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Affinity and Well-Posedness for Optimal Control Problems in Hilbert Spaces

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Quadratic optimal control problems in Hilbert spaces are considered. We show that well-posedness of problems without constraints for all desired trajectories is equivalent to affinity on the control of the dynamics in the abstract Cauchy problem.

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1. Introduction

In recent years, there has been an increasing interest in well-posedness for optimization problems. This concept is relevant to optimal control and stability analysis of problems arising on calculus of variations and mathematical programming.

Classically, two notions of well-posedness were introduced: continuous dependence of the optimal control on the desired trajectory (Hadamard well-posedness) and convergence toward the optimal control of any minimizing sequence (Tykhonov well-posedness).

A more general definition of well-posedness has been introduced in [14], i.e., the notion of well-posedness by perturbations which incorporates both the ideas of Hadamard and Tykhonov.

The well-posedness by perturbations has been related to optimal control in [15], [17], [20] and to stability analysis of problems on calculus of variations in [8] and [19]. This definition is adopted in [16], [9], [18], [20], and [7].

A similar definition appears in [6], [10], [4], [5], [11] and [12].

In [13] the author proves that the affine structure of the ordinary differential system is a necessary and sufficient condition for Hadamard and Tykhonov well-posedness for all desired trajectories.

In this paper we will prove that the results of [13] can be extended to quadratic optimal control problems in Hilbert spaces considering furthermore the well-posedness by perturbations; to this aim, we consider, in the control system, a differential equation with

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linearity on the state because a more general equation doesn't assure properties and form of solution.

In [12] optimal control problems in infinite dimensional spaces are considered and the main result establishes the generic well-posedness for two classes of problems without linearity assumption in the control system; in this paper, instead, we verify that, for quadratic problems without constraints in Hilbert spaces, the affinity on the control in the dynamics is necessary for the well-posedness for all desired trajectories.

Given two Hilbert spaces H and U, in this paper we shall consider $Z = L^2(H) \times L^2(U) \times H$ the space of desired trajectories and $X = H \times L^2(U)$ the space of controls. Given any $z^* = (y^*, w^*, \psi^*)$, we wish to minimize the quadratic functional

$$\begin{aligned} &\int_0^T [< x - y^*, P\,(x - y^*) >_H + < u - w^*, Q\,(u - w^*) >_U]\,(t)\,dt \\ &\quad + < x(T) - \psi^*, E\,(x(T) - \psi^*) >_H \end{aligned} \tag{1}$$

on the trajectories (v, u, x) of the inhomogeneous problem

$$\begin{cases} \dot{x}(t) = A\,x(t) + f(t, u(t)) & \text{a.e. } (0, T) \\ x(0) = v \end{cases} \tag{2}$$

subject to constraints of the general form

$$v \in K_1 \quad \text{and} \quad (x, u) \in K_2 \tag{3}$$

where (v, u) is the control, x is the state, K_1 and K_2 are subsets of H and $L^2(H) \times L^2(U)$ respectively.

In Section 2 we extend the previous problem to a global optimization problem and embed it in a family of perturbed problems, thus requiring the convergence to a global minimizer of every asymptotically minimizing sequence corresponding to small perturbations of the parameter which defines the global problem. Furthermore, we give assumptions about the operators A, P, Q, E and the sets K_1, K_2.

In Section 3 we obtain preliminary results about the properties of solutions of inhomogeneous problem (2), about the convexity and the Gâteaux differentiability of quadratic functional.

Finally, in Section 4 we prove the main result. We show that, under some regularity assumptions on f, the class of systems (2), such that every problem with cost (1) without constraints (3) is well-posed for all desired trajectories, is the one with dynamics f which are affine in the control variable.

2. Notation, statement of the problem and assumptions

Given two Banach spaces W and Y. Let $L(W; Y)$ be the Banach space of bounded linear operators from W into Y, let $C([0, T], W)$ be the Banach space of continuous functions defined on $[0, T]$ with values in W, let $L^p([0, T]; W)$ be the space of all (the equivalence classes of) W-valued Bochner integrable functions f defined on $[0, T]$ with

$\int_{[0,T]} \|f(t)\|_W^p \, dt < \infty$ (essentially bounded when $p = \infty$).
$L^p([0,T];W)$, $1 \le p < \infty$, is a Banach space under the norm

$$\|f\|_p = \left(\int_{[0,T]} \|f(t)\|_W^p \, dt \right)^{1/p} .$$

If W is a Hilbert space, then $L^2([0,T];W)$ with the inner product

$$< x, y >_{L^2(W)} = \int_{[0,T]} < x(t), y(t) >_W \ dt$$

is a Hilbert space [3, Chap. 4].
To simplify notation, $L(W)$ and $L^p(W)$ will be used instead of $L(W;W)$ and of $L^p([0,T];W)$.

Denote by $\to$ the strong convergence and by $\rightharpoonup$ the weak one; the symbol $< \cdot, \cdot >$ will be used for the inner product in any Hilbert space. Furthermore, let *graph* f be the graph of a mapping f and $DJ(p)$ the Gâteaux derivative of the real-valued function J evaluated at p.

A mapping f between two real vector spaces is called *affine* when

$$f(bp + (1-b)v) = bf(p) + (1-b)f(v)$$

for every $b \in \mathbf{R}$ and every p, v.

Let H and U be two Hilbert spaces; we consider the spaces

$$Z = L^2(H) \times L^2(U) \times H, \quad X = H \times L^2(U),$$

which are Hilbert spaces with the inner products

$$< z_1, z_2 >_Z \, = \, < y_1, y_2 >_{L^2(H)} + < w_1, w_2 >_{L^2(U)} + < \psi_1, \psi_2 >_H$$

$$< \varphi_1, \varphi_2 >_X \, = \, < v_1, v_2 >_H + < u_1, u_2 >_{L^2(U)}$$

where $z_i = (y_i, w_i, \psi_i)$, $\varphi_i = (v_i, u_i)$, for $i = 1, 2$.
Furthermore, we shall consider two nonempty sets

$$K_1 \subset H \quad \text{and} \quad K_2 \subset L^2(H) \times L^2(U)$$

and three operators

$$P \in L^\infty([0,T]; L(H)), \quad Q \in L^\infty([0,T]; L(U)), \quad E \in L(H) \tag{4}$$

such that E, $P(t)$ and $Q(t)$ are self adjoint operators for a.e. $t \in (0,T)$.

Given A, the generator of a strongly continuous semigroup (see [2, Chap. 2]), and a pair $(v,u) \in X$, denote by $x^{(v,u)}$ the *mild solution*, if it exists, of the inhomogeneous problem

$$\begin{cases} \dot{x}(t) = A\,x(t) + f(t, u(t)) \quad \text{a.e. } (0,T) \\ x(0) = v \end{cases}$$

where $f : [0,T] \times U \to H$.

The problem we shall consider is the following: given the trajectory

$$z^* = (y^*, w^*, \psi^*) \in Z$$

we wish to minimize the functional

$$\begin{aligned}\tilde{J}_{z^*}(v,u) = \int_0^T & [< x^{(v,u)}(t) - y^*(t), P(t)(x^{(v,u)}(t) - y^*(t)) >_H \\ & + < u(t) - w^*(t), Q(t)(u(t) - w^*(t)) >_U] dt \\ & + < x^{(v,u)}(T) - \psi^*, E(x^{(v,u)}(T) - \psi^*) >_H\end{aligned}$$

on the *admissible set*

$$\mathcal{A} = \left\{(v,u) \in X : v \in K_1,\ x^{(v,u)} \text{ solution of (2) and } (x^{(v,u)}, u) \in K_2\right\}.$$

In the following, we will assume that $\mathcal{A}$ is nonempty.

Consider the global optimization problem (X, J_{z^*}), where the functional $J_{z^*} : X \to (-\infty, +\infty]$ defined as

$$J_{z^*}(v,u) = \begin{cases} \tilde{J}_{z^*}(v,u) & \text{if } (v,u) \in \mathcal{A} \\ +\infty & \text{otherwise} \end{cases} \tag{5}$$

is to be minimized on $(v,u) \in X$ and denote by $V(z)$ the optimal value

$$V(z) = \inf\{J_z(q) : q \in X\}$$

where z belongs to the set $D = \{z \in Z : \|z - z^*\| < \delta\}$ with $\delta > 0$.

The global optimization problem (X, J_{z^*}) is called *well-posed*, if the following conditions hold:

- there exists a unique minimizer $q^* = \arg\min(X, J_{z^*})$; (6)
- the value function $V(z)$ is finite for every $z \in D$; (7)
- for every pair of sequences $z_n \in Z$, $q_n \in X$ such that
  $z_n \to z^*$ and $J_{z_n}(q_n) - V(z_n) \to 0$ (8)
  we have $q_n \to q^*$ and $V(z_n) \to V(z^*)$.

A sequence q_n with the property (8) is called *asymptotically minimizing* with respect to z_n.

The following assumptions will be used in the next sections.

(A) There exists $a > 0$ such that for every vector ξ in the appropriate space and a.e. $t \in (0,T)$ we have

$$< \xi\,, P(t)\,\xi > \geq a\,\|\xi\|^2, \qquad < \xi\,, Q(t)\,\xi > \geq a\,\|\xi\|^2, \qquad < \xi\,, E\,\xi > \geq 0.$$

(A_1) There exists $a > 0$ such that for every vector ξ in the appropriate space and a.e. $t \in (0,T)$ we have

$$< \xi\,, P(t)\,\xi > \geq a\,\|\xi\|^2, \qquad < \xi\,, Q(t)\,\xi > \geq a\,\|\xi\|^2, \qquad < \xi\,, E\,\xi > \geq a\,\|\xi\|^2.$$

(B) $K_1 \subset H$ is bounded, closed and convex,
$K_2 \subset L^2(H) \times L^2(U)$ is closed and convex.

(C) For any $(v,u) \in X$ there exists a unique mild solution of (2), $x^{(v,u)} \in C([0,T],H)$, and if $u_n \to u$ in $L^2(U)$ there exists a subsequence u_{n_k} such that

$$\|f(\cdot,u_{n_k}(\cdot)) - f(\cdot,u(\cdot))\|_{L^1(H)} \to 0.$$

(D) Let x_n, x be solutions of (2) for same (v_n,u_n), $(v,u) \in X$; if $\|x_n - x\|_{L^2(H)} \to 0$ there exists a subsequence such that $\|x_{n_k}(0) - x(0)\|_H \to 0$.

Remark 2.1. The following example proves the importance of hypothesis (D) for the well posedness of quadratic optimal control problems.

Example 2.2. Let $H = U = l^2$ be the Hilbert space of all sequences $\phi = (\phi^i)_{i\in\mathbf{N}} \in \mathbf{R}^{\mathbf{N}}$ such that $\sum_{i=1}^{\infty} |\phi^i|^2 < \infty$ with the inner product $<\phi,\varphi>_{l^2} = \sum_{i=1}^{\infty} \phi^i \varphi^i$.
Let $K_1 = \{\phi \in l^2 \,:\, \|\phi\|_{l^2} \le 1\}$ and $K_2 = L^2(l^2) \times L^2(l^2)$.
Given $z^* = (y^*, w^*, \psi^*) \in L^2(l^2) \times L^2(l^2) \times l^2$, we consider the functional

$$\tilde{J}_{z^*}(v,u) = \|x^{(v,u)} - y^*\|^2_{L^2(l^2)} + \|u - w^*\|^2_{L^2(l^2)}$$

where $x^{(v,u)}$ is the solution of the problem

$$\begin{cases} \dot{x}(t) = A\,x(t) + u(t) & \text{a.e. } (0,T) \\ x(0) = v \end{cases}$$

with

$$A : D(A) \subset l^2 \to l^2 \quad \text{defined} \quad A((\phi^i)_{i\in\mathbf{N}}) = (-i\,\phi^i)_{i\in\mathbf{N}}.$$

It is easy to prove that

$$S : [0,+\infty) \to L(l^2) \quad \text{such that} \quad S(t)\phi = (e^{-i\,t}\,\phi^i)_{i\in\mathbf{N}}$$

is a strongly continuous semigroup and

$$A\,\phi = \lim_{t\to 0^+} \frac{1}{t}\,(S(t) - I)\,\phi$$

for all $\phi \in l^2$ for which the limit exists, i.e., $\phi \in \{\phi \in l^2 : (-i\,\phi^i)_{i\in\mathbf{N}} \in l^2\} = D(A)$.
Thus, see [2, Chap. 2], A is the infinitesimal generator of the semigroup S.
If $u = 0$, $v_n = e_n \in K_1$, $n \in \mathbf{N}$, i.e., $v_n^i = 0$ for $i \ne n$ and $v_i^i = 1$, the mild solutions of previous problem are, for $t \in [0,T]$,

$$x_n(t) = S(t)\,v_n \quad \text{with } x_n^i(t) = \begin{cases} e^{-i\,t} & \text{if } i = n \\ 0 & \text{if } i \ne n \end{cases}$$

and we have

$$\|x_n\|_{L^2(l^2)} \to 0.$$

Then, $(v_n, 0)$, $n \in \mathbf{N}$, is a minimizing sequence for the functional

$$J_0(v,u) = \begin{cases} \tilde{J}_0(v,u) & \text{if } (v,u) \in \mathcal{A} \\ +\infty & \text{otherwise} \end{cases}$$

and so the global optimization problem $(l^2 \times L^2(l^2)\,,\, J_0)$ is not well posed.
We remark that in this case is not verified the hypothesis (D).

3. Preliminary results

Lemma 3.1. *If A is the generator of a strongly continuous semigroup, $B \in L^2(L(U,H))$ and $C \in L^1(H)$, then for any $(v,u) \in X$ there exists an unique mild solution $x^{(v,u)}$ of the problem*

$$\begin{cases} \dot{x}(t) = A\,x(t) + B(t)u(t) + C(t) \quad a.e.\ (0,T) \\ x(0) = v. \end{cases} \tag{9}$$

Moreover we have:

(*i*) *if $\|(v_n,u_n) - (v,u)\|_X \to 0$ then $\|x^{(v_n,u_n)} - x^{(v,u)}\|_{C([0,T],H)} \to 0$;*
(*ii*) *if $v_n \rightharpoonup v$ in H and $u_n \rightharpoonup u$ in $L^2(U)$ then $x^{(v_n,u_n)} \rightharpoonup x^{(v,u)}$ in $L^2(H)$.*

Proof. Since $u \in L^2(U)$ and $B \in L^2(L(U,H))$ it follows that $B\,u \in L^1(H)$ and, given $(v,u) \in X$, the unique mild solution of problem (9) is

$$x^{(v,u)}(t) = S(t)\,v + \int_0^t S(t-s)\,[B(s)\,u(s) + C(s)]\,ds$$

for every $0 \le t \le T$, where $S : [0,+\infty) \to L(H)$, the strongly continuous semigroup associated with A (see [2, Chap. 2]), is such that

$$\sup_{[0,T]} \|S(t)\|_{L(H)} < +\infty \quad \text{and} \quad x^{(v,u)} \in C([0,T];H).$$

Given $\{(v_n,u_n)\}_{n\in\mathbf{N}} \in X$ we write $x_n = x^{(v_n,u_n)}$ for every $n \in \mathbf{N}$.
If $(v_n,u_n) \to (v,u)$ in X it can be easily seen that

$$\|x_n(t) - x^{(v,u)}(t)\|_H \le const.\ \big(\|v_n - v\|_H + \|B\|_{L^2(L(U,H))}\,\|u_n - u\|_{L^2(U)}\big)$$

for every $t \in [0,T]$, from which

$$\sup_{t\in[0,T]} \|x_n(t) - x^{(v,u)}(t)\|_H \to 0.$$

To prove (*ii*) consider $v_n \rightharpoonup v$ in H, $u_n \rightharpoonup u$ in $L^2(U)$ and $g \in L^2(H)$. We have

$$\begin{aligned} &\int_0^T < g(t)\,,\, x_n(t) - x^{(v,u)}(t) >_H\ dt \\ = &\int_0^T < g(t)\,,\, S(t)(v_n - v) >_H dt + \int_0^T < g(t)\,, \int_0^t S(t-s)\,B(s)[u_n(s) - u(s)]\,ds >_H dt. \end{aligned}$$

Now

$$\int_0^T < g(t)\,,\, S(t)(v_n - v) >_H dt = \int_0^T < S^*(t)\,g(t)\,,\,(v_n - v) >_H\ dt$$

and

$$\int_0^T \|S^*(t)\,g(t)\,\|_H^2 dt \le \sup_{[0,T]} \|S^*(t)\|_{L(H)}^2 \int_0^T \|g(t)\|_H^2 dt < +\infty$$

since

$$\|S^*(t)\|_{L(H)} = \|S(t)\|_{L(H)}$$

for every $t \in [0,T]$.
Thus $S^*(\cdot)\, g(\cdot) \in L^2(H)$ and

$$\int_0^T < g(t)\,,\, S(t)(v_n - v) >_H dt \ \to 0. \tag{10}$$

If $t \in [0,T]$ we take

$$T_{g(t)} : H \to \mathbf{R} \quad \text{such that} \quad T_{g(t)}(p) =< g(t), p >_H$$

and

$$h_n(t,s) = \begin{cases} S(t-s)\ B(s)[u_n(s) - u(s)] & \text{if } 0 \le s \le t \\ 0 & \text{if } t < s \le T \end{cases}$$

for every $n \in \mathbf{N}$.
Then, if $t \in [0,T]$, we have $T_{g(t)} \in L(H, \mathbf{R})$,

$$\begin{aligned} \int_0^T \|h_n(t,s)\|_H\, ds &\le \sup_{[0,T]} \|S(\tau)\|_{L(H)} \int_0^T \|B(s)\,[u_n(s) - u(s)]\|_H\, ds \\ &\le const.\ \|B\|_{L^2(L(U,H))}\, \|u_n - u\|_{L^2(U)} < +\infty \end{aligned} \tag{11}$$

and

$$\begin{aligned} \int_0^T \ | < g(t), h_n(t,s) > |\, ds &\le \int_0^T \|g(t)\|_H\, \|h_n(t,s)\|_H\, ds \\ &\le \|g(t)\|_H \ \ \|h_n(t,\cdot)\|_{L^1(H)} \end{aligned} \tag{12}$$

for every $n \in \mathbf{N}$.
Therefore, from (11), (12) and [3, Theor. 6, Chap. 2], it follows

$$\begin{aligned} &\int_0^T \ < g(t)\,,\, \int_0^t S(t-s)\ B(s)\ [u_n(s) - u(s)]\, ds >_H \ dt \\ = &\int_0^T < g(t)\,,\, \int_0^T h_n(t,s)\, ds >_H \ dt = \int_0^T \int_0^T < g(t)\,,\, h_n(t,s) >_H\, ds\, dt \end{aligned}$$

with

$$\begin{aligned} &\int_0^T \int_0^T | \ < g(t)\,,\, h_n(t,s) >_H \ |\, ds\, dt \\ \le &\ \sup_{[0,T]} \|S(\tau)\|_{L(H)} \int_0^T \|g(t)\|_H dt \int_0^T \|B(s)[u_n(s) - u(s)]\|_H ds < +\infty. \end{aligned}$$

By Tonelli's theorem, we have

$$\begin{aligned} &\int_0^T \int_0^T < g(t)\,,\, h_n(t,s) >_H\, ds\, dt \\ = &\int_0^T \int_s^T < g(t)\,,\, S(t-s)\ B(s)[u_n(s) - u(s)] >_H\, dt\, ds \\ = &\int_0^T < \ \int_s^T B^*(s)\, S^*(t-s)\, g(t)\, dt\ ,\ \ u_n(s) - u(s) >_U\, ds \end{aligned} \tag{13}$$

where (13) follows from [3, Theor. 6, Chap. 2] since, if $s \in [0,T]$ and $n \in \mathbf{N}$. The operator

$$\tilde{T}_{n,s} : U \to \mathbf{R} \quad \text{with } \tilde{T}_{n,s}(q) = < q\,,\, u_n(s) - u(s)\ >_U$$

and the function

$$\tilde{h}(t,s) = \begin{cases} B^*(s)\,S^*(t-s)\,g(t) & \text{if } s \le t \le T \\ 0 & \text{if } 0 \le t < s \end{cases}$$

are such that

$$\tilde{T}_{n,s}\tilde{h}(\cdot,s) \in L^1([0,T],\mathbf{R}) \quad \text{and} \quad \tilde{h}(\cdot,s) \in L^1(U).$$

Thus, if we denote by $F(s)$ the quantity

$$F(s) = \int_s^T B^*(s)\,S^*(t-s)\,g(t)\,dt$$

we obtain

$$\begin{aligned}\int_0^T \|F(s)\|_U^2\,ds &\le \int_0^T \left(\int_s^T \|B^*(s)\,S^*(t-s)\,g(t)\|_U\,dt\right)^2 ds \\ &\le \sup_{[0,T]} \|S(\tau)\|^2_{L(H)}\,\|B^*(s)\|^2_{L^2(L(H,U))}\,\|g(t)\|^2_{L^1(H)} < +\infty\end{aligned}$$

which implies

$$\int_0^T < F(s)\,,\, u_n(s) - u(s)\ >_U\ ds\ \to 0. \tag{14}$$

Then, (ii) follows from (10) and (14).

The next lemma concerns the properties of the functional $\tilde{J}_z$.

Lemma 3.2. *Let A be the generator of a strongly continuous semigroup, $B \in L^2(L(U,H))$ and $C \in L^1(H)$. Suppose the hypotheses* (A), (B) *hold and let $f(t,u) = B(t)\,u + C(t)$ in problem* (2).
Then the functional $\tilde{J}_z$ verifies the following properties:

(i) *$\tilde{J}_z(\cdot,\cdot)$ is strictly convex for every $z \in Z$.*

(ii) *If $z_n \to \bar{z}$ in Z and $(v_n,u_n) \to (\bar{v},\bar{u})$ in X then*

$$\tilde{J}_{z_n}(v_n,u_n) \to \tilde{J}_{\bar{z}}(\bar{v},\bar{u}).$$

(iii) *Let $z = (y,w,\psi) \in Z$; for every $(v,u) \in X$ the differential $D\tilde{J}_z(v,u)$ exists and is given by*

$$\begin{aligned}< D\tilde{J}_z(v,u),(\varphi,p) > = {} & 2\int_0^T <x^{(\varphi,p)}(t) - \hat{x}_C(t), P(t)\,(x^{(v,u)}(t) - y(t))>_H\ dt \\ & + 2\int_0^T < p(t), Q(t)(u(t) - w(t)) >_U\ dt \\ & + 2\ <x^{(\varphi,p)}(T) - \hat{x}_C(T), E\,(x^{(v,u)}(T) - \psi)>_H,\end{aligned}$$

where $\hat{x}_C(t) = \int_0^t S(t-s)\,C(s)\,ds$.

Proof. Let $(v_i, u_i), i = 1, 2$, be two distinct vectors of X and $\lambda \in (0,1)$; if we set $x_i(t) = x^{(v_i,u_i)}(t)$ we have

$$\lambda x_1(t) + (1-\lambda)x_2(t) = x^{(\lambda(v_1,u_1)+(1-\lambda)(v_2,u_2))}(t)$$

for every $t \in [0,T]$.
Given $z = (y, w, \psi) \in Z$, through (B) it is easy to prove that

$$\begin{aligned} &\tilde{J}_z(\lambda(v_1,u_1) + (1-\lambda)(v_2,u_2)) \\ &\le \lambda \tilde{J}_z(v_1,u_1) + (1-\lambda)\tilde{J}_z(v_2,u_2) \\ &\qquad - \lambda(1-\lambda)\, a \left(\|x_1 - x_2\|^2_{L^2(H)} + \|u_1 - u_2\|^2_{L^2(U)} \right) \end{aligned} \tag{15}$$

which implies the strict convexity of $\tilde{J}_z(\cdot,\cdot)$ when $u_1 \neq u_2$.
On the other hand, if $v_1 \neq v_2$, take $d = \|x_1(0) - x_2(0)\|_H$, there is $\delta > 0$ such that

$$\frac{d}{2} < \|x_1(t) - x_2(t)\|_H \quad \text{for every } t \in [0,\delta]$$

from which

$$\|x_1 - x_2\|^2_{L^2(H)} \ge \|x_1 - x_2\|^2_{L^2([0,\delta],H)} > \frac{d^2}{4}\,\delta.$$

Therefore, by (15) and by the previous inequality we have again the strict convexity of $\tilde{J}_z(\cdot,\cdot)$ thus completing the proof of (i).
Now we prove (ii).
Take $z_n = (y_n, w_n, \psi_n) \to \bar{z} = (\bar{y}, \bar{w}, \bar{\psi})$ in Z and $(v_n, u_n) \to (\bar{v}, \bar{u})$ in X; if we denote by $x^n = x^{(v_n,u_n)}$ and by $\bar{x} = x^{(\bar{v},\bar{u})}$ the solutions of problem (2), we obtain $\|x^n - \bar{x}\|_{C([0,T],H)} \to 0$ by Lemma 3.1.
Thus, from the inequality

$$\begin{aligned} &|\, \tilde{J}_{z_n}(v_n,u_n) - \tilde{J}_{\bar{z}}(\bar{v},\bar{u})\, | \\ &\le \mathit{const.}\, \big(\|x_n - y_n + \bar{x} - \bar{y}\|_{L^2(H)}\, \|x_n - y_n - \bar{x} + \bar{y}\|_{L^2(H)} \\ &\qquad + \|u_n - w_n - \bar{u} + \bar{w}\|_{L^2(U)}\, \|u_n - w_n + \bar{u} - \bar{w}\|_{L^2(U)} \\ &\qquad + \|x_n(T) - \psi_n - \bar{x}(T) + \bar{\psi}\|_H\, \|x_n(T) - \psi_n + \bar{x}(T) - \bar{\psi}\|_H \,\big) \end{aligned}$$

(ii) follows.
Finally, we examine condition (iii).
If $z = (y, w, \psi) \in Z$, (v,u), $(\varphi, p) \in X$ and $\alpha \in \mathbf{R}$ we set $x_\alpha = x^{(v,u)+\alpha\,(\varphi,p)}$; Lemma 3.1 gives

$$x_\alpha(t) = x^{(v,u)}(t) + \alpha\, x^{(\varphi,p)}(t) - \alpha\, \hat{x}_C(t) \tag{16}$$

for every $t \in [0,T]$.
Now, if

$$\tilde{J}'_z\,((v,u);(\varphi,p)) = \lim_{\alpha\to 0} \frac{\tilde{J}_z\,((v,u) + \alpha(\varphi,p)) - \tilde{J}_z(v,u)}{\alpha}$$

(16) leads to

$$
\begin{aligned}
&\tilde{J}'_z(\,(v,u);(\varphi,p)\,) \\
&= 2\int_0^T < x^{(\varphi,p)}(t) - \hat{x}_C(t),\ P(t)\,(x^{(v,u)}(t) - y(t)) >_H \ dt \\
&\qquad + 2\int_0^T < p(t),\ Q(t)\,(u(t) - w(t)) >_U \ dt \\
&\qquad + 2\ < x^{(\varphi,p)}(T) - \hat{x}_C(T), E\,(x^{(v,u)}(T) - \psi) >_H .
\end{aligned} \tag{17}
$$

From the linearity of the mapping

$$(\varphi,p) \mapsto x^{(\varphi,p)} - \hat{x}_C$$

the linearity of the functional $\tilde{J}'_z((v,u);\cdot)$ follows; moreover, if $(\varphi_n,p_n) \to (\bar{\varphi},\bar{p})$ in X we have, by Lemma 3.1, $x^{(\varphi_n,p_n)} \to x^{(\bar{\varphi},\bar{p})}$ in $C([0,T],H)$ and similar procedure as in (ii) gives the continuity of the functional $\tilde{J}'_z((v,u);\cdot)$.
Then, (17) is the expression of $< D\tilde{J}_z(v,u),(\varphi,p)>$.

The proof of the following simple lemma is omitted.

Lemma 3.3. *Let W and Y be real vector spaces and $h : W \to Y$ a mapping such that its graph is convex. Then h is affine.*

4. Well-posedness and affinity on the control in inhomogeneous problem

Proposition 4.1. *Let A be the generator of a strongly continuous semigroup, $B \in L^2(L(U, H))$ and $C \in L^1(H)$. Suppose the hypotheses* (A), (B), (D) *hold and let $f(t,u) = B(t)\,u + C(t)$ in problem* (2).
Then the following properties hold:

(i) *for any $z \in Z$ there exists a unique $(v^z,u^z) = \arg\min(X,J_z)$;*
(ii) *if $z_n \to z^*$ in Z, and*

$$(v^n,u^n) = \arg\min(X,J_{z_n}), \qquad (v^*,u^*) = \arg\min(X,J_{z^*})$$

we have $(v^n,u^n) \to (v^*,u^*)$ in X.

Proof. Take $z = (y,w,\psi) \in Z$; since $\mathcal{A} \neq \emptyset$ and $\tilde{J}_z(v,u) \geq 0$ from (A), it follows that $V(z) \in \mathbf{R}$.
If (v_n,u_n) is a minimizing sequence, we have $(v_n,u_n) \in \mathcal{A}$ for n large enough and

$$\tilde{J}_z(v_n,u_n) \geq a\left(\|x_n - y\|^2_{L^2(H)} + \|u_n - w\|^2_{L^2(U)}\right)$$

where $x_n = x^{(v_n,u_n)}$ for every $n \in \mathbf{N}$.
Thus, $\|u_n\|_{L^2(U)}$ is bounded and, since $\{v_n\} \subset K_1$ and (B) hold, there exist $v_0 \in K_1$, $u_0 \in L^2(U)$ and a subsequence (v_{n_k},u_{n_k}) such that

$$v_{n_k} \rightharpoonup v_0 \text{ in } H \quad \text{and} \quad u_{n_k} \rightharpoonup u_0 \text{ in } L^2(U);$$

moreover, from Lemma 3.1 and (B) it follows that

$$x_{n_k} \rightharpoonup x_0 \text{ in } L^2(H) \quad \text{and} \quad (x_0, u_0) \in K_2$$

with $x_0 = x^{(v_0,u_0)}$.
Consequently, $(v_0, u_0) \in \mathcal{A}$ and we have

$$\liminf_{n\to\infty} J_z(v_{n_k}, u_{n_k}) = \liminf_{n\to\infty} \tilde{J}_z(v_{n_k}, u_{n_k}) \geq \tilde{J}_z(v_0, u_0) = J_z(v_0, u_0)$$

since $\tilde{J}_z$ is convex and lower semicontinuous on X, which implies that (v_0, u_0) is a minimizer. The strict convexity of $\tilde{J}_z$ allows us to obtain the validity of (i).
Now we prove (ii).
If $z_n = (y_n, w_n, \psi_n) \to z^* = (y^*, w^*, \psi^*)$ in Z and

$$(v^n, u^n) = \arg\min(X, J_{z_n}), \qquad (v^*, u^*) = \arg\min(X, J_{z^*})$$

for every $(v, u) \in X$ we have

$$\tilde{J}_{z_n}(v, u) \geq \tilde{J}_{z_n}(v^n, u^n) \geq a \left(\|x^n - y_n\|^2_{L^2(H)} + \|u^n - w_n\|^2_{L^2(U)} \right)$$

where $x^n = x^{(v^n,u^n)}$ for every $n \in \mathbf{N}$; by Lemma 3.2 we have $\tilde{J}_{z_n}(v, u) \to \tilde{J}_{z^*}(v, u)$, thus, as shown in (i), there exist $(\varphi, p) \in X$, $\varphi \in K_1$, and a subsequence $(v^{n_k}, u^{n_k}) \in \mathcal{A}$ such that

$$v^{n_k} \rightharpoonup \varphi \text{ in } H, \qquad u^{n_k} \rightharpoonup p \text{ in } L^2(U)$$

and, by Lemma 3.1,

$$x^{n_k} \rightharpoonup x^{(\varphi,p)} \text{ in } L^2(H). \tag{18}$$

For such $k \in \mathbf{N}$ we take $T_{n_k} : L^2(H) \to \mathbf{R}$ such that

$$T_{n_k}(m) = \int_0^T < m(t), P(t)(x^{(\varphi,p)}(t) - y_{n_k}(t)) >_H \, dt$$

for every $m \in L^2(H)$; we have

$$|T_{n_k}(m)| \leq const. \, \|m\|_{L^2(H)} \;\; \|x^{(\varphi,p)} - y_{n_k}\|_{L^2(H)}$$

from which

$$T_{n_k} \in (L^2(H))^* = L^2(H) \quad \text{and} \quad T_{n_k} \to T_0$$

where

$$T_0(m) = \int_0^T < m(t), P(t)(x^{(\varphi,p)}(t) - y^*(t)) >_H \, dt.$$

Thus, from (18) we obtain

$$\int_0^T < x^{(\varphi,p)-(v^{n_k},u^{n_k})}(t) - \hat{x}_C(t), P(t)(x^{(\varphi,p)}(t) - y_{n_k}(t)) >_H \, dt \;\to\; 0. \tag{19}$$

A similar procedure gives

$$\int_0^T < p(t) - u^{n_k}(t), \, Q(t)\,(p(t) - w_{n_k}(t)) >_U \, dt \;\to\; 0. \tag{20}$$

Moreover, if $h \in H$, from [3, Theor. 6, Chap. 2] we have

$$\lim_{k\to\infty} < h, \int_0^T S(T-s)\, B(s)\, [u^{n_k}(s) - p(s)]\, ds >_H$$
$$= \lim_{k\to\infty} \int_0^T < B^*(s) S^*(T-s)\, h\, ,\, u^{n_k}(s) - p(s) >_U \; ds = 0$$

since

$$\int_0^T \|B^*(s)S^*(T-s)\, h\,\|_H^2 ds \le \sup_{[0,T]} \|S(\tau)\|^2_{L(H)}\; \|B^*\|^2_{L^2(L(H,U))}\; \|h\|_H^2 < +\infty.$$

Then

$$\lim_{k\to\infty} < h,\; x^{(\varphi,p)}(T) - x^{n_k}(T) >$$
$$= \lim_{k\to\infty} < h\, ,\, S(T)(v^{n_k} - \varphi) + \int_0^T S(T-s)\, B(s)\, [u^{n_k}(s) - p(s)]\, ds >_H = 0$$

for every $h \in H$, from which

$$x^{(\varphi,p)}(T) - x^{n_k}(T) \rightharpoonup 0 \text{ in } H$$

and

$$\lim_{k\to\infty} < x^{(\varphi,p)-(v^{n_k},u^{n_k})}(T) - \hat{x}_C,\; E\,(x^{(\varphi,p)}(T) - \psi_{n_k}) >_H \;=\; 0. \tag{21}$$

Whereas (19), (20) and (21) yield

$$\lim_{k\to\infty} < D\tilde{J}_{z_{n_k}}(\varphi,p), ((\varphi,p) - (v^{n_k},u^{n_k})) >= 0. \tag{22}$$

Thus, from $(v^{n_k},u^{n_k}) = argmin(X, J_{z_{n_k}})$ we have

$$< D\tilde{J}_{z_{n_k}}(\varphi,p), ((\varphi,p) - (v^{n_k},u^{n_k})) >$$
$$\ge < D\tilde{J}_{z_{n_k}}(\varphi,p), ((\varphi,p) - (v^{n_k},u^{n_k})) > - < D\tilde{J}_{z_{n_k}}(v^{n_k},u^{n_k}), ((\varphi,p) - (v^{n_k},u^{n_k})) >$$
$$= 2\int_0^T <\; x^{(\varphi,p)}(t) - x^{n_k}(t), P(t)(x^{(\varphi,p)}(t) - x^{n_k}(t)) >_H \; dt$$
$$+\, 2\int_0^T < p(t) - u^{n_k}(t),\; Q(t)\,(p(t) - u^{n_k}(t)) >_U \; dt$$
$$+\, 2\; < x^{(\varphi,p)}(T) - x^{n_k}(T),\; E\, x^{(\varphi,p)}(T) - x^{n_k}(T) >_H$$
$$\ge 2a\left(\|x^{(\varphi,p)} - x^{n_k}\|^2_{L^2([0,T],H)} + \|p - u^{n_k}\|^2_{L^2([0,T],U)} \right)$$

which implies, by (22), that

$$u^{n_k} \to p \text{ in } L^2(U) \quad \text{and} \quad x^{n_k} \to x^{(\varphi,p)} \text{ in } L^2(H).$$

From (D), taking an another subsequence, we have $\|v_{n_{k_j}} - \varphi\|_H \to 0$. Now, the assertion (ii) of Lemma 3.2 gives

$$J_{z^*}(v,u) \ge \tilde{J}_{z^*}(v,u) = \lim_{j\to\infty} \tilde{J}_{z_{n_{k_j}}}(v,u) \ge \lim_{j\to\infty} \tilde{J}_{z_{n_{k_j}}}(v^{n_{k_j}}, u^{n_{k_j}}) = J_{z^*}(\varphi,p)$$

for every $(v,u) \in X$.
Then, since $argmin(X, J_{z^*})$ is a singleton, we get

$$(\varphi, p) = (v^*, u^*) \quad \text{and} \quad (v^n, u^n) \to (v^*, u^*).$$

Theorem 4.2. *Let A be the generator of a strongly continuous semigroup, $B \in L^2(L(U, H))$ and $C \in L^1(H)$. Suppose the hypotheses* (A), (B), (D) *hold and let $f(t,u) = B(t)\,u + C(t)$ in problem* (2).
Then, the problem (X, J_{z^*}) defined by (5) *is well-posed for every $z^* \in Z$.*

Proof. If $z^* \in Z$, the condition (6) of well-posedness follows from assertion (i) of Proposition 4.1 whereas assumption (A) ensures that $J_z \geq 0$ for every $z \in Z$, therefore (7).
Given $z = (y, w, \psi) \in Z$ we set $(v^z, u^z) = argmin(X, J_z) \in \mathcal{A}$, $x^z = x^{(v^z, u^z)}$. From part (iii) of Lemma 3.2 it follows that

$$\begin{aligned} & J_z(v,u) - V(z) \geq \tilde{J}_z(v,u) - V(z) \\ &= < D\tilde{J}_z(v^z, u^z), (v - v^z, u - u^z) > \\ &\qquad + \int_0^T [< x^{(v,u)}(t) - x^z(t), P(t)(x^{(v,u)}(t) - x^z(t)) >_H \\ &\qquad + < u(t) - u^z(t), Q(t)(u(t) - u^z(t)) >_U]dt \\ &\qquad + < x^{(v,u)}(T) - x^z(T), E(x^{(v,u)}(T) - x^z(T)) >_H \\ &\geq a\,\|u - u^z\|^2_{L^2(U)} + a\,\|x^{(v,u)} - x^z\|^2_{L^2(H)}. \end{aligned} \tag{23}$$

To prove (8) consider a sequence $z_n = (y_n, w_n, \psi_n)$ such that $z_n \to z^*$ and an asymptotically minimizing sequence (v_n, u_n) corresponding to the sequence z_n; then, by (23) we have

$$J_{z_n}(v_n, u_n) - V(z_n) \geq a\,\|u_n - u^{z_n}\|^2_{L^2(U)} + a\,\|x^{(v_n,u_n)} - x^{z_n}\|^2_{L^2(H)}$$

which implies

$$\|u_n - u^{z_n}\|_{L^2(U)} \to 0 \quad \text{and} \quad \|x^{(v_n,u_n)} - x^{z_n}\|_{L^2(H)} \to 0;$$

from (ii) of Proposition 4.1 and (i) of Lemma 3.1 we have

$$\|u_n - u^*\|_{L^2(U)} \to 0 \quad \text{and} \quad \|x^{(v_n,u_n)} - x^{(v^*,u^*)}\|_{L^2(H)} \to 0.$$

Since (D) hold, there exists a subsequence v_{n_k} such that $v_{n_k} \to v^*$ in H; then $(v_n, u_n) \to (v^*, u^*)$ in X and the proof is complete.

In what follows we will see how the well-posedness for any $z^* \in Z$ implies, in problems without constraints, the affinity of the dynamics with respect to the control variable u in the system (2).
The following definition will be used henceforth.

Definition 4.3. A set G in a Hilbert space Z is called a Čebyšev set, *if for each point $z \in Z$ there is a unique nearest point $p(z) \in G$, i.e.,*

$$\|z - y\| > \|z - p(z)\| \quad \text{if } y \in G \text{ and } y \neq p(z).$$

Remark 4.4. If the assumption (A_1) holds, then, given $z_i = (y_i, w_i, \psi_i) \in Z,\ i = 1, 2,$ the definition

$$< z_1, z_2 >_1 = \int_0^T [< y_1(t), P(t)\, y_2(t) >_H + < w_1(t), Q(t)\, w_2(t) >_U]\, dt + < \psi_1, E\psi_2 >_H$$

provides an inner product that induces on Z a Hilbert space structure equivalent to the usual one. We will denote by $\| \cdot \|_1$ the corresponding norm.

Proposition 4.5. *Let A be the generator of a strongly continuous semigroup and (X, J_{z^*}) be the problem defined by (5) with $K_1 \times K_2 = H \times L^2(H) \times L^2(U)$. Suppose that assumptions (A_1) and (C) hold. If the problem defined by (5) is well-posed for every $z^* \in Z$ then the set*

$$G = \{(m, u, \psi) \in Z : \exists\, v \in H \text{ such that } m = x^{(v,u)} \text{ and } \psi = x^{(v,u)}(T)\} \tag{24}$$

is convex.

Proof. By Remark 4.4 $(Z, < \cdot, \cdot >_1)$ is a Hilbert space.
For any $z^* \in Z$ we have by hypotheses

$$(v^*, u^*) = argmin(X, J_{z^*}) \iff \|p^* - z^*\|_1 \le \|p - z^*\|_1 \forall\, p \in G \tag{25}$$

where $p^* = (x^*, u^*, x^*(T))$ and $x^* = x^{(v^*,u^*)}$.
The well-posedness of the problem (5) for every $z^* \in Z$, together with (25), implies that G is a Čebyšev set.
If $z^* \in Z$ and $(m_n, u_n, s_n) \in G$ is such that

$$\|(m_n, u_n, s_n) - z^*\|_1 \to inf\{\|p - z^*\|_1 : p \in G\} \tag{26}$$

then, by (24), there is $v_n \in H$ such that $s_n = x^{(v_n,u_n)}(T)$ and (26) is the same as

$$J_{z^*}(v_n, u_n) \to V(z^*).$$

Consequently, (v_n, u_n) is asymptotically minimizing with respect to the sequence $z_n = z^*$ for every $n \in \mathbf{N}$. From the well-posedness of the problem (5) for every z^* it follows that $(v_n, u_n) \to (v^*, u^*)$ in X and, by assumption (C), there exists a subsequence u_{n_k} such that $\|f(\cdot, u_{n_k}(\cdot)) - f(\cdot, u^*(\cdot))\|_{L^1(H)} \to 0$.
Then, since

$$\|x^{(v_{n_k},u_{n_k})}(T) - x^{(v^*,u^*)}(T)\|_H$$
$$\le \sup_{[0,T]} \|S(t)\|_{L(H)} \left(\|(v_{n_k} - v^*)\|_H + \int_0^T \|f(s, u_{n_k}(s)) - f(s, u^*(s))\|_H\, ds \right)$$

we have

$$(m_{n_k}, u_{n_k}, s_{n_k}) \to (x^{(v^*,u^*)}, u^*, x^{(v^*,u^*)}(T)).$$

The convexity of the set G follows from Corollary 3 of [1, page 238].

Definition 4.6. Given $t_0 \in [0, T]$, let $\eta_{t_0} : X \to C([0, T - t_0], H)$ be defined by

$$\eta_{t_0}(v, u)(\tau) = x^{(v,u)}(t_0 + \tau) \quad \text{for every } \tau \in [0, T - t_0]. \tag{27}$$

Lemma 4.7. *Suppose that the same assumptions of Proposition* 4.5 *hold. Given* $t_0 \in [0,T]$ *the map* $\eta_{t_0} : X \to C([0,T-t_0],H)$ *of* (27) *is such that its graph is convex; hence* η_{t_0} *is affine.*

Proof. Let $h_i = (v_i, u_i, \eta_{t_0}(v_i,u_i)) \in graph\,\eta_{t_0}$ with $i = 1,2$; if we set $m_i = x^{(v_i,u_i)}$, from (27) we have

$$(m_i, u_i, \eta_{t_0}(v_i,u_i)(T-t_0)) = (m_i, u_i, x^{(v_i,u_i)}(T)) \in G$$

whereas Proposition 4.5 leads to

$$(\lambda\, m_1 \,+\, (1-\lambda)\, m_2 \,,\, \lambda\, u_1 \,+\, (1-\lambda)\, u_2 \,,\, \lambda\, x^{(v_1,u_1)}(T) \,+\, (1-\lambda) x^{(v_2,u_2)}(T)) \in G$$

for any fixed $\lambda \in (0,1)$. Then, there is some $v \in H$ such that

$$\lambda\, m_1(t) \,+\, (1-\lambda) m_2(t) = \lambda\, x^{(v_1,u_1)}(t) + (1-\lambda)\, x^{(v_2,u_2)}(t) = x^{(v,\lambda u_1 + (1-\lambda) u_2)}(t)$$

for every $t \in [0,T]$.
In particular, we obtain for $t = 0$

$$\lambda\, v_1 + (1-\lambda) v_2 = v,$$

and for $t = t_0 + \tau$ with $\tau \in [0, T-t_0]$

$$\lambda\, x^{(v_1,u_1)}(t_0+\tau) + (1-\lambda)\, x^{(v_2,u_2)}(t_0+\tau) = x^{(v,u)}(t_0+\tau)$$

where $u = \lambda u_1 + (1-\lambda) u_2$; hence

$$\lambda\, \eta_{t_0}(v_1,u_1)(\tau) \;+\; (1-\lambda)\, \eta_{t_0}(v_2,u_2)(\tau) = \eta_{t_0}(v,u)(\tau)$$

for every $\tau \in [0,T-t_0]$. Then, $graph\,\eta_{t_0}$ is convex and, by Lemma 3.3, η_{t_0} is affine.

Theorem 4.8. *Let A be the generator of a strongly continuous semigroup and (X, J_{z^*}) be the problem defined by* (5) *with* $K_1 \times K_2 = H \times L^2(H) \times L^2(U)$. *Suppose that assumptions* (A_1), (C) *hold and the mapping f in* (2) *is such that*

$$f(\cdot,u) \in C^1([0,T],H) \quad \textit{for every } u \in U.$$

If the problem defined by (5) *is well-posed for every $z^* \in Z$ then there exist*

$$B : [0,T] \to L(U,H) \quad \textit{and} \quad C : [0,T] \to H$$

such that $f(t,u) = B(t)\,u + C(t)$ *for every* $t \in [0,T]$ *and* $u \in U$.

Proof. If $(v_i,p_i) \in H \times U$, let $x^i = x^{(v_i,p_i)}$ be the solution of (2), with $i = 1,2$.
Given $b \in \mathbf{R}$ consider $x^3 = x^{(b(v_1,p_1)+(1-b)(v_2,p_2))}$.
If $t_0 \in [0,T]$, since the mapping η_{t_0} is affine, we have

$$\eta_{t_0}(\, b(v_1,p_1) + (1-b)(v_2,p_2)\,)(\tau) \;=\; b\,\eta_{t_0}(v_1,p_1)(\tau) \,+ (1-b)\,\eta_{t_0}(v_2,p_2)(\tau)$$

for every $\tau \in [0,T-t_0]$, from which

$$x^3(t) \;=\; b\,x^1(t) \;+\; (1-b) x^2(t) \tag{28}$$

for every $t \in [t_0, T]$.
Then, from (28) it follows that

$$\int_0^t S(t-s)\,[f(s, bp_1 + (1-b)p_2) - bf(s, p_1) - (1-b)f(s, p_2)]\,ds = 0 \tag{29}$$

for $t \in [t_0, T]$.
If we set

$$g(s) = f(s, bp_1 + (1-b)p_2) - bf(s, p_1) - (1-b)f(s, p_2)$$

for every $s \in [0, T]$, then, by Theorem 2.21 of [2], the mapping

$$\beta(t) = \int_0^t S(t-s)\,g(s)\,ds$$

is continuously differentiable on $(0, T)$ and

$$\dot\beta(t) = A\,\beta(t) + g(t) \tag{30}$$

for almost every $t \in (0, T)$.
Thus, there is $W_{b,p_1,p_2} \subset [0, T]$ such that the Lebesgue measure of the set $[0, T] \setminus W_{b,p_1,p_2}$ is zero and the equality in (30) is satisfied for every $t \in W_{b,p_1,p_2}$.
By (29) and (30) we have

$$A\,x^3(t) \,+\, f(t, b\,p_1 + (1-b)p_2) = b\,\left[A\,x^1(t) \,+\, f(t, p_1)\right] + (1-b)\,\left[A\,x^2(t) \,+\, f(t, p_2)\right]$$

for every $t \in (t_0, T) \cap W_{b,p_1,p_2}$, which gives

$$f(t, b\,p_1 + (1-b)p_2) - b\,f(t, p_1) - (1-b)\,f(t, p_2) = 0 \tag{31}$$

for every $t \in (t_0, T) \cap W_{b,p_1,p_2}$.
Let $\{t_n\} \subset (t_0, T) \cap W_{b,p_1,p_2}$, $t_n \to t_0$; from (31) and the continuity of the mapping f we have

$$f(t_0, b\,p_1 + (1-b)p_2) - b\,f(t_0, p_1) - (1-b)\,f(t_0, p_2) = 0.$$

This allows us to see that $f(t, \cdot)$ is affine for every $t \in [0, T]$ thus completing the proof.

In the next theorem we will prove the affinity on the control of the dynamics in system (2) when f is a Hölder mapping and the operator A fulfills a stronger hypothesis than that in Theorem 4.8.

Theorem 4.9. *Let A be the generator of an analytic semigroup and (X, J_{z^*}) the problem defined by* (5) *with* $K_1 \times K_2 = H \times L^2(H) \times L^2(U)$*. Suppose that assumptions* (A_1), (C) *hold and the mapping f in* (2) *is such that*

$$\|f(t, p) - f(s, p)\|_H \le C_p\,|t-s|^k \quad \textit{with } 0 < k < 1 \tag{32}$$

for every $p \in H$.
If the problem defined by (5) *is well-posed for every $z^* \in Z$ then there are*

$$B\,:[0, T] \to L(U, H) \quad \textit{and} \quad C\,:[0, T] \to H$$

such that $f(t, u) = B(t)\,u + C(t)$ *for every* $t \in [0, T]$ *and* $u \in U$.

Proof. Like in Theorem 4.8 we can obtain

$$\int_0^t S(t-s)\,[f(s, bp_1 + (1-b)p_2) - bf(s, p_1) - (1-b)f(s, p_2)]\,ds = 0$$

for every $t \in [t_0, T]$, and if we set

$$g(s) = f(s, bp_1 + (1-b)p_2) - bf(s, p_1) - (1-b)f(s, p_2)$$

for every $s \in [0, T]$, from (32) it follows that

$$\begin{aligned} \|g(t) - g(s)\|_H &\le \|f(t, bp_1 + (1-b)p_2) - f(s, bp_1 + (1-b)p_2)\|_H \\ &\qquad + |b|\,\|f(t, p_1) - f(s, p_1)\|_H + |1-b|\,\|f(t, p_2) - f(s, p_2)\|_H \\ &\le (C_3 + |b|C_1 + |1-b|C_2)\,|t-s|^k \end{aligned}$$

with $0 < k < 1$; by [2, Theor. 2.29, Chap. 2] the mapping

$$\beta(t) = \int_0^t S(t-s)\,g(s)\,ds$$

is differentiable almost everywhere in $(0, T)$ and we have

$$\dot{\beta}(t) = A\,\beta(t) + g(t)$$

for almost every $t \in (0, T)$.
Then, we complete the proof as in Theorem 4.8.

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