varepsilon, z)$, where X is an arbitrary Banach space. Since the positive cone E_+ has not a linear structure, so the result proved by them cannot be directly applied for E_+. Anyway, their idea can be applied in our case with some essential modifications. To avoid the repetition of argumentations presented by them, preserving denotations used in their proof of Theorem 2, we limit ourselves only to these modifications.

Lemma 3.1. *Let E be a Köthe space and E_+ be the positive cone of E.*

(*a*) *For each $z \in E \setminus \{0\}$ the function $\delta_{E_+}^{\rightarrow}(\cdot, z)$ is continuous on $[0, 1)$.*
(*b*) *For each $\varepsilon \in [0, 1]$ the function $\delta_{E_+}^{\rightarrow}(\varepsilon, \cdot)$ is continuous on $E \setminus \{0\}$.*

Proof. The proof of (a) is analogous to that concerning the continuity of the modulus of convexity (see [30]). To prove (b), fix $\varepsilon \in [0,1]$. Let $z \in E \setminus \{0\}$ and take an arbitrary sequence (z_n) converging to z with $\|z\|_E \le 2\|z_n\|_E$. For each $n \in \mathbb{N}$, there exist $x_n, y_n \in B(E)_+$ and $\lambda_n > 0$ such that $x_n - y_n = \lambda_n z_n$, $\|x_n - y_n\|_E \ge \varepsilon$ and

$$\frac{1}{n} + \delta_{\overrightarrow{E_+}}(\varepsilon, z_n) \ge 1 - \left\|\frac{x_n + y_n}{2}\right\|_E.$$

Moreover

$$\lambda_n = \frac{\|x_n - y_n\|_E}{\|z_n\|_E} \le \frac{4}{\|z\|_E}$$

for any $n \in \mathbb{N}$. Let $\alpha_n(t) = 0 \wedge \lambda_n(z_n - z)(t)$ and $\beta_n(t) = 0 \vee \lambda_n(z_n - z)(t)$, $t \in T$. By boundedness of (λ_n) and inequalities

$$\|\alpha_n\|_E \le \lambda_n \|z_n - z\|_E, \qquad \|\beta_n\|_E \le \lambda_n \|z_n - z\|_E,$$

we conclude that $\|\alpha_n\|_E \to 0$ and $\|\beta_n\|_E \to 0$. Putting $\widetilde{x}_n = x_n - \alpha_n$ and $\widetilde{y}_n = y_n + \beta_n$, we obtain $\widetilde{x}_n, \widetilde{y}_n \in E_+$ and

$$\widetilde{x}_n - \widetilde{y}_n = x_n - y_n - (\alpha_n + \beta_n) = \lambda_n z_n - \lambda_n(z_n - z) = \lambda_n z.$$

Define

$$\overline{x}_n = \frac{\widetilde{x}_n}{1 + \lambda_n \|z_n - z\|_E}, \qquad \overline{y}_n = \frac{\widetilde{y}_n}{1 + \lambda_n \|z_n - z\|_E}.$$

Obviously, $\overline{x}_n, \overline{y}_n \in B(E)_+$. We have also

$$\begin{aligned}\|\overline{x}_n - x_n\|_E &= \left\|\frac{x_n - \alpha_n}{1 + \lambda_n \|z_n - z\|_E} - x_n\right\|_E \\ &= \left\|\frac{\lambda_n \|z_n - z\|_E x_n + \alpha_n}{1 + \lambda_n \|z_n - z\|_E}\right\|_E \le \|\alpha_n\|_E + \lambda_n \|z_n - z\|_E \to 0\end{aligned}$$

and analogously

$$\|\overline{y}_n - y_n\|_E \le \|\beta_n\|_E + \lambda_n \|z_n - z\|_E \to 0.$$

Hence

$$b_n = \|(\overline{x}_n - \overline{y}_n) - (x_n - y_n)\|_E \to 0.$$

Moreover, $\|\overline{x}_n - \overline{y}_n\|_E \ge \varepsilon - b_n$ and $\overline{x}_n - \overline{y}_n = \lambda_n z / (1 + \lambda_n \|z_n - z\|_E)$. Now, repeating the same argumentation as in the proof of Theorem 2(b) from [21], we get the lower semicontinuity of $\delta_{\overrightarrow{E_+}}(\varepsilon, \cdot)$, i.e.

$$\liminf_{n \to \infty} \delta_{\overrightarrow{E_+}}(\varepsilon, z_n) \ge \delta_{\overrightarrow{E_+}}(\varepsilon, z).$$

To prove the upper semicontinuity of $\delta_{\overrightarrow{E_+}}(\varepsilon, \cdot)$, put

$$c_n = \frac{2\|z_n - z\|_E}{\|z\|_E} \quad \text{and} \quad \varepsilon_n = (1 + c_n)(\varepsilon + c_n).$$

Then $c_n \to 0$, $\varepsilon_n > \varepsilon$ and $\varepsilon_n \to \varepsilon$. For any $n \in \mathbb{N}$ there are $u_n, w_n \in B(E)_+$ and $\alpha_n > 0$ such that

$$\frac{1}{n} + \delta_{E_+}^{\rightarrow}(\varepsilon_n, z) \geq 1 - \left\| \frac{u_n + w_n}{2} \right\|_E$$

with $u_n - w_n = \kappa_n z$ and $\|u_n - w_n\|_E \geq \varepsilon_n$. Obviously, $|\kappa_n| \leq 2/\|z\|_E$ for any $n \in \mathbb{N}$. Let $\gamma_n(t) = 0 \wedge \kappa_n(z - z_n)(t)$ and $\rho_n(t) = 0 \vee \kappa_n(z - z_n)(t)$, $t \in T$. Similarly as above, we conclude that $\|\gamma_n\|_E \to 0$ and $\|\rho_n\|_E \to 0$. Define

$$\overline{u}_n = \frac{u_n - \gamma_n}{1 + \kappa_n \|z_n - z\|_E}, \qquad \overline{w}_n = \frac{w_n + \rho_n}{1 + \kappa_n \|z_n - z\|_E}.$$

Then $\overline{u}_n, \overline{w}_n \in B(E)_+$ and $\overline{u}_n - \overline{w}_n = \kappa_n z_n / (1 + \kappa_n \|z_n - z\|_E)$. Moreover,

$$\begin{aligned}
\|\overline{u}_n - \overline{w}_n\|_E &= \left\| \frac{u_n - w_n - (\gamma_n + \rho_n)}{1 + \kappa_n \|z_n - z\|_E} \right\|_E \\
&\geq \frac{\|u_n - w_n\|_E}{1 + \kappa_n \|z_n - z\|_E} - \frac{\kappa_n \|z_n - z\|_E}{1 + \kappa_n \|z_n - z\|_E} \\
&\geq \frac{\varepsilon_n}{1 + c_n} - c_n = \varepsilon.
\end{aligned}$$

Since $\|u_n - \overline{u}_n\|_E \to 0$ and $\|w_n - \overline{w}_n\|_E \to 0$, analogously as in [21], we get

$$\limsup_{n \to \infty} \delta_{E_+}^{\rightarrow}(\varepsilon, z_n) \leq \delta_{E_+}^{\rightarrow}(\varepsilon, z),$$

which finishes the proof. □

Lemma 3.2. *Let E be an URED Köthe space. If $z_1, z_2 \in E \setminus \{0\}$ and $|z_1| = |z_2|$, then $\delta_E^{\rightarrow}(\varepsilon, z_1) = \delta_E^{\rightarrow}(\varepsilon, z_2)$ for any $\varepsilon > 0$.*

Proof. Let $z_1, z_2 \in E \setminus \{0\}$ and $|z_1| = |z_2|$. Fix $\varepsilon > 0$. Take arbitrary $x, y \in B(E)$ with $x - y = \lambda z_1$ for some $\lambda > 0$ and $\|x - y\|_E \geq \varepsilon$. Denote

$$A = \{t \in \operatorname{supp} z_1 : z_1(t) = -z_2(t)\}.$$

Define

$$\widetilde{x}(t) = \begin{cases} x(t) & \text{for } t \in T \setminus A \\ -x(t) & \text{for } t \in A, \end{cases} \qquad \widetilde{y}(t) = \begin{cases} y(t) & \text{for } t \in T \setminus A \\ -y(t) & \text{for } t \in A. \end{cases}$$

Then $\widetilde{x} - \widetilde{y} = \lambda z_2$, $\|\widetilde{x} - \widetilde{y}\|_E \geq \varepsilon$ and

$$|\widetilde{x}(t) + \widetilde{y}(t)| = \left| (x(t) + y(t)) \chi_A(t) - (x(t) + y(t)) \chi_{T \setminus A}(t) \right| = |x(t) + y(t)|$$

for any $t \in T$. Hence

$$\left\| \frac{x + y}{2} \right\|_E = \left\| \frac{\widetilde{x} + \widetilde{y}}{2} \right\|_E \leq 1 - \delta_E^{\rightarrow}(\varepsilon, z_2).$$

Consequently,

$$\delta_E^{\rightarrow}(\varepsilon, z_2) \leq 1 - \left\| \frac{x + y}{2} \right\|_E$$

for any $x, y \in B(E)$ with $x - y = \lambda z_1$ for some $\lambda > 0$ and $\|x - y\|_E \geq \varepsilon$. By the definition of infimum, $\delta_E^{\rightarrow}(\varepsilon, z_2) \leq \delta_E^{\rightarrow}(\varepsilon, z_1)$. The opposite inequality we get by the symmetry. □

Proposition 3.3. *The following conditions are equivalent:*

(a) *The Köthe space E is $URED$.*

(b) *For any $\varepsilon \in (0,2]$ and $z \in E_+ \setminus \{0\}$ there exists $\delta(\varepsilon, z) \in (0,1)$ such that*

$$\left\| \frac{x+y}{2} \right\|_E \leq 1 - \delta(\varepsilon, z)$$

for any $x, y \in B(E)$ with $x - y = \lambda z$ for some $\lambda > 0$ and $\|x-y\|_E \geq \varepsilon$.

Proof. The implication $(a) \Rightarrow (b)$ is obvious. To prove $(b) \Rightarrow (a)$ fix $\varepsilon \in (0,2]$ and $z \in E \backslash \{0\}$. Take arbitrary $x, y \in B(E)$ with $x - y = \lambda z$ for some $\lambda > 0$ and $\|x-y\|_E \geq \varepsilon$. Denote

$$A = \{t \in \operatorname{supp} z : z(t) = |z(t)|\} \quad \text{and} \quad B = \{t \in \operatorname{supp} z : z(t) = -|z(t)|\}.$$

Obviously $A \cap B = \varnothing$ and $A \cup B = \operatorname{supp} |z|$. Define $\widetilde{x}$ and $\widetilde{y}$ as in the proof of Lemma 3.2. Then $\widetilde{x} - \widetilde{y} = \lambda |z|$, $\|\widetilde{x} - \widetilde{y}\|_E \geq \varepsilon$ and there is $\delta(\varepsilon, |z|) > 0$ such that

$$\left\| \frac{x+y}{2} \right\|_E = \left\| \frac{\widetilde{x}+\widetilde{y}}{2} \right\|_E \leq 1 - \delta(\varepsilon, |z|).$$

□

It is known that UR or even LUR of Köthe space implies OC (see [10] or [18]). However this is not the case of $URED$ since l^∞ can be renormed to be $URED$ (see [33, Counterexample 2]). On the other hand OC of E seems to be useful in study rotundity properties of E_φ (see [12], [13], [14] and [25]). It leads to the following property called $UREI$ that is slightly stronger than $URED$ in general Köthe spaces and it is equivalent to both $URED$ and OC in Köthe sequence spaces (see Theorem 3.8 below).

Definition 3.4. We say that a Köthe space E [the positive cone E_+ of E] *is uniformly rotund in every interval* ($E \in (UREI)$ for short) [resp. $E_+ \in (UREI)$] provided for any $\varepsilon > 0$ and $z \in E \backslash \{0\}$ there exists $\delta(\varepsilon, z) \in (0,1)$ such that $\|x+y\|_E \leq 2(1 - \delta(\varepsilon, z))$ for any $x, y \in B(E)$ [resp. $x, y \in B(E)_+$] with $|x-y| \leq |z|$ and $\|x-y\|_E \geq \varepsilon$.

It is easy to see that if $E \in (UREI)$, then $E \in (URED)$.

The monotonicity property of Köthe space E is crucial in order to get criteria for the respective rotundity property of E_φ. For example SM, UM, $LLUM$ are crucial for R, UR, LUR of E_φ, respectively (see [12], [14], [18] and [25]).

Definition 3.5. We say that a Köthe space E *is uniformly monotone in every interval* ($E \in (UMEI)$ for short) provided for any $\varepsilon > 0$ and $z \in E_+$ there exists $\delta(\varepsilon, z) \in (0,1)$ such that $\|x-y\|_E \leq 1 - \delta(\varepsilon, z)$ for any $x, y \in B(E)_+$ with $0 \leq y \leq x$, $\|x\|_E = 1$, $\|y\|_E \geq \varepsilon$ and $y \leq z$.

It is known that monotonicity property of E (for example SM, UM, $LLUM$) is a restriction of an appropriate rotundity property (R, UR, LUR, resp.) to couples of comparable nonnegative elements (see [18]). This is also the case with the pair $UMEI$ and $UREI$. Namely,

Lemma 3.6. *Let E be a Köthe space. If $E \in (UREI)$, then $E \in (UMEI)$. Moreover, if we restrict $UREI$ to couples of compatible nonnegative elements, then the inverse statement is true.*

Proof. The proof is analogous to that of Theorem 1 in [18]. We present the proof for the sake of convenience. Assume that $E \in (UREI)$. Let $\varepsilon > 0$, $z \in E_+ \setminus \{0\}$. Take $x, y \in B(E)_+$ with $0 \leq y \leq x$, $\|x\|_E = 1$, $\|y\|_E \geq \varepsilon$ and $y \leq z$. Set $u = x$ and $v = x - y$. Then $|u - v| \leq z$ and $\|u - v\|_E \geq \varepsilon$. By the Definition 3.4, there is a $\delta_1 = \delta(\varepsilon, z)$ such that $\|(u + v)/2\|_E \leq 1 - \delta_1$. Consequently,

$$\|x - y\|_E \leq \|x - y/2\|_E = \|(u + v)/2\|_E \leq 1 - \delta_1.$$

The proof of the inverse statement is similar. □

Lemma 3.7. *Let E be a Köthe space. If $E \in (UMEI)$, then $E \in (OC)$.*

Proof. Suppose that $E \notin (OC)$. Then, in view of the well known result of Ando [2] (see also [27]), there exists a sequence (u_n) in E_+ with $\|u_n\|_E = 1$, $\operatorname{supp} u_n \cap \operatorname{supp} u_m = \emptyset$ and a function $u \in E_+$ such that $u_n \leq u$ for each $n \in \mathbb{N}$. Set

$$v = \sum_{k=1}^{\infty} u_k, \quad z = \frac{v}{\|v\|_E}, \quad y_n = \frac{u_n}{\|v\|_E} \quad \text{and} \quad x = z.$$

It is easy to check that, taking $\varepsilon = 1/\|u\|_E$, we get $\|y_n\|_E \geq \varepsilon$. Moreover, $\|\sum_{k=1}^{\infty} u_k - u_n\|_E \to \|\sum_{k=1}^{\infty} u_k\|_E$ as $n \to \infty$ (see the proof of Proposition 2.1 in [10]), whence $\|x - y_n\|_E \to 1$. A contradiction. □

For an analogous result concerning lower local uniform monotonicity or Kadec-Klee property see [10, Proposition 2.1].

Theorem 3.8. *Let e be a Köthe sequence space. Then $e \in (UREI)$ if and only if $e \in (OC)$ and $e \in (URED)$.*

Proof. $UREI$ implies $URED$ in an obvious manner. Hence, in view of Lemma 3.6 and 3.7, only the sufficiency needs to be proved. Let $\varepsilon > 0$, $z \in e$. Take $x, y \in B(e)$ with $|x - y| \leq |z|$ and $\|x - y\|_e \geq \varepsilon$. It is enough to show that

$$\delta_I(\varepsilon, z) = \inf\{\delta_e^{\rightarrow}(\varepsilon, |w|) : |w| \leq |z|, \ \|w\|_e \geq \varepsilon\} > 0.$$

Suppose for the contrary that $\delta_I(\varepsilon, z) = 0$. Hence, we find a sequence (w_n) with $|w_n| \leq |z|$, $\|w_n\|_e \geq \varepsilon$ and $\delta_e^{\rightarrow}(\varepsilon, |w_n|) \to 0$. Since each order interval is norm compact in order continuous Köthe sequence space (see [35, Theorem 6.1]), passing to a subsequence, if necessary, we may assume that there is an element $w_0 \in [0, |z|]$ such that $\||w_n| - w_0\|_e \to 0$. Since the directed modulus of convexity $\delta_e^{\rightarrow}(\varepsilon, \cdot)$ is continuous function for each $\varepsilon > 0$ with respect to norm topology in e (see [21, Theorem 2]), we get $\delta_e^{\rightarrow}(\varepsilon, |w_n|) \to \delta_e^{\rightarrow}(\varepsilon, w_0)$. On the other hand $\|w_n\|_e \geq \varepsilon$, so $\|w_0\|_e \geq \varepsilon$. Applying the assumption that $e \in (URED)$, we get $\delta_e^{\rightarrow}(\varepsilon, w_0) > 0$. This contradiction shows that $\delta_I(\varepsilon, z) > 0$. Since

$$0 < \delta_I(\varepsilon, z) \leq \inf\{1 - \|(x + y)/2\|_e : x, y \in B(e), \ |x - y| \leq |z|, \ \|x - y\|_e \geq \varepsilon\},$$

by Definition 3.4, we finishes the proof. □

In the sequel we will need the following characterization of $UMEI$. The idea of proof has been taken from [18, Theorem 6]. However, some parts do not go the same way.

Proposition 3.9. *Let E be a Köthe space. The following assertions are equivalent:*

(a) $E \in (UMEI)$.

(b) *For each $\varepsilon > 0$ and $z \in E_+$ there is $\sigma = \sigma(\varepsilon, z) > 0$ such that for any $x, y \in E_+$ with $\|x\|_E = 1$, $\|y\|_E \geq \varepsilon$ and $y \leq z$ we have $\|x+y\|_E \geq 1+\sigma$.*

(c) *For each $\varepsilon > 0$ and $z \in E_+$ there is $\rho = \rho(\varepsilon, z) > 0$ such that for any $x, y \in E$ with $x \perp y$, $\|x\|_E = 1$, $\|y\|_E \geq \varepsilon$ and $|y| \leq z$ we have $\|x+y\|_E \geq 1+\rho$.*

(d) *For each $\varepsilon > 0$ and $z \in E_+$ there is $\eta = \eta(\varepsilon, z) > 0$ such that for any $x, y \in E_+$ with $\|x\|_E = 1$, $y \leq x$ and $y \leq z$ there holds $\|x\chi_{T\backslash B}\|_E \leq 1-\eta$ whenever $\|y\chi_B\|_E \geq \varepsilon$ and $B \in \Sigma$.*

Proof. $(a) \Rightarrow (b)$. Take $\varepsilon > 0$ and $z \in E_+$. Let $x, y \in B(E)_+$ be such that $\|x\|_E = 1$, $\|y\|_E \geq \varepsilon$ and $y \leq z$. Defining $x_1 = \frac{x+y}{a}$ and $y_1 = \frac{y}{a}$, where $a = \|x+y\|_E \geq 1$, we notice that $0 \leq y_1 \leq x_1$, $\|x_1\|_E = 1$, $\|y_1\|_E \geq \frac{\varepsilon}{2}$ (because we can assume that $a \leq 2$) and $y_1 \leq z$. Since $E \in (UMEI)$, there exists $\delta = \delta\left(\frac{\varepsilon}{2}, z\right)$ such that

$$\frac{1}{a} = \left\|\frac{x}{a}\right\|_E = \left\|\frac{x+y}{a} - \frac{y}{a}\right\|_E = \|x_1 - y_1\|_E \leq 1-\delta.$$

Hence, taking $\sigma = \sigma(\varepsilon, z) = \min\left\{1, \frac{\delta}{1-\delta}\right\} > 0$, we get

$$\|x+y\|_E = a \geq \frac{1}{1-\delta} = 1 + \frac{\delta}{1-\delta} = 1+\sigma.$$

The implication $(b) \Rightarrow (c)$ is obvious.

$(c) \Rightarrow (d)$. Let $\varepsilon > 0$ and $z \in E_+$. Take arbitrary x, y and a set $B \in \Sigma$ as in (d). Put

$$x_1 = \frac{x\chi_{T\backslash B}}{\|x\chi_{T\backslash B}\|_E}, \qquad y_1 = \frac{y\chi_B}{2\|x\chi_{T\backslash B}\|_E}.$$

Then $x_1 \perp y_1$, $\|x_1\|_E = 1$ and $\|y_1\|_E \geq \varepsilon/2$. We may clearly suppose that $\|x\chi_{T\backslash B}\|_E \geq 1/2$, whence $y_1 \leq z$. Then

$$\frac{1}{\|x\chi_{T\backslash B}\|_E} = \left\|\frac{x\chi_{T\backslash B} + x\chi_B}{\|x\chi_{T\backslash B}\|_E}\right\|_E \geq \|x_1 + y_1\|_E \geq 1+\rho,$$

where $\rho = \rho(\varepsilon/2, z)$ is from (c). So

$$\|x\chi_{T\backslash B}\|_E \leq \frac{1}{1+\rho} = 1-\eta$$

with $\eta = \eta(\varepsilon, z) = \frac{\rho}{1+\rho}$.

$(d) \Rightarrow (a)$. Let $\varepsilon > 0$ and $z \in E_+$. Take $x, y \in B(E)$ with $0 \leq y \leq x$, $\|x\|_E = 1$, $\|y\|_E \geq \varepsilon$ and $y \leq z$. Set $B = \{t \in T : y(t) \leq \varepsilon x(t)/2\}$. Then $\|y\chi_B\|_E \leq \varepsilon/2$ and consequently $\|y\chi_{T\backslash B}\|_E \geq \varepsilon/2$. Hence, by (d), we have

$$\|x-y\|_E \leq \|x\chi_B + (x - \varepsilon x/2)\chi_{T\backslash B}\|_E \leq 1 - \varepsilon/2 + \frac{\varepsilon}{2}\|x\chi_B\|_E \leq 1-\varepsilon/2 + \frac{\varepsilon}{2}(1-\eta_1),$$

where $\eta_1 = \eta(\varepsilon/2, z)$ is from (d). Now, taking $\delta = \delta(\varepsilon, z) = \frac{\varepsilon}{2}\eta_1$, we finish the proof. $\square$

It is worth mentioning that property $UMEI$ is applied in the ergodic theory (see Theorem 5.1 in [1], condition (c) is just $UMEI$ according to Proposition 3.9(b)).

The following characterization we shall need in order to prove the main result.

Proposition 3.10. *Suppose that e is a Köthe sequence space. Then $e_\varphi \in (UMEI)$ if and only if $e \in (UMEI)$, $\varphi > 0$, $\varphi(b_\varphi)\inf_i \|e_i\|_e \geq 1$ and $\varphi \in \Delta_2^e$.*

Proof. *Necessity.* The fact $e_\varphi \in (UMEI)$ implies that $e_\varphi \in (SM)$. Consequently, by Proposition 2.1(i) and Lemma 2.5 from [25], we get that $\varphi > 0$ and $\varphi(b_\varphi)\inf_i \|e_i\|_e \geq 1$. To prove that $e \in (UMEI)$ we apply Proposition 3.9. Let $\varepsilon > 0$ and $z \in e_+$. Take $x, y \in e$ with $x \perp y$, $\|x\|_e = 1$, $\|y\|_e \geq \varepsilon$ and $|y| \leq z$.
If $\|z\|_e \leq 1$, then set $w = \varphi_r^{-1} \circ |x|$, $u = \varphi_r^{-1} \circ |y|$. Then u, w are well defined by Proposition 2.1(ii) from [25]. Moreover $u \perp w$, $I_\varphi(w) = 1$ and $I_\varphi(u) \geq \varepsilon$. Hence $\|w\|_\varphi = 1$, $\|u\|_\varphi \geq \min\{\varepsilon, 1\}$. Furthermore $u = \varphi_r^{-1} \circ |y| \leq \varphi_r^{-1} \circ z$. Consequently, $\|u + w\|_\varphi \geq 1 + \rho_1$, where $\rho_1 = \rho_{e_\varphi}(\varepsilon \wedge 1, \varphi_r^{-1} \circ z)$ is from Proposition 3.9(c). Finally

$$\|x + y\|_e = \|\varphi \circ w + \varphi \circ u\|_e = \|\varphi \circ (w + u)\|_e = I_\varphi(u + w) \geq \|u + w\|_\varphi \geq 1 + \rho_1.$$

Now, consider $\|z\|_e > 1$. Then define w as above and $u = \varphi_r^{-1} \circ \left|\frac{y}{\|z\|_e}\right|$. Since

$$I_\varphi(u) = \left\|\frac{y}{\|z\|_e}\right\|_e \geq \frac{\varepsilon}{\|z\|_e},$$

we conclude that $\|u\|_\varphi \geq \min\left\{\frac{\varepsilon}{\|z\|_e}, 1\right\}$. Moreover, $u \leq \varphi_r^{-1} \circ \left(\frac{z}{\|z\|_e}\right)$. Hence $\|u + w\|_\varphi \geq 1 + \rho_2$, with $\rho_2 = \rho_{e_\varphi}\left(\frac{\varepsilon}{\|z\|_e} \wedge 1, \varphi_r^{-1} \circ \left(\frac{z}{\|z\|_e}\right)\right)$. Therefore

$$\|x + y\|_e \geq \left\|x + \frac{y}{\|z\|_e}\right\|_e = \|\varphi \circ w + \varphi \circ u\|_e \geq \|u + w\|_\varphi \geq 1 + \rho_2.$$

Taking $\rho = \min\{\rho_1, \rho_2\}$, we get (c) from Proposition 3.9, so $e \in (UMEI)$. Moreover, by Lemma 3.7, we conclude that $e \in (OC)$. Hence, applying Lemma 2.9 from [25] and Lemma 2.4 from [13], we conclude that $\varphi \in \Delta_2^e$.

Sufficiency. Let $\varepsilon > 0$ and $z \in (e_\varphi)_+ \setminus \{0\}$. Take $x, y \in e_\varphi$ with $0 \leq y \leq x$, $\|x\|_\varphi = 1$, $\|y\|_\varphi \geq \varepsilon$ and $y \leq z$. Put $u = \varphi \circ x$, $v = \varphi \circ y$. Then $\|u\|_e = 1$, $\|u\|_e \geq \eta(\varepsilon)$, by Lemma 1.3 in [25]. Furthermore $v \leq \varphi \circ z$. Since $e \in (UMEI)$, we get

$$\|u - v\|_e \leq 1 - \delta_1,$$

where $\delta_1 = \delta_e(\eta(\varepsilon), \varphi \circ z)$. Applying superadditivity φ on $\mathbb{R}_+$ we obtain

$$I_\varphi(x - y) = \|\varphi \circ (x - y)\|_e \leq \|\varphi \circ x - \varphi \circ y\|_e = \|u - v\|_e \leq 1 - \delta_1.$$

Finally, by Lemma 1.2 and 1.4 from [25], $\|x - y\|_\varphi \leq 1 - r_1$, where $r_1 = r_1(\delta_1)$ depends only on ε and z, what finishes the proof. $\square$

The natural question is whether a geometric property can be equivalently considered on positive cone E_+. Such problem has been considered for many properties (e.g. R, UR, LUR - see [18], [19], [20]) and positive answers are very useful.

Theorem 3.11. *For any Köthe sequence space e the following conditions are equivalent:*

(a) *e is UREI;*
(b) e_+ *is UREI.*

Proof. The implication $(a) \Rightarrow (b)$ is obvious. To prove the inverse, suppose (b). Then $e \in (OC)$. In view of Theorem 3.8, it remains to prove that e is $URED$. Let $\varepsilon \in (0,1)$ and $z \in S(e)$. Take arbitrary $x, y \in S(e)$ with $x - y = \varepsilon z$ and set

$$A = \{i \in \mathbb{N} : x(i)y(i) \geq 0\}.$$

Let

$$\delta_I^+(\eta, z) = \inf\left\{\delta_{e_+}^{\rightarrow}(\eta, |w|) : |w| \leq |z| \text{ and } \|w\|_e \geq \eta\right\}.$$

Applying the proof of Theorem 3.8 for the function $\delta_{e_+}^{\rightarrow}$ instead of $\delta_e^{\rightarrow}$, in virtue of Lemma 3.1, we conclude that $\delta_I^+(\eta, z) > 0$ for any $\eta > 0$. We divide the proof into two cases.

1) Suppose

$$\left\|x\chi_{\mathbb{N}\setminus A}\right\|_e \vee \left\|y\chi_{\mathbb{N}\setminus A}\right\|_e \leq \frac{1}{4}\left(\delta_I^+\left(\frac{1}{2}\varepsilon, z\right) \wedge \varepsilon\right).$$

Denote

$$\widetilde{x} = |x|\,\chi_A \quad \text{and} \quad \widetilde{y} = |y|\,\chi_A.$$

Then

$$\begin{aligned}
\varepsilon &= \|\varepsilon z\|_e = \left\|\varepsilon z\chi_A + \varepsilon z\chi_{\mathbb{N}\setminus A}\right\|_e = \left\|\varepsilon z\chi_A + (x-y)\,\chi_{\mathbb{N}\setminus A}\right\|_e \\
&\leq \|\varepsilon z\chi_A\|_e + \left\|x\chi_{\mathbb{N}\setminus A}\right\|_e + \left\|y\chi_{\mathbb{N}\setminus A}\right\|_e \leq \|\varepsilon z\chi_A\|_e + \frac{1}{2}\varepsilon.
\end{aligned}$$

Hence

$$\|\widetilde{x} - \widetilde{y}\|_e = \|\varepsilon z\chi_A\|_e \geq \frac{1}{2}\varepsilon.$$

Moreover,

$$\begin{aligned}
\left\|\frac{x+y}{2}\right\|_e &\leq \left\|\frac{\widetilde{x}+\widetilde{y}}{2} + \left|\frac{x\chi_{\mathbb{N}\setminus A} + y\chi_{\mathbb{N}\setminus A}}{2}\right|\right\|_e \\
&\leq \left\|\frac{\widetilde{x}+\widetilde{y}}{2}\right\|_e + \frac{1}{2}\left(\left\|x\chi_{\mathbb{N}\setminus A}\right\|_e + \left\|y\chi_{\mathbb{N}\setminus A}\right\|_e\right) \\
&\leq 1 - \delta_e^{\rightarrow}\left(\frac{1}{2}\varepsilon, \varepsilon z\chi_A\right) + \frac{1}{4}\delta_I^+\left(\frac{1}{2}\varepsilon, z\right) \leq 1 - \frac{3}{4}\delta_I^+\left(\frac{1}{2}\varepsilon, z\right).
\end{aligned}$$

2) Now, let

$$\left\|x\chi_{\mathbb{N}\setminus A}\right\|_e \vee \left\|y\chi_{\mathbb{N}\setminus A}\right\|_e \geq \frac{1}{4}\left(\delta_I^+\left(\frac{1}{2}\varepsilon, z\right) \wedge \varepsilon\right).$$

Notice that

$$\left|\frac{x+y}{2}\right|\chi_A = \frac{|x|+|y|}{2}\chi_A$$

and

$$\left|\frac{x+y}{2}\right| \chi_{\mathbb{N}\setminus A} = \left(\frac{|x|+|y|}{2} - (|x| \wedge |y|)\right) \chi_{\mathbb{N}\setminus A}.$$

Hence

$$\left|\frac{x+y}{2}\right| = \frac{|x|+|y|}{2} - (|x| \wedge |y|)\, \chi_{\mathbb{N}\setminus A}.$$

Similarly,

$$|x-y| = ||x|-|y|| + 2\,(|x| \wedge |y|)\, \chi_{\mathbb{N}\setminus A}.$$

Since e_+ is $URED$, the directed modulus of convexity $\delta^{\rightarrow}_{e_+}(\varepsilon, z) > 0$. To simplify denotations, put $\eta(\varepsilon, z) = \frac{1}{4}\left(\delta_I^+\left(\frac{3}{4}\varepsilon, z\right) \wedge \varepsilon\right)$. We consider the following two subcases.

(a) Suppose that

$$\left\|(|x| \wedge |y|)\, \chi_{\mathbb{N}\setminus A}\right\|_e < \eta(\varepsilon, z).$$

Divide the set $\mathbb{N} \setminus A$ into two parts D_1 and D_2 as follows

$$D_1 = \{i \in \mathbb{N} \setminus A : |x(i)| \geq |y(i)|\} \quad \text{and} \quad D_2 = \{i \in \mathbb{N} \setminus A : |x(i)| < |y(i)|\}.$$

Define

$$\overline{x}(i) = \begin{cases} |x(i)| & \text{for } i \in A \cup D_1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\overline{y}(i) = \begin{cases} |y(i)| & \text{for } i \in A \cup D_2 \\ 0 & \text{otherwise.} \end{cases}$$

Denote $v = \frac{\overline{x}-\overline{y}}{\varepsilon}$, we have that $|v(i)| = |z(i)|$ for $i \in A$ and $|v(i)| \leq |z(i)|$ for $i \in \mathbb{N} \setminus A$. Then

$$\begin{aligned} \|\overline{x}-\overline{y}\|_e &= \left\||x-y| - (|x| \wedge |y|)\, \chi_{\mathbb{N}\setminus A}\right\|_e \\ &\geq \|x-y\|_e - \left\|(|x| \wedge |y|)\, \chi_{\mathbb{N}\setminus A}\right\|_e \geq \frac{3}{4}\varepsilon. \end{aligned}$$

Since $\overline{x}, \overline{y} \in B(e)_+$ and $\overline{x}-\overline{y} \leq \varepsilon v \leq z$, it follows that

$$\begin{aligned} \left\|\frac{x+y}{2}\right\|_e &= \left\|\frac{\overline{x}+\overline{y}}{2} - (|x| \wedge |y|)\, \chi_{\mathbb{N}\setminus A}\right\|_e \\ &\leq \left\|\frac{\overline{x}+\overline{y}}{2}\right\|_e + \left\|(|x| \wedge |y|)\, \chi_{\mathbb{N}\setminus A}\right\|_e \\ &\leq 1 - \delta_I^+\left(\frac{3}{4}\varepsilon, z\right) + \eta(\varepsilon, z) \leq 1 - \frac{3}{4}\delta_I^+\left(\frac{3}{4}\varepsilon, z\right). \end{aligned}$$

(b) Let

$$\left\|(|x| \wedge |y|)\, \chi_{\mathbb{N}\setminus A}\right\|_e \geq \eta(\varepsilon, z).$$

Since

$$|x-y|\, \chi_{\mathbb{N}\setminus A} \geq 2\,(|x| \wedge |y|)\, \chi_{\mathbb{N}\setminus A},$$

we have

$$\left\|(x-y)\,\chi_{\mathbb{N}\setminus A}\right\|_e \geq 2\eta(\varepsilon,z).$$

Moreover,

$$\left\|(x-y)\,\chi_{\mathbb{N}\setminus A}\right\|_e \geq \left\|x\chi_{\mathbb{N}\setminus A}\right\|_e \vee \left\|y\chi_{\mathbb{N}\setminus A}\right\|_e \geq \frac{1}{4}\left(\delta_I^+\left(\frac{1}{2}\varepsilon,z\right)\wedge\varepsilon\right).$$

Define

$$D=\left\{i\in\mathbb{N}\setminus A:\frac{(|x|\wedge|y|)\,(i)}{|x-y|\,(i)}\geq\frac{1}{4}\eta(\varepsilon,z)\right\}$$

and $C=(\mathbb{N}\setminus A)\setminus D$. We have

$$\|(|x|\wedge|y|)\,\chi_C\|_e\leq\frac{1}{4}\eta(\varepsilon,z)\,\|(x-y)\,\chi_C\|_e\leq\frac{1}{2}\eta(\varepsilon,z).$$

Hence $\|(|x|\wedge|y|)\,\chi_D\|_e\geq\frac{1}{2}\eta(\varepsilon,z)$. Therefore

$$\left\|\frac{\varepsilon}{4}\eta(\varepsilon,z)z\chi_D\right\|_e=\left\|\frac{1}{4}\eta(\varepsilon,z)\,(x-y)\,\chi_D\right\|_e\geq\frac{1}{2}\eta(\varepsilon,z)\,\|(|x|\wedge|y|)\,\chi_D\|_e\geq\frac{1}{4}\eta^2(\varepsilon,z).$$

Moreover,

$$0\leq\frac{1}{4}\eta(\varepsilon,z)\,|x-y|\,\chi_D\leq\frac{|x|+|y|}{2}$$

and $\left\|\frac{|x|+|y|}{2}\right\|_e\leq 1$. Following the proof of Lemma 3.6, we conclude that if $e_+\in(UREI)$, then $e\in(UMEI)$. Hence we get

$$\begin{aligned}\left\|\frac{x+y}{2}\right\|_e &= \left\|\frac{|x|+|y|}{2}-(|x|\wedge|y|)\,\chi_{\mathbb{N}\setminus A}\right\|_e\leq\left\|\frac{|x|+|y|}{2}-(|x|\wedge|y|)\,\chi_D\right\|_e\\ &\leq \left\|\frac{|x|+|y|}{2}-\frac{1}{4}\eta(\varepsilon,z)\,|x-y|\,\chi_D\right\|_e\leq 1-\lambda(\varepsilon,z),\end{aligned}$$

where $\lambda(\varepsilon,z)=\delta\left(\frac{1}{4}\eta^2(\varepsilon,z),z\right)$ and δ is from Definition 3.5. Taking

$$\delta\,(\varepsilon,z)=\frac{3}{4}\delta_I^+\left(\frac{1}{2}\varepsilon,z\right)\wedge\frac{3}{4}\delta_I^+\left(\frac{3}{4}\varepsilon,z\right)\wedge\lambda(\varepsilon,z),$$

we get the thesis. □

Theorem 3.12. *Suppose that e is a symmetric Köthe sequence space. Then $e_\varphi\in(UREI)$ if and only if*

(a) $e\in(UMEI)$, $\varphi>0$, $\varphi\,(b_\varphi)\,\|e_1\|_e\geq 1$, $\varphi\in\Delta_2^e$ *and*

(b) *for any $\varepsilon>0$ and any $w\in e$ there is $\delta=\delta\,(\varepsilon,w)\in(0,1)$ such that for every $u,v\in B\,(e)_+$ with $\|u-v\|_e\geq\varepsilon$ and $|u-v|\leq|w|$ one has:*

$$\|u+v\|_e\leq 2\,(1-\delta\,(\varepsilon,w))\quad \text{or}\quad \left\|(u-v)\,\chi_{A_{\delta}(u,v)}\right\|_e\geq\delta\,(\varepsilon,w)\,,$$

where

$$A_\delta\,(u,v)=\left\{i\in\mathbb{N}:\varphi\left(\frac{x\,(i)+y\,(i)}{2}\right)\leq\frac{1-\delta\,(\varepsilon,w)}{2}\,[\varphi\,(x\,(i))+\varphi\,(y\,(i))]\right\}$$

and $x=\varphi_r^{-1}\circ u$, $y=\varphi_r^{-1}\circ v$.

To explain the idea of this theorem, note that the shape of $S(e_\varphi)$ depends on the shape of $S(e)$ and on the form of φ. It is also clear that if e is not $UREI$, then $S(e)$ contains,roughly speaking, some "almost flat areas" denoted by Flat($S(e)$). Then e_φ is $UREI$ iff e_φ is $UMEI$ (Condition (a)) and either e is $UREI$ or the function φ must be convex enough on the set $\varphi_r^{-1}(\mathrm{Flat}(S(e)))$ to improve (bring into relief) these "almost flat areas".

Proof. *Necessity.* If $e_\varphi \in (UREI)$, then, by Lemma 3.6, $e_\varphi \in (UMEI)$. Then conditions (a) hold by Proposition 3.10.

Suppose now that condition (b) is not satisfied. Then there exist a number $\varepsilon > 0$, $w \in e$ and sequences $(u_n)_{n=1}^\infty$, $(v_n)_{n=1}^\infty$ in $B(e)_+$ such that

$$\|u_n - v_n\|_e \geq \varepsilon, \qquad |u_n - v_n| \leq |w|, \qquad \|u_n + v_n\|_e > 2(1 - 1/n)$$

and

$$\|(u_n - v_n)\chi_{A_n}\|_e < 1/n$$

for every $n \in \mathbb{N}$, where

$$A_n = \left\{ i \in \mathbb{N} : \varphi\left(\frac{x_n(i) + y_n(i)}{2}\right) \leq \frac{1}{2}\left(1 - \frac{1}{n}\right)[\varphi(x_n(i)) + \varphi(y_n(i))] \right\}$$

and $x_n = \varphi_r^{-1} \circ u_n$, $y_n = \varphi_r^{-1} \circ v_n$. We will show that $e_\varphi \notin (UREI)$, i.e.

(i) $\left\|\frac{x_n + y_n}{2}\right\|_\varphi \to 1$;

(ii) $|x_n - y_n| \leq |z|$ for a certain $z \in e_\varphi \setminus \{0\}$;

(iii) $\|x_n - y_n\|_\varphi \not\to 0$.

Since $I_\varphi(f) \leq \|f\|_\varphi$ for each $f \in B(e_\varphi)$, to show (i) it is enough to prove that

$$\left\|\varphi \circ \left(\frac{x_n + y_n}{2}\right)\right\|_e \to 1.$$

We have $\|(u_n - v_n)\chi_{A_n}\|_e \to 0$. Hence $(u_n - v_n)\chi_{A_n} \to 0$ pointwisely. We claim that $(u_n - v_n)\chi_{A_n} \rightrightarrows 0$ (uniformly). If not, passing to a subsequence (if necessary), without loss of generality we can assume that there are $\gamma > 0$ and sequence $(i_n) \subset \mathbb{N}$ such that

$$|(u_n(i_n) - v_n(i_n))\chi_{A_n}(i_n)| > \gamma \tag{2}$$

for any $n \in \mathbb{N}$. The sequence (i_n) cannot contain any constant subsequence. Really, suppose that there is a subsequence (i_{n_k}) such that $i_{n_k} = i_0$ for any $k \in \mathbb{N}$. By pointwise convergence of $(u_n - v_n)\chi_{A_n}$, we have

$$\lim_{k\to\infty} \left|(u_{n_k}(i_{n_k}) - v_{n_k}(i_{n_k}))\chi_{A_{n_k}}(i_{n_k})\right| = \lim_{k\to\infty} \left|(u_{n_k}(i_0) - v_{n_k}(i_0))\chi_{A_{n_k}}(i_0)\right| = 0,$$

what contradicts the inequality (2). Therefore, without loss of generality, we can assume that $i_n \neq i_m$ for any $n \neq m$. Define $t_n = \gamma\chi_{\{i_n\}}$. Then $t_n \to 0$ pointwisely and, by (2), $t_n \leq |w|$. Hence, by the order continuity of e, $\|t_n\|_e \to 0$. On the other hand $\|t_n\|_e = c > 0$ for any $n \in \mathbb{N}$, because e is symmetric. This contradiction proves the claim. Since

$$\varphi \circ (x_n - y_n)\chi_{A_n} \leq |(\varphi \circ x_n - \varphi \circ y_n)\chi_{A_n}| = |(u_n - v_n)\chi_{A_n}|,$$

we conclude that $\varphi \circ (x_n - y_n)\, \chi_{A_n} \rightrightarrows 0$. Consequently,

$$(x_n - y_n)\, \chi_{A_n} \rightrightarrows 0, \tag{3}$$

because $\varphi > 0$. Since the symmetry of e implies the symmetry of e_φ, there exists $M > 0$ such that $|s\,(i)| \leq M$ for any $s \in B\,(e_\varphi)$ and any $i \in \mathbb{N}$. This and (3) together imply that for any $k \in \mathbb{N}$ there is $N(k)$ such that

$$\varphi \circ \left(\frac{x_n + y_n}{2}\right) \chi_{A_n} > \frac{1}{2}\left(1 - \frac{1}{k}\right) \left[\varphi \circ (x_n) + \varphi \circ (y_n)\right] \chi_{A_n}$$

whenever $n \geq N(k)$. Taking for any $k \in \mathbb{N}$ a positive integer n_k such that $n_k \geq N(k)$ and $n_k \geq k$, we have

$$\begin{aligned} \varphi \circ \left(\frac{x_{n_k} + y_{n_k}}{2}\right) &= \varphi \circ \left(\frac{x_{n_k} + y_{n_k}}{2}\right) \chi_{A_{n_k}} + \varphi \circ \left(\frac{x_{n_k} + y_{n_k}}{2}\right) \chi_{\mathbb{N} \setminus A_{n_k}} \\ &> \frac{1}{2}\left(1 - \frac{1}{k}\right) \left[\varphi \circ (x_{n_k}) + \varphi \circ (y_{n_k})\right] = \left(1 - \frac{1}{k}\right) \frac{u_{n_k} + v_{n_k}}{2}. \end{aligned}$$

Hence, by convexity of φ and our assumptions, we get

$$\begin{aligned} 1 &\geq \frac{\|u_{n_k}\|_e + \|v_{n_k}\|_e}{2} \geq \left\|\frac{u_{n_k} + v_{n_k}}{2}\right\|_e = \left\|\frac{\varphi \circ x_{n_k} + \varphi \circ y_{n_k}}{2}\right\|_e \\ &\geq \left\|\varphi \circ \left(\frac{x_{n_k} + y_{n_k}}{2}\right)\right\|_e > \left(1 - \frac{1}{k}\right) \left\|\frac{u_{n_k} + v_{n_k}}{2}\right\|_e > \left(1 - \frac{1}{k}\right)\left(1 - \frac{1}{n_k}\right). \end{aligned}$$

Consequently,

$$\left\|\varphi \circ \left(\frac{x_{n_k} + y_{n_k}}{2}\right)\right\|_e \to 1$$

as $k \to \infty$. The double extract convergence theorem finishes the proof of (i).

To show (ii) consider two cases. If $w \in B\,(e)$, notice

$$\varphi \circ (x_n - y_n) \leq |\varphi \circ x_n - \varphi \circ y_n| = |u_n - v_n| \leq |w|\,,$$

which implies that

$$|x_n - y_n| \leq \varphi_r^{-1}\,(|w|)\,,$$

as required and element $\varphi_r^{-1}\,(|w|)$ is well defined (see Proposition 2.1(ii) from [25]). If $\|w\|_e > 1$, then

$$\varphi \circ \left(\frac{x_n - y_n}{\|w\|_e}\right) \leq \frac{1}{\|w\|_e}\,|u_n - v_n| \leq \frac{|w|}{\|w\|_e}$$

and consequently

$$|x_n - y_n| \leq \|w\|_e\, \varphi_r^{-1}\left(\frac{|w|}{\|w\|_e}\right),$$

so (ii) holds true.

It remains to prove (iii). By the order continuity of e there exists a finite subset $\mathbb{N}_1 = \{i_1, i_2, ..., i_k\}$ of positive integers such that $\left\|w\chi_{\mathbb{N}\setminus\mathbb{N}_1}\right\|_e < \varepsilon/2$. Hence

$$\|(u_n - v_n)\, \chi_{\mathbb{N}_1}\|_e \geq \|u_n - v_n\|_e - \left\|(u_n - v_n)\, \chi_{\mathbb{N}\setminus\mathbb{N}_1}\right\|_e \geq \varepsilon - \left\|w\chi_{\mathbb{N}\setminus\mathbb{N}_1}\right\|_e > \frac{\varepsilon}{2}$$

for any $n \in \mathbb{N}$. By the triangle inequality, it follows that

$$\sum_{l=1}^{k} \left\| \left(u_n \left(i_l \right) - v_n \left(i_l \right) \right) e_{i_l} \right\|_e = \sum_{l=1}^{k} \left| u_n \left(i_l \right) - v_n \left(i_l \right) \right| \left\| e_1 \right\|_e > \frac{\varepsilon}{2}$$

for every $n \in \mathbb{N}$. Consequently, a positive integer $j_n \in \mathbb{N}_1$ can be find such that

$$\left| u_n \left(j_n \right) - v_n \left(j_n \right) \right| > \frac{\varepsilon}{2k \left\| e_1 \right\|_e}$$

for any $n \in \mathbb{N}$. Moreover, the fact $u_n, v_n \in B\left(e\right)_+$ implies that $\max\left\{ u_n\left(j_n\right), v_n\left(j_n\right)\right\} \leq 1/\left\|e_1\right\|_e$ for any $n \in \mathbb{N}$. By the concavity of φ_r^{-1}, it follows that

$$\begin{aligned} \left| x_n \left(j_n \right) - y_n \left(j_n \right) \right| &= \left| \varphi_r^{-1} \left(u_n \left(j_n \right) \right) - \varphi_r^{-1} \left(v_n \left(j_n \right) \right) \right| \\ &> \varphi_r^{-1} \left(\frac{1}{\left\| e_1 \right\|_e} \right) - \varphi_r^{-1} \left(\frac{1}{\left\| e_1 \right\|_e} \left(\frac{2k - \varepsilon}{2k} \right) \right) = \gamma \end{aligned}$$

for any $n \in \mathbb{N}$. Therefore

$$\left\| x_n - y_n \right\|_\varphi \geq \left\| \left(x_n - y_n \right) \chi_{\{j_n\}} \right\|_\varphi \geq \left\| \gamma e_1 \right\|_\varphi = \gamma \left\| e_1 \right\|_\varphi > 0$$

for any $n \in \mathbb{N}$, i.e. (iii). This completes the proof of necessity.

Sufficiency. Suppose that the assumptions (a) and (b) are satisfied. Fix $\varepsilon > 0$ and $z \in e_\varphi$. By Theorem 3.11, it is enough to show that $\left(e_\varphi\right)_+$ is $UREI$. Let $x, y \in B\left(e_\varphi\right)_+$ be such that $|x - y| \leq |z|$ and $\|x - y\|_\varphi \geq \varepsilon$. Define $u = \varphi \circ x$ and $v = \varphi \circ y$. Then $u, v \in B\left(e\right)_+$ and applying superadditivity φ on $\mathbb{R}_+$, we get

$$\|u - v\|_e = \|\varphi \circ x - \varphi \circ y\|_e \geq \|\varphi \circ (x - y)\|_e \geq \eta\left(\varepsilon\right),$$

because $\|x - y\|_\varphi \geq \varepsilon$, $\varphi > 0$ and $\varphi \in \Delta_2^e$ (see Lemma 1.3 in [25]). Moreover, we will prove that there exists $K > 0$ such that

$$\left|\varphi\left(x\left(i\right)\right) - \varphi\left(y\left(i\right)\right)\right| \leq K \left|x\left(i\right) - y\left(i\right)\right| \tag{4}$$

for any $i \in \mathbb{N}$. To this end we consider two cases.

I. Suppose that either $\varphi\left(b_\varphi\right) = \infty$ or $\varphi\left(b_\varphi\right) < \infty$ and $\varphi\left(b_\varphi\right)\left\|e_1\right\|_e > 1$. Take $M > 0$ with $\varphi\left(M\right)\left\|e_1\right\|_e = 1$. Then $M < b_\varphi$ and the left hand side derivative $\varphi_-'\left(M\right) < \infty$. Moreover, setting $K = \varphi_-'\left(M\right)$, we have $\left|\varphi\left(a\right) - \varphi\left(b\right)\right| \leq K\left|a - b\right|$ for any $a, b \in \left[0, M\right]$. Finally, note that $\left|s\left(i\right)\right| \leq M$ for any $s \in B\left(e_\varphi\right)$ and $i \in \mathbb{N}$.

II. Assume that $\varphi\left(b_\varphi\right)\left\|e_1\right\|_e = 1$. Note that by our assumptions $e_\varphi \in \left(UMEI\right)$ (see Proposition 3.10). We apply Proposition 3.9(d) with $\eta_1 = \eta_{e_\varphi}\left(3\varepsilon/8, |z|\right)$. By the symmetry of e, the space e_φ is also symmetric. Take $\sigma_1 \in \left(0, \frac{\varepsilon}{4\|e_1\|_\varphi}\right)$ such that $\varphi\left(b_\varphi - \sigma_1\right)\left\|e_1\right\|_e \geq 1 - \eta_1/2$.

We claim that $x\left(i\right) \wedge y\left(i\right) \leq b_\varphi - \sigma_1$ for each $i \in \mathbb{N}$. Indeed, otherwise, putting $i_0 \in \mathbb{N}$ for which $x\left(i_0\right) \wedge y\left(i_0\right) > b_\varphi - \sigma_1$ we get

$$\left\| \left(x - y\right) \chi_{\{i_0\}} \right\|_\varphi = \left|x\left(i_0\right) - y\left(i_0\right)\right| \left\|e_1\right\|_\varphi \leq \varepsilon/4.$$

Then $\left\|(x-y)\,\chi_{\{i\neq i_0\}}\right\|_{\varphi}\geq 3\varepsilon/4$. Define

$$A=\{i\neq i_0 : x(i)\geq y(i)\}\quad\text{and}\quad B=\{i\neq i_0 : x(i)<y(i)\}.$$

We may assume that $\|(x-y)\,\chi_A\|_{\varphi}\geq 3\varepsilon/8$, because in the opposite case the proof is analogous. Since $(x-y)\,\chi_A\leq x$ and $|x-y|\leq|z|$, by Proposition 3.9(d), we conclude that $\left\|x\chi_{\mathbb{N}\setminus A}\right\|_{\varphi}\leq 1-\eta_1$. Then

$$1-\eta_1/2\leq\varphi\left(b_{\varphi}-\sigma_1\right)\|e_1\|_e<I_{\varphi}\left(x\chi_{\{i_0\}}\right)\leq\left\|x\chi_{\mathbb{N}\setminus A}\right\|_{\varphi}\leq 1-\eta_1.$$

This contradiction proves the claim. Setting $K=\frac{\varphi(b_{\varphi})-\varphi(b_{\varphi}-\sigma_1)}{\sigma_1}$ we obtain (4). Hence, by (4), $|u-v|\leq K\,|z|$. Applying the assumption (b), we get $\|u+v\|_e\leq 2\,(1-\delta_1)$ or $\left\|(u-v)\,\chi_{A_{\delta_1}(u,v)}\right\|_e\geq\delta_1$, where $\delta_1=\delta\left(\eta\left(\varepsilon\right),Kz\right)$. If the first component of this disjunction is true, then

$$I_{\varphi}\left(\frac{x+y}{2}\right)=\left\|\varphi\circ\left(\frac{x+y}{2}\right)\right\|_e\leq\frac{1}{2}\left\|\varphi\circ x+\varphi\circ y\right\|_e\leq 1-\delta_1$$

whence, by Lemma 1.2 and 1.4 from [25], there is $\beta_1=\beta_1\left(\delta_1\right)>0$ such that

$$\left\|(x+y)/2\right\|_{\varphi}\leq 1-\beta_1.$$

If not, then the second one must be true. Therefore, denoting $A_{\delta_1}(u,v)=A_1$, we get

$$\begin{aligned}\varphi\circ\left(\frac{x+y}{2}\right)&\leq\frac{1}{2}\left(\varphi\circ x+\varphi\circ y\right)-\frac{\delta_1}{2}\left(\varphi\circ x+\varphi\circ y\right)\chi_{A_1}\\&\leq\frac{1}{2}\left(\varphi\circ x+\varphi\circ y\right)-\frac{\delta_1}{2}\left|\varphi\circ x-\varphi\circ y\right|\chi_{A_1}\\&=\frac{1}{2}\left(\varphi\circ x+\varphi\circ y\right)-\frac{\delta_1}{2}\left|u-v\right|\chi_{A_1}.\end{aligned}$$

Since $e\in(UMEI)$, we conclude that $\left\|\varphi\circ\left(\frac{x+y}{2}\right)\right\|_e\leq 1-\gamma$, where $\gamma=\gamma\left(\frac{\delta_1^2}{2},\frac{K|z|\delta_1}{2}\right)$. Repeating the same arguments as in the part concerning the first component, there is $\beta_2=\beta_2\left(\gamma\right)>0$ such that $\left\|(x+y)/2\right\|_{\varphi}\leq 1-\beta_2$. Thus $\left\|(x+y)/2\right\|_{\varphi}\leq 1-\beta$ with $\beta=\beta_1\wedge\beta_2$ and consequently $e_{\varphi}\in(UREI)$, as required. □

Corollary 3.13. *Let e be an order continuous symmetric Köthe sequence space. Then $e_{\varphi}\in(URED)$ if and only if (a) and (b) from Theorem 3.12 are satisfied.*

Since $e_{\varphi}\in(URED)$, e_{φ} is rotund. By Corollary 2.10 from [25], we get $\varphi>0$, $\varphi\left(b_{\varphi}\right)\|e_1\|_e\geq 1$, $\varphi\in\Delta_2^e$. This implies that $e_{\varphi}\in(OC)$ in view of Theorem 4.2 from [13]. Now applying Theorem 3.8, we get that implies that $e_{\varphi}\in(UREI)$. Therefore, by Theorem 3.12, the necessity is obvious. The sufficiency is an immediate consequence of Theorem 3.12.

4. Applications to Orlicz-Lorentz spaces

Now we consider Orlicz-Lorentz sequence spaces as a special class of Calderón-Lozanovskiĭ spaces. Recall that for any $x\in l^0$ the symbol d_x denotes the distribution function of x, that is $d_x(\gamma) = m(\{i \in \mathbb{N} : |x(i)| > \gamma\})$ for all $\gamma \geq 0$, where m is the counting measure. The decreasing rearrangement of x is denoted by x^* and is defined by

$$x^*(i) = \inf\{\gamma > 0 : d_x(\gamma) < i\}$$

for any $i \in \mathbb{N}$. A function $\omega : \mathbb{N} \to \mathbb{R}_+$ is said to be the *weight function*, if $\omega = (\omega(i))$ is a nonincreasing sequence. The *Lorentz sequence space* λ_ω is the set of all sequences $x = (x(i))$ such that $\|x\|_{\lambda_\omega} = \sum_{i=1}^{\infty} x^*(i)\omega(i) < \infty$. If $e = \lambda_\omega$, then e_φ is the Orlicz-Lorentz sequence space $(\lambda_\omega)_\varphi$ equipped with the Luxemburg norm (see [4], [17], [19] and [20]).

It is worth to formulate criteria for uniform rotundity in every interval and for uniform rotundity in every direction in $(\lambda_\omega)_\varphi$ because in this spaces they become more clear and more specified. We start with the following theorem characterizing Lorentz sequence spaces that are $UMEI$.

Theorem 4.1. *The following conditions are equivalent:*

(*i*) $\lambda_\omega \in (UMEI)$;
(*ii*) $\lambda_\omega \in (SM)$;
(*iii*) $\sum_{i=1}^{\infty} \omega(i) = \infty$.

Proof. The equivalence $(ii) \Leftrightarrow (iii)$ follows from Lemma 3.1 in [25] (see also [16]). The implication $(i) \Rightarrow (ii)$ is an immediate consequence of definitions. It remains to prove the implication $(ii) \Rightarrow (i)$ only. Take $\varepsilon \in (0,1)$ and $z \in (\lambda_\omega)_+ \setminus \{0\}$. Let $x \in S(\lambda_\omega)$ and y be such that $0 \leq y \leq x$, $y \leq z$ and $\|y\|_{\lambda_\omega} \geq \varepsilon$. By Lemma 3.2(*ii*) in [25], the Lorentz space $\lambda_\omega \in (OC)$. Hence there exists a finite subset $A \subset \mathbb{N}$ such that $\|z\chi_{\mathbb{N}\setminus A}\|_{\lambda_\omega} < \frac{\varepsilon}{2}$. Consequently, setting $y_1 = y\chi_A$ and $k = \operatorname{card} A$, we get

$$\sum_{i=1}^{k} y_1^*(i)\,\omega(i) = \|y_1\|_{\lambda_\omega} = \|y - y\chi_{\mathbb{N}\setminus A}\|_{\lambda_\omega} \geq \|y\|_{\lambda_\omega} - \|y\chi_{\mathbb{N}\setminus A}\|_{\lambda_\omega} \geq \varepsilon - \|z\chi_{\mathbb{N}\setminus A}\|_{\lambda_\omega} > \frac{\varepsilon}{2}.$$

Denoting by i_0 the smallest index in A for which $y_1(i_0) = \max_{i\in A}|y_1(i)|$, we get $y_1(i_0) = y_1^*(1)$ and $y_1^*(1)\,\omega(1) \geq \varepsilon/2k$. Hence $y_1(i_0) \geq \varepsilon/2k\omega(1)$. By (iii), we can take $N_1 \in \mathbb{N}$ such that

$$\frac{\varepsilon}{2k\omega(1)} \sum_{i=1}^{N_1-1} \omega(i) > 1.$$

We find the smallest index $i_1 \in \mathbb{N}$ such that $x(i_0) = x^*(i_1)$. It is easy to see that the number $i_1 \leq N_1$. Indeed, otherwise

$$1 = \|x\|_{\lambda_\omega} = \sum_{i=1}^{\infty} x^*(i)\,\omega(i) \geq \sum_{i=1}^{i_1} x^*(i)\,\omega(i) \geq y(i_0) \sum_{i=1}^{i_1} \omega(i) \geq \frac{\varepsilon}{2k\omega(1)} \sum_{i=1}^{N_1-1} \omega(i) > 1$$

and we get a contradiction. Recall that $x^*(i_1) = x(i_0) \geq y(i_0)$. Let $i_2 \in \mathbb{N}$ be the smallest index such that $i_2 > i_1$ and $x^*(i_1) - x^*(i_2) \geq y(i_0)/2$ (such an index i_2 exists since $\lambda_\omega \in (OC)$). Hence, for $i = i_1, i_1+1, ..., i_2-1$,

$$x^*(i_1) - x^*(i) < \frac{y(i_0)}{2}$$

and consequently

$$x^*(i) > x^*(i_1) - \frac{y(i_0)}{2} \tag{5}$$

for $i = i_1, i_1 + 1, ..., i_2 - 1$. By (iii), we can find an integer N_2 such that

$$\frac{\varepsilon}{4k\omega(1)} \sum_{i=N_1}^{N_2-1} \omega(i) > 1.$$

We claim that $i_2 \leq N_2$. If not, then

$$1 = \|x\|_{\lambda_\omega} = \sum_{i=1}^{\infty} x^*(i)\,\omega(i) \geq \sum_{i=i_1}^{i_2-1} x^*(i)\,\omega(i) > \frac{y(i_0)}{2} \sum_{i=i_1}^{i_2-1} \omega(i) \geq \frac{\varepsilon}{4k\omega(1)} \sum_{i=N_1}^{N_2-1} \omega(i) > 1.$$

This contradiction proves the claim. Therefore, applying among others (5), we get

$$\begin{aligned}
\|x-y\|_{\lambda_\omega} &\leq \|x - y(i_0)\, e_{i_0}\|_{\lambda_\omega} \leq \sum_{i=1}^{\infty} \left(x - \frac{y(i_0)}{2} e_{i_0} \right)^* (i)\,\omega(i) \\
&= \sum_{i=1}^{i_1-1} x^*(i)\,\omega(i) + \sum_{i=i_1}^{i_2-2} x^*(i+1)\,\omega(i) + \left(x^*(i_1) - \frac{y(i_0)}{2} \right) \omega(i_2-1) + \sum_{i=i_2}^{\infty} x^*(i)\,\omega(i) \\
&= \sum_{i=1}^{\infty} x^*(i)\,\omega(i) - \sum_{i=i_1}^{i_2-1} x^*(i)\,\omega(i) + \sum_{i=i_1}^{i_2-2} x^*(i+1)\,\omega(i) + \left(x^*(i_1) - \frac{y(i_0)}{2} \right) \omega(i_2-1) \\
&= 1 - \sum_{i=i_1}^{i_2-2} \left(x^*(i) - x^*(i+1) \right) \omega(i) - \left(x^*(i_2-1) - x^*(i_1) + \frac{y(i_0)}{2} \right) \omega(i_2-1) \\
&\leq 1 - \sum_{i=i_1}^{i_2-2} \left(x^*(i) - x^*(i+1) \right) \omega(N_2) - \left(x^*(i_2-1) - x^*(i_1) + \frac{y(i_0)}{2} \right) \omega(N_2) \\
&= 1 - \left(x^*(i_1) - x^*(i_2-1) \right) \omega(N_2) - \left(x^*(i_2-1) - x^*(i_1) + \frac{y(i_0)}{2} \right) \omega(N_2) \\
&= 1 - \frac{y(i_0)}{2} \omega(N_2) \leq 1 - \frac{\varepsilon\omega(N_2)}{4k\omega(1)}.
\end{aligned}$$

Thus the proof is finished with $\delta = \frac{\varepsilon\omega(N_2)}{4k\omega(1)}$ dependent only on ε and z. □

To prove the next theorem, we need the following lemma.

Lemma 4.2. *Let e be a Köthe sequence space. If $e \in (UMEI)$, then for every $\varepsilon > 0$ and $z \in e_+ \setminus \{0\}$ there is $\delta = \delta(\varepsilon, z) > 0$ such that for any sequences $(x_n), (y_n) \subset B(e)_+$ satisfying $\|x_n\|_e \to 1$, $\|y_n\|_e \to 1$ and $|x_n - y_n| \leq z$ for every $n \in \mathbb{N}$, the condition*

$$\|(x_n - y_n)\,\chi_{A_n}\|_e \geq \varepsilon \quad \textit{for any } n \in \mathbb{N}$$

implies

$$\left\|(x_n - y_n)\,\chi_{\mathbb{N}\setminus A_n}\right\|_e \geq \delta \quad \textit{for } n \textit{ sufficiently large,}$$

where $A_n = \{i \in \mathbb{N} : x_n(i) \geq y_n(i)\}$.

Proof. Define $w_n = (x_n - y_n)\chi_{A_n}$. Passing to a subsequence, if necessary, it can be assumed that $\|y_n\|_e \geq 1 - \frac{1}{n}$. Then $0 \leq w_n \leq z$, $w_n \leq x_n$ and $\|w_n\|_e \geq \varepsilon$. Since $e \in (UMEI)$, there is $\delta > 0$ such that

$$\left\|x_n\chi_{\mathbb{N}\setminus A_n} + y_n\chi_{A_n}\right\|_e = \|x_n - (x_n - y_n)\chi_{A_n}\|_e = \|x_n - w_n\|_e \leq (1-\delta)\|x_n\|_e \leq (1-\delta).$$

Hence

$$\begin{aligned}\left\|(x_n - y_n)\chi_{\mathbb{N}\setminus A_n}\right\|_e &= \left\|x_n\chi_{\mathbb{N}\setminus A_n} + y_n\chi_{A_n} - y_n\right\|_e \\ &\geq \|y_n\|_e - \left\|x_n\chi_{\mathbb{N}\setminus A_n} + y_n\chi_{A_n}\right\|_e \geq 1 - \frac{1}{n} - (1-\delta) \geq \frac{\delta}{2}\end{aligned}$$

for $n > \frac{2}{\delta}$. □

Theorem 4.3. *The following conditions are equivalent:*

(i) $(\lambda_\omega)_\varphi \in (UREI)$;
(ii) $(\lambda_\omega)_\varphi \in (URED)$;
(iii) $(\lambda_\omega)_\varphi \in (R)$;
(iv) $\sum_{i=1}^{\infty}\omega(i) = \infty$, $\varphi \in \Delta_2(0)$, $\varphi(b_\varphi)\omega(1) \geq 1$ *and* φ *is strictly convex on the interval* $\left[0, \varphi_r^{-1}\left(\frac{1}{\omega(1)+\omega(2)}\right)\right]$.

Proof. Clearly, $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ (see [4] and [25]). It remains to prove that $(iv) \Rightarrow (i)$. By Theorem 4.1, the assumption $\sum_{i=1}^{\infty}\omega(i) = \infty$ is equivalent to the fact that $\lambda_\omega \in (UMEI)$. In particular, $\lambda_\omega \in (OC)$. Since the condition (a) of Theorem 3.12 is satisfied in an obvious manner, it is enough to prove that (iv) implies the condition (b) of Theorem 3.12 with $e = \lambda_\omega$. Suppose now that condition (b) is not satisfied. Then there exist a number $\varepsilon > 0$, $w \in \lambda_\omega \setminus \{0\}$ and sequences $(u_n)_{n=1}^{\infty}$, $(v_n)_{n=1}^{\infty}$ in $B(\lambda_\omega)_+ \setminus \{0\}$ such that

$$\|u_n - v_n\|_{\lambda_\omega} \geq \varepsilon, \qquad |u_n - v_n| \leq |w|, \qquad \|u_n + v_n\|_{\lambda_\omega} > 2(1 - 1/n)$$

and

$$\|(u_n - v_n)\chi_{A_n}\|_{\lambda_\omega} < 1/n$$

for every $n \in \mathbb{N}$, where

$$A_n = \left\{i \in \mathbb{N} : \varphi\left(\frac{x_n(i) + y_n(i)}{2}\right) \leq \frac{1}{2}\left(1 - \frac{1}{n}\right)\left[\varphi(x_n(i)) + \varphi(y_n(i))\right]\right\}$$

and $x_n = \varphi_r^{-1} \circ u_n$, $y_n = \varphi_r^{-1} \circ v_n$. By the claim (iii) in the proof of necessity of Theorem 3.12, it implies that

$$\|x_n - y_n\|_\varphi \nrightarrow 0. \tag{6}$$

Moreover,

$$\|(\varphi \circ x_n - \varphi \circ y_n)\chi_{A_n}\|_{\lambda_\omega} \to 0. \tag{7}$$

Hence the inequality

$$\|\varphi \circ (x_n - y_n)\chi_{A_n}\|_{\lambda_\omega} \leq \|(\varphi \circ x_n - \varphi \circ y_n)\chi_{A_n}\|_{\lambda_\omega}$$

and assumption $\varphi \in \Delta_2(0)$ yield that $\|(x_n - y_n)\chi_{A_n}\|_\varphi \to 0$. By (6), it follows that

$$\left\|(x_n - y_n)\chi_{\mathbb{N}\setminus A_n}\right\|_\varphi \nrightarrow 0. \tag{8}$$

By (8), without loss of generality, passing to a subsequence if necessary, we can assume that there is $\eta_0 > 0$ such that

$$\left\|(x_n - y_n)\,\chi_{\mathbb{N}\setminus A_n}\right\|_\varphi > \eta_0$$

for any $n \in \mathbb{N}$. By the claim (ii) in the proof of necessity of Theorem 3.12, there is $z \in (\lambda_\omega)_\varphi$ such that $|x_n - y_n| \leq |z|$. By the order continuity of $(\lambda_\omega)_\varphi$ there exists a finite subset $\mathbb{N}_1 \subset \mathbb{N}$ such that $\left\|z\chi_{\mathbb{N}\setminus\mathbb{N}_1}\right\|_\varphi < \frac{\eta_0}{2}$. Hence

$$\begin{aligned}\left\|(x_n - y_n)\,\chi_{\mathbb{N}_1\setminus A_n}\right\|_\varphi &= \left\|(x_n - y_n)\,\chi_{(\mathbb{N}\setminus A_n)\cap\mathbb{N}_1}\right\|_\varphi \\ &\geq \left\|(x_n - y_n)\,\chi_{\mathbb{N}\setminus A_n}\right\|_\varphi - \left\|(x_n - y_n)\,\chi_{(\mathbb{N}\setminus A_n)\setminus\mathbb{N}_1}\right\|_\varphi > \eta_0 - \frac{\eta_0}{2} = \frac{\eta_0}{2}\end{aligned}$$

for any $n \in \mathbb{N}$. Let

$$B_n = \{i \in \mathbb{N}_1 \setminus A_n : x_n(i) \geq y_n(i)\} \quad \text{and} \quad C_n = \{i \in \mathbb{N}_1 \setminus A_n : x_n(i) < y_n(i)\}$$

for every $n \in \mathbb{N}$. Then, by the triangle inequality, at least one of numbers $\|(x_n - y_n)\,\chi_{B_n}\|_\varphi$ or $\|(x_n - y_n)\,\chi_{C_n}\|_\varphi$ is not smaller than $\eta_0/4$ for infinitely many $n \in \mathbb{N}$. By the "symmetry", passing to a subsequence, we can assume that

$$\|(x_n - y_n)\,\chi_{B_n}\|_\varphi > \frac{\eta_0}{4} \tag{9}$$

for any $n \in \mathbb{N}$. Define $n_1 = \mathrm{card}(\mathbb{N}_1)$. Since $\varphi \in \Delta_2(0)$, we conclude that there is $\eta_1 = \eta_1(\eta_0/4)$ such that

$$I_\varphi((x_n - y_n)\,\chi_{B_n}) \geq \eta_1$$

for every $n \in \mathbb{N}$. For any $n \in \mathbb{N}$ take an index $i_n \in B_n$ for which

$$x_n(i_n) - y_n(i_n) = \max\{(x_n(i) - y_n(i)) : i \in B_n\}.$$

Then

$$\varphi((x_n(i_n) - y_n(i_n))) \geq \frac{\eta_1}{n_1\omega(1)}$$

for every $n \in \mathbb{N}$. Since $i_n \in B_n \subset \mathbb{N}_1$ and $\mathrm{card}(\mathbb{N}_1) < \infty$, there exist $i_0 \in \mathbb{N}_1$ and a subsequence (n_k) such that $i_{n_k} = i_0$ for any $k \in \mathbb{N}$. Passing to this subsequence we can assume that there is a positive integer i_0 such that $i_0 \in B_n$ and

$$x_n(i_0) - y_n(i_0) \geq \varphi_r^{-1}\left(\frac{\eta_1}{n_1\omega(1)}\right) > 0$$

for any $n \in \mathbb{N}$. Define $w_n = x_n(i_0)$, $t_n = y_n(i_0)$. Without loss of generality, again passing to a subsequence, if necessary, we can assume that $w_n \to w_0$ and $t_n \to t_0$. We have $w_0 > t_0 \geq 0$. Set $\alpha = \varphi_r^{-1}\left(\frac{1}{\omega(1)+\omega(2)}\right)$. Consider the following cases.

a) $t_0 < \alpha$. Then, by the definition of the set $\mathbb{N} \setminus A_n$, we have

$$\frac{1}{2}[\varphi(w_n) + \varphi(t_n)] \geq \varphi\left(\frac{w_n + t_n}{2}\right) > \frac{1 - \frac{1}{n}}{2}[\varphi(w_n) + \varphi(t_n)].$$

Passing with n to ∞, we obtain

$$\varphi\left(\frac{w_0+t_0}{2}\right)=\frac{1}{2}\left[\varphi\left(w_0\right)+\varphi\left(t_0\right)\right].$$

Consequently, φ is affine on the nondegenerated interval $(t_0,w_0)\cap(t_0,\alpha)$.

b) $t_0\geq\alpha$. Then $w_0>\alpha$. Notice that $\|\varphi\circ x_n\|_{\lambda_\omega}\leq 1$ implies that for any $i\neq i_0$

$$x_n\left(i\right)\leq\alpha \tag{10}$$

for n sufficiently large. Really, if we found $i_1\neq i_0$ and a subsequence (n_l) such that $x_{n_l}\left(i_1\right)>\alpha$ for any $l\in\mathbb{N}$, then, by the fact that $x_{n_l}\left(i_0\right)\to w_0$, we have

$$\begin{aligned}\|\varphi\circ x_{n_l}\|_{\lambda_\omega} &= \sum_{i=1}^{\infty}\varphi\left(x_{n_l}\left(i\right)\right)^*\omega(i)\geq\varphi\left(x_{n_l}\left(i_1\right)\right)\omega\left(1\right)+\varphi\left(x_{n_l}\left(i_0\right)\right)\omega\left(2\right)\\ &> \varphi\left(\alpha\right)\omega\left(1\right)+\varphi\left(\alpha\right)\omega\left(2\right)=1\end{aligned}$$

for l large enough. A contradiction.

Denote

$$D_n=\left\{i\in\mathbb{N}:y_n\left(i\right)>x_n\left(i\right)\right\}.$$

Since $B_n\subset\mathbb{N}\setminus D_n$, by (9),

$$\left\|\left(x_n-y_n\right)\chi_{\mathbb{N}\setminus D_n}\right\|_\varphi\geq\left\|\left(x_n-y_n\right)\chi_{B_n}\right\|_\varphi\geq\frac{\eta_0}{4}$$

for every $n\in\mathbb{N}$. Further, by Theorem 4.1, $\lambda_\omega\in(UMEI)$ and consequently, by Proposition 3.10, $\left(\lambda_\omega\right)_\varphi\in(UMEI)$. Hence, applying Lemma 4.2, we conclude that there is $\delta_0>0$ such that

$$\left\|\left(x_n-y_n\right)\chi_{D_n}\right\|_\varphi\geq\delta_0$$

for every $n\in\mathbb{N}$. Analogously as above, a finite subset $\mathbb{N}_2$ of integers can be found such that

$$\left\|\left(x_n-y_n\right)\chi_{E_n}\right\|_\varphi\geq\frac{\delta_0}{2}$$

for sufficiently large $n\in\mathbb{N}$, where $E_n:=D_n\cap\left(\mathbb{N}\setminus A_n\right)\cap\mathbb{N}_2$. Consequently

$$I_\varphi\left(\left(x_n-y_n\right)\chi_{E_n}\right)\geq\delta_1$$

for some δ_1 depending only on δ_0. Hence $j_0\in E_n$ can be found such that

$$y_n\left(j_0\right)-x_n\left(j_0\right)\geq\varphi_r^{-1}\left(\frac{\delta_1}{n_2\omega\left(1\right)}\right)>0,$$

for any $n\in\mathbb{N}$, where $n_2=$card$\left(\mathbb{N}_2\right)$. Hence, defining $p_n=x_n\left(j_0\right)$, $r_n=y_n\left(j_0\right)$, without loss of generality, we can assume that $p_n\to p_0$ and $r_n\to r_0$. Obviously $j_0\neq i_0$. It follows, by (10), that $r_n\leq\alpha$ for n sufficiently large and consequently $p_0<r_0\leq\alpha$. Now, repeating this same arguments as in the case a) we conclude that φ is affine on (p_0,r_0). The obtained contradiction shows that (iv) implies the condition (b) from Theorem 3.12. Therefore $\left(\lambda_\omega\right)_\varphi\in(UREI)$. □

5. Remarks

The following corollary summarizes the implications that exist between the properties considered in this paper.

Corollary 5.1. *Let E be a Köthe space. Then*

$$E \in (UM) \stackrel{(1)}{\Longrightarrow} E \in (UMEI) \stackrel{(2)}{\Longrightarrow} E \in (SM)\,,$$

$$E \in (UR) \stackrel{(3)}{\Longrightarrow} E \in (UREI) \stackrel{(4)}{\Longrightarrow} E \in (URED)\,.$$

Moreover, all of these implications cannot be reversed in general.

The implications (1), (2), (3) and (4) are obvious. To show that the implication (1) cannot be reversed, it is enough to take $E = \lambda_\omega$ with ω being not regular and satisfying $\sum_{i=1}^{\infty} \omega(i) = +\infty$ (see Example 1 from [14]). Recall that ω is *regular* if there is a number $K > 1$ such that $S(2n) \geq KS(n)$ for any $n \in \mathbb{N}$, where $S(n) = \sum_{i=1}^{n} \omega(i)$. Then $\lambda_\omega \in (UMEI)$ by Theorem 4.1. On the other hand $\lambda_\omega \notin (UM)$ (see [16]). Now, let $E = L_\varphi$ be an Orlicz space with the Amemiya-Orlicz norm generated by an Orlicz function φ that does not satisfy the suitable Δ_2-condition and vanishes only at zero. Then, by Theorem 5 from [9], $E \in (SM)$ and $E \notin (OC)$. Hence, by Lemma 3.7, $E \notin (UMEI)$. Consequently, the converse of the implication (2) is not true. Further, let $E = (\lambda_\omega)_\varphi$ be such that φ and ω satisfy conditions in (iv) of Theorem 4.3, but φ is not uniformly convex on the interval $\left[0, \varphi_r^{-1}\left(\frac{1}{\omega(1)+\omega(2)}\right)\right]$. Then $E \in (UREI)$ and $E \notin (UR)$ (see [4]), so the implication (3) cannot be reversed. Finally, take $E = l^\infty$ equipped with the norm introduced by M. A. Smith in [33] and defined for any $x = (x_i) \in l^\infty$ by the formula

$$\|x\|_S = \left[\|x\|_\infty^2 + \sum_{j=2}^{\infty} a_j^2 \left(|x_1| + |x_j|\right)^2\right]^{1/2},$$

where (a_j) is a sequence of positive real numbers such that $\sum_{j=2}^{\infty} a_j^2 = 1$ and $\|\cdot\|_\infty$ is the natural norm on l^∞. M. A. Smith proved in [33] that $(l^\infty, \|x\|_S) \in (URED)$ and that the norms $\|\cdot\|_S$ and $\|\cdot\|_\infty$ are equivalent. Hence $(l^\infty, \|x\|_S) \notin (OC)$. It follows, by Theorem 3.8 that $(l^\infty, \|x\|_S) \notin (UREI)$. Consequently, the converse implication to (4) does not hold.

Remark. The property $UREI$ is related to the property $URWC$ (uniform rotundity in weakly compact sets of directions) considered by M. A. Smith in [32]. Namely, for any Köthe space E we have

$$E \in (OC) \quad \text{and} \quad E \in (URWC) \Rightarrow E \in (UREI)\,.$$

It is an immediate consequence of Theorem 3.8 and the fact that $URWC$ implies $URED$. The converse implication is not true. It is enough to take $E = l^2$ with the norm $\|\cdot\|_A$ defined by M. A. Smith in [32]. The norm $\|\cdot\|_A$ is equivalent to the natural norm $\|\cdot\|_2$. Hence $(l^2, \|\cdot\|_A) \in (OC)$. Moreover, M. A. Smith proved in [33] that $(l^2, \|\cdot\|_A) \in (URED)$. By Theorem 3.8, it implies that $(l^2, \|\cdot\|_A) \in (UREI)$. On the other hand, $(l^2, \|\cdot\|_A) \notin (URWC)$. Consequently, properties $UREI$ and $URWC$ are distinct.

Open problem. Find full criteria for $URED$ in the space E_φ with E being Köthe function space.

Some partial results in this direction have been obtained in [31]. To solve this problem we cannot use the similar way as in Theorem 3.8 because each order interval in order continuous Köthe function space is only weakly compact. Then the main difficulty is the fact that a weakly convergent sequence need not converge in measure.

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