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Balanced Complex Polytopes and Related Vector and Matrix Norms*

Nicola Guglielmi
Dipartimento di Matematica, Università di L'Aquila, via Vetoio, loc. Coppito, 67100 l'Aquila, Italy
guglielm@matematica.univaq.it

Marino Zennaro
Dipartimento di Matematica e Informatica, Università di Trieste, via A. Valerio 12/1, 34100 Trieste, Italy
zennaro@univ.trieste.it

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In this paper we study the notion of *balanced complex polytope* as a generalization of a symmetric real polytope to the complex space $\mathbb{C}^n$. We pay particular attention to the geometric properties of such complex polytopes and of their counterparts in the *adjoint* form. In particular, we stress the differences occurring with respect to the well-known real case. We also introduce and discuss the related definitions of *complex polytope norm* and *adjoint complex polytope norm*.

1. Introduction

The notion of polytope in the real space $\mathbb{R}^n$ is very well known and developed (see, e.g., Ziegler [5]).

When a polytope is symmetric, it is the unit ball of a norm on $\mathbb{R}^n$. Such kind of norms, called *polytope norms*, constitute an important subset of all the norms which include, as particular cases, the ∞-norm (or maximum norm) defined by $\|x\|_\infty = \max_{1\le i\le n} |x_i|$ and the 1-norm defined by $\|x\|_1 = \sum_{i=1}^n |x_i|$.

As is known, the real case is not always satisfactory in that it may sometimes be needed to consider the complex space $\mathbb{C}^n$ instead of $\mathbb{R}^n$. In particular, if one wants to keep on using polytope norms also in $\mathbb{C}^n$, then one has to adapt the notion of polytope to the complex case. An example of such a situation may be found when trying to find an extremal norm for a family of matrices, as is explained by Guglielmi, Wirth and Zennaro [2].

In this paper we define the *balanced complex polytopes* as generalizations of symmetric real polytopes to the complex case (see Section 2) and introduce the related definition of *complex polytope norm* (see Section 5).

We pay particular attention to the geometric properties of such complex polytopes (see Section 3) and of their counterparts in the *adjoint* form (see again Section 2 and Section 4). In particular, we stress the differences occurring with respect to the well-known real case.

Although most of the results are more or less straightforward particularizations or generalizations of other results which can be found in the literature, here we prefer to always

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give a direct proof, in order to make the paper as selfcontained as possible.

2. Complex polytopes

We begin this section by recalling the concepts of *convex*, *balanced* and *absolutely convex* set (see, e.g., Heuser [3]).

Definition 2.1. We say that a set $\mathcal{X} \subset \mathbb{C}^n$ ($\mathcal{X} \subset \mathbb{R}^p$) is *convex* if, for all $x', x'' \in \mathcal{X}$ and $\lambda', \lambda'' \in \mathbb{R}$ such that $\lambda', \lambda'' \geq 0$ and $\lambda' + \lambda'' = 1$, it holds that $\lambda' x' + \lambda'' x'' \in \mathcal{X}$.

Definition 2.2. Let $\mathcal{X} \subset \mathbb{C}^n$ ($\mathcal{X} \subset \mathbb{R}^p$). Then the intersection of all convex sets containing $\mathcal{X}$ is called the *convex hull* of $\mathcal{X}$ and is denoted by $\mathrm{co}(\mathcal{X})$.

It is well-known that $\mathrm{co}(\mathcal{X})$ is the set of all the finite convex linear combinations of vectors of $\mathcal{X}$, i.e., $x \in \mathrm{co}(\mathcal{X})$ if and only if there exist $x^{(1)}, \ldots, x^{(k)} \in \mathcal{X}$ with $k \geq 1$ such that

$$x = \sum_{i=1}^{k} \lambda_i x^{(i)} \quad \text{with } \lambda_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i = 1.$$

In particular, if $\mathcal{X} = \{x^{(i)}\}_{1 \leq i \leq p}$ is a finite set of vectors, then

$$\mathrm{co}(\mathcal{X}) = \Big\{ x \in \mathbb{C}^n \;\Big|\; x = \sum_{i=1}^{p} \lambda_i x^{(i)} \text{ with } \lambda_i \geq 0 \text{ and } \sum_{i=1}^{p} \lambda_i = 1 \Big\}.$$

Definition 2.3. We say that a set $\mathcal{X} \subset \mathbb{C}^n$ is *balanced* if, for all $x \in \mathcal{X}$ and $\lambda \in \mathbb{C}$ such that $|\lambda'| \leq 1$, it holds that $\lambda x \in \mathcal{X}$.

Definition 2.4. Let $\mathcal{X} \subset \mathbb{C}^n$. Then the intersection of all balanced sets containing $\mathcal{X}$ is called the *balanced hull* of $\mathcal{X}$ and is denoted by $\mathrm{bal}(\mathcal{X})$.

It is well-known that

$$\mathrm{bal}(\mathcal{X}) = \Big\{ \lambda x \;\Big|\; x \in \mathcal{X} \text{ and } \lambda \in \mathbb{C} \text{ with } |\lambda| \leq 1 \Big\}.$$

Definition 2.5. We say that a set $\mathcal{X} \subset \mathbb{C}^n$ is *absolutely convex* if, for all $x', x'' \in \mathcal{X}$ and $\lambda', \lambda'' \in \mathbb{C}$ such that $|\lambda'| + |\lambda''| \leq 1$, it holds that $\lambda' x' + \lambda'' x'' \in \mathcal{X}$.

Definition 2.6. Let $\mathcal{X} \subset \mathbb{C}^n$. Then the intersection of all absolutely convex sets containing $\mathcal{X}$ is called the *absolutely convex hull* of $\mathcal{X}$ and is denoted by $\mathrm{absco}(\mathcal{X})$.

It is well-known that $\mathrm{absco}(\mathcal{X})$ is the set of all the finite absolutely convex linear combinations of vectors of $\mathcal{X}$, i.e., $x \in \mathrm{absco}(\mathcal{X})$ if and only if there exist $x^{(1)}, \ldots, x^{(k)} \in \mathcal{X}$ with $k \geq 1$ such that

$$x = \sum_{i=1}^{k} \lambda_i x^{(i)} \quad \text{with } \lambda_i \in \mathbb{C} \quad \text{and} \quad \sum_{i=1}^{k} |\lambda_i| \leq 1.$$

In particular, if $\mathcal{X} = \{x^{(i)}\}_{1 \leq i \leq m}$ is a finite set of vectors, then

$$\mathrm{absco}(\mathcal{X}) = \Big\{ x \in \mathbb{C}^n \;\Big|\; x = \sum_{i=1}^{m} \lambda_i x^{(i)} \text{ with } \lambda_i \in \mathbb{C} \text{ and } \sum_{i=1}^{m} |\lambda_i| \leq 1 \Big\}. \tag{1}$$

The next results are also well-known and easy to prove.

Proposition 2.7. *A set $\mathcal{X} \subset \mathbb{C}^n$ is absolutely convex if and only if it is balanced and convex. Moreover, it holds that*

$$\mathrm{absco}(\mathcal{X}) = \mathrm{co}(\mathrm{bal}(\mathcal{X})) = \mathrm{bal}(\mathrm{co}(\mathcal{X})).$$

From here upwards, if $\mathcal{X}'$ and $\mathcal{X}''$ are two subsets of $\mathbb{C}^n$, we shall write $\mathcal{X}' \subset \mathcal{X}''$ ($\mathcal{X}' \supset \mathcal{X}''$) to mean that $\mathcal{X}' \subseteq \mathcal{X}''$ ($\mathcal{X}' \supseteq \mathcal{X}''$) and $\mathcal{X}' \neq \mathcal{X}''$ (*proper* inclusions).

The forthcoming definition extends the usual definition of symmetric polytope in the real space $\mathbb{R}^n$.

Definition 2.8. We say that a bounded set $\mathcal{P} \subset \mathbb{C}^n$ is a *balanced complex polytope* (in short, b.c.p.) if there exists a finite set $\mathcal{X} = \{x^{(i)}\}_{1 \le i \le m}$ of vectors such that $\mathrm{span}(\mathcal{X}) = \mathbb{C}^n$ and

$$\mathcal{P} = \mathrm{absco}(\mathcal{X}). \tag{2}$$

Moreover, if $\mathrm{absco}(\mathcal{X}') \subset \mathrm{absco}(\mathcal{X})$ for all $\mathcal{X}' \subset \mathcal{X}$, then $\mathcal{X}$ is called an *essential system of vertices* for $\mathcal{P}$, whereas any vector $ux^{(i)}$ with $u \in \mathbb{C}$, $|u| = 1$, is called a *vertex* of $\mathcal{P}$.

The following property characterizes the essential systems of vertices.

Proposition 2.9. *Assume that $\mathcal{P}$ is a b.c.p. and that $\mathcal{X} = \{x^{(i)}\}_{1 \le i \le m}$ is an essential system of vertices for $\mathcal{P}$. Then for each $x^{(i)} \in \mathcal{X}$ the equality*

$$x^{(i)} = \sum_{j=1,\ j \neq i}^{m} \lambda_j x^{(j)}$$

implies

$$\sum_{j=1,\ j \neq i}^{m} |\lambda_j| > 1. \tag{3}$$

Proof. Assume, by contradiction, that (3) does not hold. Then it turns out that

$$x^{(i)} \in \mathrm{absco}(\mathcal{X} \setminus \{x^{(i)}\}).$$

This yields $\mathrm{absco}(\mathcal{X} \setminus \{x^{(i)}\}) = \mathrm{absco}(\mathcal{X})$, which is against the essentiality of $\mathcal{X}$. □

Remark that, geometrically speaking, a b.c.p. $\mathcal{P}$ is not a classical polytope. In fact, if we identify the complex space $\mathbb{C}^n$ with the real space $\mathbb{R}^{2n}$, we can easily see that $\mathcal{P}$ is not bounded by a finite number of hyperplanes. In general, even the intersection $\mathcal{P} \bigcap \mathbb{R}^n$ is not a classical polytope. However, if the b.c.p. $\mathcal{P}$ admits an essential system of real vertices, then $\mathcal{P} \bigcap \mathbb{R}^n$ is a classical polytope.

Now we consider the concept that, in the literature, is often referred to as *polarity* or *duality* (see again [5] and [3]). However, in view of the forthcoming Theorem 5.6, we prefer to change the terminology. We shall use the usual Euclidean scalar product in $\mathbb{C}^n$ defined by $\langle x, y \rangle = \sum_{j=1}^{n} x_j \hat{y}_j$.

Definition 2.10. Let $\mathcal{X} \subset \mathbb{C}^n$. Then the set

$$\operatorname{adj}(\mathcal{X}) = \left\{ y \in \mathbb{C}^n \;\middle|\; |\langle y, x\rangle| \le 1 \text{ for all } x \in \mathcal{X} \right\} \tag{4}$$

is called the *adjoint* of $\mathcal{X}$.

It is immediately seen that $\operatorname{adj}(\mathcal{X})$ is closed and absolutely convex.

Definition 2.11. We say that a bounded set $\mathcal{P}^* \subset \mathbb{C}^n$ is a *b.c.p. of adjoint type* if there exists a finite set $\mathcal{X} = \{x^{(i)}\}_{1\le i\le m}$ of vectors such that $\operatorname{span}(\mathcal{X}) = \mathbb{C}^n$ and

$$\mathcal{P}^* = \operatorname{adj}(\mathcal{X}) = \left\{ y \in \mathbb{C}^n \;\middle|\; |\langle y, x^{(i)}\rangle| \le 1,\ i = 1, \dots, m \right\}. \tag{5}$$

Moreover, if $\operatorname{adj}(\mathcal{X}') \supset \operatorname{adj}(\mathcal{X})$ for all $\mathcal{X}' \subset \mathcal{X}$, then $\mathcal{X}$ is called an *essential system of facets* for $\mathcal{P}^*$, whereas any vector $ux^{(i)}$ with $u \in \mathbb{C}$, $|u| = 1$, is called a *facet* of $\mathcal{P}^*$.

The following property characterizes the essential systems of facets.

Proposition 2.12. *Assume that $\mathcal{P}^*$ is a b.c.p. of adjoint type and that $\mathcal{X} = \{x^{(i)}\}_{1\le i\le m}$ is an essential system of facets for $\mathcal{P}^*$. Then for each $x^{(i)} \in \mathcal{X}$ there exists $y^{(i)} \in \mathcal{P}^*$ such that*

$$|\langle y^{(i)}, x^{(i)}\rangle| = 1 \quad \textit{and} \quad |\langle y^{(i)}, x^{(j)}\rangle| < 1 \ \textit{ for all } j \ne i. \tag{6}$$

Proof. The essentiality of the system $\mathcal{X}$ immediately implies that, for each $x^{(i)} \in \mathcal{X}$, there exists $\hat{y}^{(i)} \in \mathbb{C}^n$ such that

$$|\langle \hat{y}^{(i)}, x^{(i)}\rangle| > 1 \quad \text{and} \quad |\langle \hat{y}^{(i)}, x^{(j)}\rangle| \le 1 \ \text{ for all } j \ne i.$$

Therefore, with $y^{(i)} = \hat{y}^{(i)}/\langle \hat{y}^{(i)}, x^{(i)}\rangle$, (6) is proved. □

Unlike it happens for classical polytopes in $\mathbb{R}^n$, it is not true that the class of b.c.p.'s coincides with the class of b.c.p.'s of adjoint type. Indeed, for every b.c.p. $\mathcal{P}$, the equality $\mathcal{P} = \operatorname{adj}(\mathcal{X})$ implies that $\mathcal{X}$ is an infinite set of vectors and, analogously, the same implication holds whenever we express a b.c.p. of adjoint type $\mathcal{P}^*$ in the form $\mathcal{P}^* = \operatorname{absco}(\mathcal{X})$.

Intuitively, this may be explained by observing that, whereas in the real case the number of vertices and facets of a symmetric polytope is always finite, in the complex case this is not true. In fact, although the essential system of vertices (facets) $\mathcal{X}$ of a b.c.p. $\mathcal{P}$ (of a b.c.p. of adjoint type $\mathcal{P}^*$) is finite, the total number of vertices (facets) is infinite. However, these aspects will be investigated more deeply later.

Now we examine the mutual relationships between b.c.p.'s and b.c.p.'s of adjoint type.

Proposition 2.13. *Let $\mathcal{X} \subset \mathbb{C}^n$. Then*

$$\operatorname{adj}(\operatorname{absco}(\mathcal{X})) = \operatorname{adj}(\mathcal{X}).$$

Proof. Let $y \in \operatorname{adj}(\mathcal{X})$. Now, if $x \in \operatorname{absco}(\mathcal{X})$, then $x = \sum_{i=1}^k \lambda_i x^{(i)}$ with $x^{(1)}, \dots, x^{(k)} \in \mathcal{X}$ and $\sum_{i=1}^k |\lambda_i| \le 1$. Therefore, we have that

$$|\langle y, x\rangle| = |\langle y, \sum_{i=1}^k \lambda_i x^{(i)}\rangle| \le \Big(\sum_{i=1}^k |\lambda_i| \Big) \max_{1\le i\le k} |\langle y, x^{(i)}\rangle| \le 1.$$

Thus $y \in \mathrm{adj}(\mathrm{absco}(\mathcal{X}))$, too.

Viceversa, let $y \in \mathrm{adj}(\mathrm{absco}(\mathcal{X}))$. Since $\mathcal{X} \subseteq \mathrm{absco}(\mathcal{X})$, it is clear that $y \in \mathrm{adj}(\mathcal{X})$, too. □

Corollary 2.14. *Let $\mathcal{P}$ be a b.c.p. and let $\mathcal{X} = \{x^{(i)}\}_{1 \le i \le m}$ be a finite set of vectors such that $\mathcal{P} = \mathrm{absco}(\mathcal{X})$. Then $\mathrm{adj}(\mathcal{P})$ is a b.c.p. of adjoint type and it holds that*

$$\mathrm{adj}(\mathcal{P}) = \mathrm{adj}(\mathcal{X}). \tag{7}$$

The adjoint version of Proposition 2.13 holds. For the proof, we need the following lemma regarding real convex sets, which is known as *the theorem of the separating hyperplanes.* It is a well-known result and its (easy) proof can be found, e.g., in Luenberger [4].

Lemma 2.15. *Given any closed convex set $\mathcal{S} \subset \mathbb{R}^p$ and a point $\hat{x} \notin \mathcal{S}$, there exists $\hat{z} \in \mathbb{R}^p$ such that*

$$(\hat{z}, \hat{x}) < \min_{x \in \mathcal{S}} (\hat{z}, x), \tag{8}$$

where $(\cdot, \cdot)$ denotes the Euclidean scalar product in $\mathbb{R}^p$.

The geometric meaning of Lemma 2.15 is that there exists a hyperplane, represented by $\hat{z}$ and containing $\hat{x}$, such that all of $\mathcal{S}$ is contained in one open half-space produced by that hyperplane.

Lemma 2.16. *Let $\mathcal{X} \subset \mathbb{C}^n$ be compact. Then $\mathrm{bal}(\mathcal{X})$ is compact too.*

Proof. It is clear that, $\mathcal{X}$ being bounded, then so is $\mathrm{bal}(\mathcal{X})$.

Now consider a sequence $\{z_k\}_{k \ge 1}$ of vectors of $\mathrm{bal}(\mathcal{X})$ such that there exists

$$\lim_{k \to \infty} z_k = z \in \mathbb{C}^n.$$

In order to prove that $z \in \mathrm{bal}(\mathcal{X})$, observe that, for each k, we have $z_k = \lambda_k x_k$, where $\lambda_k \in \mathbb{C}$ with $|\lambda_k| \le 1$ and $x_k \in \mathcal{X}$.

Since $\mathcal{X}$ is compact, there exists a sequence of integers k_s such that

$$\lim_{s \to \infty} \lambda_{k_s} = \lambda \quad \text{with } |\lambda| \le 1$$

and

$$\lim_{s \to \infty} x_{k_s} = x \in \mathcal{X}.$$

Therefore, it must be $z = \lambda x \in \mathrm{bal}(\mathcal{X})$. □

Lemma 2.17. *Let $\mathcal{X} \subset \mathbb{R}^p$ and $x \in \mathrm{co}(\mathcal{X})$. If $\dim(\mathrm{span}(\mathcal{X} - x)) = q$, then there exist $q+1$ vectors $x^{(1)}, \ldots, x^{(q+1)} \in \mathcal{X}$ such that $x \in \mathrm{co}(x^{(1)}, \ldots, x^{(q+1)})$.*

Proof. By hypothesis, there exist $x^{(1)}, \ldots, x^{(k)} \in \mathcal{X}$ with $k \ge 1$ such that

$$x = \sum_{i=1}^{k} \lambda_i x^{(i)} \quad \text{with } \lambda_i > 0 \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i = 1.$$

If $k \le q+1$, the theorem is proved. Otherwise, we have $k \ge q+2$ and hence there exist k real numbers $\delta_1, \dots, \delta_k$, not all of them equal to zero, such that

$$\sum_{i=1}^{k} \delta_i (x^{(i)} - x) = 0 \quad \text{and} \quad \sum_{i=1}^{k} \delta_i = 0. \tag{9}$$

Therefore, for all $\epsilon > 0$, we get

$$x = \sum_{i=1}^{k} (\lambda_i + \epsilon\delta_i) x^{(i)} \quad \text{and} \quad \sum_{i=1}^{k} (\lambda_i + \epsilon\delta_i) = 1.$$

Since at least one of the δ_i's is nonzero and $\lambda_i > 0$ for all $i = 1, \dots, k$, there exists $\epsilon_0 \neq 0$ such that $\lambda_i + \epsilon\delta_i \ge 0$ for all $i = 1, \dots, k$ whenever $-|\epsilon_0| \le \epsilon \le |\epsilon_0|$ and such that $\lambda_j + \epsilon_0\delta_j = 0$ for a certain index j. Then, by (9) for $\epsilon = \epsilon_0$, we have that

$$x \in \mathrm{co}(x^{(1)}, \dots, x^{(j-1)}, x^{(j+1)}, \dots, x^{(k)}).$$

It is clear that we can repeat the same procedure as many times as necessary to eliminate $k - q - 1$ vectors among $x^{(1)}, \dots, x^{(k)}$. So the result is proved. □

Remark that (9) is equivalent to

$$\sum_{i=1}^{k} \delta_i x^{(i)} = 0 \quad \text{and} \quad \sum_{i=1}^{k} \delta_i = 0.$$

Lemma 2.18. *Let $\mathcal{X} \subset \mathbb{C}^n$ be compact. Then* $\mathrm{co}(\mathcal{X})$ *is compact too.*

Proof. It is clear that, $\mathcal{X}$ being bounded, also $\mathrm{co}(\mathcal{X})$ is so.

Now consider a sequence $\{z_k\}_{k \ge 1}$ of vectors of $\mathrm{co}(\mathcal{X})$ such that there exists

$$\lim_{k \to \infty} z_k = z \in \mathbb{C}^n.$$

In order to prove that $z \in \mathrm{co}(\mathcal{X})$, observe that $\mathbb{C}^n$ may be identified with $\mathbb{R}^{2n}$. Therefore, by Lemma 2.17, for each k, we have $z_k = \sum_{i=1}^{2n+1} \lambda_{k,i} x_k^{(i)}$ with $\lambda_i > 0$, $\sum_{i=1}^{2n+1} \lambda_{k,i} = 1$ and $x_k^{(i)} \in \mathcal{X}$, $i = 1, \dots, 2n+1$.

Since $\mathcal{X}$ is compact, there exists a sequence of integers k_s such that, for $i = 1, \dots, 2n+1$,

$$\lim_{s \to \infty} \lambda_{k_s,i} = \lambda_i \quad \text{with} \quad \sum_{i=1}^{2n+1} \lambda_i = 1$$

and

$$\lim_{s \to \infty} x_{k_s}^{(i)} = x^{(i)} \in \mathcal{X}.$$

Therefore, it must be $z = \sum_{i=1}^{2n+1} \lambda_i x^{(i)} \in \mathrm{co}(\mathcal{X})$. □

Proposition 2.19. *Let $\mathcal{X} \subset \mathbb{C}^n$ be compact. Then*

$$\mathrm{adj}(\mathrm{adj}(\mathcal{X})) = \mathrm{absco}(\mathcal{X}).$$

Proof. First of all, for convenience of notation, define $\mathcal{S} = \mathrm{absco}(\mathcal{X})$ and $\mathcal{S}^* = \mathrm{adj}(\mathcal{X})$. The inclusion $\mathcal{X} \subseteq \mathrm{adj}(\mathcal{S}^*)$ is obvious and, therefore, the absolute convexity of $\mathrm{adj}(\mathcal{S}^*)$ implies that $\mathcal{S} \subseteq \mathrm{adj}(\mathcal{S}^*)$, too.

In order to prove the opposite inclusion, we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by the standard vector space isomorphism $\Phi : \mathbb{C}^n \longrightarrow \mathbb{R}^{2n}$ such that a vector $x = \Re(x) + \mathrm{i}\Im(x) \in \mathbb{C}^n$ is mapped into the vector $\Phi(x) = \begin{bmatrix} \Re(x) \\ \Im(x) \end{bmatrix} \in \mathbb{R}^{2n}$.

For simplicity of notation, since no misunderstanding is possible, we use the notation x for a vector if it is thought as an element of either $\mathbb{C}^n$ or $\mathbb{R}^{2n}$, and we give up to use the notation $\Phi(x)$ in the second case.

Now assume that $\hat{x} \notin \mathcal{S}$. Clearly, $\mathcal{S}$ is a convex subset of $\mathbb{R}^{2n}$ and, since $\mathcal{X}$ is compact, by Proposition 2.7 and Lemmas 2.16 and 2.18, also $\mathcal{S}$ is so. Therefore, Lemma 2.15 applies.

If we set $c = (\hat{z}, \hat{x})$ and $d = \min_{x \in \mathcal{S}} (\hat{z}, x)$ in (8), by the symmetry of $\mathcal{S}$, we have that $c < d < 0$. Hence, by defining the vector $\hat{y} = \begin{bmatrix} \hat{y}^R \\ \hat{y}^I \end{bmatrix} = d^{-1}\hat{z} \in \mathbb{R}^{2n}$, we obtain

$$(\hat{y}, \hat{x}) = \frac{c}{d} > 1 \tag{10}$$

and

$$(\hat{y}, x) \leq 1 \quad \text{for all } x \in \mathcal{S}. \tag{11}$$

By interpreting $\hat{y} = \hat{y}^R + \mathrm{i}\hat{y}^I$ as a vector of $\mathbb{C}^n$ and by considering the usual scalar product $\langle \cdot, \cdot \rangle$ in $\mathbb{C}^n$, we easily obtain

$$(\hat{y}, x) = \Re(\langle \hat{y}, x \rangle) \quad \text{for all } x \in \mathbb{C}^n(\mathbb{R}^{2n}). \tag{12}$$

Since $\mathcal{S}$ is balanced, we have that $x \in \mathcal{S}$ implies also $e^{\mathrm{i}\varphi}x \in \mathcal{S}$ for all $\varphi \in \mathbb{R}$. Thus, by (11) and (12), it holds that

$$(\hat{y}, e^{\mathrm{i}\varphi}x) = \Re(\langle \hat{y}, e^{\mathrm{i}\varphi}x \rangle) = \Re(e^{-\mathrm{i}\varphi}\langle \hat{y}, x \rangle) \leq 1 \tag{13}$$

for all $x \in \mathcal{S}$ and $\varphi \in \mathbb{R}$.

Consequently, by choosing $\varphi(x) = \arg(\langle \hat{y}, x \rangle)$, we get

$$|\langle \hat{y}, x \rangle| = (\hat{y}, e^{\mathrm{i}\varphi(x)}x) \leq 1 \quad \text{for all } x \in \mathcal{S},$$

i.e., $\hat{y} \in \mathrm{adj}(\mathcal{S})$, and therefore, by Proposition 2.13, $\hat{y} \in \mathcal{S}^*$.

On the other hand, (10) and (12) imply $|\langle \hat{y}, \hat{x} \rangle| > 1$, i.e., $\hat{x} \notin \mathrm{adj}(\mathcal{S}^*)$, and thus $\mathrm{adj}(\mathcal{S}^*) \subseteq \mathcal{S}$. □

Corollary 2.20. *Let $\mathcal{P}^*$ be a b.c.p. of adjoint type and let $\mathcal{X} = \{x^{(i)}\}_{1 \leq i \leq m}$ be a finite set of vectors such that $\mathcal{P}^* = \mathrm{adj}(\mathcal{X})$. Then $\mathrm{adj}(\mathcal{P}^*)$ is a b.c.p. and it holds that*

$$\mathrm{adj}(\mathcal{P}^*) = \mathrm{absco}(\mathcal{X}). \tag{14}$$

Corollaries 2.14 and 2.20 yield immediately the following result. In the literature, it is often referred to as the *bipolar theorem*.

Theorem 2.21. *Let $\mathcal{P}$ be a b.c.p. and let $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$. Then it holds that*

$$\mathcal{P} = \mathrm{adj}(\mathcal{P}^*). \tag{15}$$

Conversely, let $\mathcal{P}^*$ be a b.c.p. of adjoint type and let $\mathcal{P} = \mathrm{adj}(\mathcal{P}^*)$. Then it holds that

$$\mathcal{P}^* = \mathrm{adj}(\mathcal{P}). \tag{16}$$

The next two propositions state that, given a b.c.p. $\mathcal{P}$ (a b.c.p. of adjoint type $\mathcal{P}^*$), the essential system of vertices (facets) $\mathcal{X}$ is uniquely determined, but for scalar factors of unitary modulus.

Proposition 2.22. *Assume that $\mathcal{X} = \{x^{(i)}\}_{1\le i\le m}$ and $\hat{\mathcal{X}} = \{\hat{x}^{(i)}\}_{1\le i\le k}$ are two essential systems of vertices for a b.c.p. $\mathcal{P}$. Then $k = m$ and there exists a permutation $\{j_1, \ldots, j_m\}$ of $\{1, \ldots, m\}$ such that, for each $i = 1, \ldots, m$, $\hat{x}^{(i)} = u_i x^{(j_i)}$, where $u_i \in \mathbb{C}$ with $|u_i| = 1$.*

Proof. By hypothesis, it holds that

$$\hat{x}^{(i)} = \sum_{j=1}^{m} \lambda_{ij} x^{(j)} \quad \text{with} \ \sum_{j=1}^{m} |\lambda_{ij}| \le 1, \quad i = 1, \ldots, k, \tag{17}$$

and

$$x^{(h)} = \sum_{i=1}^{k} \mu_{hi} \hat{x}^{(i)} \quad \text{with} \ \sum_{i=1}^{m} |\mu_{hi}| \le 1, \quad h = 1, \ldots, m. \tag{18}$$

Therefore, it follows that

$$x^{(h)} = \sum_{j=1}^{m} \pi_{hj} x^{(j)}, \quad h = 1, \ldots, m, \tag{19}$$

where

$$\pi_{hj} = \sum_{i=1}^{k} \mu_{hi} \lambda_{ij} \quad \text{with} \ \sum_{j=1}^{m} |\pi_{hj}| \le 1, \quad h = 1, \ldots, m, \tag{20}$$

and

$$\hat{x}^{(i)} = \sum_{h=1}^{k} \sigma_{ih} \hat{x}^{(h)}, \quad i = 1, \ldots, k, \tag{21}$$

where

$$\sigma_{ih} = \sum_{j=1}^{m} \lambda_{ij} \mu_{jh} \quad \text{with} \ \sum_{h=1}^{k} |\sigma_{ih}| \le 1, \quad i = 1, \ldots, k. \tag{22}$$

Now, fix an index h, $1 \le h \le m$. If $\pi_{hh} \ne 1$ in (19), in which case $x^{(h)}$ is linearly dependent of the other vertices $\mathcal{X}' = \{x^{(j)}\}_{1\le j\le m}^{j\ne h}$, the essentiality of $\mathcal{X}$ implies that $x^{(h)} \notin \mathrm{absco}(\mathcal{X}')$ and, therefore, it must be

$$|1 - \pi_{hh}| < \sum_{j=1, j\ne h}^{m} |\pi_{hj}|.$$

On the other hand, if we add the quantity $|\pi_{hh}|$ to both sides, by the inequality in (20) we obtain

$$|\pi_{hh}| + |1 - \pi_{hh}| < 1,$$

which is clearly impossible. Therefore, it must be

$$\pi_{hh} = 1 \tag{23}$$

in (19) and hence, again by the inequality in (20),

$$\pi_{hj} = 0 \quad \text{for all } j = 1, \ldots, m, \ j \neq h. \tag{24}$$

Analogously, by using (21) and (22) and by the essentiality of $\hat{\mathcal{X}}$, for any $i = 1, \ldots, k$, we obtain

$$\sigma_{ii} = 1 \tag{25}$$

and

$$\sigma_{ih} = 0 \quad \text{for all } h = 1, \ldots, k, \ h \neq i. \tag{26}$$

Summarizing, if we define the matrices $L = [\lambda_{ij}]_{1\le i\le k, 1\le j\le m}$ and $M = [\mu_{hi}]_{1\le h\le m, 1\le i\le k}$, then (20), (23), (24) and (22), (25), (26) imply

$$LM = I_m \quad \text{and} \quad ML = I_k,$$

where I_m and I_k are the identity $m \times m$- and $k \times k$-matrix, respectively. Thus it must be $k = m$ and $M = L^{-1}$.

In order to conclude the proof, observe that the inequalities in (17) and (18) imply $\|L\|_\infty = \|L^{-1}\|_\infty = 1$, i.e., L is an isometry in the ∞-norm which, for all $z \in \mathbb{C}^m$, is defined by $\|z\|_\infty = \max_{1\le i\le m} |z_i|$.

This implies that all the columns of L are vectors of unitary ∞-norm, as they are the images in L of the vectors of the canonical basis. So, for all $j = 1, \ldots, m$, there exists an index i_j such that $|\lambda_{i_j j}| = 1$. On the other hand, the inequality in (17) implies also $\lambda_{ij} = 0$ for all $i \neq i_j$.

In conclusion, the matrix L must be of the form $L = UP$, where P is a permutation matrix and $U = \mathrm{diag}(u_1, \ldots, u_m)$ with $|u_i| = 1$, $i = 1, \ldots, m$. So the proof is complete. □

Proposition 2.23. *Assume that $\mathcal{X} = \{x^{(i)}\}_{1\le i\le m}$ and $\hat{\mathcal{X}} = \{\hat{x}^{(i)}\}_{1\le i\le k}$ are two essential systems of facets for a b.c.p. of adjoint type $\mathcal{P}^*$. Then $k = m$ and there exists a permutation $\{j_1, \ldots, j_m\}$ of $\{1, \ldots, m\}$ such that, for each $i = 1, \ldots, m$, $\hat{x}^{(i)} = u_i x^{(j_i)}$, where $u_i \in \mathbb{C}$ with $|u_i| = 1$.*

Proof. By Corollary 2.20, we have that

$$\mathrm{adj}(\mathcal{P}^*) = \mathrm{absco}(\mathcal{X}) = \mathrm{absco}(\hat{\mathcal{X}})$$

and that both $\mathcal{X}$ and $\hat{\mathcal{X}}$ are essential systems of vertices for the b.c.p. $\mathrm{adj}(\mathcal{P}^*)$. Thus the result follows by Proposition 2.22. □

Proposition 2.24. *Assume that $\mathcal{P}^*$ is a b.c.p. of adjoint type and that $\mathcal{X} = \{x^{(i)}\}_{1\le i\le m}$ is an essential system of facets for $\mathcal{P}^*$. Then $\mathcal{X}$ is an essential system of vertices for the b.c.p. $\mathrm{adj}(\mathcal{P}^*)$.*

Proof. Assume, by contradiction, that $\mathcal{X}$ is not an essential system of vertices for $\mathrm{adj}(\mathcal{P}^*)$. By Proposition 2.9, this means that, for at least one vertex, say $x^{(1)}$, we have

$$x^{(1)} = \sum_{j=2}^{m} \lambda_j x^{(j)} \quad \text{with} \quad \sum_{j=2}^{m} |\lambda_j| \leq 1. \tag{27}$$

Then consider formula (6) for $i = 1$ in Proposition 2.12, so that (27) yields the absurde chain of inequalities

$$1 = |\langle y^{(1)}, x^{(1)} \rangle| \leq \sum_{j=2}^{m} |\lambda_j| \cdot |\langle y^{(1)}, x^{(j)} \rangle| < 1.$$

□

Proposition 2.25. *Assume that $\mathcal{P}$ is a b.c.p. and that $\mathcal{X} = \{x^{(i)}\}_{1 \leq i \leq m}$ is an essential system of vertices for $\mathcal{P}$. Then $\mathcal{X}$ is an essential system of facets for the b.c.p. of adjoint type* $\mathrm{adj}(\mathcal{P})$.

Proof. Assume, by contradiction, that $\mathcal{X}$ is not an essential system of facets for $\mathrm{adj}(\mathcal{P}) = \mathrm{adj}(\mathcal{X})$ (see Corollary 2.14). Then there exists a proper subset $\mathcal{X}' \subset \mathcal{X}$ which fulfils this property. So, by Corollary 2.20 and Theorem 2.21, we have that

$$\mathcal{P} = \mathrm{absco}(\mathcal{X}) = \mathrm{absco}(\mathcal{X}')$$

and that both $\mathcal{X}$ and $\mathcal{X}'$ are essential systems of vertices for $\mathcal{P}$. This contradicts Proposition 2.22 and the proof is complete. □

Given a b.c.p. $\mathcal{P}$ and the b.c.p. of adjoint type $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$, the foregoing Propositions 2.24 and 2.25 imply that

$$F(\mathcal{P}^*) = V(\mathcal{P}), \tag{28}$$

where $V(\mathcal{P})$ denotes the set of the vertices of $\mathcal{P}$ and $F(\mathcal{P}^*)$ denotes the set of the facets of $\mathcal{P}^*$.

We conclude this section by observing that, by Definitions 2.8 and 2.11 and by Theorem 2.21, we can immediately get the following results.

Proposition 2.26. *Let $\mathcal{P}^*$ be a b.c.p. of adjoint type, $\mathcal{X} = \{x^{(i)}\}_{1 \leq i \leq m}$ be an essential system of facets, $\mathcal{P} = \mathrm{adj}(\mathcal{P}^*)$ and $y \in \partial \mathcal{P}^*$. Then the following condition is satisfied:*

(C1) $|\langle y, x \rangle| \leq 1 \ \forall x \in \mathcal{P}$ *and* $\exists \hat{x} \in \partial \mathcal{P}$, $\hat{x} = \hat{u} x^{(j)}$ *for some j with $\hat{u} \in \mathbb{C}$, $|\hat{u}| = 1$, such that* $\langle y, \hat{x} \rangle = 1$.

Remark that $\hat{x}$ in (*C1*) is a facet of $\mathcal{P}^*$, i.e., a vertex of $\mathcal{P} = \mathrm{adj}(\mathcal{P}^*)$.

Proposition 2.27. *Let $\mathcal{P}$ be a b.c.p., $\mathcal{X} = \{x^{(i)}\}_{1 \leq i \leq m}$ be an essential system of vertices, $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$ and $x \in \partial \mathcal{P}$. Then the following conditions are satisfied:*

(C2) $|\langle y, x \rangle| \leq 1 \ \forall y \in \mathcal{P}^*$ *and* $\exists \hat{y} \in \partial \mathcal{P}^*$ *such that* $\langle \hat{y}, x \rangle = 1$;

(C3) $\sum_{i=1}^{m} |\mu_i| \geq 1$ *whenever* $x = \sum_{i=1}^{m} \mu_i x^{(i)}$ *and* $\exists \lambda_1, \ldots, \lambda_m \in \mathbb{C}$ *such that* $x = \sum_{i=1}^{m} \lambda_i x^{(i)}$ *with* $\sum_{i=1}^{m} |\lambda_i| = 1$.

3. The geometry of the b.c.p.'s

In this section we want to analyze some geometric features of the b.c.p.'s. More precisely, we are interested in a description of their boundaries. As usually done, we shall write $\partial\mathcal{P}$ ($\partial\mathcal{P}^*$) to denote the boundary of $\mathcal{P}$ ($\mathcal{P}^*$).

In doing our geometric analysis, we shall try to use as much as possible the customary terminology which can be found, for example, in Grünbaum [1].

Definition 3.1. Let $\mathcal{P}$ be a b.c.p., $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$ and $y \in \partial\mathcal{P}^*$. Then the convex set

$$F_y = \{x \in \mathcal{P} \;\Big|\; \langle y, x\rangle = 1\} \tag{29}$$

is called a *(geometric) face* of $\mathcal{P}$.

Clearly,

$$F_y \subseteq \partial\mathcal{P} \tag{30}$$

and, by property (*C1*) in Proposition 2.26, the face F_y is never empty.

Definition 3.2. Let $\mathcal{P}$ be a b.c.p. and $y \in \partial\mathcal{P}^*$, where $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$. Then any vertex of $\mathcal{P}$ belonging to F_y is called a *vertex* of F_y.

Theorem 3.3. *Let $\mathcal{P}$ be a b.c.p. and $y \in \partial\mathcal{P}^*$, where $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$. Then the set $\mathcal{X}_y$ of all the vertices of F_y is nonempty and*

$$\mathcal{X}_y \subseteq \mathcal{X},$$

where $\mathcal{X} = \{x^{(i)}\}_{1\le i\le m}$ is a suitable essential system of vertices for $\mathcal{P}$.

Moreover, it holds that

$$F_y = \mathrm{co}(\mathcal{X}_y). \tag{31}$$

Proof. By Theorem 2.21, we have that $\mathcal{P} = \mathrm{adj}(\mathcal{P}^*)$ and, thus, property (*C1*) in Proposition 2.26 implies that F_y has at least one vertex.

By (29) the set F_y can not have two vertices which are proportional to each other. Therefore, all the vertices of F_y belong to an essential system of vertices $\mathcal{X} = \{x^{(i)}\}_{1\le i\le m}$ for $\mathcal{P}$.

In order to prove (31), we can assume without restriction that the vertices of F_y are the first p elements of $\mathcal{X}$.

Then let $x \in F_y$. If $p = m$, the inclusion (30) and the property (*C3*) in Proposition 2.27 imply that

$$1 = \langle y, x\rangle = \sum_{i=1}^{m} \bar{\lambda}_i \langle y, x^{(i)}\rangle = \sum_{i=1}^{m} \bar{\lambda}_i \le \sum_{i=1}^{m} |\lambda_i| = 1,$$

so that

$$\lambda_i \ge 0,\ \ i = 1, \ldots, m,$$

and hence $x \in \mathrm{co}(\mathcal{X}_y)$.

On the other hand, if $p < m$, it holds that

$$\sigma = \max_{p+1 \le i \le m} |\langle y, x^{(i)} \rangle| < 1. \tag{32}$$

Consequently, we have that

$$\begin{aligned} 1 &= \langle y, x \rangle = \sum_{i=1}^{m} \bar{\lambda}_i \langle y, x^{(i)} \rangle = \sum_{i=1}^{p} \bar{\lambda}_i + \sum_{i=p+1}^{m} \bar{\lambda}_i \langle y, x^{(i)} \rangle \\ &\le \sum_{i=1}^{p} |\lambda_i| + \sigma \sum_{i=p+1}^{m} |\lambda_i| \le 1 \end{aligned}$$

and hence, by (32), we can conclude that

$$\lambda_i \ge 0, \; i = 1, \ldots, p \quad \text{and} \quad \lambda_i = 0, \; i = p+1, \ldots, m,$$

Therefore, once again, $x \in \operatorname{co}(\mathcal{X}_y)$.

Conversely, since F_y is a convex set, it immediately follows that $\operatorname{co}(\mathcal{X}_y) \subseteq F_y$. □

Proposition 3.4. *Let $\mathcal{P}$ be a b.c.p., F_y be a (geometric) face of $\mathcal{P}$ and $\mathcal{X}'_y \subset \mathcal{X}_y$. Then*

$$\operatorname{absco}(\mathcal{X}'_y) \bigcap F_y = \operatorname{co}(\mathcal{X}'_y). \tag{33}$$

Proof. It is obvious that $\operatorname{co}(\mathcal{X}'_y) \subseteq \operatorname{absco}(\mathcal{X}'_y) \bigcap F_y$.

In order to prove the opposite inclusion, it is not restrictive to assume that $\mathcal{X}'_y = \{x^{(1)}, \ldots, x^{(k)}\}$. Then consider $x \in \operatorname{absco}(\mathcal{X}'_y) \bigcap F_y$. It holds that

$$x = \sum_{i=1}^{k} \lambda_i x^{(i)} \quad \text{with} \quad \sum_{i=1}^{k} |\lambda_i| = 1.$$

On the other hand, by (29) we get

$$1 = \langle y, x \rangle = \sum_{i=1}^{k} \bar{\lambda}_i \langle y, x^{(i)} \rangle = \sum_{i=1}^{k} \bar{\lambda}_i,$$

which implies that $\lambda_i \ge 0$, $i = 1, \ldots, k$. This means that $x \in \operatorname{co}(\mathcal{X}'_y)$. □

The next theorem states that a nonempty intersection of faces still is a face.

Lemma 3.5. *Let $\mathcal{P}$ be a b.c.p., $\mathcal{P}^* = \operatorname{adj}(\mathcal{P})$, $Y \subseteq \partial \mathcal{P}^*$,*

$$F_Y = \bigcap_{y \in Y} F_y \tag{34}$$

and

$$\mathcal{X}_Y = \bigcap_{y \in Y} \mathcal{X}_y, \tag{35}$$

where, for each $y \in Y$, $\mathcal{X}_y$ is given by Theorem 3.3.

Then it holds that

$$F_Y = \operatorname{co}(\mathcal{X}_Y). \tag{36}$$

Proof. The inclusion

$$\operatorname{co}(\mathcal{X}_Y) \subseteq F_Y$$

is obvious.

If $F_Y = \emptyset$, then (36) is already proved. Otherwise, in order to prove the opposite inclusion, consider $x \in F_Y$. Then let $\hat{y} \in Y$ so that we immediately have

$$x \in F_{\hat{y}} = \operatorname{co}(\mathcal{X}_{\hat{y}}).$$

It is not restrictive to assume that $x^{(1)}, \dots, x^{(k)} \in \mathcal{X}_{\hat{y}}$ are such that

$$x = \sum_{i=1}^{k} \lambda_i x^{(i)} \quad \text{with} \quad \sum_{i=1}^{k} \lambda_i = 1 \quad \text{and} \quad \lambda_i > 0, \; i = 1, \dots, k.$$

Then, by (34), for any $y \in Y$ we have

$$1 = \langle y, x \rangle = \langle y, \sum_{i=1}^{k} \lambda_i x^{(i)} \rangle \leq \sum_{i=1}^{k} \lambda_i |\langle y, x^{(i)} \rangle| \leq \sum_{i=1}^{k} \lambda_i = 1,$$

implying

$$\langle y, x^{(i)} \rangle = 1, \quad i = 1, \dots, k.$$

We can conclude that

$$x^{(1)}, \dots, x^{(k)} \in \mathcal{X}_y \quad \text{for all } y \in Y$$

and hence, by (35), that

$$x \in \operatorname{co}(\mathcal{X}_Y).$$

□

Theorem 3.6. *Let $\mathcal{P}$ be a b.c.p., $\mathcal{P}^* = \operatorname{adj}(\mathcal{P})$, $Y \subseteq \partial \mathcal{P}^*$ and*

$$F_Y = \bigcap_{y \in Y} F_y. \tag{37}$$

If $F_Y \neq \emptyset$, then there exists $\tilde{y} \in \operatorname{co}(Y)$ such that

$$F_Y = F_{\tilde{y}}.$$

Proof. Since all the sets of vertices $\mathcal{X}_y$ have cardinality $\leq m$, by virtue of the foregoing Lemma 3.5 there always exists a finite subset

$$\tilde{Y} = \{\tilde{y}^{(1)}, \dots, \tilde{y}^{(k)}\} \subseteq Y$$

with $1 \leq k \leq m$ such that

$$F_Y = \operatorname{co}(\mathcal{X}_Y) = \operatorname{co}(\mathcal{X}_{\tilde{Y}}) = F_{\tilde{Y}}.$$

As a consequence, it holds that

$$\langle \tilde{y}^{(j)}, x \rangle = 1, \; j = 1, \dots, k \quad \Longleftrightarrow \quad x \in F_Y.$$

Thus we consider the vector

$$\tilde{y} = \frac{1}{k} \sum_{j=1}^{k} \tilde{y}^{(j)}$$

that clearly belongs to $\mathrm{co}(Y) \subset \mathcal{P}^*$.

Let $x \in F_Y$ (there exists such an x since $F_Y \neq \emptyset$). Then we have

$$\langle \tilde{y}, x \rangle = \frac{1}{k} \sum_{j=1}^{k} \langle \tilde{y}^{(j)}, x \rangle = 1,$$

so that $\tilde{y} \in \partial\mathcal{P}^*$ and

$$x \in F_{\tilde{y}}.$$

Vice versa, let $x \in F_{\tilde{y}}$. Then

$$1 = \langle \tilde{y}, x \rangle = \frac{1}{k} \sum_{j=1}^{k} \langle \tilde{y}^{(j)}, x \rangle \leq \frac{1}{k} \sum_{j=1}^{k} |\langle \tilde{y}^{(j)}, x \rangle| \leq 1,$$

implying

$$\langle \tilde{y}^{(j)}, x \rangle = 1, \quad j = 1, \dots, k,$$

that is, $x \in F_Y$. □

The foregoing theorem allows us to give the following definition.

Definition 3.7. Let $\mathcal{P}$ be a b.c.p. and $x \in \partial\mathcal{P}$. Then we say that the set

$$G_x = \bigcap_{x \in F_y} F_y, \tag{38}$$

is the face of $\mathcal{P}$ *generated* by x, that is, the smallest face of $\mathcal{P}$ that contains x.

Observe that, in this case, $Y = \{y \in \partial\mathcal{P}^* \mid x \in F_y\}$, which is always nonempty by virtue of property (*C2*) in Proposition 2.27 (see also Definition 4.1 and formula (56) in Section 4).

Remark 3.8. The face G_x generated by x, which can be written also as F_y for a suitable y, uniquely determines the set $\mathcal{X}_y$ of its vertices, that we also denote by $\mathcal{X}_x$.

Proposition 3.9. *Let $\mathcal{P}$ be a b.c.p. and $x \in \partial\mathcal{P}$. Then x is a vertex if and only if $G_x = \{x\}$.*

Proof. Since Theorem 3.3 states that any face contains at least one vertex, if $G_x = \{x\}$, then x must be a vertex.

Vice versa, let $x \in \partial\mathcal{P}$ be a vertex and, by contradiction, assume that $G_x \supset \{x\}$. Again Theorem 3.3 implies that G_x must contain another vertex, say $\hat{x}$, not proportional to x. Thus both x and $\hat{x}$ belong to an essential system of vertices of $\mathcal{P}$ and, by (38), it holds that, for any $y \in \partial\mathcal{P}^*$,

$$\langle y, x \rangle = 1 \quad \Longrightarrow \quad \langle y, \hat{x} \rangle = 1.$$

Now the above implication clearly contradicts Proposition 2.12 and, therefore, the proof is complete. □

It is useful to introduce the notion of *dimension* of a face. To this aim, we state the following lemma.

Lemma 3.10. *Let $\mathcal{P}$ be a b.c.p. and F_y be a face of $\mathcal{P}$. Then it holds that*

$$\mathrm{span}(F_y - x) = \mathrm{span}(F_y - u) \tag{39}$$

and

$$\dim(\mathrm{span}(F_y - x)) = \dim(\mathrm{span}(F_y)) - 1 \tag{40}$$

for all $x, u \in F_y$.

Proof. Let $x, u \in F_y$. Then we have

$$F_y - x = F_y - u - (x - u) \subset \mathrm{span}(F_y - u),$$

that implies

$$\mathrm{span}(F_y - x) \subseteq \mathrm{span}(F_y - u).$$

Obviously, the opposite inclusion holds too and, thus, (39) is proved.

In order to prove also (40), set

$$\delta = \dim(\mathrm{span}(F_y))$$

and consider δ vectors $\tilde{x}^{(1)}, \ldots, \tilde{x}^{(\delta)} \in F_y$, not necessarily vertices, which are linearly independent. Then any other vector $z \in F_y$ can be written in the form

$$z = \sum_{i=1}^{\delta} \alpha_i \tilde{x}^{(i)}.$$

Since scalar multiplication by y yields

$$\sum_{i=1}^{\delta} \alpha_i = 1,$$

we have that

$$z - \tilde{x}^{(1)} = \sum_{i=2}^{\delta} \alpha_i (\tilde{x}^{(i)} - \tilde{x}^{(1)}).$$

We can conclude that the set of $\delta - 1$ linearly independent vectors $\{\tilde{x}^{(2)} - \tilde{x}^{(1)}, \ldots, \tilde{x}^{(\delta)} - \tilde{x}^{(1)}\}$ is a basis of the space $\mathrm{span}(F_y - \tilde{x}^{(1)})$. Therefore, by virtue of (39), the proof is complete. □

Definition 3.11. Let $\mathcal{P}$ be a b.c.p. and F_y be a face of $\mathcal{P}$. Then we say that the number

$$\dim(F_y) = \dim(\mathrm{span}(F_y)) - 1$$

is the *dimension* of the face F_y.

We have the following result.

Proposition 3.12. *Let $\mathcal{P}$ be a b.c.p. and F_v, F_y be two faces of $\mathcal{P}$ such that $F_v \subset F_y$. Then it holds that*

$$\dim(F_v) < \dim(F_y).$$

Proof. Since $F_v \subset F_y$, it holds that

$$\langle v - y, x \rangle = 0 \quad \text{for all } x \in F_v. \tag{41}$$

It is obvious that $\dim(F_v) \leq \dim(F_y)$. If we assume that $\dim(F_v) = \dim(F_y)$, then it easily follows that

$$F_y \subset \operatorname{span}(F_v).$$

Now, in view of (41), the above inclusion implies

$$\langle v - y, x \rangle = 0 \quad \text{for all } x \in F_y$$

and, consequently, the absurde inclusion $F_y \subseteq F_v$. $\square$

Definition 3.13. Let $\mathcal{P}$ be a b.c.p. and $\mathcal{P}^* = \operatorname{adj}(\mathcal{P})$. Then we say that a vector $y \in \partial\mathcal{P}^*$ is a *facet* of $\mathcal{P}$ if there exist n linearly independent vertices $x^{(i_1)}, \ldots, x^{(i_n)}$ belonging to an essential system of vertices $\mathcal{X} = \{x^{(i)}\}_{1 \leq i \leq m}$ of $\mathcal{P}$ such that

$$\langle y, x^{(i_j)} \rangle = 1, \quad j = 1, \ldots, n,$$

that is,

$$x^{(i_j)} \in F_y, \quad j = 1, \ldots, n.$$

Definition 3.14. Let $\mathcal{P}$ be a b.c.p. and y a facet of $\mathcal{P}$. Then the set F_y is called a *(geometric) facet* of $\mathcal{P}$ as well.

Remark 3.15. The dimension of a face F_y of a b.c.p. $\mathcal{P}$ can vary between 0 and $n - 1$. In particular, a face F_y is a facet if and only if

$$\dim(F_y) = n - 1,$$

whereas, by Proposition 3.9, a face F_y is a vertex if and only if

$$\dim(F_y) = 0.$$

The next theorem gives an important upper bound for computational purposes.

Theorem 3.16. *Let $\mathcal{P}$ be a b.c.p. and let F_y be a face of $\mathcal{P}$ of dimension d. Then, for each $x \in F_y$, there exist s vertices $x^{(i_1)}, \ldots, x^{(i_s)} \in \mathcal{X}_y$ with*

$$s \leq 2d + 1 \tag{42}$$

such that

$$x \in \operatorname{co}\left(x^{(i_1)}, \ldots, x^{(i_s)}\right).$$

Proof. Observe that Theorem 3.3 easily implies

$$F_y - x = \mathrm{co}(\mathcal{X}_y - x) \tag{43}$$

and, hence, in $\mathbb{C}^n$ it holds that

$$\dim(\mathrm{span}(\mathcal{X}_y - x)) = d.$$

Therefore, if we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by the same standard vector space isomorphism (which preserves convexity) we used in the proof of Proposition 2.19, we can conclude that in $\mathbb{R}^{2n}$

$$\dim(\mathrm{span}(\mathcal{X}_y - x)) = 2d.$$

Consequently, by Lemma 2.17 the result is proved. □

It is wellknown that, if the b.c.p. $\mathcal{P}$ is real, then (42) is replaced by the more stringent inequality

$$s \leq d + 1. \tag{44}$$

On the other hand, in the general complex case (44) is not true and the upper bound (42) can be actually attained. This is illustrated by the following example.

Example 3.17. Consider the b.c.p.

$$\mathcal{P} = \mathrm{absco}\left(x^{(1)}, x^{(2)}, x^{(3)}\right)$$

included in $\mathbb{C}^2$, where

$$x^{(1)} = [1, \mathrm{i}]^T, \qquad x^{(2)} = [1, 1-\mathrm{i}]^T, \qquad x^{(3)} = [1, 1]^T.$$

Note that, although these vectors belong to the same line in $\mathbb{C}^2$ (of equation $x_1 = 1$), they form an essential system of vertices.

Now observe that the vector $y = [1, 0]^T$ satisfies

$$\langle y, x^{(i)} \rangle = 1, \quad i = 1, 2, 3,$$

so that $y \in \partial\mathcal{P}^*$,

$$\dim(F_y) = 1$$

and

$$\mathcal{X}_y = \{x^{(1)}, x^{(2)}, x^{(3)}\}.$$

If we consider

$$x = \frac{1}{3}x^{(1)} + \frac{1}{3}x^{(2)} + \frac{1}{3}x^{(3)} = \left[1, \frac{2}{3}\right]^T \in F_y,$$

a few simple calculations show that

$$x \notin \mathrm{absco}\left(x^{(1)}, x^{(2)}\right), \qquad x \notin \mathrm{absco}\left(x^{(1)}, x^{(3)}\right), \qquad x \notin \mathrm{absco}\left(x^{(2)}, x^{(3)}\right).$$

We can conclude that, in this case, the upper bound (42) is actually attained.

We also observe that the b.c.p. $\mathcal{P}$ has also some facets with only two vertices. For example, it turns out that the vector $\tilde{y} = \left[\frac{-1+2\mathrm{i}}{5}, \frac{2-4\mathrm{i}}{5}\right]^T$ satisfies

$$\langle \tilde{y}, -x^{(1)} \rangle = \langle \tilde{y}, x^{(2)} \rangle = 1 \quad \text{and} \quad |\langle \tilde{y}, x^{(3)} \rangle| = \frac{1}{\sqrt{5}} < 1,$$

so that $\tilde{y} \in \partial \mathcal{P}^*$ and

$$\mathcal{X}_{\tilde{y}} = \{-x^{(1)}, x^{(2)}\}.$$

Indeed, by some routine calculations it is possible to show that the b.c.p. $\mathcal{P}$ has only one facet (modulo scalar factors of unitary modulus) with three vertices other than F_y. It is determined by the vector $\hat{y} = \left[\frac{2-\mathrm{i}}{5}, \frac{3+\mathrm{i}}{5}\right]^T$ and the set of vertices is

$$\mathcal{X}_{\hat{y}} = \left\{ \frac{3-4\mathrm{i}}{5} x^{(1)}, \frac{4+3\mathrm{i}}{5} x^{(2)}, x^{(3)} \right\}. \quad \diamond$$

In what follows, we describe some characterizing properties of the internal points of a face.

Lemma 3.18. *Let $\mathcal{P}$ be a b.c.p., F_y be a face of $\mathcal{P}$ that does not reduce to a vertex and $\{x^{(i)}\}_{1 \le i \le k} = \mathcal{X}_y$ be the set of vertices of F_y. Then $x \in F_y \setminus \partial F_y$ if and only if it admits a representation of the form*

$$x = \sum_{i=1}^{k} \lambda_i x^{(i)} \quad \textit{with} \; \sum_{i=1}^{k} \lambda_i = 1 \quad \textit{and} \quad \lambda_i > 0, \; i = 1, \ldots, k. \tag{45}$$

Proof. Assume that $x \in \mathbb{C}^n$ admits the representation (45) with all positive coefficients. Therefore, by virtue of (43) all the vectors of the form

$$x + \sum_{i=1}^{k} \epsilon_i (x^{(i)} - x) \quad \text{with } \epsilon_i \in \mathbb{R}, \; i = 1, \ldots, k \quad \text{and} \quad \max_{1 \le i \le k} |\epsilon_i| < \min_{1 \le i \le k} \lambda_i$$

constitute a nonempty open neighbourhood of x in F_y, implying that $x \notin \partial F_y$.

Vice versa, let $x \in F_y \setminus \partial F_y$. Then, for any vertex $x^{(i)} \in \mathcal{X}_y$, there exists $\tilde{x}^{(i)} \in \partial F_y$ such that

$$x = \alpha_i x^{(i)} + (1 - \alpha_i) \tilde{x}^{(i)} \quad \text{with } \alpha_i > 0, \; i = 1, \ldots, k.$$

Since $\tilde{x}^{(i)} \in \mathrm{co}(\mathcal{X}_y)$, $i = 1, \ldots, k$, it turns out that the representation

$$x = \frac{1}{k} \sum_{i=1}^{k} \left(\alpha_i x^{(i)} + (1 - \alpha_i) \tilde{x}^{(i)} \right)$$

is of the form (45) with all positive coefficients. □

Corollary 3.19. *Let $\mathcal{P}$ be a b.c.p., F_y be a face of $\mathcal{P}$ that does not reduce to a vertex, $x \in F_y$ and $\{x^{(i)}\}_{1 \le i \le k} = \mathcal{X}_y$ be the set of vertices of F_y. Then $x \in \partial F_y$ if and only if there exists an index $j \in \{1, \ldots, k\}$ such that*

$$\lambda_j = 0$$

whenever x is written in the form

$$x = \sum_{i=1}^{k} \lambda_i x^{(i)} \quad \textit{with} \ \sum_{i=1}^{k} \lambda_i = 1 \quad \textit{and} \quad \lambda_i \geq 0, \ i = 1, \ldots, k. \tag{46}$$

Proof. The sufficiency is an obvious consequence of Lemma 3.18.

Vice versa, if for any $j \in \{1, \ldots, k\}$ there existed a representation of $x \in \partial F_y$ of the form (46) with $\lambda_j > 0$, then we could make the average of all such representations and get one of the form (45) with all positive coefficients, that would contradict Lemma 3.18. □

Proposition 3.20. *Let $\mathcal{P}$ be a b.c.p., F_y be a face of $\mathcal{P}$ that does not reduce to a vertex and $x \in \partial\mathcal{P}$. Then*

$$x \in F_y \setminus \partial F_y \quad \Longrightarrow \quad G_x = F_y.$$

Proof. If $x \in F_y \setminus \partial F_y$ the inclusion $G_x \subseteq F_y$ is obvious.

Vice versa, let $z \in \partial\mathcal{P}^*$ such that $x \in F_z$. Then Lemma 3.18 yields

$$1 = \langle z, x \rangle = \sum_{i=1}^{k} \lambda_i \langle z, x^{(i)} \rangle \leq \sum_{i=1}^{k} \lambda_i |\langle z, x^{(i)} \rangle| \leq 1$$

with $\lambda_i > 0$, $i = 1, \ldots, k$, yielding

$$\langle z, x^{(i)} \rangle = 1, \quad i = 1, \ldots, k.$$

Therefore, $\mathcal{X}_y \subseteq F_z$ and, consequently, by (38) it follows that $\mathcal{X}_y \subseteq G_x$. Thus we can conclude that $F_y \subseteq G_x$. □

Remark 3.21. Unlike the case of a real polytope $\mathcal{P}$, in the general complex case the equality $G_x = F_y$ is not a sufficient condition to assure that the point x is internal to the face F_y.

The following example illustrates the previous remark.

Example 3.22. Consider the b.c.p. $\mathcal{P}$ and the face F_y of it which were introduced in the previous Example 3.17.

It is easy to check that the vector

$$\hat{x} = \frac{1}{2} x^{(1)} + \frac{1}{2} x^{(2)} = \left[1, \frac{1}{2}\right]^T \in F_y$$

has no representations of the form (46) with a positive coefficient for $x^{(3)}$. Therefore, by Corollary 3.19 we can conclude that $\hat{x} \in \partial F_y$.

Nevertheless, it can also be easily seen that the only face of $\mathcal{P}$ which includes $\hat{x}$ is F_y, so that $G_{\hat{x}} = F_y$.

More in general, by using Corollary 3.19 we can prove that

$$\partial F_y = \mathrm{co}\left(x^{(1)}, x^{(2)}\right) \bigcup \mathrm{co}\left(x^{(2)}, x^{(3)}\right) \bigcup \mathrm{co}\left(x^{(1)}, x^{(3)}\right)$$

and that $G_{\hat{x}} = F_y$ for any $\hat{x} \in \partial F_y \setminus \mathcal{X}_y$. ◊

The next theorem states the existence of a (geometric) facet including a given point on the boundary of a b.c.p.

Theorem 3.23. *Let $\mathcal{P}$ be a b.c.p. and let $x \in \partial\mathcal{P}$. Then there exists a (geometric) facet F_y such that $x \in F_y$.*

Proof. By property (*C2*) in Proposition 2.27, there exists $\hat{y} \in \partial\mathcal{P}^*$, $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$, such that $\langle \hat{y}, x\rangle = 1$, i.e., $x \in F_{\hat{y}}$.

Now, let $\{x^{(i)}\}_{1\le i\le m}$ be an essential system of vertices for $\mathcal{P}$. We can assume without restriction that $x^{(1)}, \ldots, x^{(p)}$ (with $p \ge 1$) are the vertices of the set $F_{\hat{y}}$ assured by Theorem 3.3, i.e.,

$$\langle \hat{y}, x^{(i)}\rangle = 1, \ i = 1, \ldots, p,$$

and that the remaining vertices $x^{(p+1)}, \ldots, x^{(m)}$ are such that

$$|\langle \hat{y}, x^{(i)}\rangle| < 1, \ i = p+1, \ldots, m. \tag{47}$$

If $\mathrm{span}(\{x^{(1)}, \ldots, x^{(p)}\}) = \mathbb{C}^n$, then $\hat{y}$ is a facet of $\mathcal{P}$ and the theorem is proved.

Otherwise, we have that

$$\dim(\mathrm{span}(\{x^{(1)}, \ldots, x^{(p)}\})) \le n-1. \tag{48}$$

In this case we consider the set

$$Y = \{y \in \partial\mathcal{P}^* \,\Big|\, \langle y, x^{(i)}\rangle = 1, \ i = 1, \ldots, p\}$$

and hence, since Theorem 3.3 implies that $x \in \mathrm{co}(\{x^{(1)}, \ldots, x^{(p)}\})$, it turns out that

$$x \in F_y \quad \text{for all } y \in Y. \tag{49}$$

Now we can assume without restriction that all the other vertices that are linearly dependent on $x^{(1)}, \ldots, x^{(p)}$, if any, are exactly $x^{(p+1)}, \ldots, x^{(q)}$ for some $q \le m-1$, whereas all the other vertices that are linearly independent of $x^{(1)}, \ldots, x^{(p)}$ are exacltly $x^{(q+1)}, \ldots, x^{(m)}$.

Then, on the set Y, we define the function

$$f(y) = \max_{q+1\le i\le m} |\langle y, x^{(i)}\rangle|. \tag{50}$$

Since Y is compact and the function f is continuous on Y, it achieves its maximum M at some vector $\tilde{y} \in Y$. We can assume without restriction that

$$M = f(\tilde{y}) = |\langle \tilde{y}, x^{(q+1)}\rangle| = \langle \tilde{y}, x^{(q+1)}\rangle.$$

If we suppose that $M < 1$, because of (48) we can find $z \in \mathbb{C}^n$ sufficiently small such that

- $\langle z, x^{(i)}\rangle = 0, \quad i = 1, \ldots, p,$
  and, therefore, also for $i = p+1, \ldots, q$;
- $\langle z, x^{(q+1)}\rangle > 0$;
- $|\langle \tilde{y} + z, x^{(i)}\rangle| \le 1, \quad i = p+1, \ldots, m.$

Therefore, it turns out that $\tilde{y} + z \in Y$ and that

$$M \geq f(\tilde{y} + z) \geq \langle \tilde{y} + z, x^{(q+1)} \rangle = M + \langle z, x^{(q+1)} \rangle > M,$$

which is an absurde inequality.

We can conclude that $M = 1$, that is, the set $F_{\tilde{y}}$ has at least $p+1$ vertices and has higher dimension than $F_{\hat{y}}$. Moreover, by (49), it holds that $x \in F_{\tilde{y}}$.

It is clear that, if $\tilde{y}$ is not a facet itself, we can repeat the procedure a suitable number of times until we find a (geometric) facet F_y which includes x. □

Corollary 3.24. *Let $\mathcal{P}$ be a b.c.p. and let F_y be a face. Then there exists a facet F_z such that $F_y \subseteq F_z$.*

Proof. Let $x \in \partial\mathcal{P}$ be such that $G_x = F_y$ (see Proposition 3.20). Since Theorem 3.23 states that there exists a facet F_z such that $x \in F_z$, formula (38) concludes the proof. □

The foregoing Theorem 3.23 and the inclusion (30) allow us to conclude that

$$\partial\mathcal{P} = \bigcup_{y \in F^*(\mathcal{P})} F_y, \tag{51}$$

where $F^*(\mathcal{P})$ is the set of all the facets of $\mathcal{P}$. In other words, $\partial\mathcal{P}$ is the union of all the (geometric) facets of $\mathcal{P}$.

Another straightforward consequence of Theorem 3.23 is that

$$\mathcal{P} = \operatorname{adj}(F^*(\mathcal{P})). \tag{52}$$

However, it is not easy to determine an *essential system of facets*, i.e., a minimal subset Y of $F^*(\mathcal{P})$ such that $\mathcal{P} = \operatorname{adj}(Y)$. In any case, such a minimal subset of facets Y exists by virtue of Zorn's lemma and, clearly, it is infinite. Moreover, we have the following result.

Proposition 3.25. *Let $\mathcal{P}$ be a b.c.p. Then the set $F^*(\mathcal{P})$ of its facets is compact.*

Proof. Since $F^*(\mathcal{P}) \subset \mathcal{P}^*$, $F^*(\mathcal{P})$ is bounded. In order to prove that it is also closed, consider a sequence $\{y_k\}_{k \geq 0}$ of its elements such that there exists

$$\lim_{k \to \infty} y_k = y \in \mathbb{C}^n.$$

Since $\mathcal{P}^*$ is closed, also $y \in \mathcal{P}^*$.

Now consider a vertex x of $\mathcal{P}$ and let $\epsilon > 0$. Then there exists k_ϵ such that, for all $k \geq k_\epsilon$, we have

$$\begin{aligned} |\langle y_k, x \rangle| &\leq |\langle y, x \rangle| + |\langle y_k - y, x \rangle| \\ &\leq |\langle y, x \rangle| + \|x\|_2 \|y_k - y\|_2 \\ &\leq |\langle y, x \rangle| + \epsilon, \end{aligned}$$

where $\|\cdot\|_2$ denotes the Euclidean norm.

Given an essential system of vertices $\mathcal{X} = \{x^{(i)}\}_{1\le i\le m}$, if we assume that y is not a facet, there exist p of such vertices, say $x^{(1)}, \ldots, x^{(p)}$, such that

$$|\langle y, x^{(i)}\rangle| < 1, \quad i = 1, \ldots, p,$$

and such that the remaining vertices $x^{(p+1)}, \ldots, x^{(m)}$, if any, do not span $\mathbb{C}^n$. Therefore, there exist a sufficiently small $\epsilon > 0$ and corresponding k_ϵ such that, for all $k \ge k_\epsilon$ and $i = 1, \ldots, p$, we have

$$|\langle y_k, x^{(i)}\rangle| < 1$$

as well. But this clearly contradicts the assumption that the y_k's are facets of $\mathcal{P}$. □

4. The geometry of the b.c.p.'s of adjoint type and duality relationships

In this section we describe the boundary of a b.c.p. of adjoint type $\mathcal{P}^*$ and point out some useful relationships between b.c.p.'s and b.c.p.'s of adjoint type.

Definition 4.1. Let $\mathcal{P}^*$ be a b.c.p. of adjoint type, $\mathcal{P} = \mathrm{adj}(\mathcal{P}^*)$ and $x \in \partial\mathcal{P}$. Then the convex set

$$F_x^* = \{y \in \mathcal{P}^* \;\Big|\; \langle y, x\rangle = 1\} \tag{53}$$

is called a *(geometric) face* of $\mathcal{P}^*$.

Clearly,

$$F_x^* \subseteq \partial\mathcal{P}^* \tag{54}$$

and, by property (*C2*) in Proposition 2.27, the face F_x^* is never empty. Moreover, given $x \in \partial\mathcal{P}$ and $y \in \partial\mathcal{P}^*$, it holds that

$$y \in F_x^* \quad \Longleftrightarrow \quad x \in F_y. \tag{55}$$

The above equivalence allows us to rewrite the face generated by a given vector $x \in \partial\mathcal{P}$ defined in (38) as

$$G_x = \bigcap_{y\in F_x^*} F_y. \tag{56}$$

In turn, the above formulae yield the following result.

Proposition 4.2. *Let $\mathcal{P}$ be a b.c.p., $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$, $x \in \partial\mathcal{P}$ and G_x be the face of $\mathcal{P}$ generated by x. Then, for any $u \in \partial\mathcal{P}$, it holds that*

$$u \in G_x \quad \Longleftrightarrow \quad F_x^* \subseteq F_u^* \tag{57}$$

and, consequently, that

$$F_x^* = \bigcap_{u\in G_x} F_u^*. \tag{58}$$

Proof. The equality (56) yields

$$u \in G_x \quad \Longleftrightarrow \quad u \in F_y \quad \text{for all } y \in F_x^*.$$

Then the equivalence (55) (for $x = u$) yields

$$u \in G_x \quad \Longleftrightarrow \quad y \in F_u^* \quad \text{for all } y \in F_x^*,$$

that is, (57).

Finally, since $x \in G_x$, (58) is obvious. □

Remark 4.3. By virtue of Proposition 3.20, it turns out that formulae (56) and (58) implicitely define a one-to-one correspondence between the set of the faces of $\mathcal{P}$ and the set of the faces of $\mathcal{P}^*$.

Moreover, this correspondence is clearly strictly decreasing, that is,

$$G_u \subset G_x \qquad \Longleftrightarrow \qquad F_u^* \supset F_x^* \tag{59}$$

for all $u, x \in \partial\mathcal{P}$.

Definition 4.4. Let $\mathcal{P}^*$ be a b.c.p. of adjoint type and x be a facet of $\mathcal{P}^*$. Then the face F_x^* is called a *(geometric) facet* of $\mathcal{P}^*$ as well.

In view of Definitions 4.1 and 4.4, property (*C1*) in Proposition 2.26 obviously yields the following result.

Proposition 4.5. *Let $\mathcal{P}^*$ be a b.c.p. of adjoint type and let $y \in \partial\mathcal{P}^*$. Then there exists a (geometric) facet F_x^* such that $y \in F_x^*$.*

By Proposition 4.5 and formula (54), we can conclude that

$$\partial\mathcal{P}^* = \bigcup_{x \in F(\mathcal{P}^*)} F_x^*. \tag{60}$$

In other words, $\partial\mathcal{P}^*$ is the union of all the (geometric) facets of $\mathcal{P}^*$. Moreover, $\partial\mathcal{P}^*$ is *piecewise quadratic* in the following sense.

Proposition 4.6. *Let $\mathcal{P}^*$ be a b.c.p. of adjoint type and $\mathcal{X} = \{x^{(i)}\}_{1 \le i \le m}$ be an essential system of facets for $\mathcal{P}^*$. Moreover, let $\Phi : \mathbb{C}^n \longrightarrow \mathbb{R}^{2n}$ be the standard vector space isomorphism such that, for each $x = \Re(x) + \mathrm{i}\Im(x) \in \mathbb{C}^n$, $\Phi(x) = \begin{bmatrix} \Re(x) \\ \Im(x) \end{bmatrix}$.*

Then the boundary of $\Phi(\mathcal{P}^*)$ is contained in the union of the zero sets of m quadratic polynomials $p_i(z_1, \ldots, z_{2n}) \in \mathbb{R}[z_1, \ldots, z_{2n}]$ such that $p_i(0, \ldots, 0) \neq 0$, $i = 1, \ldots, m$.

Proof. It is straightforward to see that

$$\partial\Phi(\mathcal{P}^*) = \Phi(\partial\mathcal{P}^*).$$

Thus, by denoting the vectors $z \in \mathbb{R}^{2n}$ in the form $\begin{bmatrix} z^R \\ z^I \end{bmatrix}$ where $z^R, z^I \in \mathbb{R}^n$, Proposition 2.26 implies that each $z \in \partial\Phi(\mathcal{P}^*)$ satisfies the second degree algebraic equation

$$|\langle z^R + \mathrm{i}z^I, x^{(i)} \rangle|^2 = 1$$

for at least one index $i \in \{1, \ldots, m\}$.

We can conclude that $\partial\Phi(\mathcal{P}^*)$ is included in the union of the zero sets of the quadratic polynomials

$$p_i = \left(\sum_{j=1}^{n} \Re(x_j^{(i)}) z_j^R + \Im(x_j^{(i)}) z_j^I \right)^2 + \left(\sum_{j=1}^{n} \Re(x_j^{(i)}) z_j^I - \Im(x_j^{(i)}) z_j^R \right)^2 - 1$$

in the $2n$ variables $z_1^R, \ldots, z_n^R, z_1^I, \ldots, z_n^I$.

Finally, observe that $p_i(0, \ldots, 0) = -1$, $i = 1, \ldots, m$. □

Now we give the adjoint version of Definition 3.2.

Definition 4.7. Let $\mathcal{P}^*$ be a b.c.p. of adjoint type and $x \in \partial\mathcal{P}$, where $\mathcal{P} = \mathrm{adj}(\mathcal{P}^*)$. Then any (geometric) facet of $\mathcal{P}^*$ that contains F_x^* is called a *(geometric) facet* passing through F_x^*.

The following theorem contains the adjoint version of Theorem 3.3.

Theorem 4.8. *Let $\mathcal{P}^*$ be a b.c.p. of adjoint type and $x \in \partial\mathcal{P}$, where $\mathcal{P} = \mathrm{adj}(\mathcal{P}^*)$. Then the set of all the facets u of $\mathcal{P}^*$ such that F_u^* passes through F_x^* is nonempty and coincides with the set $\mathcal{X}_x$ of the vertices of the face G_x of $\mathcal{P}$ generated by x.*

Moreover, it holds that

$$F_x^* = \bigcap_{u \in \mathcal{X}_x} F_u^*. \tag{61}$$

Proof. By Proposition 2.24, the equivalence (57) implies that all the facets passing through F_x^* are necessarily vertices of G_x. Vice versa, by (58) it is obvious that

$$F_x^* \subseteq \bigcap_{u \in \mathcal{X}_x} F_u^*.$$

Therefore, by Proposition 2.25, the vertices of G_x are necessarily facets passing through F_x^* and, thus, the first part of the proof is complete.

In order to reverse the above inclusion, observe that

$$x = \sum_{i=1}^{k} \lambda_i x^{(i)} \quad \text{with} \ \sum_{i=1}^{k} \lambda_i = 1 \quad \text{and} \quad \lambda_i \geq 0, \ i = 1, \ldots, k,$$

where $\{x^{(1)}, \ldots, x^{(k)}\} = \mathcal{X}_x$.

Now let

$$y \in \bigcap_{u \in \mathcal{X}_x} F_u^* = \bigcap_{i=1}^{k} F_{x^{(i)}}^*,$$

so that

$$\langle y, x^{(i)} \rangle = 1, \quad i = 1, \ldots, k, \tag{62}$$

and, hence,

$$\langle y, x \rangle = \sum_{i=1}^{k} \lambda_i \langle y, x^{(i)} \rangle = \sum_{i=1}^{k} \lambda_i = 1.$$

This implies that $y \in F_x^*$ and (61) is proved, too. □

The next theorem states that a nonempty intersection of faces still is a face.

Theorem 4.9. *Let $\mathcal{P}^*$ be a b.c.p. of adjoint type, $\mathcal{P} = \mathrm{adj}(\mathcal{P}^*)$, $X \subset \partial\mathcal{P}$ and*

$$F_X^* = \bigcap_{x \in X} F_x^*.$$

If $F_X^* \neq \emptyset$, then there exists $\tilde{x} \in \mathrm{co}(X)$ such that

$$F_X^* = F_{\tilde{x}}^*.$$

Proof. Theorem 4.8 allows us to say that

$$F_X^* = \bigcap_{u \in \mathcal{X}_X} F_u^*, \tag{63}$$

where

$$\mathcal{X}_X = \bigcup_{x \in X} \mathcal{X}_x$$

is a set of facets of $\mathcal{P}^*$.

If $F_X^* \neq \emptyset$, then there exists $y \in F_X^*$. Thus the equivalence (55) (for $x = u$) and (63) imply that $\mathcal{X}_X \subseteq \mathcal{X}_y$, the set of vertices of the face F_y of $\mathcal{P}$. So we can conclude that the set $\mathcal{X}_X$ is finite and, consequently, that there exists a finite subset of faces

$$\tilde{X} = \{\tilde{x}^{(1)}, \ldots, \tilde{x}^{(s)}\} \subseteq X$$

with $1 \leq s \leq m$ such that

$$F_X^* = \bigcap_{i=1}^{s} F_{\tilde{x}^{(i)}}^*.$$

As a consequence, it holds that

$$\langle y, \tilde{x}^{(i)} \rangle = 1, \ i = 1, \ldots, s \qquad \Longleftrightarrow \qquad y \in F_X^*.$$

Thus we consider the vector

$$\tilde{x} = \frac{1}{s} \sum_{i=1}^{s} \tilde{x}^{(i)}$$

that clearly belongs to $\mathrm{co}(X) \subset \mathcal{P}$.

Let $y \in F_X^*$. Then we have

$$\langle y, \tilde{x} \rangle = \frac{1}{s} \sum_{i=1}^{s} \langle y, \tilde{x}^{(i)} \rangle = 1,$$

so that $\tilde{x} \in \partial\mathcal{P}$ and

$$y \in F_{\tilde{x}}^*.$$

Vice versa, let $y \in F^*_{\tilde{x}}$. Then

$$1 = \langle y, \tilde{x} \rangle = \frac{1}{s} \sum_{i=1}^{s} \langle y, \tilde{x}^{(i)} \rangle \le \frac{1}{s} \sum_{i=1}^{s} |\langle y, \tilde{x}^{(i)} \rangle| \le 1,$$

implying

$$\langle y, \tilde{x}^{(i)} \rangle = 1, \quad i = 1, \ldots, s,$$

that is, $y \in F^*_X$. □

The foregoing theorem allows us to give the adjoint version of Definition 3.7.

Definition 4.10. Let $\mathcal{P}^*$ be a b.c.p. of adjoint type and $y \in \partial \mathcal{P}^*$. Then we say that the set

$$G^*_y = \bigcap_{y \in F^*_x} F^*_x, \tag{64}$$

is the face of $\mathcal{P}^*$ *generated* by y, that is, the smallest face of $\mathcal{P}^*$ that contains y.

Then the adjoint version of formula (56) and Proposition 4.2 hold, too. We have

$$G^*_y = \bigcap_{x \in F_y} F^*_x. \tag{65}$$

Proposition 4.11. *Let $\mathcal{P}^*$ be a b.c.p. of adjoint type, $\mathcal{P} = \mathrm{adj}(\mathcal{P}^*)$, $y \in \partial \mathcal{P}^*$ and G^*_y be the face of $\mathcal{P}^*$ generated by y. Then, for any $v \in \partial \mathcal{P}^*$, it holds that*

$$v \in G^*_y \quad \Longleftrightarrow \quad F_y \subseteq F_v \tag{66}$$

and, consequently, that

$$F_y = \bigcap_{v \in G^*_y} F_v. \tag{67}$$

Remark 4.12. Formulae (65) and (67) implicitely define the same one-to-one correspondence between the set of the faces of $\mathcal{P}$ and the set of the faces of $\mathcal{P}^*$ mentioned in Remark 4.3.

Moreover,

$$G^*_v \subset G^*_y \quad \Longleftrightarrow \quad F_v \supset F_y \tag{68}$$

for all $v, y \in \partial \mathcal{P}^*$.

It is of interest to point out also the following (obvious) result.

Proposition 4.13. *Let $\mathcal{P}$ be a b.c.p., $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$, $x \in \partial \mathcal{P}$ and $y \in \partial \mathcal{P}^*$. Then*

$$G_x = F_y \quad \Longleftrightarrow \quad G^*_y = F^*_x.$$

Inspired by Proposition 3.9, we give the following definition.

Definition 4.14. Let $\mathcal{P}^*$ be a b.c.p. of adjoint type. Then a vector $y \in \partial \mathcal{P}^*$ is called a vertex of $\mathcal{P}^*$ if $G^*_y = \{y\}$.

We can characterize all the vertices of a b.c.p. of adjoint type $\mathcal{P}^*$ as follows.

Proposition 4.15. *Let $\mathcal{P}^*$ be a b.c.p. of adjoint type. Then a vector $y \in \partial\mathcal{P}^*$ is a vertex if and only if there exist n linearly independent facets $x^{(i_1)}, \ldots, x^{(i_n)}$, belonging to an essential system of facets $\mathcal{X} = \{x^{(i)}\}_{1 \le i \le m}$ of $\mathcal{P}^*$, such that*

$$\langle y, x^{(i_j)} \rangle = 1, \quad j = 1, \ldots, n,$$

that is,

$$y \in F^*_{x^{(i_j)}}, \quad j = 1, \ldots, n.$$

Proof. Let $y \in \partial\mathcal{P}^*$ and $G^*_y = \{y\}$. If we assume that F_y is not a facet of $\mathcal{P} = \mathrm{adj}(\mathcal{P}^*)$, then Corollary 3.24 implies the existence of a facet F_z such that $F_y \subset F_z$. Consequently, (68) makes the absurde and, hence, we can conclude that F_y is necessarily a facet. Eventually, Proposition 2.24 and Definition 3.13 complete the proof of the only-if-part.

Vice versa, let $y \in \partial\mathcal{P}^*$ be the (necessarily unique) solution of the system of n linearly independent equations $\langle v, x^{(i_j)} \rangle = 1$, $j = 1, \ldots, n$, and assume that $\{y\} \subset G^*_y$. Then there exists $z \in G^*_y$, $z \neq y$ and, consequently, (66) implies that such a system has the solution z too, that makes the absurde. □

Remark that, in view of Proposition 2.24, Definition 3.13 and the foregoing Proposition 4.15, the vertices of $\mathcal{P}^*$ are the facets of $\mathcal{P} = \mathrm{adj}(\mathcal{P}^*)$. In other words, if we denote by $V^*(\mathcal{P}^*)$ the set of the vertices of $\mathcal{P}^*$, we have that

$$V^*(\mathcal{P}^*) = F^*(\mathcal{P}).$$

Finally, observe that, by (52) and Propositions 2.19 and 3.25, it holds that

$$\mathcal{P}^* = \mathrm{absco}(V^*(\mathcal{P}^*)). \tag{69}$$

Analogously to the case of the equality (52), it is not easy to determine an *essential system of vertices*, i.e., a minimal subset Y of $V^*(\mathcal{P}^*)$ such that $\mathcal{P}^* = \mathrm{absco}(Y)$, which exists by virtue of Zorn's lemma and is clearly infinite.

In what follows, we describe some characterizing properties of the internal points of a face.

Lemma 4.16. *Let $\mathcal{P}^*$ be a b.c.p. of adjoint type, F^*_x be a face of $\mathcal{P}^*$ that does not reduce to a vertex and $y \in F^*_x$. Then $y \in \partial F^*_x$ if and only if there exists a facet $\hat{u} \in F(\mathcal{P}^*) \setminus \mathcal{X}_x$ such that $y \in F^*_{\hat{u}}$.*

Proof. Let there exists a facet $\hat{u} \in F(\mathcal{P}^*) \setminus \mathcal{X}_x$ such that $y \in F^*_{\hat{u}}$ and assume, by contradiction, that $y \notin \partial F^*_x$. Then for any $z \in F^*_x$, $z \neq y$, there exists $\tilde{z} \in \partial F^*_x$ such that

$$y = \alpha z + (1 - \alpha)\tilde{z} \quad \text{with } 0 < \alpha < 1.$$

Thus we have

$$1 = \langle y, \hat{u} \rangle = \langle \alpha z + (1 - \alpha)\tilde{z}, \hat{u} \rangle \le \alpha |\langle z, \hat{u} \rangle| + (1 - \alpha)|\langle \tilde{z}, \hat{u} \rangle| \le 1,$$

which implies

$$\langle z, \hat{u} \rangle = \langle \tilde{z}, \hat{u} \rangle = 1.$$

We can conclude that $F_x^* \subseteq F_{\hat{u}}^*$, which is an absurde inclusion since $\hat{u} \notin \mathcal{X}_x$. Therefore, $y \in \partial F_x^*$.

In order to prove the opposite implication, assume that $y \notin F_u^*$ for all $u \in F(\mathcal{P}^*) \setminus \mathcal{X}_x$.

If $\mathcal{X}_x = \{x^{(i)}\}_{1 \le i \le k}$, then we consider other $m-k$ vertices $x^{(k+1)}, \ldots, x^{(m)}$ such that $\mathcal{X} = \{x^{(i)}\}_{1 \le i \le m}$ is an essential system of facets for $\mathcal{P}^*$. Note that, since F_x^* does not reduce to a vertex, we have necessarily $k < m$. It follows that

$$\max_{k+1 \le i \le m} |\langle y, x^{(i)} \rangle| < 1$$

and, hence, that there exists an open neighbourhood V of y in $\mathbb{C}^n$ such that

$$|\langle v, x^{(i)} \rangle| < 1, \; i = k+1, \ldots, m, \quad \text{for all } v \in V.$$

Then consider the affine subspace

$$S = \bigcap_{i=1}^{k} \{ v \in \mathbb{C}^n \;\Big|\; \langle v, x^{(i)} \rangle = 1 \},$$

which is nonempty since $F_x^* \subset S$. Now it is clear that $V \bigcap S$ is nonempty and is an open neighbourhood of y in F_x^*. Therefore, we can conclude that $y \notin \partial F_x^*$. □

Proposition 4.17. *Let $\mathcal{P}^*$ be a b.c.p. of adjoint type, F_x^* be a face of $\mathcal{P}^*$ that does not reduce to a vertex and $y \in \partial \mathcal{P}^*$. Then*

$$y \in F_x^* \setminus \partial F_x^* \quad \Longleftrightarrow \quad G_y^* = F_x^*.$$

Proof. Assume that $y \in F_x^* \setminus \partial F_x^*$. Then it necessarily holds that $G_y^* \subseteq F_x^*$ and, on the other hand, Lemma 4.16 implies that $y \notin F_u^*$ for all $u \in F(\mathcal{P}^*) \setminus \mathcal{X}_x$. Therefore, by Theorem 4.8 we can conclude that $G_y^* = F_x^*$ as they are two faces which have the same set of facets passing through them.

Vice versa, assume that $G_y^* = F_x^*$. Clearly, it holds that $y \in F_x^*$. Since Theorem 4.8 implies that $F_x^* \bigcap F_{\hat{u}}^* \subset F_x^*$ whenever $\hat{u} \in F(\mathcal{P}^*) \setminus \mathcal{X}_x$, such an $\hat{u}$ for which $y \in F_{\hat{u}}^*$ cannot exist. Eventually, Lemma 4.16 yields $y \notin \partial F_x^*$. □

We remark that, unlike the case of the faces of a b.c.p. $\mathcal{P}$ (see Remark 3.21), the equality $G_y^* = F_x^*$ is a sufficient condition to assure that the point y is internal to the face F_x^*.

The proof of the following lemma is the same as that of Lemma 3.10.

Lemma 4.18. *Let $\mathcal{P}^*$ be a b.c.p. of adjoint type and F_x^* be a face of $\mathcal{P}^*$. Then it holds that*

$$\operatorname{span}(F_x^* - y) = \operatorname{span}(F_x^* - v) \tag{70}$$

and

$$\dim(\operatorname{span}(F_x^* - y)) = \dim(\operatorname{span}(F_x^*)) - 1 \tag{71}$$

for all $y, v \in F_x^*$.

Definition 4.19. Let $\mathcal{P}^*$ be a b.c.p. of adjoint type and F_x^* be a face of $\mathcal{P}^*$. Then we say that the number

$$\dim(F_x^*) = \dim(\mathrm{span}(F_x^*)) - 1$$

is the *dimension* of the face F_x^*.

Now we can state two further important relationships between the faces of $\mathcal{P}$ and the faces of $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$.

Theorem 4.20. *Let $\mathcal{P}$ be a b.c.p., $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$, $x \in \partial\mathcal{P}$, $y \in F_x^*$ and $u \in G_x$. Then it holds that*

$$\mathrm{span}(F_x^* - y) = \Big(\mathrm{span}(G_x)\Big)^{\perp}. \tag{72}$$

and

$$\mathrm{span}(G_x - u) = \Big(\mathrm{span}(F_x^*)\Big)^{\perp}. \tag{73}$$

Proof. Formula (56) implies that $\langle v, u\rangle = 1$ for all $v \in F_x^*$ and $u \in G_x$. Consequently, $u \perp F_x^* - y$ and $v \perp G_x - u$. Therefore, the inclusions

$$\mathrm{span}(F_x^* - y) \subseteq \Big(\mathrm{span}(G_x)\Big)^{\perp} \tag{74}$$

and

$$\mathrm{span}(G_x - u) \subseteq \Big(\mathrm{span}(F_x^*)\Big)^{\perp} \tag{75}$$

are proved.

The inclusion opposite to (74) is obvious if $\Big(\mathrm{span}(G_x)\Big)^{\perp} = \{0\}$. Otherwise, let $\mathcal{X}_x = \{x^{(i)}\}_{1\le i\le k}$ be the set of the vertices of G_x and consider $v \in F_x^*$ such that $F_v = G_x$. Then v is such that

$$\langle v, x^{(i)}\rangle = 1, \quad i = 1, \ldots, k,$$

and

$$|\langle v, x^{(i)}\rangle| < 1, \quad i = k+1, \ldots, m,$$

where $x^{(k+1)}, \ldots, x^{(m)}$ are the remaining elements of an essential system of vertices of $\mathcal{P}$ that includes $\mathcal{X}_x$.

Now it is clear that there exists a sufficiently small neighbourhood of the null vector 0 in $\Big(\mathrm{span}(G_x)\Big)^{\perp}$ such that, for any $z \in V$, we have

$$\langle v + z, x^{(i)}\rangle = 1, \quad i = 1, \ldots, k,$$

and

$$|\langle v + z, x^{(i)}\rangle| < 1, \quad i = k+1, \ldots, m.$$

This means that $F_{v+z} = G_x$ (since they have the same vertices), and hence $v + z \in F_x^*$, for all of such $z \in V$, that implies

$$V \subseteq F_x^* - v.$$

Since $\Big(\mathrm{span}(G_x)\Big)^{\perp} = \mathrm{span}(V)$, we can conclude that

$$\Big(\mathrm{span}(G_x)\Big)^{\perp} \subseteq \mathrm{span}(F_x^* - v).$$

Eventually, (70) completes the proof of (72).

In order to prove also (73), observe that Lemma 3.10, Lemma 4.18 and the equality (72) imply

$$\begin{aligned}
\dim(\mathrm{span}(G_x - u)) &= \dim(\mathrm{span}(G_x)) - 1 \\
&= \dim\Big(\Big(\mathrm{span}(F_x^* - y)\Big)^{\perp}\Big) - 1 \\
&= n - \dim(\mathrm{span}(F_x^* - y)) - 1 \\
&= n - \dim(\mathrm{span}(F_x^*)) \\
&= \dim\Big(\Big(\mathrm{span}(F_x^*)\Big)^{\perp}\Big).
\end{aligned}$$

Eventually, (75) concludes the proof. □

Corollary 4.21. *Let $\mathcal{P}^*$ be a b.c.p. of adjoint type, $\mathcal{P} = \mathrm{adj}(\mathcal{P}^*)$, $y \in \partial\mathcal{P}^*$, $x \in F_y$ and $v \in G_y^*$. Then it holds that*

$$\mathrm{span}(G_y^* - v) = \Big(\mathrm{span}(F_y)\Big)^{\perp}.$$

and

$$\mathrm{span}(F_y - x) = \Big(\mathrm{span}(G_y^*)\Big)^{\perp}.$$

Proof. In view of the foregoing Theorem 4.20, this result is an obvious consequence of Propositions 3.20, 4.13 and 4.17. □

Also the relationships between the dimensions of the faces of $\mathcal{P}$ and the faces of $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$ are now obvious.

Corollary 4.22. *Let $\mathcal{P}^*$ be a b.c.p. of adjoint type, $\mathcal{P} = \mathrm{adj}(\mathcal{P}^*)$, $x \in \partial\mathcal{P}$ and $y \in \partial\mathcal{P}^*$. Then it holds that*

$$\dim(G_x) + \dim(F_x^*) = n - 1$$

and

$$\dim(G_y^*) + \dim(F_y) = n - 1.$$

Remark 4.23. The dimension of a face F_x^* of a b.c.p. of adjoint type $\mathcal{P}^*$ can vary between 0 and $n-1$. In particular, the previous corollary implies that the a face F_x^* is a facet if and only if

$$\dim(F_x^*) = n - 1,$$

whereas a face F_x^* is a vertex if and only if

$$\dim(F_x^*) = 0.$$

The foregoing Corollary 4.22 along with formula (59) yields the analogue of Proposition 3.12.

Proposition 4.24. *Let $\mathcal{P}^*$ be a b.c.p. of adjoint type and F_u^*, F_x^* be two faces of $\mathcal{P}^*$ such that $F_u^* \subset F_x^*$. Then it holds that*

$$\dim(F_u^*) < \dim(F_x^*).$$

We conclude the section with a corollary to Theorem 4.8.

Corollary 4.25. *Let $\mathcal{P}^*$ be a b.c.p. of adjoint type and F_x^* be a face of $\mathcal{P}^*$ of dimension d. Then there exist s facets $F^*_{x^{(i_1)}}, \ldots, F^*_{x^{(i_s)}}$, with $s \le n-d$, such that $x^{(i_1)}, \ldots, x^{(i_s)}$ are linearly independent and*

$$F_x^* = \bigcap_{j=1}^{s} F^*_{x^{(i_j)}}. \tag{76}$$

Proof. Corollary 4.22 implies that $\dim(G_x) = n-d-1$. Therefore, there exist $n-d$ linearly independent vertices $x^{(i_1)}, \ldots, x^{(i_{n-d})} \in \mathcal{X}_x$ and, by Theorem 4.8, they are such that

$$F_x^* \subseteq \bigcap_{j=1}^{n-d} F^*_{x^{(i_j)}}. \tag{77}$$

On the other hand, it is clear that the dimension of the face $\bigcap_{j=1}^{n-d} F^*_{x^{(i_j)}}$ cannot exceed d. Hence, by virtue of Proposition 4.24, the inclusion (77) is indeed an equality.

Finally, we observe that it might well be that the first s facets, for some $s < n-d$, already satisfy the equality (76) (see the forthcoming Example 4.26). □

Example 4.26. Consider the b.c.p.

$$\mathcal{P} = \text{absco}\left(x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}\right)$$

included in $\mathbb{C}^3$, where

$$x^{(1)} = [1,1,1]^T, \qquad x^{(2)} = [-1,1,1]^T, \qquad x^{(3)} = [-1,-1,1]^T, \qquad x^{(4)} = [1,-1,1]^T,$$

and then consider its adjoint

$$\mathcal{P}^* = \text{adj}\left(x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}\right).$$

As it is illustrated by Figure 1 (where the intersection with the real space $\mathbb{R}^3$ is depicted), choosing $x = [0,0,1]^T$ yields

$$G_x = \text{co}\left(x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}\right)$$

and

$$F_x^* = \{y\}, \quad \text{where } y = [0,0,1]^T.$$

Moreover, accordingly with Corollary 4.22, it holds that

$$\dim(G_x) = 2 \quad \text{and} \quad \dim(F_x^*) = 0.$$

On the other hand, the equality (76) holds with

$$s = 2 < n - \dim(F_x^*) = 3,$$

since we have

$$F_x^* = F_{x^{(1)}}^* \bigcap F_{x^{(3)}}^* = F_{x^{(2)}}^* \bigcap F_{x^{(4)}}^*. \ \diamond$$

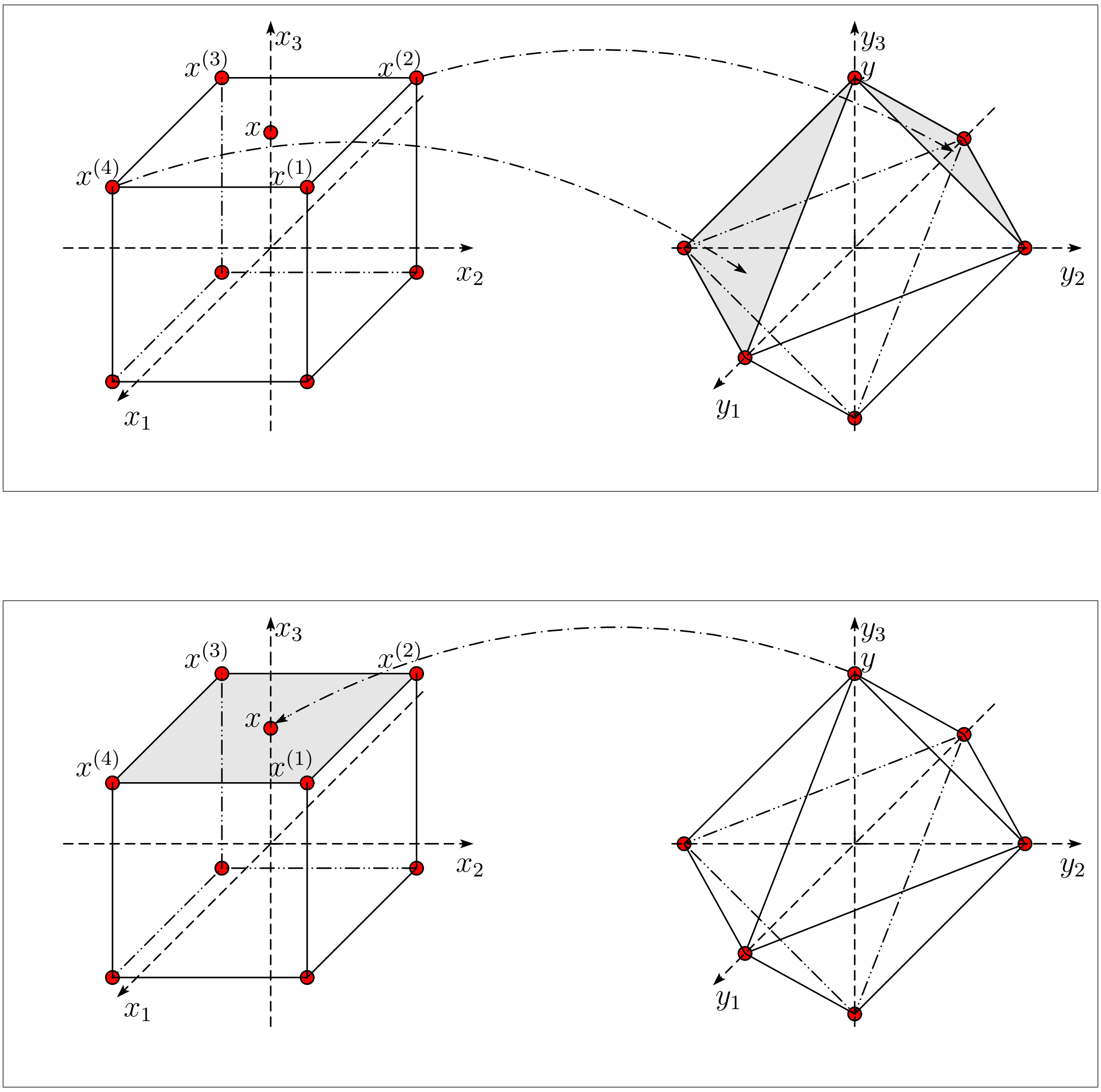

Figure 4.1: Illustration of Example 4.26

5. Complex polytope norms

Now we extend the concept of *polytope norm* to the complex case in a straightforward way.

Lemma 5.1. *Any b.c.p.* $\mathcal{P}$ *is the unit ball of a norm* $\| \cdot \|_{\mathcal{P}}$ *on* $\mathbb{C}^n$.

Proof. Since $\mathrm{span}(\mathcal{X}) = \mathbb{C}^n$, the set $\mathcal{P}$ is *absorbing*. Therefore, since it is absolutely convex and bounded, the Minkowski functional associated to $\mathcal{P}$, defined for all $z \in \mathbb{C}^n$ by

$$\|z\|_{\mathcal{P}} = \inf\{\rho > 0 \mid z \in \rho\mathcal{P}\}, \tag{78}$$

is indeed a norm on $\mathbb{C}^n$ (see again [3]). □

Definition 5.2. We shall call *complex polytope norm* any norm $\|\cdot\|_{\mathcal{P}}$ whose unit ball is a b.c.p. $\mathcal{P}$.

Due to the substantial difference between a b.c.p. $\mathcal{P}$ and a b.c.p. of adjoint type $\mathcal{P}^*$, we need to introduce also the adjoint version of the concept of complex polytope norm.

Lemma 5.3. *Any b.c.p. of adjoint type $\mathcal{P}^*$ is the unit ball of a norm $\|\cdot\|_{\mathcal{P}^*}$ on $\mathbb{C}^n$.*

Proof. Since also $\mathcal{P}^*$ is absorbing and absolutely convex, as in the proof of Lemma 5.1, we can conclude that the Minkowski functional defined by

$$\|z\|_{\mathcal{P}^*} = \inf\{\rho > 0 \mid z \in \rho\mathcal{P}^*\} \tag{79}$$

is a norm on $\mathbb{C}^n$. □

Definition 5.4. We shall call *adjoint complex polytope norm* any norm $\|\cdot\|_{\mathcal{P}^*}$ whose unit ball is a b.c.p. of adjoint type $\mathcal{P}^*$.

Now we illustrate an important link between polytope norms and adjoint polytope norms.

Theorem 5.5. *Let $\mathcal{P}$ be a b.c.p. and let $\|\cdot\|_{\mathcal{P}}$ be the corresponding complex polytope norm. Then, for any $z \in \mathbb{C}^n$, it holds that*

$$\|z\|_{\mathcal{P}} = \min\Big\{ \sum_{i=1}^{m} |\lambda_i| \;\Big|\; z = \sum_{i=1}^{m} \lambda_i x^{(i)} \Big\} = \max_{y \in \partial\mathcal{P}^*} |\langle y, z\rangle|, \tag{80}$$

where $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$ and $\mathcal{X} = \{x^{(i)}\}_{1\le i\le m}$ is an essential system of vertices for $\mathcal{P}$.

Analogously, let $\mathcal{P}^*$ be a b.c.p. of adjoint type and let $\|\cdot\|_{\mathcal{P}^*}$ the corresponding adjoint complex polytope norm. Then, for any $z \in \mathbb{C}^n$, it holds that

$$\|z\|_{\mathcal{P}^*} = \max_{1\le i\le m} |\langle z, x^{(i)}\rangle| = \max_{x \in \partial\mathcal{P}} |\langle z, x\rangle|, \tag{81}$$

where $\mathcal{P} = \mathrm{adj}(\mathcal{P}^*)$ and $\mathcal{X} = \{x^{(i)}\}_{1\le i\le m}$ is an essential system of facets for $\mathcal{P}^*$.

Proof. The first equality in (80) is got just by rewriting (78) taking Definition 2.8 into account. Then, by using some standard arguments, it can be easily seen that the second equality in (80) follows from (78) as well.

Analogously, the two equalities in (81) are yielt by (79) and Definition 2.11. □

Remark that, even if $m \ge 2n$, as a consequence of Theorem 3.16, formula (80) can be rewritten in the form

$$\|z\|_{\mathcal{P}} = \min\Big\{ \sum_{j=1}^{2n-1} |\lambda_{i_j}| \;\Big|\; z = \sum_{j=1}^{2n-1} \lambda_{i_j} x^{(i_j)} \text{ and } \{i_1, \ldots, i_{2n-1}\} \subset \{1, \ldots, m\} \Big\}. \tag{82}$$

Corollary 5.6. *Let $\mathcal{P}$ be a b.c.p. and let $\|\cdot\|_{\mathcal{P}}$ the corresponding complex polytope norm. Moreover, let $\mathcal{P}^* = \operatorname{adj}(\mathcal{P})$ and let $\|\cdot\|_{\mathcal{P}^*}$ the corresponding adjoint complex polytope norm. Then, for any complex $n \times n$-matrix A and its adjoint A^*, it holds that*

$$\|A\|_{\mathcal{P}^*} = \|A^*\|_{\mathcal{P}}. \tag{83}$$

Proof. By (80) and (81) and by the definition of adjoint matrix A^*, we have

$$\begin{aligned} \|A\|_{\mathcal{P}^*} &= \max_{y \in \partial\mathcal{P}^*} \|Ay\|_{\mathcal{P}^*} = \max_{y \in \partial\mathcal{P}^*} \max_{x \in \partial\mathcal{P}} |\langle Ay, x\rangle| = \\ &= \max_{x \in \partial\mathcal{P}} \max_{y \in \partial\mathcal{P}^*} |\langle y, A^*x\rangle| = \max_{x \in \partial\mathcal{P}} \|A^*x\|_{\mathcal{P}} = \|A^*\|_{\mathcal{P}}. \end{aligned}$$

□

The next theorem shows that the set of the complex polytope norms is *dense* in the set of all norms defined on $\mathbb{C}^n$ and that, consequently, the corresponding set of induced matrix complex polytope norms is *dense* in the set of all induced $n \times n$-matrix norms.

Theorem 5.7. *Let $\|\cdot\|$ be a norm on $\mathbb{C}^n$. Then for any $\epsilon > 0$ there exists a b.c.p. $\mathcal{P}_\epsilon$ whose corresponding complex polytope norm $\|\cdot\|_\epsilon$ satisfies the inequalities*

$$\|x\| \le \|x\|_\epsilon \le (1+\epsilon)\|x\| \quad \text{for all } x \in \mathbb{C}^n. \tag{84}$$

Moreover, denoting by $\|\cdot\|$ and $\|\cdot\|_\epsilon$ also the corresponding induced matrix norms, it holds that

$$(1+\epsilon)^{-1}\|A\| \le \|A\|_\epsilon \le (1+\epsilon)\|A\| \quad \text{for all } A \in \mathbb{C}^{n\times n}. \tag{85}$$

Proof. Let $\mathcal{B}$ be the unit ball of the norm $\|\cdot\|$ and let $x_1, \ldots, x_n \in \partial\mathcal{B}$ such that $\operatorname{span}(\{x_1, \ldots, x_n\}) = \mathbb{C}^n$. Then define the b.c.p.

$$\mathcal{P}_\diamond = \operatorname{absco}(\{x_1, \ldots, x_n\}).$$

If $\|\cdot\|_\diamond$ is the vector norm corresponding to $\mathcal{P}_\diamond$, by the equivalence of all the norms in $\mathbb{C}^n$, there exists a constant $K > 0$ such that

$$\|x\|_\diamond \le K\|x\| \quad \text{for all } x \in \mathbb{C}^n. \tag{86}$$

Since $\partial\mathcal{B}$ is compact, given $\epsilon > 0$, there exist a finite number of vectors $y_1, \ldots, y_{k_\epsilon} \in \partial\mathcal{B}$ such that

$$\partial\mathcal{B} \subset \bigcup_{i=1}^{k_\epsilon} \mathcal{S}_i, \tag{87}$$

where $\mathcal{S}_i = \{x \in \mathbb{C}^n \mid \|x - y_i\| < \epsilon/K\}$. Then define the b.c.p.

$$\mathcal{P}_\epsilon = \operatorname{absco}(\{x_1, \ldots, x_n, y_1, \ldots, y_{k_\epsilon}\}).$$

Since $\mathcal{B}$ is absolutely convex, we clearly have

$$\mathcal{P}_\diamond \subseteq \mathcal{P}_\epsilon \subseteq \mathcal{B}. \tag{88}$$

Therefore, if $\|\cdot\|_\epsilon$ is the vector norm corresponding to $\mathcal{P}_\epsilon$, then

$$\|x\| \le \|x\|_\epsilon \le \|x\|_\diamond \quad \text{for all } x \in \mathbb{C}^n. \tag{89}$$

On the other hand, (87) implies that, for any $y \in \partial\mathcal{B}$, there exists $y_\epsilon \in \partial\mathcal{P}_\epsilon$ such that

$$\|y - y_\epsilon\| < \epsilon/K. \tag{90}$$

Now consider an arbitrary $x \in \mathbb{C}^n$, $x \neq 0$, and define $y = x/\|x\|$. By (86), (89) and (90), it follows that

$$\begin{aligned} \|x\|_\epsilon \leq (1 + \|y - y_\epsilon\|_\epsilon)\|x\| &\leq (1 + \|y - y_\epsilon\|_\diamond)\|x\| \\ &\leq (1+\epsilon)\|x\|, \end{aligned} \tag{91}$$

that is,

$$\mathcal{B} \subseteq (1+\epsilon)\mathcal{P}_\epsilon. \tag{92}$$

proving (84).

Moreover, for any $n \times n$-matrix A, (89) and (91) yield

$$\|A\|_\epsilon = \max_{\|x\|_\epsilon=1} \|Ax\|_\epsilon \leq \max_{\|x\|=1} \|Ax\|_\epsilon \leq (1+\epsilon) \max_{\|x\|=1} \|Ax\| = (1+\epsilon)\|A\|$$

and

$$\|A\| = \max_{\|x\|=1} \|Ax\| \leq (1+\epsilon) \max_{\|x\|_\epsilon=1} \|Ax\| \leq (1+\epsilon) \max_{\|x\|=1} \|Ax\|_\epsilon = (1+\epsilon)\|A\|_\epsilon,$$

that is, (85). □

The adjoint version of the previous *density* theorem holds.

Theorem 5.8. *Let $\|\cdot\|$ be a norm on $\mathbb{C}^n$. Then for any $\epsilon > 0$ there exists a b.c.p. of adjoint type $\mathcal{P}^*_\epsilon$ whose corresponding adjoint complex polytope norm $\|\cdot\|_{*,\epsilon}$ satisfies the inequalities*

$$(1+\epsilon)^{-1}\|x\| \leq \|x\|_{*,\epsilon} \leq \|x\| \quad \textit{for all } x \in \mathbb{C}^n. \tag{93}$$

Moreover, denoting by $\|\cdot\|$ and $\|\cdot\|_{*,\epsilon}$ also the corresponding induced matrix norms, it holds that

$$(1+\epsilon)^{-1}\|A\| \leq \|A\|_{*,\epsilon} \leq (1+\epsilon)\|A\| \quad \textit{for all } A \in \mathbb{C}^{n\times n}. \tag{94}$$

Proof. Let $\mathcal{B}$ be the unit ball of the norm $\|\cdot\|$ and consider the set

$$\mathcal{B}^* = \mathrm{adj}(\mathcal{B}),$$

which is the unit ball of a norm on $\mathbb{C}^n$ as well. Then fix $\epsilon > 0$ and apply the same arguments of the proof of Theorem 5.7 to obtain the analogue of formulae (88) and (92) for $\mathcal{B}^*$, that is,

$$\mathcal{P}_\epsilon \subseteq \mathcal{B}^* \subseteq (1+\epsilon)\mathcal{P}_\epsilon,$$

where $\mathcal{P}_\epsilon$ is a suitable b.c.p.

By Proposition 2.19, we can write the adjoint inclusions

$$(1+\epsilon)^{-1}\mathcal{P}^*_\epsilon \subseteq \mathcal{B} \subseteq \mathcal{P}^*_\epsilon,$$

where $\mathcal{P}^*_\epsilon = \mathrm{adj}(\mathcal{P}_\epsilon)$. Therefore, the corresponding norm $\|\cdot\|_{*,\epsilon}$ satisfies the inequalities (93).

At this point, the proof of (94) is the same as that of (85). □

Now we consider the so called ∞-*norm* and 1-*norm* which, for all $z \in \mathbb{C}^n$, are defined by $\|z\|_\infty = \max_{1\le i\le n} |z_i|$ and $\|z\|_1 = \sum_{i=1}^n |z_i|$, respectively.

As is wellknown in the real case, $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are adjoint to each other. In fact, they are associated to the n-dimensional *complex hypercube*

$$\mathcal{H}^* = \mathrm{adj}(\{e^{(1)}, \ldots, e^{(n)}\}) \tag{95}$$

and to its adjoint b.c.p., the n-dimensional *complex crosspolytope*

$$\mathcal{H} = \mathrm{absco}(\{e^{(1)}, \ldots, e^{(n)}\}), \tag{96}$$

respectively, where the $e^{(i)}$'s are the vectors of the canonical basis of $\mathbb{C}^n$. Note that $\mathcal{H}^* \bigcap \mathbb{R}^n$ is the classical n-dimensional hypercube and that $\mathcal{H} \bigcap \mathbb{R}^n$ is the classical n-dimensional crosspolytope.

By the way, it may be interesting to remark that, for $n = 3$, $\mathcal{H}^* \bigcap \mathbb{R}^3$ is equal to the b.c.p. $\mathcal{P}$ of Example 4.26 and $\mathcal{H} \bigcap \mathbb{R}^3$ is equal to its adjoint $\mathcal{P}^*$. Nevertheless, as it is easy to verify, in the complex space $\mathbb{C}^3$ we have that $\mathcal{H} \subset \mathcal{P}^*$ and $\mathcal{H}^* \subset \mathcal{P}$.

The above two special norms can be used to express conveniently all the other complex and adjoint complex polytope norms as follows. Given a b.c.p. $\mathcal{P}$ and an essential system of vertices $\mathcal{X} = \{x^{(i)}\}_{1\le i\le m}$, define the *vertex matrix*

$$V = \left[x^{(1)} \ldots x^{(m)}\right]$$

and, for the b.c.p. of adjoint type $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$, the *facet matrix*, adjoint of V,

$$F = V^*.$$

Then the first equality in (81) yields

$$\|z\|_{\mathcal{P}^*} = \|Fz\|_\infty, \tag{97}$$

whereas, with $\lambda = [\lambda_1 \ldots \lambda_m]^T$, the first equality in (80) leads to

$$\|z\|_{\mathcal{P}} = \min_{V\lambda = z} \|\lambda\|_1. \tag{98}$$

Note that, if $m = n$, then (98) reduces to $\|z\|_{\mathcal{P}} = \|V^{-1}z\|_1$.

Now consider the case $m > n$. In order to compute $\|z\|_{\mathcal{P}}$, assume without any restriction that the first n columns of the vertex matrix V are linearly independent and define the matrices

$$V_1 = \left[x^{(1)} \ldots x^{(n)}\right] \quad \text{and} \quad V_2 = \left[x^{(n+1)} \ldots x^{(m)}\right].$$

Then, if $\lambda \in \mathbb{C}^m$, define also the $(m-n)$-vector

$$\mu = [\lambda_{n+1} \ldots \lambda_m]^T,$$

so that any solution of the equation $V\lambda = z$ may be written in the form

$$\lambda = \begin{bmatrix} V_1^{-1}(z - V_2\mu) \\ \mu \end{bmatrix}.$$

In conclusion, we obtain

$$\|z\|_{\mathcal{P}} = \min_{\mu \in \mathbb{C}^{m-n}} \left\| \begin{bmatrix} V_1^{-1}(z - V_2\mu) \\ \mu \end{bmatrix} \right\|_1 , \tag{99}$$

that is, the computation of $\|z\|_{\mathcal{P}}$ requires the solution of a minimization problem in $\mathbb{C}^{m-n}$. Therefore, in general, $\|z\|_{\mathcal{P}^*}$ is clearly much easier to compute. However, even if $m \geq 2n$, formula (82) reveals that a minimizing λ in (98) can be found among those which have at most $2n-1$ nonzero entries.

As far as the computation of the induced matrix norms $\|A\|_{\mathcal{P}}$ and $\|A\|_{\mathcal{P}^*}$ is concerned, Theorem 5.6 indicates that the problem is substantially the same. However, in order to find a practical formula, it seems to us that it is easier to look first for $\|A\|_{\mathcal{P}}$ and, subsequently, to compute $\|A\|_{\mathcal{P}^*}$ as $\|A^*\|_{\mathcal{P}}$. So, let us do this.

First of all, observe that, by Definition 2.8, we immediately have

$$\|A\|_{\mathcal{P}} = \max_{z \in \partial\mathcal{P}} \|Az\|_{\mathcal{P}} = \max_{1 \leq i \leq m} \|Ax^{(i)}\|_{\mathcal{P}}. \tag{100}$$

Thus formula (99) yields

$$\|A\|_{\mathcal{P}} = \max_{1 \leq i \leq m} \min_{\mu \in \mathbb{C}^{m-n}} \left\| \begin{bmatrix} V_1^{-1}(Ax^{(i)} - V_2\mu) \\ \mu \end{bmatrix} \right\|_1 . \tag{101}$$

In conclusion, by defining the $(m-n) \times m$-matrix

$$M = \left[\mu^{(1)} \ldots \mu^{(m)}\right],$$

where, for each $i = 1, \ldots, m$, $\mu^{(i)}$ gives the minimum in the right-hand side of (101), we get the equality

$$\|A\|_{\mathcal{P}} = \left\| \begin{bmatrix} V_1^{-1}(AV - V_2M) \\ M \end{bmatrix} \right\|_1 . \tag{102}$$

As a consequence, in view of Theorem 5.6, with

$$F_1 = V_1^* \quad \text{and} \quad F_2 = V_2^*,$$

we have also

$$\begin{aligned} \|A\|_{\mathcal{P}^*} &= \|A^*\|_{\mathcal{P}} = \left\| \begin{bmatrix} V_1^{-1}(A^*V - V_2M) \\ M \end{bmatrix} \right\|_1 \\ &= \left\| \left[(FA - M^*F_2)F_1^{-1} \; M^*\right] \right\|_\infty . \end{aligned} \tag{103}$$

We can conclude with the following results.

Theorem 5.9. *Let $\mathcal{P}$ be a b.c.p. and let V be the corresponding vertex $n \times m$-matrix (with $m > n$). Then, for any $n \times n$-matrix A there exists an $m \times m$-matrix $\tilde{A}$ of the block lower triangular form*

$$\tilde{A} = \begin{bmatrix} A & O \\ A_{21} & A_{22} \end{bmatrix} \tag{104}$$

such that

$$\|A\|_{\mathcal{P}} = \|\tilde{V}^{-1}\tilde{A}\tilde{V}\|_1, \tag{105}$$

where $\tilde{V}$ *is the block upper triangular* $m \times m$*-matrix*

$$\tilde{V} = \begin{bmatrix} V_1 & V_2 \\ O & I \end{bmatrix}, \tag{106}$$

I *being the* $(m-n) \times (m-n)$*-identity matrix.*

Proof. The result follows easily from (102) with

$$A_{21} = M_1 V_1^{-1} \quad \text{and} \quad A_{22} = M_2 - M_1 V_1^{-1} V_2, \tag{107}$$

where

$$M_1 = \left[\mu^{(1)} \dots \mu^{(n)}\right] \quad \text{and} \quad M_2 = \left[\mu^{(n+1)} \dots \mu^{(m)}\right].$$

□

Theorem 5.10. *Let* $\mathcal{P}^*$ *be a b.c.p. of adjoint type and let* F *be the corresponding facet* $m \times n$*-matrix (with* $m > n$*). Then, for any* $n \times n$*-matrix* A *there exists an* $m \times m$*-matrix* $\tilde{A}$ *of the block upper triangular form*

$$\tilde{A} = \begin{bmatrix} A & A_{12} \\ O & A_{22} \end{bmatrix} \tag{108}$$

(which prolongs A *to* $\mathbb{C}^m$*) such that*

$$\|A\|_{\mathcal{P}^*} = \|\tilde{F}\tilde{A}\tilde{F}^{-1}\|_\infty, \tag{109}$$

where $\tilde{F}$ *is the block lower triangular* $m \times m$*-matrix*

$$\tilde{F} = \begin{bmatrix} F_1 & O \\ F_2 & I \end{bmatrix}, \tag{110}$$

I *being the* $(m-n) \times (m-n)$*-identity matrix.*

Proof. The result follows easily from (103) with

$$A_{12} = F_1^{-1} M_1^* \quad \text{and} \quad A_{22} = M_2^* - F_2 F_1^{-1} M_1^*. \tag{111}$$

□

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