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A Regularity Result in a Shape Optimization Problem with Perimeter

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We consider optimal shapes of the functional

$$\mathcal{E}_\lambda(\Omega) = J(\Omega) + P(\Omega) + \lambda||\Omega| - m|$$

among all the measurable subsets Ω of a given open bounded domain $D \subset \mathbf{R}^d$ where $J(\Omega)$ is some Dirichlet energy associated with Ω, $P(\Omega)$ and $|\Omega|$ being respectively the perimeter and the Lebesgue measure of Ω. We prove here that for some optimal shape, the state function associated with the Dirichlet energy is Lipschitz-continuous. Then we deduce the same regularity properties for the boundary of the optimal shape as in the pure isoperimetric problem (case $J \equiv 0$). We also consider the minimization of $\mathcal{E}_0$ with Lebesgue measure constraint $|\Omega| = m$.

1. Introduction

We prove here some regularity results for the optimal shape of two optimization problems involving the Dirichlet energy of shapes together with their perimeter: the first one is with a Lebesgue measure constraint, the other one is a penalized version.

To be more precise, let D be an open subset of $\mathbf{R}^d$ and $f \in L^2(D)$. For any measurable subset Ω of D with finite Lebesgue measure $|\Omega|$, we denote by $P(\Omega)$ its perimeter (see Appendix E for definition) and by $J(\Omega)$ its "Dirichlet energy", defined as follows:

$$J(\Omega) = \inf\{\ G(v)\ :\ v \in H_0^1(\Omega)\ \} \tag{1}$$

where $G(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 - f.v$. In the case where Ω is open, it is well-known that $J(\Omega) = G(u_\Omega)$ where u_Ω is the solution to the Dirichlet problem

$$u_\Omega \in H_0^1(\Omega),\ -\Delta u_\Omega = f \text{ in } \Omega. \tag{2}$$

The definition of $H_0^1(\Omega)$ may be extended to any measurable set Ω (see e.g. [10]) and u_Ω may then be defined as the unique function $u_\Omega \in H_0^1(\Omega)$ such that $J(\Omega) = G(u_\Omega)$.

Now we fix $m \in (0, |D|)$, and for $\lambda > 0$, we set

$$\mathcal{E}(\Omega) = J(\Omega) + P(\Omega),\ \mathcal{E}_\lambda(\Omega) = J(\Omega) + P(\Omega) + \lambda||\Omega| - m|. \tag{3}$$

Our goal is to study the regularity of optimal shapes Ω^*, Ω^*_λ solutions of

$$\mathcal{E}(\Omega^*) = \min\{\mathcal{E}(\Omega)\ :\ \Omega \text{ measurable}, \Omega \subset D, |\Omega| = m\} \tag{4}$$

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$$\mathcal{E}_\lambda(\Omega^*_\lambda) = \min\{\mathcal{E}_\lambda(\Omega) \,:\, \Omega \text{ measurable},\ \Omega \subset D,\ |\Omega| \text{ finite }\}. \tag{5}$$

Note that if $f = 0$, these problems are nothing but the classical isoperimetric problems. The regularity of the corresponding optimal shapes is a difficult question, but now completely solved. Assuming $f = 0$, one has

$$\text{if } d \leq 7,\ \partial\Omega^* \text{ and } \partial\Omega^*_\lambda \text{ are analytic hypersurfaces,} \tag{6}$$

$$\begin{array}{l}\text{and if } d \geq 8, \text{ there exist singular sets } \Sigma, \Sigma_\lambda \text{ of } s\text{-Hausdorff} \\ \text{measure zero for all } s > d-8 \text{ such that } \partial\Omega^*\backslash\Sigma \text{ and } \partial\Omega^*_\lambda\backslash\Sigma_\lambda \\ \text{are analytic hypersurfaces}\end{array} \tag{7}$$

These results are particular cases of the known facts about minimal surfaces and may be found for instance in [7], [8], [11] and in their references.

On the other hand, if one drops the perimeter term in $\mathcal{E}(\Omega)$ and $\mathcal{E}_\lambda(\Omega)$ to consider the pure Dirichlet problem (i.e., minimizing $J(\Omega)$ with prescribed $|\Omega|$ or $J(\Omega) + \lambda||\Omega| - m|$), similar regularity results may be obtained in the case when f is bounded and nonnegative. Complete regularity of the boundary holds if $d = 2$ while regularity up to a singular set of $(d-1)$-Hausdorff measure zero holds for $d \geq 3$ (see [4] and [5]). Note that, even when $d = 2$, singularities like cusps may occur if f changes sign.

Here, we prove in particular the following

Theorem 1.1. *Assume f is Hölder continuous, bounded and nonnegative. Then, for any optimal shape Ω^*_λ of (5), the state function $u_{\Omega^*_\lambda}$ is locally Lipschitz continuous in D. As a consequence, the reduced boundary of Ω^*_λ is at least $\mathcal{C}^{1,\alpha}$ and so is the reduced boundary of any solution Ω^* of (4) when both problems are equivalent. If $d \leq 7$, the same regularity applies to the measure-theoretic boundary of the optimal shape.*

One knows that many shape optimization problems with Lebesgue measure constraint of type (4) are actually equivalent to their penalized version of type (5) for λ large enough. This is also proved here too under the technical assumption that D is star-shaped and bounded (this assumption is not optimal, but it allows us to provide a direct proof). As a consequence, regularity results of Theorem 1.1 may also be applied to optimal shapes of problem (4).

Before reaching the Lipschitz continuity of the state function, we provide here a rather simple proof of the fact that it is $\frac{1}{2}$-Hölder continuous. Although we do it here for simplicity only for nonnegative state functions, it may be extended to any $f \in L^\infty$ without sign by adding the use of the Alt-Caffarelli-Friedman Monotonicity Lemma (see [1]).

The hardest part of this paper is to reach the Lipschitz continuity of the state function. The proof uses very much the ideas and techniques in [2]. Once this is done, we may easily deduce that the optimal shapes are perimeter *quasi-minimizers* (See Definition 5.1), and then apply known regularity results for these quasi-minimizers (see [3], [12])

Let us mention that problems of type (4) and (5) often appears in applications: the term $P(\Omega)$ generally corresponds to a surface tension energy, while $J(\Omega)$ is a potential energy (like an electromagnetic energy). It is strongly related to Bernoulli type problems for fluids. Note for instance that the Euler-Lagrange equation leads to the following equation for (Ω^*, u_{Ω^*}) on $\partial\Omega^*$

$$\frac{1}{2}|\nabla u|^2 + \mathcal{C} = \text{constant}$$

where $\mathcal{C}$ is the mean curvature of $\partial\Omega^*$.

To conclude, let us mention, among related topics, the work of A. Chambolle and C.J. Larsen ([6]) where a functional with Neumann energy, perimeter and measure penalization is studied and regularity is proved in dimension 2.

This paper is organized as follows:

- Equivalence between (4) and (5) is treated in Section 2 together with extra remarks on existence.
- Section 3 presents a proof of $\frac{1}{2}$-Hölder continuity together with extra properties on the state function and its Laplacian.
- The Lipschitz continuity of the state function is established in Section 4.
- Then, we recall in Section 5 how the theory of quasi-minimizers leads to the regularity of $\partial\Omega^*$.
- Finally, we collect more of less known lemmas in an appendix.

2. Equivalence results

2.1. Existence of minimizers

The existence of minimizers does not always hold. Even when f is the null function, i.e., in the problem of minimization of the perimeter with a volume constraint, we know cases of unbounded D where there is no minimizer. An example where D is an open connected unbounded set is given in [10]. Nevertheless, the following proposition gives a sufficient condition.

Proposition 2.1. *If D is bounded, the problems* (4) *and* (5) *both have optimal shapes.*

One can find details upon such results in [10].

2.2. The equivalence result

Theorem 2.2. *Suppose D is bounded and star-shaped and $f \in L^2(D)$. Then, there exists $\lambda^* = \lambda^*(m, f, D)$ such that any solution Ω^* of* (4) *is also a solution of* (5) *with $\lambda = \lambda^*$, that is,*

$$\forall \omega \subset D \text{ measurable}, J(\Omega^*) + P(\Omega^*) \leq J(\omega) + P(\omega) + \lambda^* ||\omega| - m|. \tag{8}$$

Moreover, for any $\lambda \geq \lambda^*$, any solution Ω^*_λ of (5) *satisfies $|\Omega^*_\lambda| = m$ and is therefore solution of* (4).

Proof of the theorem. Let Ω^* be a solution of (4). To prove that it satisfies (8), it is sufficient to prove that any solution Ω^*_λ of (5) satisfies $|\Omega^*_\lambda| = m$ for λ large enough (i.e., for $\lambda \geq \lambda^*$ to be determined). Indeed, we then have

$$\begin{aligned} J(\Omega^*) + P(\Omega^*) &= \mathcal{E}(\Omega^*) \leq \mathcal{E}(\Omega^*_\lambda) = \mathcal{E}_\lambda(\Omega^*_\lambda) \\ &\leq \mathcal{E}_\lambda(\omega) = J(\omega) + P(\omega) + \lambda||\omega| - m| \end{aligned}$$

for any $\omega \subset D$ measurable.

Let us consider $\Omega = \Omega^*_\lambda$ a solution of (5) (which exists by Proposition 2.1). We will prove that if $|\Omega| \neq m$, then λ is necessarily lower than some $\lambda^* = \lambda^*(f, m, D)$. This will give the result.

The proof will be split in two parts. We will first assume that $|\Omega| < m$: the estimate on λ will rely on the monotonicity of J and on a result from I. Tamanini [11] valid for $P(\Omega)$ and providing "good" exterior perturbations of Ω. Then the case $|\Omega| > m$ will be treated by simple perturbations of the form $t\Omega$, $t < 1$: this is where the geometric assumption on D is mainly needed.

Let us first note that there exists $C_0 = C_0(f, m, D)$ (independent of λ) such that

$$\max\{|\Omega|, P(\Omega), -J(\Omega), \int_\Omega |\nabla u_\Omega|^2\} \leq C_0. \tag{9}$$

Indeed first $|\Omega| \leq D$. Then, J being nonincreasing, $-J(\Omega) \leq -J(D) < \infty$. Then, since $m < |D|$, we may construct $\omega \subset D$ measurable such that $|\omega| = m$ and $P(\omega) < \infty$. It follows that

$$J(\Omega) + P(\Omega) + \lambda||\Omega| - m| \leq J(\omega) + P(\omega) \leq P(\omega)$$

since $J(\omega) \leq G(0) = 0$. And so $P(\Omega) \leq P(\omega) - J(D)$. The last estimate finally comes from the equality $J(\Omega) = -\frac{1}{2}\int_\Omega |\nabla u_\Omega|^2$.

2.3. Proof of the equivalence: study of the case $|\Omega| < m$

Assume $|\Omega| < m$. We use the following lemma which is a simple generalisation of a lemma from I. Tamanini ([11]).

Lemma 2.3. *Assume D is a bounded, open and star-shaped set. Let us take some measurable set $L \subset D$, and some constants $\alpha > 0$, $\beta > 0$ such that*

$$P(L) \leq \alpha,$$

$$|L| \geq \beta, \ |D \backslash L| \geq \beta.$$

Then, there exist two positive constants b, C_1 depending only on d, α and β such that: for all $\delta \in (0, b)$, one may find a measurable set G_δ satisfying:

$$L \subset G_\delta, \ G_\delta \backslash L \subset D,$$

$$|G_\delta \backslash L| = \delta,$$

$$P(G_\delta) \leq P(L) + C_1\delta.$$

Now since $\lambda||\Omega| - m| \leq -J(D) + P(\omega) = C$ we can find $\lambda_1(m, D)$ large enough so that, if $\lambda \geq \lambda_1$, we have

$$\min\{|\Omega|, |D \backslash \Omega|\} \geq \frac{1}{2}\min\{m, |D| - m\}.$$

We now assume that $\lambda \geq \lambda_1$ and apply Lemma 2.3 to $L = \Omega$, $\alpha = C_0$, $\beta = \frac{1}{2}\min\{m, |D| - m\}$. We obtain G_δ satisfying the conditions of the lemma and $|G_\delta| < m$ for some δ

positive. Note here that the constant C_1 given by the lemma depends only on the data of the problem. By optimality of Ω,

$$\begin{aligned} J(\Omega) + P(\Omega) + \lambda(m - |\Omega|) &= \mathcal{E}_\lambda(\Omega) \leq \mathcal{E}_\lambda(G_\delta) \\ &\leq J(G_\delta) + P(G_\delta) + \lambda(m - |G_\delta|). \end{aligned}$$

Since J is nonincreasing for the inclusion, $J(G_\delta) \leq J(\Omega)$ and using the properties of G_δ, we deduce

$$\lambda\delta = \lambda(|G_\delta| - |\Omega|) \leq C_1\delta, \text{ and so } \lambda \leq C_1.$$

Thus we have proved, with $\Lambda_1 = \max\{C_1, \lambda_1\}$,

$$(|\Omega| < m) \Rightarrow (\lambda \leq \Lambda_1).$$

2.4. Proof of the equivalence: study of the case $|\Omega| > m$

Assume now $|\Omega| > m$. To make interior variations of the minimizer, the use of scaling works well when the open set D is star-shaped. Up to translating, we may suppose that D is star-shaped with respect to 0.

Recall here that Ω is an optimal shape of (5) with the extra assumption $|\Omega| > m$. If t is in $(1-\varepsilon, 1)$ for ε sufficiently small, we still have $|\Omega_t| > m$, and so, using the optimality condition:

$$J(\Omega) + P(\Omega) + \lambda(|\Omega| - m) \leq J(t\Omega) + P(t\Omega) + \lambda(|t\Omega| - m),$$

that is to say, since we know explicitly the expressions of the perimeter and measure of $t\Omega$,

$$(1 - t^{d-1})P(\Omega) + \lambda(1 - t^d)|\Omega| \leq J(t\Omega) - J(\Omega).$$

Since $t < 1$ and $|\Omega| \geq m$,

$$\lambda(1 - t^d)m \leq J(t\Omega) - J(\Omega).$$

To conclude, let us divide by $1 - t$ and make t tend to 1, to obtain, using Lemma D.1

$$\begin{aligned} d\lambda m \leq \limsup_{t\to 1} \frac{J(t\Omega) - J(\Omega)}{1-t} &\leq (C(d) + ||f(x)x||_{L^2(\mathbf{R}^d)})\sqrt{-2J(\Omega)} \\ &\leq C_2 = C_2(f, m, D). \end{aligned}$$

Thus we have proved, with $\Lambda_2 = \frac{C_2}{dm}$,

$$(|\Omega| > m) \Rightarrow (\lambda \leq \Lambda_2).$$

Taking $\lambda^* = \max\{\Lambda_1, \Lambda_2\}$ concludes the proof. □

3. First results on the minimizers and Hölder continuity of the state function

Until the end of the paper, we will suppose that Ω *is an optimal shape of the problem* (5). Since from now on, all the results will be purely local, no assumption needs to be made upon the shape of D. In particular, D does not need to be bounded. Note that, according to Theorem 2.2, all the regularity results attached to Ω and u carry over to the solution of the constrained problem (4).

In the sequel, we will often denote u_Ω by u. We will suppose from now on that u *is nonnegative*, but all the results in this section can be generalized to the case where no sign is given. Notice that since an optimal shape is an admissible shape, Ω has a finite measure.

Finally, let us assume in this section that $f \in L^q(D)$, for some $q > d$.

3.1. First properties of the optimal shape

Let us enumerate some properties of the optimal shape. First of all, using the well-known inequality, $P(\Omega \cup B) + P(\Omega \cap B) \leq P(\Omega) + P(B)$, we have the following:

Lemma 3.1. *Let B be an open ball included in D and let v belong to $H_0^1(B \cup \Omega)$. Then:*

$$J(\Omega) \leq G(v) + P(B) - P(\Omega \cap B) + \lambda|\Omega^c \cap B|.$$

And as a corollary:

Lemma 3.2. *Let B be an open ball included in D and $\varphi \in \mathcal{C}_0^\infty(B)$. Then*

$$| < \Delta u + f, \varphi > | \leq 2 \left(\int |\nabla \varphi|^2 \right)^{\frac{1}{2}} (P(B) - P(B \cap \Omega) + \lambda |B \cap \Omega^c|)^{\frac{1}{2}}.$$

Proof. Let $t \in (0, +\infty)$. From Lemma 3.1 applied with $v = u + t\varphi$ we have:

$$G(u) \leq G(u + t\varphi) + P(B) - P(B \cap \Omega) + \lambda |B \cap \Omega^c|.$$

Expanding and dividing by t leads to

$$< \Delta u + f, \varphi > \leq t \int |\nabla \varphi|^2 + \frac{1}{t}(P(B) - P(B \cap \Omega) + \lambda |B \cap \Omega^c|),$$

and finally, minimizing in t, we obtain the result. □

Now we can prove the regularity of u in the "interior" of Ω in the following sense

Lemma 3.3. *Let B be a ball included in D such that $|B \cap \Omega^c| = 0$. Then u is a solution of*

$$-\Delta u = f \text{ in } B.$$

Proof. Since $|B \cap \Omega^c| = 0$, we have $P(\Omega \cap B) = P(B)$. We then apply Lemma 3.2. □

Whereas the preceding lemma gives Δu in the interior of Ω, the following lemma gives some *key* information about the nature of Δu on the boundary of Ω.

Lemma 3.4. *There exist a positive measure μ such that*

$$\Delta u + f \mathbf{1}_{[u>0]} = \mu. \tag{10}$$

Proof. We define $p_n, q_n : \mathbf{R} \to \mathbf{R}$ as

$$p_n(x) = n.x \mathbf{1}_{(0,\frac{1}{n})}(x) + \mathbf{1}_{[\frac{1}{n},\infty)}(x), \; q_n(x) = \int_0^x p_n(s) ds.$$

Let $\psi \in \mathcal{C}_0^\infty(D)$ and $t \in (0, \infty)$. Since $v = u + t\psi p_n(u) \in H_0^1(\Omega)$, we have $G(u) \leq G(v)$, i.e.,

$$t \int \nabla u \nabla (\psi p_n(u)) + t^2 \int |\nabla(\psi p_n(u))|^2 \leq t \int f.\psi p_n(u).$$

Dividing by t and making t tend to 0, we have

$$\int p_n(u) \nabla u . \nabla \psi + \int p_n'(u) \psi |\nabla u|^2 - \int f p_n(u) \psi = 0,$$

i.e., in the sense of distribution on D, (we say in $\mathcal{D}'(D)$):

$$n|\nabla u|^2 \mathbf{1}_{[0<u<1/n]} - \Delta(q_n(u)) - f.p_n(u) = 0.$$

But $f.p_n(u)$ tends to $f\mathbf{1}_{[0<u]}$ in $L^1(D)$, and so in $\mathcal{D}'(D)$. Similarly, $q_n(u)$ tends to u in $L^2(D)$. If now we call μ^n the positive measure $n|\nabla u|^2 \mathbf{1}_{[0<u<1/n]}$, we have

$$\Delta(q_n(u)) + f.p_n(u) = \mu^n.$$

Since the left term converges in $\mathcal{D}'$, then so does μ^n. Finally, the distribution $\Delta u + f\mathbf{1}_{[0<u]}$ is positive, and so is a positive Radon measure, that we call μ. And we have:

$$\Delta u + f\mathbf{1}_{[0<u]} = \mu.$$

□

Now, from the preceding lemma, we have $-\Delta u \leq |f|$ whence comes the following L^∞ estimate:

Lemma 3.5. *The function u belongs to $L^\infty(D)$ and $||u||_{L^\infty(D)} \leq C(d, q, |\Omega|, ||f||_{L^q(D)})$.*

3.2. 1/2-Hölder continuity of u

Theorem 3.6. *The state function u is locally $\frac{1}{2}$-Hölder continuous.*

This regularity result comes from the study of the measure Δu on the boundary of Ω. The following estimation is the key to the Hölder continuity:

Lemma 3.7. *There exists a constant $C = C(d, f, \lambda)$ such that for any ball $B(x_0, r) \subset D$, with $r \leq 1$,*

$$|\Delta u|(\overline{B(x_0, r/2)}) \leq C r^{d-1-\frac{1}{2}}.$$

Proof. In Lemma 3.2, consider the open ball $B(x_0, r)$ with radius $r \leq 1$ and $\varphi \in \mathcal{C}_0^\infty(B(x_0, r))$. We have:

$$| < \Delta u + f, \varphi > | \leq C_1 ||\nabla\varphi||_{L^2} (r^{d-1} + \lambda r^d)^{\frac{1}{2}} \leq C ||\nabla\varphi||_{L^2} r^{\frac{d-1}{2}}.$$

We now take φ as follows:

$$\varphi = 1 \text{ in } B(x_0, r/2),\ \varphi = 0 \text{ out of } B(x_0, r),$$

$$0 \leq \varphi \leq 1,\ ||\nabla\varphi||_{L^\infty} \leq \frac{C}{r},$$

and obtain the result. □

Remark 3.8. When u does not have sign, we can get the same conclusion by adding the tool of the *Monotonicity Lemma* (Cf. [1]) but this requires more work as in the proof of Lemma 9 in [4]. The preceding lemma is the point where the *Monotonicity Lemma* has to be employed when we do not know the sign of u.

Integrating the result of the previous lemma, we find that

Lemma 3.9. *If $r \leq 1$ and $B(x_0, 2r) \subset D$, we have*

$$\int_0^r s^{1-d} \int_{B(x_0,s)} d(|\Delta u|) ds \leq C\sqrt{r}.$$

Using now the remark to Lemma A.1, we can take the following representation of u:

$$\forall x \in D, u(x) = \lim_{r\to 0} \fint_{\partial B(x,r)} u,$$

where we use the notation $\fint_{\partial B(x,r)} u$ to denote the average of u over $\partial B(x, r)$. One verifies that according to this particular definition, we also have

$$\forall x \in D, u(x) = \lim_{r\to 0} \fint_{B(x,r)} u.$$

In what follows, $\partial\Omega$ will always denote the measure-theoretic boundary of Ω, i.e.,

$$\partial\Omega = \{x \in D \,:\, \forall r > 0, 0 < |B(x, r) \cap \Omega| < |B(x, r)|\}.$$

Moreover, let us define $d(x) = d(x, \partial\Omega)$. First of all, we show that u is zero outside the measure-theoretic interior of Ω.

Lemma 3.10. *Let us take x_0 in D such that $|B(x_0, r) \cap \Omega^c| > 0$ for all $r > 0$. Then $u(x_0) = 0$.*

Proof. Consider $r > 0$ such that $B(x_0, 4r) \subset D$ and $x_1 \in B(x_0, r)$ such that $u(x_1) = 0$ (such a point exists because $u = 0$ almost everywhere outside Ω). From Lemma A.3, we have

$$||u||_{L^\infty(B(x_0,r))} \leq ||u||_{L^\infty(B(x_1,2r))} \leq C(3r + \fint_{\partial B(x_1,2r)} u)$$

but thanks to Lemmas A.1 and 3.9:

$$\fint_{\partial B(x_1,2r)} u = \fint_{\partial B(x_1,2r)} u - u(x_1) \leq C\sqrt{r}$$

whence $||u||_{L^\infty(B(x_0,r))} \leq C\sqrt{r}$. Finally

$$0 \leq u(x_0) \leq \liminf_{r\to 0} ||u||_{L^\infty(B(x_0,r))} = 0.$$

□

Proof of the Hölder continuity. Let $\delta \in (0, \frac{1}{3})$. Let us call $D_\delta = \{x \in D \,:\, d(x, \partial D) \geq 6\delta\}$.

Lemma 3.11. *There exists $C_\delta > 0$ such that for any $x_0 \in D_\delta$, $u(x_0) \leq C_\delta d(x_0)^{\frac{1}{2}}$.*

Proof. Take x_0 in D_δ. First suppose $d(x_0) \geq \delta$. Then, since u is bounded

$$u(x_0) \leq \frac{||u||_\infty}{\sqrt{\delta}} d(x_0)^{\frac{1}{2}} \leq C_\delta d(x_0)^{\frac{1}{2}}.$$

Now suppose $r_0 = d(x_0) < \delta$ and take $y_0 \in \partial\Omega$ such that $r_0 = d(x_0, y_0)$. In Lemma 3.10, we saw that $u(y_0) = 0$, so that, applying Lemma A.1, we get

$$\begin{aligned} \fint_{\partial B(y_0,2r_0)} u &= \fint_{\partial B(y_0,2r_0)} u - u(y_0) \\ &= (d\omega_d)^{-1} \int_0^{2r_0} s^{1-d} \int_{B(z_0,s)} \Delta u ds \leq C\sqrt{r_0}. \end{aligned}$$

and applying point (ii) of Lemma A.3, we get

$$||u||_{L^\infty(B(x_0,r_0))} \leq ||u||_{L^\infty(B(y_0,2r_0))} \leq C(3r_0 + \fint_{\partial B(y_0,2r_0)} u)$$

and so $u(x_0) \leq Cd(x_0)^{\frac{1}{2}}$. □

Lemma 3.12. *There exists C'_δ such that for any $x_0 \in D_\delta$ with $d(x_0) > 0$, we have*

$$||\nabla u||_{L^\infty(B(x_0,\frac{d(x_0)}{4}))} \leq C'_\delta \max\{1, \frac{1}{d(x_0)^{\frac{1}{2}}}\}.$$

Proof. First suppose that $|B(x_0, d(x_0)) \cap \Omega| = 0$. Thanks to Lemma 3.10, we know that $u \equiv 0$ in $B(x_0, d(x_0))$. In particular, $|\nabla u(x_0)| = 0$.

Now suppose that $|B(x_0, d(x_0) \cap \Omega^c| = 0$. We have $-\Delta u = f$ in $B(x_0, d(x_0))$ (by Lemma 3.3) so that, applying point (ii) of Lemma A.1,

$$\begin{aligned} ||\nabla u||_{L^\infty(B(x_0,\frac{d(x_0)}{4}))} &\leq C\left[1 + \frac{1}{d(x_0)}||u||_{L^\infty\left(B\left(x_0,\frac{d(x_0)}{2}\right)\right)}\right] \\ &\leq C\left[1 + \frac{1}{d(x_0)^{\frac{1}{2}}}\right] \end{aligned}$$

□

To conclude the proof of the Hölder continuity, take x and y in D_δ. Denote $\Omega_{\text{int}} = \{x \in D : \exists r > 0, B(x,r) \subset D, |B(x,r) \cap \Omega^c| = 0\}$. Note that Ω_{int} is open. Suppose x or y belong to Ω_{int}^c. Then Lemmas 3.10 and 3.11 show that there exists a constant C such that

$$|u(x) - u(y)| \leq Cd(x,y)^{\frac{1}{2}}.$$

Now suppose both x and y belong to Ω_{int}. First suppose $d(x,y) \leq d(x)/4$. Using that u is regular in $B(x, d(x))$, and the estimate on ∇u given by Lemma 3.12, we find that

$$\begin{aligned} |u(x) - u(y)| &\leq C'_\delta \max\{1, \frac{1}{d(x)^{\frac{1}{2}}}\}d(x,y) \\ &\leq C \max\{d(x,y), d(x,y)^{\frac{1}{2}}\} \leq Cd(x,y)^{\frac{1}{2}}. \end{aligned}$$

If $d(x,y) \leq d(y)/4$, the result is the same by symmetry.

Now if $d(x,y) \geq \max\{d(y), d(x)\}/4$

$$\begin{aligned} |u(x) - u(y)| &\leq 2\max\{u(x), u(y)\} \\ &\leq 2C\max\{d(x)^{\frac{1}{2}}, d(y)^{\frac{1}{2}}\} \leq Cd(x,y)^{\frac{1}{2}}. \end{aligned}$$

And so there exists $C > 0$ such that for any x, y in D_δ, $|u(x) - u(y)| \leq Cd(x,y)^{\frac{1}{2}}$. □

4. Lipschitz continuity of the state function

This section is devoted to the proof of Theorem 1.1. First, we prove a density result which will be needed at the very end of the second part of this section. Then we will prove the local Lipschitz continuity of the state function.

4.1. A density result

The following result is true in general, i.e., no hypothesis is needed on f other than $f \in L^2(D)$.

Lemma 4.1. *There exist some constants $C(d)$ and $r_0(d,\lambda)$ such that, for all $x \in \partial\Omega$ and $r \in (0, \min\{r_0, d(x,\partial D)\})$,*

$$|\Omega^c \cap B(x,r)| \geq Cr^d.$$

Proof. The proof is classical and can be found for example in [7]. We consider perturbations of Ω of the form $\widetilde{\Omega} = \Omega \cup B(x,r)$ for balls $B(x,r)$ included in D. Using that Ω is an optimal shape of (5) and the monotonicity of J, we have

$$P(\Omega^c, B(x,r)) \leq \mathcal{H}^{d-1}(\partial B(x,r) \cap \Omega^c) + \lambda|\Omega^c \cap B(x,r)|.$$

But, since $P(\Omega^c, \partial B(x,r))$ is null for almost any r (because Ω has finite perimeter) and thanks to the inequality

$$P(\Omega^c \cap B(x,r)) \leq P(\Omega^c, B(x,r)) + \mathcal{H}^{d-1}(\partial B(x,r) \cap \Omega^c) + P(\Omega^c, \partial B(x,r)),$$

we have that for almost any r:

$$P(\Omega^c \cap B(x,r)) \leq 2\mathcal{H}^{d-1}(\partial B(x,r) \cap \Omega^c) + \lambda|\Omega^c \cap B(x,r)|,$$

i.e., using an isoperimetric inequality on the left term:

$$C(d)|\Omega^c \cap B(x,r)|^{\frac{d-1}{d}} - \lambda|\Omega^c \cap B(x,r)| \leq 2\mathcal{H}^{d-1}(\partial B(x,r) \cap \Omega^c).$$

Note that $\dfrac{d}{dr}|\Omega^c \cap B(x,r)| = \mathcal{H}^{d-1}(\partial B(x,r) \cap \Omega^c)$. And so, for almost any r,

$$C(d)|\Omega^c \cap B(x,r)|^{\frac{d-1}{d}} - \lambda|\Omega^c \cap B(x,r)| \leq 2\frac{d}{dr}|\Omega^c \cap B(x,r)|.$$

We can study the function $g(x) = C(d)x^{\frac{d-1}{d}} - \lambda x$, and see that $\frac{g(x)}{x^{\frac{d-1}{d}}} \to C(d)$, when $x \to 0$ from above. Then, for r small enough (i.e., lower than some $r_0(d,\lambda)$ and than $d(x,\partial D)$), and still for almost all r,

$$\frac{C(d)}{4}|\Omega^c \cap B(x,r)|^{\frac{d-1}{d}} \leq \frac{d}{dr}|\Omega^c \cap B(x,r)|,$$

i.e., dividing by $|\Omega^c \cap B(x,r)|^{\frac{d-1}{d}}$ and integrating from 0 to $s < \min\{r_0(d,\lambda), d(x,\partial D)\}$: $\hat{C}(d)s \leq |\Omega^c \cap B(x,r)|^{\frac{1}{d}}$, i.e., $|\Omega^c \cap B(x,s)| \geq \hat{\hat{C}}(d)s^d$. □

4.2. Lipschitz continuity, setting of the proof

Our main objective is to prove some regularity result on the boundary of Ω. In order to do this, one may want to apply known results on the quasi-minimizers. But the regularity already known on u is not sufficient to do this. In the study of the Hölder-continuity, we made variations of Ω by taking unions with balls. The following method uses another kind of perturbations which will be described in 4.3.

From now on, we suppose that the function f is *locally Hölder continuous, bounded and nonnegative*. As a consequence, so is the state function u. From the previous section, we know that the function u is continuous.

We want to study the local Lipschitz continuity of u in D, i.e., to prove that for any point x of D, there exists a neighborhood of x in D such that u is Lipschitz continuous in this neighborhood. For a point x that lies in the measure-theoretic interior or exterior of Ω, this property is clearly satisfied, u satisfying some Poisson's equation in some neighborhood of the point.

Now take $\overline{x} \in \partial\Omega$ (Recall that $\partial\Omega$ denotes the measure-theoretic boundary). We suppose $B(\overline{x}, \delta) \subset D$. We are going to prove the Lipschitz continuity in the ball $B(\overline{x}, \frac{\delta}{32})$. In particular, the property will be local and we may suppose, up to renormalizing, that $\delta = 1$ and denote $B_1 = B(\overline{x}, 1)$. Let φ be a smooth function, such that $0 \leq \varphi \leq 1$, $\varphi = 0$ in $B(\overline{x}, \frac{1}{2})^c$, $\varphi = 1$ in $B(\overline{x}, \frac{1}{4})$. Let $\varepsilon > 0$. We call

$$w_\varepsilon = [u - \varepsilon]^+.$$

As in [2], let M_ε be the smallest constant such that

$$\forall x \in B_1,\ M_\varepsilon d(x) \geq w_\varepsilon(x)\varphi(x),$$

where $d(x) = d(x, \partial\Omega)$. We want to estimate M_ε from above independently of ε. The result then comes from the following lemma:

Lemma 4.2. *We suppose the existence of a constant $\mathcal{M} < \infty$ such that, for all $\varepsilon > 0$, $M_\varepsilon < \mathcal{M}$. Then, u is Lipschitz continuous in $B(\overline{x}, \frac{1}{32})$.*

Proof. Let y and z belong to $B(\overline{x}, \frac{1}{32})$.

If y or z belong to $\partial\Omega$, we have $0 = |u(y) - u(z)| \leq M d(y, z)$.

If both $d(y)$ and $d(z)$ are smaller than $d(y, z)/6$, we find that

$$|u(y) - u(z)| \leq 2M \max\{d(y), d(z)\} \leq \frac{2M}{6} d(y, z).$$

Finally, if, for example $r = d(y) > d(y, z)/6$, we have the following estimates:

$$||\nabla u||_{L^\infty(B(y, \frac{r}{6}))} \leq C + \frac{C'}{r} ||u||_{L^\infty(B(y, \frac{r}{3}))}.$$

Let $y_0 \in \partial\Omega$ be such that $r = d(y) = d(y, y_0)$. Then, $B(y, \frac{r}{3}) \subset B(y_0, \frac{2r}{3})$. So $||u||_{L^\infty(B(y, \frac{r}{3}))} \leq 2\mathcal{M}\frac{r}{3}$, whence

$$||\nabla u||_{L^\infty(B(y, \frac{r}{6}))} \leq C + C'\frac{\mathcal{M}}{3}, \quad \text{and} \quad |u(y) - u(z)| \leq C(\mathcal{M})|y - z|.$$

□

We can easily see the two following properties of M_ε:

Lemma 4.3. $\forall \varepsilon > 0, M_\varepsilon < \infty$.

Lemma 4.4. *If* $M_\varepsilon > 0$*,* $\exists x_0 \in B(0,1),\ M_\varepsilon d(x_0) = w_\varepsilon(x_0)\varphi(x_0) > 0$.

Let us fix ε positive (and small). Take the x_0 given by the previous lemma. Take now y_0 on $\partial\Omega$ such that $d(x_0) = d(x_0, y_0)$. Up to some rotations and translations, we can suppose that $y_0 = 0$ and $x_0 = d(x_0)e_1$, $(e_1, \ldots, e_d)$ being the standard basis of $\mathbf{R}^d$. Notice that, since $\varphi(x_0) > 0$, x_0 is in $B(x, \frac{1}{2})$ and so $d(x_0) \leq d(x_0, x) < \frac{1}{2}$, whence y_0 is finally in $B(x, 1)$. Our goal here is to estimate M_ε from above independently of ε. So we will suppose that $M_\varepsilon \geq 1$, for in the other case we have the good estimate.

In the sequel, we will suppose that ε is fixed and call $M = M_\varepsilon$ and $w = w_\varepsilon$. The aim of the following is to estimate M from above independently of ε.

4.3. Description of the perturbations

We suppose (and we will prove later that it is indeed the case) that there exists a $\mathcal{C}^2$ surface $S = \{(x_1, x') \in \mathcal{V} : x_1 = \psi(x')\}$ in some neighborhood $\mathcal{V}$ (included in B_1) of $y_0 = 0$ such that ψ can be expanded in the following sense

$$\psi(x') = Q'(x') + o_{x' \to 0}(|x'|^2)$$

where Q' is some quadratic form, satisfying the following mean curvature condition

$$\kappa(S)(x') \leq \frac{1}{d-1}\Delta\psi(x') \leq \frac{C}{M} + \frac{C}{\varphi(x_0)}$$

where the constants C do not depend on ε, nor on x_0, but may depend on the uniform norm of φ and of its derivates (so will be any constant denoted by C in the following). And we suppose that

$$\partial\Omega \cap \mathcal{V} \subset \{(x_1, x') \in \mathcal{V} : x_1 \leq \psi(x')\}$$

We are going to perturb Ω by doing slight perturbations of the surface S:

$$S_t^- = \left\{(x_1, x') \in \mathcal{V} : x_1 = \psi_t^-(x') = \psi(x') + \frac{\delta_0}{\varphi(x_0)}|x'|^2 - t\right\}$$

and

$$S_t^+ = \left\{(x_1, x') \in \mathcal{V} : x_1 = \psi_t^+(x') = \psi(x') + t\right\}$$

where $t \geq 0$ and $\delta_0 > 0$ are both *small*. Notice that, in $\mathcal{V}$

$$\kappa(S_t^\pm)(x) \leq \frac{C}{M} + \frac{C}{\varphi(x_0)}.$$

Let Z_t be the domain between S_t^+ and S_t^-, i.e.,

$$Z_t = \{(x_1, x') \in \mathcal{V} : \psi_t^-(x') < x_1 < \psi_t^+(x')\}.$$

As S_t^+ and S_t^- intersect for $|x'|^2 = 2\frac{\varphi(x_0)}{\delta_0}t$, we will have $\overline{Z_t} \subset \mathcal{V}$ if t is small enough (δ_0 being fixed). Let

$$\Omega_t = \Omega \cup Z_t,$$

and

$$V_t = \{x_1 > \psi_t^-(x')\}\backslash\Omega \quad (= Z_t\backslash\Omega \text{ for } t \text{ small enough }).$$

Let u_t be the solution to the following problem (with $u_t = u$ outside Z_t):

$$\begin{cases} -\Delta u_t = f \ \text{ in } \ Z_t \\ u_t \in u + H_0^1(Z_t) \end{cases}$$

The set Ω being an optimal shape of (5), we have

$$J(\Omega) + P(\Omega) + \lambda||\Omega| - m| \leq J(\Omega_t) + P(\Omega_t) + \lambda||\Omega_t| - m|$$

so that (since $J(\Omega) = G(u)$ and $J(\Omega_t) \leq G(u_t)$)

$$G(u) - G(u_t) \leq P(\Omega_t) - P(\Omega) + \lambda|V_t|.$$

4.4. Using the perturbations

In this section, we will estimate the differences $P(\Omega_t) - P(\Omega)$ from above and $G(u) - G(u_t)$ from below with the help of $|V_t|$ and $|V_{t/2}|$ and thus reach a bound on M.

4.4.1. Variations of the perimeter

Lemma 4.5.

$$P(\Omega_t) - P(\Omega) \leq \frac{C}{\varphi(x_0)}|V_t| + \frac{C}{M}|V_t|.$$

Proof. We first give an expression of both perimeters:

$$P(\Omega_t, B_1) = \mathcal{H}^{d-1}(S_t^- - \Omega) + P(\Omega, B_1 - \overline{Z_t})$$

$$P(\Omega, B_1) \geq P(\Omega, Z_t) + P(\Omega, B_1 - \overline{Z_t}).$$

Let now $K = \sup|K_j(y)|$ where the sup is taken w.r.t. $j \in \{1, \ldots, d-1\}$ and $y \in B_{1/2} \cap S_0$, and K_j denotes the j-th curvature of the surface S_0 at the point y. Then, if $d_t(.) = d(., S_t^-)$, we have (see [9], Chapter 14.6), if $d_t(x) = |x - y|$ for some $y \in S_t^-$, and t being small enough,

$$\Delta d_t(x) = -\sum_{j=1}^{d-1} \frac{K_j(y)}{1 - K_j(y)d_t(x)} \geq -\frac{C}{\varphi(x_0)} - \frac{C}{M}.$$

We thus have, ν being the exterior generalized normal on the reduced boundary of V_t ($\partial_{red}V_t$)

$$\begin{aligned} -\left(\frac{C}{\varphi(x_0)} + \frac{C}{M}\right)|V_t| \;&\leq\; \int_{V_t} \Delta d_t(x)dx = \int_{\partial_{red}V_t} < \nabla d_t, \nu > d\mathcal{H}^{d-1} \\ &\leq\; -\mathcal{H}^{d-1}(S_t^- - \Omega) + P(\Omega, Z_t). \end{aligned}$$

which concludes the proof. □

4.4.2. Variations of J

Before computing the energy variation, we need some estimates on u and u_t given by the following lemmas. First of all, the next result easily follows by a straightforward scaling:

Lemma 4.6. *Let v and g be functions and R be a positive real number such that*

$$\begin{cases} -\Delta v = g \text{ in } B_R \\ v \in H_0^1(B_R). \end{cases}$$

Then $v \in L^\infty(B_R)$ and $||v||_{L^\infty(B_R)} \leq C(d)||g||_{L^\infty} R^2$.

Next we obtain an estimate on u:

Lemma 4.7. *There exists a neighborhood $\overline{\mathcal{V}}$ of 0 in $\mathbf{R}^d$ such that*

$$\forall x \in \overline{\mathcal{V}}, u(x) \geq \frac{C(d,f)M}{\varphi(x_0)}(x_1 - \psi(x')).$$

Proof. For any x on S, calling n_x the unit normal vector to S that points toward x_0, we consider the ball $B^x = B(x + \frac{d(x_0)}{2}n_x, \frac{d(x_0)}{2})$. In some neighborhood of 0 in S, any such ball B^x is included in Ω.

(i) *There exist two positive constants $\alpha_0 = \alpha_0(d, ||f||_{L^\infty}) < 1/4$ and $C_0 = C_0(d)$ such that*

$$\inf_{B(x_0,\alpha_0 d(x_0))} u \geq C_0 M \frac{d(x_0)}{\varphi(x_0)}.$$

Indeed, take $\alpha < 1/4$ so that $r = \alpha d(x_0) < d(x_0)/4$ and consider w the solution of

$$\begin{cases} -\Delta w = f & \text{in } B(x_0, 2r) \\ w = 0 & \text{on } \partial B(x_0, 2r). \end{cases}$$

Lemma 4.6 tells us that $||w||_{L^\infty} \leq C(d)||f||_{L^\infty} 4r^2$. Now notice that $u - w$ is a harmonic function equal to u on $\partial B(x_0, 2r)$ so that, applying Harnack inequality (on $u-w$ between $B(x_0, r)$ and $B(x_0, 2r)$)

$$\begin{aligned} \inf_{B(x_0,r)} u \geq \inf_{B(x_0,r)} (u-w) &\geq C(d)(u(x_0) - w(x_0)) \\ &\geq C(d)(M\frac{d(x_0)}{\varphi(x_0)} - 4r^2 C(d)||f||_{L^\infty}). \end{aligned}$$

But since $r \leq 4M\frac{d(x_0)}{\varphi(x_0)}$, we get

$$\inf_{B(x_0,r)} u \geq C(d)M\frac{d(x_0)}{\varphi(x_0)}(1 - 4rC(d)||f||_{L^\infty}),$$

and so, as $r = \alpha d(x_0)$ and $d(x_0) < 1$,

$$\inf_{B(x_0,r)} u \geq C(d)M\frac{d(x_0)}{\varphi(x_0)}(1 - 4\alpha C(d)||f||_{L^\infty}),$$

which gives α_0.

(ii) Now let us define $r_0 = d(x_0) - \alpha_0 d(x_0)/2$. For a point $x \in S$, we call n_x the unit normal to S in x oriented toward x_0. Then *there exists a neighborhood $\mathcal{V}'$ of 0 in S such that for any $x \in \mathcal{V}'$, we have*

(α) $B(x + r_0 n_x, r_0) \subset \Omega$,
(β) $x + r_0 n_x \in B(x_0, \alpha_0 d(x_0))$.

Indeed, this follows from the regularity of S. Now, as in (i), we find the existence of a positive constant $C_1(d, f)$ such that

$$\forall x \in \mathcal{V}', \inf_{B(x+r_0 n_x, \alpha_0 r_0)} u \geq C_1 M \frac{d(x_0)}{\varphi(x_0)}.$$

Take v the solution of

$$\begin{cases} \Delta v = 0 & \text{in } B(x + r_0 n_x, r_0) \backslash B(x + r_0 n_x, \alpha r_0) \\ v = C_1 M \dfrac{d(x_0)}{\varphi(x_0)} & \text{on } \partial B(x + r_0 n_x, \alpha r_0) \\ v = 0 & \text{on } \partial B(x + r_0 n_x, r_0). \end{cases}$$

As $u \geq v$, and computing explicitly v, we get the existence of a positive constant $C_2(d, f)$ such that

$$\forall x \in \mathcal{V}', \forall h \in (0, d(x_0)/4), u(x + h n_x) \geq C_2 M \frac{h}{\varphi(x_0)}.$$

That is to say, there exists a neighborhood $\mathcal{V}''$ of 0 in $\mathbf{R}^d$ such that

$$\forall x \in \mathcal{V}'' \cap \{x_1 > \psi(x')\}, u(x) \geq C_2 \frac{M}{\varphi(x_0)} d(x, S),$$

but if x is small enough, $d(x, S) \geq C(d)(x_1 - \psi(x'))$ which yields the result. □

Lemma 4.8.

$$\inf_{Z_{t/2}} u_t \geq \frac{CM}{\varphi(x_0)} t.$$

Proof. We can apply Lemma 4.7 to obtain, on the surface $S^+_{t/2}$ the estimate

$$\inf_{S^+_{t/2}} u \geq \frac{CM}{\varphi(x_0)} t.$$

Now consider any point $x = (x_1, x')$ in $Z_{t/2}$ and build the points $x^- = (\psi^-_{t/2}(x'), x')$ and $x^+ = (\psi^+_{t/2}(x'), x')$. Consider $r^+ = d(x^+, \partial Z_t)$ and $r^- = d(x^-, \partial Z_t)$.

One has that, t being small enough, $r^+ \geq t/4$, $r^- \geq t/4$. One shows furthermore that $d(x^+, x^-) \leq t$. So consider the set C defined as the smallest convex set containing $B(x^+, t/4) \cup B(x^-, t/4)$ and C' defined as the smallest convex set containing $B(x^+, t/8) \cup B(x^-, t/8)$. We can apply some uniform Harnack inequality to the harmonic replacement of u_t upon these sets and prove that (u_t being super-harmonic is greater than its harmonic replacement, and we also have $u_t \geq u$)

$$u_t(x) \geq \inf_{C'} u_t \geq C(d) \sup_{C'} u_t \geq C(d) u_t(x^+) \geq C(d) u(x^+),$$

so that finally $u_t(x) \geq C(d) \dfrac{CM}{\varphi(x_0)} t$. □

Lemma 4.9.

$$G(u) - G(u_t) \geq \frac{CM^2}{\varphi(x_0)} |V_{t/2}|.$$

Proof. First recall that, thanks to the definition of u_t, we have $\int_{Z_t} \nabla u_t . \nabla (u - u_t) = \int_{Z_t} f.(u - u_t)$, and so

$$G(u) - G(u_t) \geq \frac{1}{2} \int |\nabla(u - u_t)|^2.$$

Now, for y in S_t^-, let l_y be the line from y and following the vector e_1. Let S_t' be the set of all the points y of S_t^- such that $l_y \cap V_{t/2}$ is not empty. For such a point y, let l_y' be the set $(y, y + s_y e_1)$ where $s_y = \sup\{s : y + se_1 \in \overline{V}_{t/2}\}$. Finally, let

$$V'_{t/2} = \{y + se_1 : 0 < s < s_y,\ y \in S_t'\}.$$

Integrating on the lines, we find that

$$\begin{aligned} \int_{V'_{t/2}} |D_{e_1}(u_t - u)| dx &\geq \int_{S_t'} dy \int_{l_y'} D_{e_1}(u_t - u) ds \\ &\geq \int_{S_t'} dy (u_t - u)(y + s_y e_1) \\ &= \int_{S_t'} dy u_t(y + s_y e_1). \end{aligned}$$

But,

$$u_t(y + s_y e_1) \geq \frac{CM}{\varphi(x_0)} t \geq \frac{CM}{\varphi(x_0)} |l_y'|,$$

and so

$$\int_{V'_{t/2}} |D_{e_1}(u_t - u)| dx \geq \frac{CM}{\varphi(x_0)} |V'_{t/2}| \geq \frac{CM}{\varphi(x_0)} |V_{t/2}|.$$

But, thanks to Schwarz inequality,

$$\int_{V'_{t/2}} |D_{e_1}(u - u_t)| \leq |V'_{t/2}|^{\frac{1}{2}} \left(\int_{V'_{t/2}} |D_{e_1}(u - u_t)|^2 \right)^{\frac{1}{2}}$$

and so

$$\begin{aligned} \int_\Omega |\nabla(u - u_t)|^2 &\geq \int_{V'_{t/2}} |D_{e_1}(u - u_t)|^2 \\ &\geq \frac{1}{|V'_{t/2}|} \left(\int_{V'_{t/2}} |D_{e_1}(u - u_t)| \right)^2 \geq \frac{CM^2}{\varphi(x_0)^2} |V'_{t/2}|. \end{aligned}$$

And so, as $V_{t/2} \subset V'_{t/2}$, and as $\varphi(x_0) \leq 1$,

$$\int_\Omega |\nabla(u - u_t)|^2 \geq \frac{CM^2}{\varphi(x_0)} |V_{t/2}|.$$

□

4.4.3. Conclusion

The optimality of Ω now leads, using all the previous estimates, to:

$$|V_{t/2}|\frac{CM^2}{\varphi(x_0)} \leq \left(\frac{C}{\varphi(x_0)} + CM + \lambda\right)|V_t|.$$

But for any t small enough, $B(0,t/2) \subset Z_t$, and so, thanks to Lemma 4.1, $|V_t| \geq |\Omega^c \cap B(0,t/2)| \geq Ct^d$. Thus, using Lemma B.1, there exist a constant $C(d)$ and a sequence $t_i \downarrow 0$ such that

$$\forall i, |V_{2t_i}| \leq C(d)|V_{t_i}|.$$

Putting all the terms in the minimization inequality for t_i sufficiently small, and after dividing by $|V_{t_i}|$, we find that

$$\frac{CM^2}{\varphi(x_0)} \leq \frac{C}{\varphi(x_0)} + \lambda + \frac{C}{M}$$

whence we can conclude to $M \leq C$ and thus to the local Lipschitz-continuity of u in D.

4.5. Construction of the surface S

Since f is locally Hölder continuous and $u(x_0) > \varepsilon$, u can be expanded at the order 2 around x_0. We thus have, for all y near x_0,

$$Md(y) \geq Md(x_0) + w(y)\varphi(y) - w(x_0)\varphi(x_0) \text{ (by definition of } x_0) \tag{11}$$

For a point $x' \in \mathbf{R}^{d-1}$, one can consider the point $x = (d(x_0), x')$ which lies in the ball $B(x_0, d(x_0))$ if x' is small enough. And so, defining

$$\psi(x') = -\frac{w(x)\psi(x) - w(x_0)\psi(x_0)}{M},$$

we find

$$d(x) \geq d(x_0) - \psi(x'),$$

whence we deduce that the boundary of Ω is below the surface

$$\mathcal{S} = \{(x_1, x') \,:\, x_1 = \psi(x')\}.$$

The surface $\mathcal{S}$ is $\mathcal{C}^2$ in some neighborhood of 0 since the function f is locally Hölder continuous.

By definition, $\psi(0) = 0$ so that we only have to prove that $\nabla\psi(0) = 0$ and to obtain the curvature condition. In order to obtain the desired bound on this surface's mean curvature, one has to study the value of the Laplacian of ψ at 0.

(i) First, using the technical Lemma C.4 in x_0 with the inequality (11), we find that d is differentiable in x_0, that

$$\nabla d(x_0) = \frac{x_0 - y_0}{|x_0 - y_0|} = e_1,$$

and that

$$M\nabla d(x_0) = \nabla(w\varphi)(x_0) = \varphi(x_0)\nabla w(x_0) + w(x_0)\nabla\varphi(x_0). \tag{12}$$

From this equality we first find that $\nabla(w\varphi)(x_0) = Me_1$ so that $\nabla\psi(x') = 0$.

(ii) Before studying $\Delta\psi(0)$, we can study $\Delta(-w\varphi)(x_0)$ and get

$$\Delta(-w\varphi)(x_0) = -\Delta w(x_0)\varphi(x_0) - \nabla w(x_0).\nabla\varphi(x_0) - w(x_0)\Delta\varphi(x_0).$$

The function φ being bounded, $-\Delta w(x_0)\varphi(x_0) \leq C||f||_{L^\infty}$. Then, using equation (12), one finds that (using that $\nabla\varphi$ and u are bounded and that $M \geq 1$):

$$\begin{aligned} -\nabla w(x_0).\nabla\varphi(x_0) &= -\frac{M\nabla d(x_0)}{\varphi(x_0)} + \frac{w(x_0)|\nabla\varphi(x_0)|^2}{\varphi(x_0)} \\ &\leq \frac{M}{\varphi(x_0)} + \frac{||u||_{L^\infty}CM}{\varphi(x_0)}. \end{aligned}$$

Now, since $\Delta\varphi$ is bounded, and x_0 lies in B_1, one finds that

$$-\Delta\varphi(x_0)w(x_0) \leq ||\Delta\varphi||_{L^\infty}\frac{Md(x_0)}{\varphi(x_0)}$$

so that, finally,

$$\Delta(-w\varphi)(x_0) \leq C||f||_{L^\infty} + \frac{CM}{\varphi(x_0)}.$$

(iii) Now we must make the link between $\Delta(-w\varphi)(x_0)$ and $\Delta\psi(0)$. Let us call $Q = D^2(-w\varphi)(x_0)$. So that (identifying the $d\times d$ matrix Q with the quadratic form associated)

$$\Delta(-w\varphi)(x_0) = \Delta Q = \text{ Tr } (Q).$$

Now, we have

$$\Delta\psi(0) = \frac{1}{M}\sum_{i=2}^{d} Q_{i,i} = \frac{\Delta Q - Q_{1,1}}{M}.$$

so that in order to obtain the bound on the curvature of $\mathcal{S}$, one just have to prove that $Q_{1,1} \geq 0$.

(iv) To do so, one uses technical Lemma C.1 at the point x_0 with the inequality (11) and finds that

$$< -Q(x_0).\nabla d(x_0), \nabla d(x_0) > \leq 0.$$

That is to say, using (i) that

$$Q_{1,1} =< Q.e_1, e_1 > \geq 0.$$

(v) Finally, we proved that, in some neighborhood of 0,

$$\kappa(S)(x) = \frac{1}{d-1}\Delta\psi(x') \leq \frac{C}{M} + \frac{C}{\varphi(x_0)}.$$

5. Quasi-Minimizers and some consequences

First, we recall some results upon quasi-minimizers.

Definition 5.1. A measurable subset E of the open set D ($\subset \mathbf{R}^d$) is said to be a (local) α-quasi-minimizers (for some $\alpha \in (0, \frac{1}{2}]$) if for any subset $A \subset\subset D$ (i.e., such that $\overline{A}$ is bounded and included in D), there exist some $R \in (0, \text{dist}(A, \partial D))$ and $C > 0$ such that

$$P(E, B_r(x)) \leq P(E', B_r(x)) + Cr^{d-1+2\alpha}$$

for every $x \in A$, every $r \in (0, R)$ and every E' with $E\Delta E' \subset\subset B_r(x)$ (taking $E\Delta E' = (E\backslash E') \cup (E'\backslash E)$).

We are going to use the following

Theorem 5.2. *Suppose E is an α-quasi-minimizer (for some $\alpha \in (0, \frac{1}{2}]$). Then,*

(i) $\partial^* E \cap D$ *is a* $\mathcal{C}^{1,\alpha}$ *hypersurface,*
(ii) $\mathcal{H}^s[(\partial E \backslash \partial^* E) \cap D] = 0$ *for each* $s > d - 8$.

The proof of Theorem 1.1 then comes from this result upon quasi-minimizers and from the following Theorem:

Theorem 5.3. *Suppose f is bounded and Ω is an optimal shape of (5) such that $u = u_\Omega$ is locally Lipschitz continuous in D. Then Ω is a $\frac{1}{2}$-quasi-minimizer.*

Proof. Take $A \subset\subset D$ and $R \in (0, \text{dist}(A, \partial D))$. Take some Lipschitz constant L of u on $\overline{A + B_R(0)}$. Now take some $x \in A$, some $r \in (0, R)$ and some measurable set Ω' such that $\Omega'\Delta\Omega \subset\subset B_r(x)$.

The case where $B_r(x) \subset \Omega$ is easy, since we have $P(\Omega', B_r(x)) \geq 0 = P(\Omega, B_r(x))$.

Now suppose that $B_r(x)$ intersects $\partial\Omega$. Then, $||u||_{L^\infty(B_{2r}(x))} \leq 4rL$. Take some smooth function φ such that $\varphi = 0$ in $B_r(x)$, $\varphi = 1$ in $B_{2r}(x)^c$, and satisfying $||\nabla\varphi||_{L^\infty} \leq \frac{C}{r}$ (C being universal) and $||\varphi||_{L^\infty} \leq 1$. Now we can estimate

$$\begin{aligned} J(\Omega') - J(\Omega) &\leq G(\varphi u) - G(u) \\ &\leq \frac{1}{2}\int (\varphi^2 - 1)|\nabla u|^2 + \int (\nabla u.\nabla\varphi)u\varphi + \int (1-\varphi)uf \\ &\leq C(d, ||f||_{L^\infty}, L)r^d \end{aligned}$$

And so Ω is a $\frac{1}{2}$-quasi-minimizer in D. □

A. Technical lemmas, first part

The proofs of the following two lemmas on the Laplacian may be found in [4]. We denote by $\fint_E$ the average over the set E.

Lemma A.1. *Let $B(x_0, r_0)$ be an open ball and $U \in \mathcal{C}^2(B(x_0, r_0))$. Then, for all $r \in (0, r_0)$,*

$$\fint_{\partial B(x_0,r)} U - U(x_0) = (d\omega_d)^{-1} \int_0^r ds\ s^{1-d} \int_{B(x_0,s)} d(\Delta U).$$

This remains valid for all $U \in H^1(B(x_0, r_0))$ such that ΔU is a measure satisfying

$$\int_0^r ds\, s^{1-d} \int_{B(x_0,s)} d|\Delta U| < \infty, \tag{13}$$

and such that

$$U(x_0) = \lim_{r\to 0} \fint_{\partial B(x_0,r)} U. \tag{14}$$

Moreover, (13) is satisfied if $U \in L^\infty(B(x_0, r_0))$ and there exists $g \in L^q(B(x_0, r_0))$ with $q > d/2$ such that $\Delta U^+ \geq -g$ and $\Delta U^- \geq -g$.

Remark A.2. The proof shows furthermore that the condition (13) implies the existence of the limit in (14) for any x_0 whence we can take some precise representation of U defined thanks to (14).

Lemma A.3. *Let $B(x_0, r_0)$ be an open ball, $r_0 \leq 1$, $F \in L^q(B(x_0, r_0))$, $q > d$. Then, there exists some constant $C = C(||F||_{L^q(B(x_0,r_0))}, d)$ such that, for $r \in (0, r_0)$,*

(i) if $\Delta U = F$ on $B(x_0, r_0)$, then

$$||\nabla U||_{L^\infty(B(x_0,r/2))} \leq C[1 + r^{-1}||U||_{L^\infty(B(x_0,r))}], \tag{15}$$

(ii) if $\Delta U \geq F$ and $U \geq 0$ on $B(x_0, r_0)$, then

$$||U||_{L^\infty(B(x_0,2r/3))} \leq C[r + \fint_{\partial B(x_0,r)} U], \tag{16}$$

B. Technical lemmas, part two

Lemma B.1. *Take a function $f : (0, r) \to \mathbf{R}^+$ such that there exist some positive constants C_0 and α with*

$$\forall x \in (0, r),\ f(x) \geq C_0 x^\alpha.$$

Then, there exist some constant $C(\alpha)$ and a sequence $(t_n)_n$ in $(0, r/2)$ and converging to 0 such that

$$\forall n, f(t_n) \leq C(\alpha) f(t_n/2).$$

Proof. Let us take $C(\alpha) = 2^\alpha + 1$. Take some $r_0 \in (0, r)$ and suppose that for any $t \in (0, r_0)$, $f(t) \geq C(\alpha)f(t/2)$. Then $f(t) \geq C(\alpha)f(t/2) \geq C(\alpha)^k f(t/2^k) \geq C(\alpha)^k C_0 t^\alpha/2^{\alpha k}$ and this last expression goes to infinity as k goes to infinity, whence there must exist some $t_0 \in (0, r_0)$ such that $f(t_0) \leq C(\alpha)f(t_0/2)$. The same construction applied to $r_k = r_0/2^k$ gives us some t_k. The sequence $(t_n)_n$ obtained satisfies the required properties. □

C. Technical lemmas, part three

In this part, we study the properties of the distance to the boundary of a set. In what follows, let $d(x) = d(x, \partial\Omega)$.

Lemma C.1. *In the interior of Ω, the function $d(.)$ is a super-solution of viscosity of the equation $- < \nabla^2 d.\nabla d, \nabla d >= 0$.*

Corollary C.2. *Let x_0 be an interior point of Ω, and f be a function of class $\mathcal{C}^2$ in some neighborhood of x_0 with*

$$d(.) \geq f(.) \text{ near } x_0, \text{ and } d(x_0) = f(x_0).$$

$$\text{Then } < \nabla^2 f(x_0).\nabla f(x_0), \nabla f(x_0) > \leq 0.$$

Lemma C.3. *Let $x_0 \in \mathbf{R}^d$. We suppose that $d(x_0) = r_0 > 0$. Then, $d(.)$ is differentiable at x_0 if and only if the set $\partial B(x_0, r_0) \cap \partial\Omega$ is a singleton $\{a\}$. Moreover, in this case, we have $\nabla d(x_0) = \frac{x_0 - a}{|x_0 - a|}$.*

Lemma C.4. *Let f be a function of class $\mathcal{C}^1$ in some neighborhood of a point x_0 interior to Ω and satisfying*

$$d(.) \geq f(.) \text{ near } x_0, \text{ and } d(x_0) = f(x_0).$$

Then the function d is differentiable in x_0 and $\nabla f(x_0) = \nabla d(x_0)$.

D. Technical lemmas, part four

Lemma D.1. *Assume that D is star-shaped with respect to the origin and that $x \mapsto xf(x)$ belongs to $L^2(D)$. Take $L \subset D$ measurable with finite measure. Then*

$$\limsup_{t \to 1, t<1} \frac{J(tL) - J(L)}{1-t} \leq (C(d) + ||f(x)x||_{L^2(D)})\sqrt{-2J(L)}.$$

Proof. For $u = u_L$, define $u_t(.) = u(./t)$. First we compute $\int |\nabla u_t|^2 = t^{d-2} \int |\nabla u|^2$. Then we derivate $g : t \mapsto \int f.u_t$ and get $g'(1) = \int f(x)(x.\nabla u(x))dx$. □

E. About the perimeter

We recall here the definition and the main properties of the perimeter (see for example [7]):

Definition E.1. Let D and Ω be (respectively) an open and a measurable subset of $\mathbf{R}^d$. We define

$$P(\Omega, D) = \inf \left\{ \int_\Omega \text{div}\varphi \,:\, \varphi \in \mathcal{C}_0^1(D),\ ||\varphi||_{L^\infty} \leq 1 \right\}.$$

And we denote $P(.) = P(., \mathbf{R}^d)$.

Proposition E.2. *Let us take two measurable subsets A and B of $\mathbf{R}^d$. Then, $P(A \cup B) + P(A \cap B) \leq P(A) + P(B)$.*

Proposition E.3. *Take Ω such that $P(\Omega, D) < \infty$. Then one can extend the function $P(\Omega, .)$ to any measurable subset of D and the function*

$$A \in \{B \subset D \,:\, B \text{ measurable }\} \mapsto P(\Omega, A)$$

defined in this way is a measure.

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