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Two Semiconic Duality Theorems

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This paper continues the study of semiconic duality of sets and functions that was started in the previous papers by the author. We obtain formulas for negative polars of intersections of n closed convex semiconic sets and for dual functions of sums of n convex lower semicontinuous semihomogeneous functions. A particular attention will be paid to the case of polyhedral sets and functions.

1. Introduction

This paper is concerned with the study of semiconic duality of sets and functions that was started earlier in [2, 3, 4]. The semiconic duality has applications in infinite linear programming and in geometric theory of convex sets [3, 4]. Also, it can be of use in some aspects of mathematical economics.

Let us recall some definitions and facts from [2, 3][1] that will be used throughout the paper. A set C in a linear space that coincides with its *semiconic hull*, $\operatorname{scon} C := \{\lambda x : x \in C, \lambda \geq 1\}$, is called *semiconic*. Clearly, for a convex semiconic set C one has $C + C \subseteq C$. (Moreover, a convex set C is semiconic if and only if $C + C \subseteq C$ and $\lambda C \subseteq C$ for every $\lambda > 1$.) A set C in a locally convex space is called *strictly semiconic* if it is semiconic and its closed convex hull does not contain the origin: $0 \notin \operatorname{cl}\operatorname{co} C$. In what follows, E is a locally convex space, and E^* is its topological dual space endowed with the weak-star topology. The value that a functional $x' \in E^*$ takes at an element $x \in E$ is denoted as $\langle x, x' \rangle$. Given a nonempty subset C in E, by its *negative polar* we mean the set

$$C^{\#} := \{x' \in E^* : \langle x, x' \rangle \leq -1 \quad \forall x \in C\}.$$

The set $C^{\#}$ is nonempty if and only if $0 \notin \operatorname{cl}\operatorname{co} C$, in what case

$$\operatorname{scon}\operatorname{cl}\operatorname{co} C = \operatorname{cl}\operatorname{scon}\operatorname{co} C = \operatorname{cl}\operatorname{co}\operatorname{scon} C$$

(see [3, Theorem 2.1]), $C^{\#}$ is a closed convex strictly semiconic subset in E^*, and $C^{\#} = (\operatorname{scon}\operatorname{cl}\operatorname{co} C)^{\#}$. Moreover, the *negative bipolar theorem* (see [2, Corollary 1], [3, Corollary 3.1]) holds true as follows.

Theorem 1.1. *Given a nonempty subset $C \subset E$ such that $0 \notin \operatorname{cl}\operatorname{co} C$, the following assertions are equivalent:*

(a) *C is closed, convex and semiconic (hence strictly semiconic);*

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[1]Unfortunately, the quality of the English translation of [3] is too low.

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(b) *C coincides with its negative bipolar*

$$C^{\#\#} := (C^{\#})^{\#} = \{x \in E : \langle x, x' \rangle \le -1 \quad \forall x' \in C^{\#}\}.$$

The semiconic duality of sets generates some duality of strictly semihomogeneous convex functions. A nonnegative function $f : E \to \mathbb{R} \cup \{+\infty\}$ is called *semihomogeneous* (resp. *strictly semihomogeneous*) if its epigraph $\operatorname{epi} f = \{(x, \alpha) : f(x) \le \alpha\}$ is a semiconic (resp. strictly semiconic) subset in $E \times \mathbb{R}$. We say that $f : E \to \mathbb{R}_+ \cup \{+\infty\}$ is a *majorant function* if $f \not\equiv +\infty$ and there is a strictly semihomogeneous function $h : E \to \mathbb{R}_+ \cup \{+\infty\}$ such that $h(x) \le f(x)$ for all $x \in E$.

We consider the duality between $E \times \mathbb{R}$ and $E^* \times \mathbb{R}$ determined by the bilinear form

$$\langle (x, \alpha), (x', \beta) \rangle_1 := \langle x, x' \rangle - \alpha\beta, \quad (x, \alpha) \in E \times \mathbb{R},\ (x', \beta) \in E^* \times \mathbb{R},$$

and we will use this duality when dealing with negative polars of subsets C in $E \times \mathbb{R}$. The next result is proved in [3, Theorem 6.2].

Theorem 1.2. *Suppose $f : E \to \mathbb{R} \cup \{+\infty\}$ is not identically equal to ∞, the following statements are then equivalent:*

(a) *the function f is majorant;*
(b) *the negative polar of the epigraph of f is the epigraph of some function $Df : E^* \to \mathbb{R} \cup \{+\infty\}$, which is not identically equal to ∞.*

When these equivalent assertions hold true, the function Df is convex, lower semicontinuous and strictly semihomogeneous, and it is given by

$$Df(x') = \inf\{\beta > 0 : \langle x, x' \rangle + 1 \le \beta f(x)\ \forall x \in E\} = \sup_{x \in E} \frac{\langle x, x' \rangle + 1}{f(x)}$$

(here, we assume by definition that $0/0 = 0$).

The next result is a direct consequence of Theorem 1.2; see [3, Corollary 6.1].

Corollary 1.3. *Suppose a function $f : E \to \mathbb{R} \cup \{+\infty\}$ is convex lower semicontinuous, $f \not\equiv +\infty$, and its epigraph is a strictly semiconic set. Then $(\operatorname{epi} f)^{\#}$ is the epigraph of some function $E^* \to \mathbb{R} \cup \{+\infty\}$ if and only if f is nonnegative.*

Given a majorant function $f : E \to \mathbb{R} \cup \{+\infty\}$, the function Df defined in Theorem 1.2 will be referred to as *dual* to f. In such a case, the *second dual function* $D^2 f := D(Df) : E \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$D^2 f(x) = \sup_{x' \in E^*} \frac{\langle x, x' \rangle + 1}{Df(x')},$$

and the equalities hold

$$D^2 f(x) = \inf_{\lambda \ge 1} \lambda f^{**}\left(\frac{x}{\lambda}\right) = \sup_{h \in H(f)} h^{**}(x) \quad \forall x \in E,$$

where f^{**} is the second Legendre-Fenchel conjugate function[2] of f, and $H(f)$ is the set of all strictly semihomogeneous functions $h : E \to \mathbb{R} \cup \{+\infty\}$ satisfying $h(x) \leq f(x)$ for all $x \in E$; see Theorem 6.4 and the proof of Theorem 6.1 in [3]. It follows that $D^2(D^2 f) = D^2 f \leq f$.

The next result (see [3, Corollary 6.4]) may be considered as a *negative bipolar theorem for functions.*

Theorem 1.4. *Given a majorant function $f : E \to \mathbb{R} \cup \{+\infty\}$, the following assertions are equivalent:*

(a) *f is convex, lower semicontinuous, and strictly semihomogeneous;*
(b) $f = D^2 f$.

In [3, 4][3], basing on the negative bipolar theorems, duality formulas for negative polars and dual functions for particular sets and functions were given and duality of various operations that do not lead out of the classes of closed convex strictly semiconic sets and lower semicontinuous convex strictly semihomogeneous functions was examined.

In this paper we prove two representation theorems giving formulas for negative polars $(C_1 \cap \cdots \cap C_n)^{\#}$ and dual functions $D(f_1 + \cdots + f_n)$. The statements of the theorems are in Section 2, and their proofs are in Sections 3 and 4. Section 5 is devoted to the case of polyhedral sets and functions.

2. Main results

Lemma 2.1. *Suppose C is a closed convex semiconic set which is not strictly semiconic, then it is necessarily a cone.*

Proof. Since C is closed convex and is not strictly semiconic, $0 \in \operatorname{cl} \operatorname{co} C = C$. We have to show that for every $x \in C$ and $\lambda > 0$ one has $\lambda x \in C$. Indeed, when $\lambda \geq 1$, this holds because C is semiconic, and when $0 < \lambda < 1$, this holds because 0 and x are in C and C is convex. □

Theorem 2.2. *1. Suppose $C_1, \ldots, C_n$ are closed convex semiconic sets in E, at least one of them is strictly semiconic*[4]*, and their intersection is nonempty; then*

$$(C_1 \cap \cdots \cap C_n)^{\#} = \operatorname{cl} \bigcup_{(t_1,\ldots,t_n)\in\Lambda} (M(t_1, C_1) + \cdots + M(t_n, C_n)), \tag{1}$$

where $\Lambda = \{(t_1, \ldots, t_n) \in \mathbb{R}^n_+ : t_1 + \cdots + t_n = 1\}$,

$$M(t, C) := tC^{\#},\ t > 0,$$
$$M(0, C) := (\operatorname{con} C)^0,$$

[2]The dual function Df is connected with the conjugate one f^*, $f^*(x') = \sup\{\langle x, x'\rangle - f(x) : x \in E\}$, via the equality $Df(x') = \inf\{\beta > 0 : f^*(\frac{x'}{\beta}) \leq -\frac{1}{\beta}\}$, which implies the inequality $Df(x')f^*(\frac{x'}{Df(x')}) \leq -1$ when $0 < Df(x') < +\infty$. (In [3, Proposition 6.1] the latter inequality was erroneously given with the equality sign.)

[3]Also see [2] where most results of [3] were announced without proofs.

[4]If none of C_i is strictly semiconic, then $(C_1 \cap \cdots \cap C_n)^{\#} = \varnothing$.

$\operatorname{con} C = \{\alpha x : x \in C, \alpha \geq 0\}$ is the conic hull of C, $(\operatorname{con} C)^0 = \{x' : \langle x, x' \rangle \leq 0 \quad \forall x \in \operatorname{con} C\} = \{x' : \langle x, x' \rangle \leq 0 \quad \forall x \in C\}$ is the conjugate (polar) cone of $\operatorname{con} C$.

Furthermore, if all C_i are strictly semiconic, then

$$(C_1 \cap \cdots \cap C_n)^{\#} = \operatorname{cl} \operatorname{co} \left(C_1^{\#} \cup \cdots \cup C_n^{\#} \right), \tag{2}$$

and if some C_i are not strictly semiconic, then

$$(C_1 \cap \cdots \cap C_n)^{\#} = \operatorname{cl} \left(\sum_{i \notin J} C_i^0 + \operatorname{co} \bigcup_{i \in J} C_i^{\#} \right), \tag{3}$$

where $J = \{i : C_i$ is strictly semiconic$\}$.

2. Suppose, in addition, that the intersection of all C_i, $i = 1, \ldots, n$, contains a point which is interior for $n-1$ of them or E is finite-dimensional and there is a point $x_0 \in C_1 \cap \cdots \cap C_n$ such that[5] $x_0 \in \operatorname{ri} C_i$ for $n-1$ of C_i; then

$$(C_1 \cap \cdots \cap C_n)^{\#} = \bigcup_{(t_1, \ldots, t_n) \in \Lambda} \left(M(t_1, C_1) + \cdots + M(t_n, C_n) \right). \tag{4}$$

Furthermore, this may be rewritten as

$$(C_1 \cap \cdots \cap C_n)^{\#} = \begin{cases} \operatorname{cl} \operatorname{co} \left(C_1^{\#} \cup \cdots \cup C_n^{\#} \right), & \text{if } J = \{1, \ldots, n\}; \\ \sum\limits_{i \notin J} C_i^0 + \operatorname{cl} \operatorname{co} \bigcup\limits_{i \in J} C_i^{\#}, & \text{if } J \neq \{1, \ldots, n\}. \end{cases} \tag{5}$$

The following result is a particular case of Theorem 2.2.

Corollary 2.3. *Suppose K is a closed convex cone, C is a closed convex semiconic set, and their intersection is nonempty, then*

$$(K \cap C)^{\#} = \operatorname{cl} (K^0 + C^{\#}). \tag{6}$$

(When C is not strictly semiconic, equality (6) remains true since in this case both its sides are empty sets.) If, in addition, $K \cap C$ contains a point, which is interior for K or for C, then

$$(K \cap C)^{\#} = K^0 + C^{\#}. \tag{7}$$

As follows from the next example, the closure in formulas (2) and (6) generally cannot be omitted.

Example 2.4. Let us consider two sets in $E = \mathbb{R}^2$ as follows:

$$C = \{(x, y) : 0 \leq x \leq 4,\ y \geq 1 - \sqrt{x}\} \cup \{(x, y) : x \geq 4,\ y \geq -x/4\},$$

$$K = \{(x, y) : x \leq 0,\ y \geq 0\}.$$

[5] Here, $\operatorname{ri} C_i$ stands for the relative interior of C_i.

Clearly, $C = \operatorname{scon}\{(x,y) : x \geq 0,\ y \geq 1 - \sqrt{x}\}$ is closed, convex, and strictly semiconic, K is a closed convex cone, and $K \cap C = \{(x,y) : x = 0,\ y \geq 1\}$. An easy calculation gives us

$$C^{\#} = \left\{ (x', y') : x' \leq \frac{y'^2}{4(1+y')},\ y' < -1 \right\},$$

$$K^0 = \{(x',y') : x' \geq 0,\ y' \leq 0\};$$

therefore $K^0 + C^{\#} = \{(x',y') : y' < -1\}$. At the same time, $(K \cap C)^{\#} = \{(x',y') : y' \leq -1\}$.

Similarly, let us take C as above and define $C_1 = C$, $C_2 = \{(x,y) : (-x,y) \in C\}$, then $C_1 \cap C_2 = \{(x,y) : x = 0,\ y \geq 1\}$, $(C_1 \cap C_2)^{\#} = \{(x',y') : y' \leq -1\}$, but $\operatorname{co}(C_1^{\#} \cup C_2^{\#}) = \{(x',y') : y' < -1\}$.

Remark 2.5. Some results on negative polars of intersections of semiconic sets have been stated earlier in [2, Theorem 2 (ii) and Corollary 3] and [3, Theorem 3.3 and Corollary 3.4] (see also [4, Theorem 3.2]); however, as may be seen from Example 2.4, there were defects in their formulations. The correct formulations are given in Theorem 2.2(*2*) and Corollary 2.3.

Theorem 2.6. *Let $f = f_1 + \cdots + f_n$, where $f_1, \ldots, f_n$ are convex lower semicontinuous functions from E into $\mathbb{R}_+ \cup \{+\infty\}$. Suppose that $f_1, \ldots, f_m$ are strictly semihomogeneous, $f_{m+1}, \ldots, f_n$ are semihomogeneous but not strictly semihomogeneous, and there is a point $x_0 \in \operatorname{dom} f_1 \cap \cdots \cap \operatorname{dom} f_n$ where at least $n-1$ of the functions are continuous. Then for every $x' \in E^*$ the equality holds*

$$\begin{aligned} Df(x') = \min\Big\{ \beta : x' = \sum_{m+1}^{n} x'_i + \lim_{\nu}\Big(\sum_{1}^{m} t_i^{\nu} x'_{i\nu}\Big),\ (t_1^{\nu}, \ldots, t_m^{\nu}) \in \Lambda_{0m}, \\ \beta = \lim_{\nu} t_i^{\nu} \beta_{i\nu},\ (x'_{i\nu}, \beta_{i\nu}) \in \operatorname{epi} Df_i\ (i = 1, \ldots, m), \\ \beta \geq d_i(x'_i)\,(i = m+1, \ldots, n) \Big\}, \end{aligned} \tag{8}$$

where $\Lambda_{0m} = \{(t_1, \ldots, t_m) : t_i > 0\ (i = 1, \ldots, m),\ t_1 + \cdots + t_m = 1\}$, $d_i(z') := \inf\{\beta > 0 : \frac{z'}{\beta} \in \operatorname{dom} f_i^*\}$[6] for each $z' \in E^*$, and by definition, $\min \varnothing := +\infty$.

A formula for $Df(x')$ is substantially simplified when $m = 1$.

Corollary 2.7. *Let $f = f_1 + \cdots + f_n$, where $f_1, \ldots, f_n$ are semihomogeneous convex lower semicontinuous functions on E, exactly one of them, namely f_1, is strictly semihomogeneous, and there is a point $x_0 \in \bigcap_1^n \operatorname{dom} f_i$ where at least $n-1$ of the functions are continuous; then for each $x' \in E^*$,*

$$Df(x') = \min_{x'_1 + \cdots + x'_n = x'} \max\left(Df_1(x'_1), d_2(x'_2), \ldots, d_n(x'_n)\right).$$

Remark 2.8. Note that a formula for $D(f_1 + \cdots + f_n)$ as given in [3, Theorem 6.7] is not correct.

[6]When $0 \in \operatorname{int} \operatorname{dom} f_i^*$, d_i becomes the Minkowski gauge functional of $\operatorname{dom} f_i^*$.

3. Proof of Theorem 2.2

1. The inclusion $\supseteq$ in (1) is obvious. In order to establish the opposite inclusion, fix any point $x'_0 \in (C_1 \cap \cdots \cap C_n)^\#$ and consider the sequence $x'_k := (1+\frac{1}{k})x'_0$. Since it converges to x'_0 as k tends to ∞, the inclusion $\subseteq$ will follow if we show that x'_k belongs to the right-hand side of (1) whenever $k = 1, 2, \ldots$. Note that $C^\# = \{x' : s(C, x') \le -1\}$ where $s(C)$ is the support function of C, $s(C, x') = \sup\{\langle x, x'\rangle : x \in C\}$. Since $\delta^*(C) = s(C)$ where $\delta(C)$ is the indicator function of C (i.e. $\delta(C, x) = 0$ for $x \in C$ and $\delta(C, x) = +\infty$ for $x \notin C$), one gets

$$\begin{aligned} \delta^*(C_1 \cap \cdots \cap C_n, x'_k) &= s(C_1 \cap \cdots \cap C_n, x'_k) \\ &= \left(1+\frac{1}{k}\right) s(C_1 \cap \cdots \cap C_n, x'_0) \le -\left(1+\frac{1}{k}\right). \end{aligned} \tag{9}$$

Furthermore, for every $x \in E$

$$\delta(C_1 \cap \cdots \cap C_n, x) = \delta(C_1, x) + \cdots + \delta(C_n, x),$$

and since the conjugate function of the sum $\delta(C_1, x) + \cdots + \delta(C_n, x)$ is the closure of the infimal convolution $\delta^*(C_1)\square\ldots\square\delta^*(C_n)$,

$$(\delta(C_1) + \cdots + \delta(C_n))^* = \mathrm{cl}\,(\delta^*(C_1)\square\ldots\square\delta^*(C_n))$$

(see e.g. [5, Theorem 16.4] where the finite-dimensional case is considered; in general case the proof is practically the same), one gets

$$\begin{aligned} \delta^*(C_1 \cap \cdots \cap C_n, x'_k) &= \mathrm{cl}\ (\delta^*(C_1)\square\ldots\square\delta^*(C_n))\,(x'_k) \\ &= \mathrm{cl}\,(s(C_1)\square\ldots\square s(C_n))\,(x'_k) \\ &= \lim_V \inf_{x'_V \in V} (s(C_1)\square\cdots\square s(C_n))\,(x'_V), \end{aligned} \tag{10}$$

where V runs a base $\mathcal{V}(x'_k)$ of weak* neighborhoods of x'_k. Let us consider pairs $\nu = (V, \varepsilon)$ where $V \in \mathcal{V}(x'_k)$, $0 < \varepsilon < \frac{1}{k}$, and write $\nu \succ \nu'$ if and only if $V \subset V'$, $\varepsilon < \varepsilon'$. It follows from (10) that for every $\nu = (V, \varepsilon)$ one can find $x'_\nu \in V$ and $x'_{1\nu}, \ldots, x'_{n\nu}$, such that

$$x'_{1\nu} + \cdots + x'_{n\nu} = x'_\nu, \tag{11}$$

$$s(C_1, x'_{1\nu}) + \cdots + s(C_n, x'_{n\nu}) \le (s(C_1)\square\cdots\square s(C_n))\,(x'_\nu) + \frac{\varepsilon}{2}, \tag{12}$$

$$(s(C_1)\square\ldots\square s(C_n))\,(x'_\nu) \le \inf_{x'_V \in V} (s(C_1)\square\ldots\square s(C_n))\,(x'_V) + \frac{\varepsilon}{2}. \tag{13}$$

Also, it follows from (10) that

$$\delta^*(C_1 \cap \cdots \cap C_n, x'_k) \ge \inf_{x'_V \in V} (s(C_1)\square\ldots\square s(C_n))\,(x'_V), \tag{14}$$

and we derive from (12)-(14) along with (9) that

$$s(C_1, x'_{1\nu}) + \cdots + s(C_n, x'_{n\nu}) \le \delta^*(C_1 \cap \cdots \cap C_n, x'_k) + \varepsilon < -1. \tag{15}$$

Recall now that, according to [3, Theorem 2.2], a closed convex set C is semiconic if and only if $s(C, x')$ is nonpositive or equals $+\infty$ whenever $x' \in E^*$. Since all C_i are closed convex semiconic sets, (15) implies then $s(C_i, x'_{i\nu}) \leq 0$, $i = 1, \dots, n$, hence $x'_{i\nu} \in M(0, C_i)$ for all i.

If we could find $(t_1, \dots, t_n) = (t_{1\nu}, \dots, t_{n\nu}) \in \Lambda$ such that $x'_{i\nu} \in M(t_i, C_i)$, $i = 1, \dots, n$, then $x'_\nu = x'_{1\nu} + \dots + x'_{n\nu}$ will lie in the right-hand side of (1), and $x'_k = \lim_\nu x'_\nu$ will lie there as well. Thus, (1) will be established if we show that $x'_{i\nu} \in M(t_i, C_i)$, $i = 1, \dots, n$, where $(t_1, \dots, t_n) \in \Lambda$.

If $s(C_i, x'_{i\nu}) \leq -1$ for some i, we take $t_i = 1$ and $t_j = 0$, $j \neq i$. Then $x'_{i\nu} \in C_i^\# = M(t_i, C_i)$, $x'_{j\nu} \in M(0, C_j) = M(t_j, C_j)$, $j \neq i$, and the result follows.

If $s(C_i, x'_{i\nu}) > -1$ for all i, then either

$$s(C_1, x'_{1\nu}) + \dots + s(C_n, x'_{n\nu}) = -1 \tag{16}$$

or there is $m < n$ such that

$$\begin{aligned} &s(C_1, x'_{1\nu}) + \dots + s(C_m, x'_{m\nu}) \geq -1, \\ &s(C_1, x'_{1\nu}) + \dots + s(C_m, x'_{m\nu}) + s(C_{m+1}, x'_{m+1\nu}) < -1. \end{aligned} \tag{17}$$

We take $t_i = -s(C_i, x'_{i\nu})$ for all i when (16) holds, and $t_i = -s(C_i, x'_{i\nu})$ for $i = 1, \dots, m$, $t_{m+1} = 1 - (t_1 + \dots + t_m)$, and $t_i = 0$ for $i > m + 1$ when (17) holds. Since in both cases $(t_1, \dots, t_n) \in \Lambda$ and $x'_{i\nu} \in M(t_i, C_i)$ for all i, the result follows, and formula (1) is thus completely established.

To prove formulas (2) and (3), notice that $C_i = \operatorname{con} C_i$ for $i \notin J$ (see Lemma 2.1) hence the right-hand sides of (2) and (3) are contained in the right-hand side of (1). Therefore, it suffices to show that every $x' \in \bigcup\{M(t_1, C_1) + \dots M(t_n, C_n) : (t_1, \dots, t_n) \in \Lambda\}$ lies in $\operatorname{co} \bigcup_{i \in J} C_i^\#$ if $J = \{1, \dots, n\}$ or in $\operatorname{cl}\left(\sum_{i \notin J} C_i^0 + \operatorname{co} \bigcup_{i \in J} C_i^\#\right)$ if $J \neq \{1, \dots, n\}$.

Let $x' = x'_1 + \dots + x'_n$, where $x'_i \in M(t_i, C_i)$, $i = 1, \dots, n$, $(t_1, \dots, t_n) \in \Lambda$. If all $t_i > 0$, then $M(t_i, C_i) = t_i C_i^\#$, $i = 1, \dots, n$, hence $J = \{1, \dots, n\}$ and $x' \in t_1 C_1^\# + \dots + t_n C_n^\# \subset \operatorname{co}(C_1^\# \cup \dots \cup C_n^\#)$.

Now consider the case where some t_i equal 0. Let $J(x') = \{i : t_i = 0\}$, hence $J(x') \supseteq \{1, \dots, n\} \setminus J$ and

$$\sum_{i \notin J(x')} t_i = 1.$$

If $i \notin J(x')$ then $x'_i = t_i z'_i$ where $z'_i \in C_i^\#$. If $i \in J(x') \cap J$, we take an arbitrary element $y'_i \in C_i^\#$ ($C_i^\#$ is nonempty because C_i is strictly semiconic) and set

$$z'_i(\varepsilon) = y'_i + \varepsilon^{-1} x'_i,$$

where $\varepsilon > 0$. Since $z'_i(\varepsilon) \in C_i^\# + (\operatorname{con} C_i)^0 = C_i^\#$ whenever $i \in J(x') \cap J$, $\varepsilon > 0$, we get for every ε, $0 < \varepsilon < 1$,

$$\varepsilon \sum_{i \in J(x') \cap J} z'_i(\varepsilon) + (1 - \varepsilon) \sum_{i \notin J(x')} t_i z'_i \in \operatorname{co} \bigcup_{i \in J} C_i^\#.$$

Taking into account that

$$\varepsilon \sum_{i\in J(x')\cap J} z_i'(\varepsilon) + (1-\varepsilon)\sum_{i\notin J(x')} t_i z_i'$$

tends to

$$\sum_{i\in J(x')\cap J} x_i' + \sum_{i\notin J(x')} x_i' = \begin{cases} x', & \text{if } J=\{1,\dots,n\},\\ x' - \sum_{i\notin J} x_i', & \text{if } J\neq\{1,\dots,n\}\end{cases}$$

as ε tends to 0, it follows that $x' \in \operatorname{cl}\operatorname{co}\bigcup_{i\in J} C_i^{\#}$ if $J=\{1,\dots,n\}$ or $x' \in \sum_{i\notin J} C_i^0 + \operatorname{cl}\operatorname{co}\bigcup_{i\in J} C_i^{\#}$ if $J\neq\{1,\dots,n\}$. Now, since obviously $\sum_{i\notin J} C_i^0 + \operatorname{cl}\operatorname{co}\bigcup_{i\in J} C_i^{\#} \subseteq \operatorname{cl}\left(\sum_{i\notin J} C_i^0 + \operatorname{co}\bigcup_{i\in J} C_i^{\#}\right)$, formulas (2) and (3) are completely proved.

2. Note that under the supposition on $C_1,\dots,C_n$ the duality formula

$$(\delta(C_1)+\dots+\delta(C_n))^* = \delta^*(C_1)\Box\dots\Box\delta^*(C_n) = s(C_1)\Box\dots\Box s(C_n)$$

holds true, and the infimal convolution $s(C_1)\Box\cdots\Box s(C_n)$ is exact that is for every $x'\in E^*$ there are $x_1',\dots,x_n'\in E^*$ such that $x' = x_1'+\dots x_n'$ and

$$(s(C_1)\Box\cdots\Box s(C_n))(x') = s(C_1,x_1')+\dots+s(C_n,x_n');$$

see [5, Theorem 16.4], [1, Theorem 3.4.1 on p. 189]. Therefore, given $x_0'\in(C_1\cap\dots\cap C_n)^{\#}$, we find $x_1',\dots,x_n'$ such that $x_0' = x_1'+\dots x_n'$ and

$$s(C_1\cap\dots\cap C_n, x_0') = (s(C_1)\Box\cdots\Box s(C_n))(x_0') = s(C_1,x_1')+\dots+s(C_n,x_n'),$$

and since $s(C_1\cap\dots\cap C_n,x_0')\le -1$, we get

$$s(C_1,x_1')+\dots+s(C_n,x_n')\le -1. \tag{18}$$

The reminder of the proof is as above provided that one uses (18) instead of (15) and replaces x_ν' and $x_{i\nu}'$ with x_0' and x_i'. □

Remark 3.1. As follows from the above proof, Λ in (1) may be replaced with a smaller set $\Lambda_0=\{(t_1,\dots,t_n)\in\Lambda : t_i>0,\ i=1,\dots,n\}$.

Remark 3.2. A different proof of (1) is as follows: general formula (3) hence (1) results from its particular cases (2) and (6) if one takes into account a well-known formula for the dual cone of an intersection of closed convex cones: $(\bigcap_{i\notin J} C_i)^0 = \operatorname{cl}(\sum_{i\notin J} C_i^0)$. Furthermore, formulas (2) and (6) can be established independently by using a separation theorem. An advantage of this method is its shortness, but we prefer to give a direct proof of formula (1) in general situation and then to get (2) and (6) as its consequences.

4. Proofs of Theorem 2.6 and Corollary 2.7

Proof of Theorem 2.6. One has $\operatorname{epi} f = AC$, where

$$C = \left\{(x,\alpha,\alpha_1,\dots,\alpha_n) : f_i(x)\le\alpha_i\ (i=1,\dots,n),\ \sum_1^n \alpha_i\le\alpha\right\},$$

A is a linear operator (projector) of $E \times \mathbb{R}^{n+1}$ onto $E \times \mathbb{R}$ acting by formula

$$A(x, \alpha, \alpha_1, \dots, \alpha_n) = (x, \alpha).$$

According to [3, Proposition 3.4], the equality holds

$$\operatorname{epi} Df = (\operatorname{epi} f)^{\#} = (AC)^{\#} = A^{*-1}(C^{\#}), \tag{19}$$

hence

$$Df(x') = \inf\{\beta \geq 0 : (x', \beta) \in A^{*-1}(C^{\#})\}, \tag{20}$$

and for every $x' \in \operatorname{dom} Df$ this infimum is attained at some β. We consider the duality between $E \times \mathbb{R}^{n+1}$ and $E^* \times \mathbb{R}^{n+1}$ determined by the bilinear form

$$\langle (x, \alpha, \alpha_1, \dots, \alpha_n), (x', \beta, \beta_1, \dots, \beta_n) \rangle_2 := \langle x, x' \rangle - \alpha\beta - \sum_1^n \alpha_i \beta_i,$$

therefore the conjugate operator $A^* : E^* \times \mathbb{R} \to E^* \times \mathbb{R}^{n+1}$ acts by formula

$$A^*(x', \beta) = (x', \beta, 0, \dots, 0). \tag{21}$$

Our nearest goal is to calculate $C^{\#}$. One has

$$C = C_0 \cap C_1 \cap \cdots \cap C_n,$$

where

$$C_0 = \{(x, \alpha, \alpha_1, \dots, \alpha_n) : \alpha_1 + \cdots + \alpha_n \leq \alpha\},$$
$$C_i = \{(x, \alpha, \alpha_1, \dots, \alpha_n) : f_i(x) \leq \alpha_i\},\ i = 1, \dots, n.$$

Clearly, C_i, $i = 1, \dots, m$, are strictly semiconic, C_i, $i = m+1, \dots, n$, and C_0 are semiconic but not strictly semiconic, and the point $(x_0, f(x_0)+2n, f_1(x_0)+1, f_2(x_0)+1, \dots, f_n(x_0)+1)$ belongs to all C_i $(i = 0, 1, \dots, n)$ and is interior for at least n of them. Applying statement 2 of Theorem 2.2 yields then

$$C^{\#} = C_0^0 + \sum_{m+1}^n C_i^0 + \operatorname{cl} \operatorname{co} \bigcup_1^m C_i^{\#}. \tag{22}$$

Now, an easy calculation gives

$$C_0^0 = \{(x', \beta, \beta_1, \dots, \beta_n) : x' = 0, \beta \geq 0, \beta_1 = \cdots = \beta_n = -\beta\},$$
$$C_i^0 = \{(x', \beta, \beta_1, \dots, \beta_n) : \beta = 0, \beta_j = 0\ (j \neq i),$$
$$\beta_i > 0, f_i^*(\frac{x'}{\beta_i}) \leq 0 \quad \text{or } \beta_i = 0, x' = 0\},\ i = m+1, \dots, n,$$

and since $f_i^*(\frac{x'}{\beta_i}) \leq 0$ is equivalent to $\frac{x'}{\beta_i} \in \operatorname{dom} f_i^*$ (see [3, Proposition 4.4]), the last equality may be rewritten as

$$C_i^0 = \{(x', \beta, \beta_1, \dots, \beta_n) : \beta = 0, \beta_j = 0\ (j \neq i),$$
$$\beta_i > 0, \frac{x'}{\beta_i} \in \operatorname{dom} f_i^* \text{ or } \beta_i = 0, x' = 0\},\ i = m+1, \dots, n.$$

Furthermore, since f_i is nonnegative, one has $0 \in \operatorname{dom} f_i^*$, and the relation $[\beta_i > 0, \frac{x'}{\beta_i} \in \operatorname{dom} f_i^*$ or $\beta_i = 0, x' = 0]$ is rewritten as $\beta_i \geq d_i(x')$. We get

$$C_i^0 = \{(x', \beta, \beta_1, \dots, \beta_n) : \beta = 0, \beta_j = 0 \ (j \neq i), \beta_i \geq d_i(x')\},$$

$i = m+1, \dots, n$.

Next, it is easily seen that

$$C_i^\# = \{(x', \beta, \beta_1, \dots, \beta_n) : \beta = \beta_j = 0 \ (j \neq i), (x', \beta_i) \in \operatorname{epi} Df_i\}$$

for $i = 1, \dots, m$. It follows that $(x', \beta, \beta_1, \dots, \beta_n) \in \operatorname{cl} \operatorname{co} \bigcup_1^m C_i^\#$ if and only if there exist generalized sequences $(t_1^\nu, \dots, t_m^\nu) \subset \Lambda_{0m}$ and $(x'_{i\nu}, \beta_{i\nu}) \subset \operatorname{epi} Df_i$, $i = 1, \dots, m$, such that $x' = \lim_\nu \sum_1^m t_i^\nu x'_{i\nu}$, $\beta_i = \lim_\nu t_i^\nu \beta_{i\nu}$ $(i = 1, \dots, m)$, $\beta = \beta_{m+1} = \dots = \beta_n = 0$.

Now, taking into account (22) along with the above equalities for C_0^0 and C_i^0, $i = m+1, \dots, n$, we derive from (19) and (21) that $\operatorname{epi} Df$ comprises the pairs (x', β) such that $x' = \lim_\nu \sum_1^m t_i^\nu x'_{i\nu} + \sum_{m+1}^n x'_i$, $\beta = \lim_\nu t_i^\nu \beta_{i\nu}$ $(i = 1, \dots, m)$, and $\beta = \beta_i$ $(i = m+1, \dots, n)$, where $\beta_{i\nu} \geq Df_i(x'_{i\nu})$ $(i = 1, \dots, m)$, $\beta_i \geq d_i(x'_i)$ $(i = m+1, \dots, n)$, and the result follows. □

Proof of Corollary 2.7. Considering the same sets C_i $(i = 0, 1, \dots, n)$ and C as in the proof of Theorem 2.6 and applying to them statement 2 of Theorem 2.2 yields

$$C^\# = C_0^0 + \sum_2^n C_i^0 + C_1^\#.$$

Next, arguing as in the proof of Theorem 2.6 and applying (19) we get

$$\operatorname{epi} Df = A^{*-1} \left(C_0^0 + \sum_{i=2}^n C_i^0 + C_1^\# \right). \tag{23}$$

Now, since $C_0^0 + \sum_2^n C_i^0 + C_1^\#$ comprises the vectors

$$(x'_1 + \dots + x'_n, \beta, -\beta + \beta_1, -\beta + \beta_2, \dots, -\beta + \beta_n) \in E^* \times \mathbb{R}^{n+1},$$

for which $\beta \geq 0$, $\beta_1 \geq Df_1(x'_1)$, $\beta_i \geq d_i(x'_i)$ $(i = 2, \dots, n)$, (23) along with (21) implies that $\operatorname{epi} Df$ consists of (x', β) such that $x' = x'_1 + \dots + x'_n$, $\beta \geq \max(Df_1(x'_1), d_2(x'_2), \dots, d_n(x'_n))$, and the result follows. □

Remark 4.1. A similar argument shows that if hypotheses of Theorem 2.6 are satisfied and the set

$$C_0^0 + \sum_{m+1}^n C_i^0 + \operatorname{co} \bigcup_1^m C_i^\# \tag{24}$$

is closed, then the formula holds true

$$Df(x') = \min \max \left(t_1 Df_1(x'_1), \dots, t_m Df_m(x'_m), d_{m+1}(x'_{m+1}), \dots, d_n(x'_n) \right), \tag{25}$$

where minimum is taken over all $x'_1, \dots, x'_n \in E^*$, $(t_1, \dots, t_m) \in \Lambda_{0m}$ such that $t_1 x'_1 + \dots + t_m x'_m + x'_{m+1} + \dots + x'_n = x'$.

5. A case of polyhedral sets and functions

Recall (see [5]), that a set C in E is called *polyhedral* if it can be represented in form

$$C = \{x \in E : \langle x, x'_k \rangle \le \alpha_k,\ k = 1, \dots, N\},$$

where N is a positive integer, $x'_1, \dots, x'_N \in E^*$, $\alpha_1, \dots, \alpha_N \in \mathbb{R}$.

A function $f : E \to \mathbb{R} \cup \{+\infty\}$ is called *polyhedral* if its epigraph is a polyhedral set in $E \times \mathbb{R}$.

It is clear that polyhedral sets are convex closed and polyhedral functions are convex lower semicontinuous.

Theorem 5.1. *Suppose $C_1, \dots, C_n$ are polyhedral semiconic sets in E, at least one of them is strictly semiconic, and their intersection is nonempty; then*

$$(C_1 \cap \dots \cap C_n)^{\#} = \begin{cases} \operatorname{co}\left(C_1^{\#} \cup \dots \cup C_n^{\#}\right), & \text{if } J = \{1, \dots, n\}; \\ \sum\limits_{i \notin J} C_i^0 + \operatorname{co} \bigcup\limits_{i \in J} C_i^{\#}, & \text{if } J \ne \{1, \dots, n\}, \end{cases}$$

where $J = \{i : C_i \text{ is strictly semiconic}\}$.

Corollary 5.2. *Suppose K is a polyhedral cone, C is a polyhedral semiconic set, and their intersection is nonempty, then*

$$(K \cap C)^{\#} = K^0 + C^{\#}.$$

If C is strictly semiconic, this is a particular case of Theorem 5.1, otherwise both $(K \cap C)^{\#}$ and $K^0 + C^{\#}$ are empty.

To prove Theorem 5.1, the following lemma will be needed.

Lemma 5.3. *Given positive integers $m < n$ and n nonzero linear functionals $y'_1, \dots, y'_n \in E^*$, the set*

$$M = \left\{ x' = \sum_1^n \lambda_i y'_i : \lambda_i \ge 0\ (i = 1, \dots, n),\ \sum_1^m \lambda_i \ge 1 \right\}$$

is closed.

Proof. Consider the set

$$M_0 = \left\{ x' = \sum_1^n \lambda_i y'_i : \lambda_i \ge 0\ (i = 1, \dots, n),\ \sum_1^m \lambda_i = 1 \right\}.$$

It is closed by [5, Theorem 19.1] (this theorem may be applied since M_0 is a subset of finite-dimensional space $Lin(y'_1, \dots, y'_n)$ spanned over $y'_1, \dots, y'_n$); and as $M = \operatorname{scon} M_0$, it follows from [3, Proposition 2.2 (2)] that M is closed too. □

Proof of Theorem 5.1. Since C_i are polyhedral, there is a representation

$$C_i = \{x \in E : \langle x, x'_{ik} \rangle \le \alpha_{ik},\ k = 1, \dots, N_i\},\ i = 1, \dots, n, \tag{26}$$

such that all x'_{ik} are nonzero. We can assume without loss of generality that all α_{ik} are nonpositive. Indeed, since $\lambda C_i \subseteq C_i$ whenever $\lambda > 1$, it follows that every positive α_{ik} in (26) may be replaced with 0. Finally, taking into account Lemma 2.1, we'll assume that $\alpha_{i1} = \cdots = \alpha_{iN_i} = 0$ whenever $i \notin J$.

A direct calculation then yields

$$C_i^0 = \left\{ \sum_{k=1}^{N_i} \lambda_{ik} x'_{ik} : \lambda_{ik} \geq 0 \, (k = 1, \ldots, N_i) \right\}, \ i \notin J.$$

Further, fix $i \in J$, denote $J_i := \{k \in \{1, \ldots, N_i\} : \alpha_{ik} < 0\}$, and consider N_i closed half-spaces $H_{ik} = \{x \in E : \langle x, x'_{ik} \rangle \leq \alpha_{ik}\}$, $k = 1, \ldots, N_i$. Clearly, H_{ik} is strictly semiconic if and only if $k \in J_i$. An easy calculation yields $H_{ik}^0 = \{\lambda x'_{ik} : \lambda \geq 0\}$ for $k \notin J_i$, and $H_{ik}^{\#} = \{\lambda x'_{ik} : \lambda \geq 1/|\alpha_{ik}|\}$ for $k \in J_i$. Now, applying Theorem 2.2 (see (3)) along with Lemma 5.3 to the set $C_i = H_{i1} \cap \cdots \cap H_{iN_i}$ gives us

$$C_i^{\#} = \left\{ \sum_{k \notin J_i} \lambda_{ik} x'_{ik} + \sum_{k \in J_i} \lambda_{ik} y'_{ik} : \lambda_{ik} \geq 0 \, (k = 1, \ldots, N_i), \right. \\ \left. \sum_{k \in J_i} \lambda_{ik} \geq 1, \ y'_{ik} = x'_{ik}/|\alpha_{ik}| \, (k \in J_i) \right\}, \ i \in J.$$

We get

$$\mathrm{co} \bigcup_{i=1}^{n} C_i^{\#} = \left\{ \sum_{i=1}^{n} \sum_{k \notin J_i} t_{ik} x'_{ik} + \sum_{i=1}^{n} \sum_{k \in J_i} t_{ik} y'_{ik} : t_{ik} \geq 0 \, (k = 1, \ldots, N_i; \, i = 1, \ldots, n), \right. \\ \left. \sum_{i=1}^{n} \sum_{k \in J_i} t_{ik} \geq 1, \ y'_{ik} = x'_{ik}/|\alpha_{ik}| \, (k \in J_i, \, i = 1, \ldots, n) \right\}$$

when $J = \{1, \ldots, n\}$, and

$$\sum_{i \notin J} C_i^0 + \mathrm{co} \bigcup_{i \in J} C_i^{\#} = \left\{ \sum_{i \notin J} \sum_{k=1}^{N_i} t_{ik} x'_{ik} + \sum_{i \in J} \sum_{k \notin J_i} t_{ik} x'_{ik} + \sum_{i \in J} \sum_{k \in J_i} t_{ik} y'_{ik} : \right. \\ t_{ik} \geq 0 \, (k = 1, \ldots, N_i; \, i = 1, \ldots, n), \\ \left. \sum_{i \in J} \sum_{k \in J_i} t_{ik} \geq 1, \ y'_{ik} = x'_{ik}/|\alpha_{ik}| \, (k \in J_i; \, i \in J) \right\}$$

when $J \neq \{1, \ldots, n\}$. According to Lemma 5.3, both sets are closed, and applying Theorem 2.2 (see (2) and (3)) completes the proof. □

Theorem 5.4. *Let $f = f_1 + \cdots + f_n$, where $f_1, \ldots, f_n$ are polyhedral functions on E. Suppose that $f_1, \ldots, f_m$ are strictly semihomogeneous, $f_{m+1}, \ldots, f_n$ are semihomogeneous but not strictly semihomogeneous, and the effective domains of all the functions have a common point (*$\mathrm{dom}\, f_1 \cap \cdots \cap \mathrm{dom}\, f_n \neq \varnothing$*); then $Df(x')$ is given by formula* (25).

Proof of Theorem 5.4. Considering the same sets C_i $(i = 0, 1, \dots, n)$ and C as in the proof of Theorem 2.6 and applying to them Theorem 5.1, we see that the corresponding set (24) is closed. Now, taking into account Remark 4.1, the result follows. □

Remark 5.5. There are some defects in my paper [3] (and in its short version [2]). The main of them are indicated in Remarks 2.5 and 2.8 and in a footnote on p. 857. Yet one omission: I forgot to say that the function φ in [3, Proposition 4.6] is semihomogeneous.

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