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A Holomorphic Function with given almost all Boundary Values on a Domain with Holomorphic Support Function

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We give a new construction of holomorphic function with given almost all boundary values on a bounded strictly pseudoconvex domain Ω with the boundary of class C^2. Next we use this construction to solve Radon inversion problem and describe exceptional sets for square integrals of holomorphic functions along complex directions.

Keywords: Inner function, Radon inversion problem

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1. Introduction

In the papers [1, 10, 11], the authors constructed an inner function on the domain with a holomorphic support function. In this paper we use a similar holomorphic support function as in [11] but we give a new construction of inner function. In fact we describe our inner function in terms of a given Borel probability measure. As an application we solve Radon inversion problem for holomorphic functions integrable with square along complex directions. We also describe exceptional sets in terms of any Borel probability measure. For more information about exceptional sets see [3, 4, 5, 6, 7, 8].

1.1. Geometric notions

Assume that $\Omega \subset \mathbb{C}^d$ is a bounded domain with the diameter $2R := \sup_{z,w\in\partial\Omega} \|z-w\|$. Let σ be a Borel probability measure on $\partial\Omega$.

First, we define so called holomorphic support functions (see also similar definition in [11]).

Let S be a Borel subset of $\partial\Omega$. We shall say that $\Phi : \overline{\Omega} \times S \to \mathbb{C}$ is a holomorphic support function on S if:

(1) $\Phi(\cdot, z)$ is holomorphic on Ω and continuous on $\overline{\Omega}$.
(2) There exist constants $c_1, c_2 > 0$ such that for $(z, w) \in \partial\Omega \times S$ we have

$$\exp\left(-c_2 \|z-w\|^2\right) \le |\Phi(z,w)| \le \exp\left(-c_1 \|z-w\|^2\right);$$

(3) If T is a compact subset of Ω then $\sup_{(z,w)\in T\times S} |\Phi(z,w)| < 1$.

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We say that Ω admits holomorphic support function on S if there exists a sequence $\{S_i\}_{i\in\mathbb{N}}$ of Borel subsets of $\partial\Omega$ such that $S=\bigcup_{i\in\mathbb{N}} S_i$ and for a given $i\in\mathbb{N}$ there exists a holomorphic support function on S_i.

For $I:=\{1,..,k\}$ and $\xi\in\bigcup_{i=1}^k S_i$, let $p_k(\xi)=\min\{i\in I:\xi\in S_i\}$. Let us observe that if $\Phi_i:\overline{\Omega}\times S_i\to\mathbb{C}$ is a holomorphic support function on S_i for $i=1,...,k$ then

$$\Phi:\overline{\Omega}\times\bigcup_{i=1}^k S_i\ni(z,\xi)\to\Phi_{p_k(\xi)}(z,\xi)$$

is a holomorphic support function on $\bigcup_{i=1}^k S_i$.

Let $z\in\partial\Omega$ and $\gamma:[0,1]\ni t\to\gamma(t)\in\overline{\Omega}$ be a continuous curve such that $\gamma(1)=z$. We say that γ crosses $\partial\Omega$ transversally at z if $\gamma([0,1))\subset\Omega$ and there exists $1\geq\alpha>0$ such that for all $t\in[0,1]$ we have

$$\operatorname{dist}(\partial\Omega,\gamma(t)):=\inf_{w\in\partial\Omega}\|\gamma(t)-w\|>\alpha\,\|\gamma(t)-z\|\,.$$

The set of all such continuous curves is denoted by $\Gamma_\Omega(z)$. Moreover, we denote $B(x,r)=\left\{y\in\mathbb{R}^{2d}:\|x-y\|<r\right\}$ for $x\in\mathbb{R}^{2d}$.

Let $\pi_d\in\mathbb{R}$ be such that $\mathfrak{L}^{2d}(B(z,r))=\pi_d r^{2d}$ where $\mathfrak{L}^m$ is the m-dimensional Lebesgue measure.

We say that $A\subset\mathbb{C}^d$ is α-separated set if $\|z-w\|>\alpha$ for $z,w\in A$ and $z\neq w$.

2. Preliminary calculations

In this section we begin with a natural example of domain Ω which has a holomorphic support function on $\partial\Omega$.

Example 2.1. Let Ω be a bounded strictly pseudoconvex domain with the boundary of class C^2. Then Ω has a holomorphic support function on $\partial\Omega$.

Proof. By Fornaess' embedding [2] theorem, there exists a neighbourhood U of $\overline{\Omega}$, a strictly convex, bounded domain $\widetilde{\Omega}\subset\mathbb{C}^{\widetilde{d}}$ with the boundary of class C^2 and a holomorphic mapping $\psi:U\to\mathbb{C}^{\widetilde{d}}$ such that ψ maps U biholomorphically onto some complex submanifold $\psi(U)$ of $\mathbb{C}^{\widetilde{d}}$ and

(1) $\psi(\Omega)\subset\widetilde{\Omega}$,
(2) $\psi(\partial\Omega)\subset\partial\widetilde{\Omega}$,
(3) $\psi(U\setminus\overline{\Omega})\subset\mathbb{C}^N\setminus\overline{\widetilde{\Omega}}$,
(4) $\psi(U)$ intersects $\partial\widetilde{\Omega}$ transversally.

Due to [9, Lemma 3.1.6] there exists a defining function ρ of class C^2 for $\widetilde{\Omega}$ and constants $c_3,c_4>0$ such that for $\xi\in\partial\Omega$ and $w\in\mathbb{C}^d$ we have

$$c_3\|w\|^2\leq H_\rho(\xi,w)\leq c_4\|w\|^2$$

where $H_\rho(\xi,w):=\frac{1}{2}\sum_{j,k=1}^d\frac{\partial^2\rho(\xi)}{\partial z_j\partial z_k}w_jw_k+\frac{1}{2}\sum_{j,k=1}^d\frac{\partial^2\rho(\xi)}{\partial\bar{z}_j\partial\bar{z}_k}\overline{w}_j\overline{w}_k+\sum_{j,k=1}^d\frac{\partial^2\rho(\xi)}{\partial z_j\partial\bar{z}_k}w_j\overline{w}_k$.

Assume that we have proved the following fact: there exist $c_1,c_2>0$ such that for $z,\xi\in\partial\widetilde{\Omega}$ we have the following inequality:

$$-c_2\|z-\xi\|^2\leq\Re\langle z-\xi,\overline{\rho_\xi}\rangle\leq-c_1\|z-\xi\|^2 \tag{1}$$

where $\rho_\xi = \left(\frac{\partial\rho}{\partial z_1}(\xi), ..., \frac{\partial\rho}{\partial z_{\tilde{d}}}(\xi)\right)$. Let us observe that $g(z,\xi) := \exp\left(\langle z-\xi, \overline{\rho_\xi}\rangle\right)$ is now a holomorphic support function on $\partial\widetilde{\Omega}$ and due to Fornaess' embedding [2] theorem the following function $\overline{\Omega} \times \partial\Omega \ni (z,w) \to g(\psi(z), \psi(w))$ is a holomorphic support function on $\partial\Omega$.

So it suffices to prove (1). Let

$$\phi(\xi,h) := \langle h, \overline{\rho_\xi}\rangle + \overline{\langle h, \overline{\rho_\xi}\rangle} - \rho_\xi(\xi+h).$$

Since ρ_ξ is of class C^2 therefore we have $\rho_\xi(\xi+h) = \rho_\xi(\xi) + \langle h, \overline{\rho_\xi}\rangle + \overline{\langle h, \overline{\rho_\xi}\rangle} + H_\rho(\xi,h) + f(\xi,h)\left\|h\right\|^2$, where f is a continuous function such that $f(\xi,0) = 0$. Observe that $2\Re\langle z-\xi, \overline{\rho_\xi}\rangle = \phi(\xi, z-\xi)$ for $z \in \partial\widetilde{\Omega}$ and $\xi \in \partial\widetilde{\Omega}$. In particular we may estimate

$$\frac{2\Re\langle z-\xi, \overline{\rho_\xi}\rangle}{\left\|z-\xi\right\|^2} = \frac{-H_\rho(\xi, z-\xi) - f(\xi, z-\xi)\left\|z-\xi\right\|^2}{\left\|z-\xi\right\|^2} \leq -c_3 - f(\xi, z-\xi)$$

and

$$\frac{2\Re\langle z-\xi, \overline{\rho_\xi}\rangle}{\left\|z-\xi\right\|^2} \geq -c_4 - f(\xi, z-\xi).$$

The above inequalities imply that there exist constants $c_1, c_2 > 0$ such that (1) holds. □

In order to control values of the constructed functions we need some information about α-separated sets,

Lemma 2.2. *Suppose that $A = \{\xi_1, .., \xi_s\}$ is a 2α-separated subset of $\partial\Omega$. For $z \in \partial\Omega$ let*

$$A_k(z) := \{\xi \in A : \alpha k \leq \|z-\xi\| \leq \alpha(k+1)\}.$$

Then the set $A_k(z)$ has at most $(k+2)^{2d}$ elements i.e. $\#A_k(z) \leq (k+2)^{2d}$. The set A_0 has at most 1 element and $s \leq \max\left\{1, \left(\frac{2R}{\alpha}\right)^{2d}\right\}$.

Proof. Observe that $B(\xi_1;\alpha) \cap B(\xi_1;\alpha) = \emptyset$ for $\xi_1 \neq \xi_2 \in A$. Moreover

$$\bigcup_{\xi \in A_k(z)} B(\xi;\alpha) \subset B(z;\alpha(k+2)).$$

Let d_k be a number of elements of $A_k(z)$. In particular

$$d_k \pi_d \alpha^{2d} = \sum_{\xi \in A_k(z)} \mathfrak{L}^{2d}\left(B(\xi;\alpha)\right) \leq \mathfrak{L}^{2d}\left(B(z;\alpha(k+2))\right) = \pi_d(\alpha(k+2))^{2d}.$$

We conclude that $d_k \leq (k+2)^{2d}$. Moreover, if $\xi_j, \xi_k \in A_0(z)$ then $\rho(\xi_j, \xi_k) \leq \rho(z, \xi_j) + \rho(z, \xi_k) < 2\alpha$ so $\xi_j = \xi_k$ and $d_0 \leq 1$.

Since $\Omega \subset B(0,R)$ therefore if $\alpha > R$ then $s \leq 1$ so we may assume that $\alpha \leq R$. In particular

$$\bigcup_{\xi \in A_k(z)} B(\xi;\alpha) \subset B(0;2R)$$

and we may estimate

$$s \pi_d \alpha^{2d} \leq \sum_{\xi \in A} \mathfrak{L}^{2d}\left(B(\xi;\alpha)\right) \leq \mathfrak{L}^{2d}\left(B(0;2R)\right) \leq \pi_d(2R)^{2d}.$$ □

Lemma 2.3. *Let $\beta > \alpha > 0$. There exists an integer $N = N(\alpha, \beta)$ such that: if $t > 0$ and $A \subset \partial\Omega$ is αt-separated set then A can be partitioned into N disjoint βt-separated sets.*

Proof. Let us select from A a maximal βt-separated subset A_1. Next from $A \setminus A_1$ we select a maximal βt-separated subset A_2. We proceed this way till we exhaust A. Let A_s be the last non-empty set in this procedure. Let $\xi \in A_s$. Observe that $B(\xi, \beta t) \cap A_k \neq \emptyset$ for $k = 1, ..., s-1$. In particular $B(\xi, \beta t)$ contains at least s different elements $\{\xi_1, ..., \xi_s\}$ from A. Since $B(\xi_j, \alpha t) \cap B(\xi_k, \alpha t) = \emptyset$ for $j \neq k$ and $B(\xi_j; \alpha t) \subset B(\xi; (\beta + \alpha)t)$ therefore

$$s\pi_d(\alpha t)^{2d} \leq \sum_{j=1}^{s} \mathfrak{L}^{2d}\left(B(\xi_j; \alpha t)\right) \leq \mathfrak{L}^{2d}\left(B(\xi; (\beta+\alpha)t)\right) \leq \pi_d((\beta+\alpha)t)^{2d}.$$

We can conclude that $s \leq \left(\frac{\beta}{\alpha} + 1\right)^{2d}$. Now it suffices to choose a natural number N so that $\left(\frac{\beta}{\alpha} + 1\right)^{2d} \leq N$. □

3. Inner function

In this section we construct an inner function for a domain Ω.

We prove the following fact.

Lemma 3.1. *Let S be a Borel subset of $\partial\Omega$ and $\Phi : \overline{\Omega} \times S \to \mathbb{C}$ be a holomorphic support function on S. Let $\theta \in (0,1)$. There exist constants $C > c > 0$ and $n \in \mathbb{N}$ (C, c, n dependent only on θ, S) such that, if $\varepsilon \in (0,1)$, T is a compact subset of Ω, H is a continuous strictly positive function on $\partial\Omega$ and g is a complex continuous function on $\partial\Omega$, then we can choose $m_0 \in \mathbb{N}$ such that for $m \geq m_0$ and each $\frac{C}{\sqrt{m}}$-separated subset A of S the functions[1] $f_{A,m,k}(z) := \sum_{\xi \in A} H(\xi)\Phi^m(z,\xi) \exp\left(\left(\frac{2k\pi}{n} + \arg g(\xi)\right) i\right)$ have the following properties:*

(1) $\|f_{A,m,k}\|_T \leq \varepsilon$;
(2) $|(f_{A,m,k} + g)(z)| - |g(z)| \leq |f_{A,m,k}(z)| < (1+2\theta)H(z)$ *for all* $z \in \partial\Omega$;
(3) $\max_{k=0,...n-1} |(f_{A,m,k} + g)(z)| - |g(z)| > (1-2\theta)H(z)$ *for each* $z \in \partial\Omega$ *such that* $\|z - \xi\| \leq \frac{c}{\sqrt{m}}$ *for some* $\xi \in A$.

Proof. There exist constants $c_2 \geq c_1 > 0$ such that for $(z, w) \in \partial\Omega \times S$ we have

$$\exp\left(-c_2 \|z-w\|^2\right) \leq |\Phi(z,w)| \leq \exp\left(-c_1 \|z-w\|^2\right).$$

There exists $n \in \mathbb{N}$ such that $\cos\left(\frac{\pi}{n}\right) > 1 - \theta$. Observe that for $\psi \in \mathbb{R}$ we have

$$\max_{k=0,\ldots,n-1} \cos\left(\frac{2k\pi}{n} + \psi\right) > 1 - \theta.$$

Let $0 < \delta < 1$ be such that $(1-\delta)(1-\theta) - 5\delta > 1 - 2\theta$ and $2\delta < \theta$. Let $c > 0$ be such that $e^{-cc_2} > 1 - \theta$. There exists $C > c$ such that for $k \in \mathbb{N} \setminus \{0\}$ we have

$$(k+2)^{2d} e^{-\frac{c_1 C^2 k^2}{4}} \leq \delta 2^{-k}.$$

[1] $\arg 0 = 0$

Due to Lemma 2.2 the set A has at most $\left(\frac{4R\sqrt{m}}{C}\right)^{2d}$ elements i.e. $\#A \leq \left(\frac{4R\sqrt{m}}{C}\right)^{2d}$.

Let $t := \sup_{(z,w)\in T\times S} |\Phi(z,w)|$. Since $0 < t < 1$, for $w \in T$, m_0 high enough and $m \geq m_0$ we may estimate

$$|f_{A,m,k}(w)| \leq \sum_{\xi\in A} \|H\|\, t^m \leq \left(\frac{4R\sqrt{m}}{C}\right)^{2d} \|H\|\, t^m \leq \varepsilon$$

and conclude the property (*1*).

Let us denote for $z \in \partial\Omega$

$$A_k(z) := \left\{\xi \in A : \frac{Ck}{2\sqrt{m}} \leq \|z-\xi\| \leq \frac{C(k+1)}{2\sqrt{m}}\right\}.$$

Let now $s > 0$ be so small that $\|\eta - \xi\| \leq Cs \Longrightarrow (1-\delta)H(\eta) \leq H(\xi) \leq (1+\delta)H(\eta)$ and $|g(\eta) - g(\xi)| < \delta H(\xi)$. We may assume that m_0 is so large that $s \geq \frac{C}{2\sqrt{m}} + \frac{c}{\sqrt{m}}$ for $m \geq m_0$. Observe that we may estimate

$$\sum_{k\geq[2s\sqrt{m}]+1} (k+2)^{2d} e^{-\frac{c_1C^2k^2}{4}} \leq \sum_{k\geq[2s\sqrt{m}]+1} 2^{-k} \leq \frac{1}{2^{2s\sqrt{m}-1}}.$$

Let m_0 be so large that $\|H\| \leq \delta 2^{2s\sqrt{m}-1}H(z)$ for $m \geq m_0$ and $z \in \partial\Omega$.

Let now $z \in \partial\Omega$. First assume that $A_0(z) = \emptyset$. In particular due to Lemma 2.2 for $m \geq m_0$ we may estimate

$$\begin{aligned}
|(f_{A,m,k}+g)(z)| - |g(z)| &\leq |f_{A,m,k}(z)| \leq \sum_{k=1}^{\infty}\sum_{\xi\in A_k(z)} H(\xi)\,|\Phi(z,\xi)|^m \\
&\leq \sum_{k=1}^{\infty}\sum_{\xi\in A_k(z)} H(\xi)e^{-mc_1\|z-\xi\|^2} \\
&\leq (1+\delta)H(z)\sum_{k=1}^{[2s\sqrt{m}]} (k+2)^{2d} e^{-\frac{c_1C^2k^2}{4}} + \frac{\|H\|}{2^{2s\sqrt{m}-1}} \\
&\leq (1+\delta)\delta H(z) + \delta H(z) \leq 3\delta H(z).
\end{aligned}$$

Now assume that $A_0(z) \neq \emptyset$. Due to Lemma 2.2 we have $A_0(z) = \{\xi_0\}$ for some $\xi_0 \in A$ where $\|z-\xi_0\| \leq \frac{C}{2\sqrt{m}} \leq Cs$. In particular we can obtain the property (*2*):

$$\begin{aligned}
|(f_{A,m,k}+g)(z)| - |g(z)| &\leq |f_{A,m,k}(z)| \leq H(\xi_0) + 3\delta H(z) \\
&\leq (1+\delta)H(z) + 3\delta H(z) < (1+2\theta)H(z)
\end{aligned}$$

for $z \in \partial\Omega$ and $m \geq N$.

Let now $\xi \in A$ be such that $\|z-\xi\| \leq \frac{c}{\sqrt{m}} \leq Cs$. Let $r = |\Phi(z,\xi)|^m$ and $\psi_k := \frac{2\pi k}{n} + \arg\Phi^m(z,\xi) + \arg g(\xi)$, $\widetilde{\psi}_k := \frac{2\pi k}{n} + \arg\Phi^m(z,\xi)$. We have $r \geq e^{-mc_2\|z-\xi\|^2} \geq e^{-cc_2} > 1-\theta$. Moreover, there exists $k_0 \in \{0,1,...,n-1\}$ such that $\cos\widetilde{\psi}_{k_0} \geq 1-\theta$. In particular

$$\begin{aligned}
\left|H(\xi)re^{i\psi_{k_0}} + g(\xi)\right|^2 &= \left|H(\xi)re^{i\widetilde{\psi}_{k_0}} + |g(\xi)|\right|^2 \\
&= r^2H(\xi)^2 + 2H(\xi)\,|g(\xi)|\cos\widetilde{\psi}_{k_0} + |g(\xi)|^2 \\
&\geq (1-\theta)^2 H(\xi)^2 + 2(1-\theta)H(\xi)\,|g(\xi)| + |g(\xi)|^2 \\
&\geq ((1-\theta)H(\xi) + |g(\xi)|)^2 .
\end{aligned}$$

Now we may conclude the property (*3*):

$$\begin{aligned} |(f_{A,m,k_0}+g)(z)| &\geq |f_{A,m,k_0}(z)+g(\xi)|-|g(z)-g(\xi)| \\ &\geq \left|H(\xi)re^{i\psi_{k_0}}+g(\xi)\right|-3\delta H(z)-|g(z)-g(\xi)| \\ &\geq (1-\theta)\,H(\xi)+|g(\xi)|-4\delta H(z) \\ &\geq (1-\theta)\,(1-\delta)H(z)+|g(z)|-5\delta H(z) \\ &> (1-2\theta)H(z)+|g(z)|\,. \end{aligned}$$ □

Theorem 3.2. *Let S be a Borel subset of $\partial\Omega$. Assume that there exists a holomorphic support function on S. Let $a\in(0,1)$. There exists a natural number $N=N(a,S)$ such that, if $\varepsilon\in(0,1)$, T is a compact subset of Ω, H is a continuous, strictly positive function on $\partial\Omega$, g is a complex continuous function on $\partial\Omega$, then there exists a relatively open neighbourhood U of S in $\partial\Omega$ and there exist holomorphic functions $f_1,...,f_N$ on Ω and continuous on $\overline{\Omega}$ such that $\|f_j\|_T\leq\varepsilon$ and:*

(*1*) $|(f_j+g)(z)|-|g(z)|\leq|f_j(z)|<H(z)$ *for* $j=1,...,N$ *and* $z\in\partial\Omega$*;*
(*2*) $aH(z)<\max_{j=1,...,N}|(f_j+g)(z)|-|g(z)|$ *for* $z\in U$*.*

Proof. Let $\Phi:\overline{\Omega}\times S\to\mathbb{C}$ be a holomorphic support function on S. Let $\theta\in(0,1)$ be such that $\frac{1-2\theta}{1+2\theta}=a$. Now constants $C>c>0$ and $n\in\mathbb{N}$ can be chosen from Lemma 3.1. Due to Lemma 2.3 there exists a natural number N_0 such that each $\frac{c}{\sqrt{m}}$-separated subset of $\partial\Omega$ can be partitioned into N_0 disjoint $\frac{C}{\sqrt{m}}$-separated sets. Let us define $N=nN_0$.

Let A be a maximal $\frac{c}{\sqrt{m}}$-separated subset of S. It can be partitioned into $A_1,...,A_{N_0}$ disjoint $\frac{C}{\sqrt{m}}$-separated sets. Now due to Lemma 3.1 there exists m and holomorphic functions f_j on Ω and continuous on $\overline{\Omega}$ such that $\|f_j\|_T\leq\varepsilon$ and

(1) $|(f_j+g)(z)|-|g(z)|\leq|f_j(z)|<\frac{1+2\theta}{1+2\theta}H(z)$ for $z\in\partial\Omega$, $j=1,...,N$;
(2) $\max_{k=0,1,...,n-1}|(f_{nj+k}+g)(z)|-|g(z)|>\frac{1-2\theta}{1+2\theta}H(z)$ for each $z\in\partial\Omega$ such that $\|z-\xi\|\leq\frac{c}{\sqrt{m}}$ for some $\xi\in A_j$ and $j=1,...,N_0$.

Let $K=\bigcup_{j=1}^{N_0}\bigcup_{\xi\in A_j}\overline{B(\xi,\frac{c}{\sqrt{m}})}$. Since A is a maximal $\frac{c}{\sqrt{m}}$-separated subset of S, one has $S\subset K$. Moreover, if $z\in K$ then there exist $j_0\in\{1,...,N_0\}$ and $\xi_{j_0}\in A_{j_0}$ such that $\|z-\xi_{j_0}\|\leq\frac{c}{\sqrt{m}}$. In particular

$$aH(z)<\max_{k=0,1,...,n-1}|(f_{nj_0+k}+g)(z)|-|g(z)|\leq\max_{j=1,...,N}|(f_j+g)(z)|-|g(z)|$$

for $z\in K$. Since K is a compact set and functions f_j,g are continuous on $\partial\Omega$, there exists U, an open neighbourhood of K, such that inequality (2) is fulfilled. □

Recall that σ is a Borel probability measure on $\partial\Omega$.

Proposition 3.3. *Assume that U is a relatively open subset of $\partial\Omega$ and K is a compact subset of U. There exists a relatively open subset V of $\partial\Omega$ such that $K\subset V\subset\overline{V}\subset U$ and $\sigma(V)=\sigma(\overline{V})$.*

Proof. Let us denote $V_\varepsilon:=\{z\in U:\inf_{w\in K}\|z-w\|<\varepsilon\}$. We may observe that there exists $\varepsilon_0>0$ such that $V_{\varepsilon_0}\subset U$. In particular $K\subset V_\varepsilon\subset\overline{V}_\varepsilon\subset U$ for $0<\varepsilon<\varepsilon_0$. Let $A_k:=\left\{t\in(0,\varepsilon_0):\sigma(\overline{V}_t\setminus V_t)\geq\frac{1}{k}\right\}$. Since A_k has at most k elements, $\bigcup_{k=1}^{\infty}A_k$ is a countable set and we may conclude that there exists $r\in(0,\varepsilon_0)\setminus\bigcup_{k\in\mathbb{N}}A_k$. Now it is enough to observe that $\sigma(V_r)=\sigma(\overline{V}_r)$ and set $V:=V_r$. □

Lemma 3.4. *Let S be a Borel subset of $\partial\Omega$ such that $\sigma(S) = 1$ and Ω admits holomorphic support function on S. Let $\varepsilon, \theta \in (0,1)$, T be a compact subset of Ω. If H is a continuous strictly positive function on $\partial\Omega$ and g is a complex continuous function on $\partial\Omega$ then there exists V, a relatively open subset of $\partial\Omega$, f holomorphic on Ω and continuous on $\overline{\Omega}$ such that:*

(1) $\|f\|_T \leq \varepsilon$.
(2) $|(f+g)(z)| - |g(z)| < H(z)$ *for* $z \in \partial\Omega$.
(3) $|(f+g)(z)| - |g(z)| > \theta H(z)$ *for* $z \in V$.
(4) $\sigma(\overline{V}) = \sigma(V) > 1 - 2\varepsilon$.

Proof. Let $a, \delta \in (0,1)$ be such that $a - 2\delta = \theta$ and $(1-a)|g(z)| \leq \delta H(z)$ for all $z \in \partial\Omega$.

There exists a sequence $\{S_i\}_{i\in\mathbb{N}}$ of Borel sets so that $S = \bigcup_{i\in\mathbb{N}} S_i$ and for a given $i \in \mathbb{N}$ there exists a holomorphic support function on S_i. Moreover, we can choose $m_0 \in \mathbb{N}$ such that $\sigma\left(\bigcup_{i=1}^{m_0} S_i\right) > 1 - \varepsilon$. Since there exists a holomorphic support function on $\bigcup_{i=1}^{m_0} S_i$, let us choose a natural number $N = N\left(a, \bigcup_{i=1}^{m_0} S_i\right)$ given by Theorem 3.2.

Moreover let us define a new Borel probability measure on $\partial\Omega$ in the following way $\widetilde{\sigma}(W) = \sigma(W \cap \bigcup_{i=1}^{m_0} S_i)/\sigma(\bigcup_{i=1}^{m_0} S_i)$.

Now we construct $\{V_k\}_{k\in\mathbb{N}}$, a sequence of relatively open subsets in $\partial\Omega$ and $\{f_k\}_{k\in\mathbb{N}}$, a sequence of holomorphic functions on Ω and continuous on $\overline{\Omega}$ such that

(a) $|f_k(z)| \leq \frac{\varepsilon}{2^k}$ for $z \in T$;
(b) $\omega_{k+1}(z) - \omega_k(z) \leq |f_k(z)| < \frac{\delta}{2^k} H_k(z)$ for $z \in \bigcup_{j=1}^{k-1} \overline{V}_j$;
(c) $\omega_{k+1}(z) - \omega_k(z) \leq |f_k(z)| < H_k(z)$ for $z \in \partial\Omega$;
(d) $\omega_{k+1}(z) - \omega_k(z) > aH_k(z)$ for $z \in V_k$;
(e) $\overline{V}_k \subset \partial\Omega \setminus \bigcup_{j=1}^{k-1} \overline{V}_j$ and $\widetilde{\sigma}(\overline{V}_k) = \widetilde{\sigma}(V_k) > \frac{1-\sum_{j=1}^{k-1}\widetilde{\sigma}(V_j)}{N+1}$.

where $\omega_1(z) = |g(z)|$, $\omega_{m+1}(z) = \left|\left(g + \sum_{j=1}^{m} f_j\right)(z)\right|$, $H_1 = H$ and $H_{m+1}(z) = H_m(z) - \omega_{m+1}(z) + \omega_m(z)$.

Let $k = 1$. Due to Theorem 3.2 there exist f_1, V_1 such that (a)-(d) hold and $\widetilde{\sigma}(V_1) \geq \frac{1}{N}$. Due to Proposition 3.3 we can decrease V_1 so that (e) is also fulfilled.

Now assume that $f_1, ..., f_{k-1}, V_1, ..., V_{k-1}$ are already constructed. There exists an open set U such that $\overline{U} \subset \partial\Omega \setminus \bigcup_{j=1}^{k-1} \overline{V}_j$ and $\widetilde{\sigma}(U) = \widetilde{\sigma}(\overline{U}) > N\frac{1-\sum_{j=1}^{k-1}\widetilde{\sigma}(V_j)}{N+1}$. There exists G, a continuous function on $\partial\Omega$ such that $G(z) \leq H(z)$ for $z \in \partial\Omega$, $G(z) \leq \frac{\delta}{2^k} H(z)$ for $z \in \bigcup_{j=1}^{k-1} \overline{V}_j$ and $G(z) = H(z)$ for $z \in \overline{U}$. Now due to Theorem 3.2 there exist f_k, V_k such that (a)-(d) hold and $V_k \subset U$, $\widetilde{\sigma}(V_k) \geq \frac{1}{N}\widetilde{\sigma}(U) > \frac{1-\sum_{j=1}^{k-1}\widetilde{\sigma}(V_j)}{N+1}$. If $\widetilde{\sigma}(\overline{V_k}) = \widetilde{\sigma}(V_k)$ then (e) also holds. Now suppose that $\widetilde{\sigma}(\overline{V_k}) > \widetilde{\sigma}(V_k)$. There exists K a compact subset of V_k such that $\widetilde{\sigma}(K) > \frac{1-\sum_{j=1}^{k-1}\widetilde{\sigma}(V_j)}{N+1}$. Due to Propositin 3.3 there exists an open set $\widetilde{V}_k$ such that $K \subset \widetilde{V}_k \subset V_k$ and $\widetilde{\sigma}(\overline{\widetilde{V}_k}) = \widetilde{\sigma}(\widetilde{V}_k)$. We can redefine $V_k := \widetilde{V}_k$ and observe that conditions (a)-(e) are fulfilled.

Now we use the constructed functions $\{f_k\}_{k\in\mathbb{N}}$ and relatively open sets $\{V_k\}_{k\in\mathbb{N}}$.

Since $\sum_{j=1}^{\infty} \widetilde{\sigma}(V_j) \leq 1$, we may obtain $\lim_{k\to\infty} \frac{1-\sum_{j=1}^{k-1}\widetilde{\sigma}(V_j)}{N+1} \leq 0$. In particular there exists $m \in \mathbb{N}$ large enough such that $1 - \varepsilon < \sum_{j=1}^{m} \widetilde{\sigma}(V_j)$. Let now define $V = \bigcup_{j=1}^{m} V_j$ and

$f = \sum_{j=1}^{m} f_j$.

First we prove the properties (*1*), (*4*):

$$\sigma(V) \geq \sum_{j=1}^{m} \sigma(V_j) \geq \sum_{j=1}^{m} \widetilde{\sigma}(V_j)\sigma(\bigcup_{i=1}^{m_0} S_i) > (1-\varepsilon)^2 > 1-2\varepsilon$$

and $|f(z)| \leq \sum_{j=1}^{m} |f_j(z)| \leq \sum_{j=1}^{m} \varepsilon_j < \varepsilon$ for $z \in T$. If $\sigma(\overline{V}) > \sigma(V)$ then we can slightly decrease all V_j so that (*4*) is now fulfilled.

Observe that $H_m - H_1 = H_m - H = -\omega_m + \omega_1$. In particular the property (*2*) is obvious: $\omega_{m+1}(z) - \omega_1(z) < H_m(z) + \omega_m(z) - \omega_1(z) = H(z)$ for $z \in \partial\Omega$.

If $k < j$ and $z \in V_k$ then we may estimate

$$\omega_{j+1}(z) - \omega_j(z) \geq \left| \left(g + \sum_{i=1}^{j} f_i \right)(z) \right| - \left| \left(g + \sum_{i=1}^{j-1} f_i \right)(z) \right| \geq -|f_j(z)| > -\frac{\delta H(z)}{2^j}.$$

Now let $z \in V$. There exists $k \in \{1, ..., m\}$ such that $z \in V_k$. Since $H_k = H - \omega_k + \omega_1$ therefore we may obtain the property (*3*):

$$\begin{aligned} \omega_{n+1}(z) - \omega_1(z) &= \sum_{j=k+1}^{n} (\omega_{j+1}(z) - \omega_j(z)) + \omega_k(z) - \omega_1(z) + \omega_{k+1}(z) - \omega_k(z) \\ &> -\sum_{j=k+1}^{\infty} -\frac{\delta H(z)}{2^j} + \omega_k(z) - \omega_1(z) + aH_k(z) \\ &\geq -\delta H(z) + (1-a)\left(\omega_k(z) - \omega_1(z)\right) + aH(z) \\ &\geq -\delta H(z) - (1-a)|g(z)| + aH(z) \geq (a-2\delta)H(z) \geq \theta H(z). \quad \square \end{aligned}$$

Theorem 3.5. *Let S be a Borel subset of $\partial\Omega$ such that $\sigma(S) = 1$ and such that Ω admits a holomorphic support function on S. Assume that G is a lower semicontinuous, strictly positive function on $\partial\Omega$. Then there exists $\widetilde{S}$ a Borel subset of $\partial\Omega$ and a non constant holomorphic function f on Ω with the following properties:*

- *There exists $\{g_n\}_{n\in\mathbb{N}}$, a sequence of holomorphic functions on Ω and continuous on $\overline{\Omega}$ such that g_n converges uniformly to f on compact subsets of Ω, $|g_n(z)| < G(z)$ for $z \in \partial\Omega$, $\lim_{n\to\infty} |g_n(z)| = G(z)$;*
- *$\sigma(\widetilde{S}) = \sigma(S) = 1$;*
- *If $\gamma : [0,1] \ni t \to \gamma(t) \in \overline{\Omega}$ is a continuous curve crossing $\partial\Omega$ transversally at $\gamma(1) = z \in \widetilde{S}$, then there exists a sequence $\{t_n\}_{n\in\mathbb{N}} \subset (0,1)$ such that $\lim_{n\to\infty} t_n = 1$ and $\lim_{n\to\infty} |f(\gamma(t_n))| = G(z)$.*

Proof. There exists $\{G_m\}_{m\in\mathbb{N}}$, a sequence of strictly positive continuous functions on $\partial\Omega$ such that $G(z) = \sum_{m=1}^{\infty} G_m(z)$.

For a given $\alpha \in (0,1)$ and U, a relatively open subset of $\partial\Omega$ let us denote

$$\Gamma_\varepsilon(U) := \{\gamma : \gamma \in \Gamma_\Omega(z), z \in U, \mathrm{dist}(\partial\Omega, \gamma(0)) > \varepsilon, \quad \mathrm{dist}(\partial\Omega, \gamma(t)) > \varepsilon \|\gamma(t) - z\|\}.$$

We construct a sequence $\{\varepsilon_m\}_{m\in\mathbb{N}}$ of positive numbers, a sequence $\{f_m\}_{m\in\mathbb{N}}$ of holomorphic functions on $\overline{\Omega}$, and a sequence $\{V_m\}_{m\in\mathbb{N}}$ of relatively open subsets of $\partial\Omega$ such that the following properties are fulfilled:

(1) $0 < 4m\varepsilon_{m+1} \leq \varepsilon_m \leq 1$,
(2) If $z, w \in \overline{\Omega}$ and $\|z - w\| \leq m\varepsilon_{m+1}$ then $|\omega_m(z) - \omega_m(w)| \leq \varepsilon_m$,
(3) $\varepsilon_{m+1} H_m \leq G_m$,
(4) $\|f_m\|_{T_m} \leq \varepsilon_{m+1}$ where $T_m := \left\{z \in \overline{\Omega} : \mathrm{dist}(\partial\Omega, z) \geq \varepsilon_{m+1}\right\}$,
(5) $\omega_{m+1}(z) - \omega_m(z) < H_m(z)$ for $z \in \partial\Omega$,
(6) $\omega_{m+1}(z) - \omega_m(z) > (1 - \varepsilon_{m+1})H_m(z)$ for $z \in V_m$,
(7) $\sigma(\overline{V}_m) = \sigma(V_m) > 1 - 2\varepsilon_{m+1}$,

where $\omega_1 = 0$, $\omega_{m+1} = \left|\sum_{j=1}^m f_j\right|$, $H_1 = G_1$ and

$$H_{m+1} := H_m - \omega_{m+1} + \omega_m + G_{m+1} > G_{m+1}.$$

If $m = 1$ then we can set $\varepsilon_1 = 1$ and $\varepsilon_2 \in (0, \frac{1}{4})$ so that (3) is fulfilled. Now due to Lemma 3.4 there exist f_1, V_1 so that the properties (4)-(7) are fulfilled. Since $\omega_1 = 0$, we can observe that properties (1)-(7) are fulfilled. Assume that $(\varepsilon_j, \varepsilon_{j+1}, f_j, V_j)$ are constructed for $j = 1, ..., m$ so that (1)-(7) are fulfilled. Since ω_{m+1} is continuous and depends only on $f_1, ..., f_m$, there exists $\varepsilon_{m+2} > 0$ such that: (1)-(3) are fulfilled. Now we use again Lemma 3.4 to construct f_{m+1} and V_{m+1} so that (4)-(7) are fulfilled, which finishes the construction of $(\varepsilon_{m+1}, \varepsilon_{m+2}, f_{m+1}, V_{m+1})$.

Let $D_k := \bigcap_{m=k}^{\infty} V_m$ and $D = \bigcup_{k\in\mathbb{N}} D_k$. Observe that $\sigma(D_k) > 1 - \sum_{m=k}^{\infty} 2\varepsilon_{m+1} \geq 1 - \varepsilon_k$. In particular $\sigma(D) = 1$.

Let us observe that

$$H_{m+1} - H_1 = \sum_{k=1}^{m}(H_{k+1} - H_k) = \sum_{k=1}^{m}(\omega_k - \omega_{k+1} + G_{k+1}) = -\omega_{m+1} + \sum_{k=1}^{m} G_{k+1}$$

In particular

$$\omega_{m+1} = -H_{m+1} + \sum_{k=0}^{m} G_{k+1} < \sum_{k=0}^{\infty} G_{k+1} = G. \tag{2}$$

Let now $z \in D$. There exists $m_z \in \mathbb{N}$ such that $z \in V_m$ for $m \geq m_z$.

We may estimate

$$\begin{aligned} H_{m+1}(z) &= H_m(z) - \omega_{m+1}(z) + \omega_m(z) + G_{m+1}(z) \leq \varepsilon_{m+1} H_m(z) + G_{m+1}(z) \\ &\leq G_m(z) + G_{m+1}(z). \end{aligned}$$

Due to (2) we can conclude that $\sum_{k=0}^{m-2} G_{k+1}(z) \leq \omega_{m+1}(z) \leq \sum_{k=0}^{\infty} G_{k+1}(z) = G(z)$ and therefore

$$\lim_{m\to\infty} \omega_m(z) = G(z) \tag{3}$$

for $z \in D$. In particular we can choose $g_m := \sum_{k=1}^m f_k$.

Let $f = \sum_{m=1}^{\infty} f_m$. Now we prove that the holomorphic function f has the required properties.

Assume that $\gamma : [0, 1] \ni t \to \gamma(t) \in \overline{\Omega}$ is a continuous curve crossing $\partial\Omega$ transversally at $\gamma(1) = z \in D$. There exists $\alpha > 0$ such that $\gamma \in \Gamma_\alpha(V_m)$ for $m \geq m_z$. We can additionally assume that $\alpha > \frac{1}{m}$ for $m \geq m_z$. In particular $\gamma(0) \in T_m$ for $m \geq m_z$.

Now we can define $t_m := \sup\{t \in [0,1] : \gamma(t) \in T_m\}$ and $z_m := \gamma(t_m)$. In particular $\lim_{m\to\infty} t_m = 1$ and $\operatorname{dist}(\partial\Omega, z_m) = \varepsilon_{m+1}$. Since $z_m \in T_j$ for $j \geq m \geq m_z$, we have $|f_j(z_m)| \leq \varepsilon_{j+1}$ for $j \geq m$. Now we may estimate

$$||f(z_m)| - \omega_m(z_m)| \leq \sum_{j=m}^{\infty} |f_j(z_m)| \leq \sum_{j=m}^{\infty} \varepsilon_{j+1} \leq \varepsilon_m.$$

Since $\gamma \in \Gamma_\alpha(V_m)$ therefore $\varepsilon_{m+1} = \operatorname{dist}(\partial\Omega, z_m) \geq \alpha \|z - z_m\|$. Observe that $\|z - z_m\| \leq \frac{\varepsilon_{m+1}}{\alpha} \leq m\varepsilon_{m+1}$ for $m \geq m_z$. In particular due to property (2) we have $|\omega_m(z) - \omega_m(z_m)| \leq \varepsilon_m$ and $||f(z_m)| - \omega_m(z)| \leq 2\varepsilon_m$. Now due to (3) we may conclude that

$$\lim_{m\to\infty} |f(z_m)| = \lim_{m\to\infty} \omega_m(z) = G(z).$$

Since $\sigma(D) = \sigma(S) = 1$, we can define $\widetilde{S} := D$, which finishes the proof. $\square$

It is known (see [9, Theorem 8.4.1]) that a bounded holomorphic function f on a bounded domain Ω with the boundary of class C^2 has non tangent limits $f^*(z)$ for almost all $z \in \partial\Omega$. In particular we can obtain a classical result about inner function:

Theorem 3.6. *Let $\Omega \subset\subset \mathbb{C}^n$ be a domain with the boundary of class C^2 and G be a bounded, strictly positive and lower semicontinuous function on $\partial\Omega$. Let η denote $(2n-1)$-dimensional Hausdorff measure on $\partial\Omega$. Assume that Ω admits a holomorphic support function on a Borel set S with full η measure. Then there exists f, a holomorphic bounded function on Ω such that $\|f\|_\Omega \leq \|G\|_{\partial\Omega}$ and $|f^*(z)| = G(z)$ for η-almost all $z \in \partial\Omega$.*

Proof. We can define a Borel probability measure $\sigma := \frac{1}{\eta(\partial\Omega)}\eta$. Let f be a holomorphic function from Theorem 3.5. In particular we have $\|f\|_\Omega \leq \|G\|_{\partial\Omega}$. Moreover, since f has $f^*(z)$ for almost all $z \in \partial\Omega$, therefore the last part of Theorem 3.5 gives us the following equality: $|f^*(z)| = G(z)$ for η-almost all $z \in \partial\Omega$. $\square$

4. Applications for Radon inversion problem and exceptional sets.

Assume that $\Omega \subset \mathbb{C}^n$ is a circular bounded domain such that $\gamma : [0,1] \ni t \to tz \in \overline{\Omega}$ crosses Ω transversally at $z \in \partial\Omega$. We can define a projective $(n-1)$-dimensional space $\mathbb{P}^{n-1} = \mathbb{P}(\partial\Omega) := \{[z] : z \in \partial\Omega\}$ where $[z] := \partial\mathbb{D}z$ and $\mathbb{D} := \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. In particular, we can identify $\partial\Omega = \partial\mathbb{D}\mathbb{P}^{n-1}$. Now we can consider the following Radon inversion problem:

Assume that G is a lower semicontinuous function on $\mathbb{P}^{n-1}$. Let us construct a holomorphic function f on Ω such that

$$G(z) = \int_{\mathbb{D}} |f(\lambda z)|^2 \, d\mathfrak{L}^2(\lambda).$$

We solve this problem in terms of a given Borel probability measure:

Theorem 4.1. *Let η be a Borel probability measure on $\mathbb{P}^{n-1}$ and G be a strictly positive and lower semicontinuous function on $\mathbb{P}^{n-1}$. Assume that Ω admits a holomorphic support function on $\partial\Omega$. Then there exists h a holomorphic function on Ω such that $G(z) = \int_{\mathbb{D}} |h(\lambda z)|^2 \mathfrak{L}^2(\lambda)$ for η-almost all $z \in \mathbb{P}^{n-1}$ and $\int_{\mathbb{D}} |h(\lambda z)|^2 \mathfrak{L}^2(\lambda) \leq G(z)$ for all $z \in \partial\Omega$.*

Proof. For a given set $A \subset \partial\Omega$ we can define $[A] := \{[z] : z \in A\}$ and $A(z) := \{t \in [0,1] : e^{2\pi i t}z \in A\}$ for $z \in \partial\Omega$. Now we define a Borel probability measure σ on $\partial\Omega$ by putting $\sigma(A) = \int_{[A]}\int_{A(z)} dt d\eta(z)$.

Let $\widetilde{G}(\lambda z) = \sqrt{G(z)}$. Observe that $\widetilde{G}$ is a strictly positive and lower semicontinuous function on $\partial\Omega$. Due to Theorem 3.5 there exists S a Borel subset of $\partial\Omega$ and a holomorphic function f on Ω with the following properties:

- There exists $\{g_n\}_{n\in\mathbb{N}}$, a sequence of holomorphic functions on Ω and continuous on $\overline{\Omega}$ such that g_n converges uniformly to f on compact subsets of Ω, $|g_n(z)| < \widetilde{G}(z)$ for $z \in \partial\Omega$, $\lim_{n\to\infty} |g_n(z)| = \widetilde{G}(z)$;
- $\sigma(S) = \sigma(\partial\Omega) = 1$,
- If $z \in S$ then there exists a sequence $\{r_n\}_{n\in\mathbb{N}} \subset (0,1)$ such that $\lim_{n\to\infty} r_n = 1$ and $\lim_{n\to\infty} |f(r_n z)| = \widetilde{G}(z)$.

Since $\sigma(S) = 1$, there exists a Borel set $S_2 \subset S$ such that $\eta([S_2]) = 1$ and $\int_{S_2(z)} dt = 1$ for $z \in S_2$. In particular $\sigma(S_2) = 1$.

Using the maximum property for the holomorphic function $\mathbb{D} \ni \lambda \to f(\lambda z)$ we may conclude that $\limsup_{r\to 1^-} |f(rz)| = \widetilde{G}(z)$ for $z \in S_2$ and $\limsup_{r\to 1^-} |f(rz)| \le \widetilde{G}(z)$ for $z \in \partial\Omega$.

Now there exists a sequence $\{p_m\}_{m\in\mathbb{N}}$ of homogeneous polynomials such that $f(w) = \sum_{m\in\mathbb{N}} p_m(w)$ for $w \in \Omega$. Let a_m be such that $\int_{\mathbb{D}} |a_m p_m(\lambda z)|^2 d\mathfrak{L}^2(\lambda) = |p_m(z)|^2$. Now we can define a holomorphic function $h := \sum_{m\in\mathbb{N}} a_m p_m$.

In particular we may estimate for $z \in S_2$:

$$\begin{aligned}
\int_{\mathbb{D}} |h(\lambda z)|^2 d\mathfrak{L}^2(\lambda) &= \sum_{m\in\mathbb{N}} |p_m(z)|^2 = \limsup_{r\to 1^-} \sum_{m\in\mathbb{N}} |p_m(rz)|^2 \\
&= \limsup_{r\to 1^-} \int_0^1 \left|f(re^{2\pi i s}z)\right|^2 ds \\
&= \int_0^1 \limsup_{r\to 1^-} \left|f(re^{2\pi i s}z)\right|^2 ds = \left(\widetilde{G}(z)\right)^2 = G(z).
\end{aligned}$$

In the similar way we may obtain for $z \in \partial\Omega$ the following inequality:

$$\int_{\mathbb{D}} |h(\lambda z)|^2 d\mathfrak{L}^2(\lambda) = \int_0^1 \limsup_{r\to 1^-} \left|f(re^{2\pi i s}z)\right|^2 ds \le G(z). \qquad \square$$

Using the above theorem we can describe exceptional sets:

Theorem 4.2. *Let E be a set of type G_δ in $\mathbb{P}^{n-1}$ and η be a probability measure on E. Then there exists a holomorphic function f such that $\int_{\Omega\setminus\mathbb{D}E} |f|^2 d\mathfrak{L}^{2n} < \infty$ and $E^2_\Omega(f) \subset E$, $\eta(E) = \eta(E^2_\Omega(f))$ where*

$$E^2_\Omega(f) := \left\{ z \in \mathbb{P}^{n-1} : \int_{\mathbb{D}} |f(\lambda z)|^2 d\mathfrak{L}^2(\lambda) = \infty \right\}.$$

Proof. We can assume that η is a Borel probability measure on $\mathbb{P}^{n-1}$. Let ν be $(2n-1)$-dimensional Hausdorff measure on $\partial\Omega$. There exists a sequence $\{U_m\}_{m\in\mathbb{N}}$ in $\partial\Omega$ of open, circular sets such that

$$\sum_{m\in\mathbb{N}} \nu\left(U_m \setminus E\right) \leq 1$$

and $\partial\mathbb{D}E = \bigcap_{m\in\mathbb{N}} U_m$. Let

$$\chi_m(z) := \begin{cases} 1 & \text{for } z \in U_m \\ 0 & \text{for } z \in \partial\Omega \setminus U_m. \end{cases}$$

Obviously χ_m is a lower semicontinuous function. Let $G = 1 + \sum_{m\in\mathbb{N}} \chi_m$. The function G is also a lower semicontinuous function such that $G(\lambda z) = G(z) > 0$ for $|\lambda| = 1$ and $z \in \partial\Omega$ so we can assume that G is defined on $\mathbb{P}^{n-1}$. On the basis of Theorem 4.1 there exists a holomorphic function $f \in \mathbb{O}(\Omega)$ such that

(1) $G(z) = \int_{\mathbb{D}} |f(\lambda z)|^2 \mathfrak{L}^2(\lambda)$ for η-almost all $z \in \mathbb{P}^{n-1}$.
(2) $\int_{\mathbb{D}} |f(\lambda z)|^2 \mathfrak{L}^2(\lambda) \leq G(z)$ for all $z \in \partial\Omega$.

Observe now that

$$\int_{\partial\Omega\setminus\partial\mathbb{D}E} G d\nu = 1 + \sum_{m\in\mathbb{N}} \nu\left(U_m \setminus E\right) \leq 2.$$

There exists a constant $C > 0$ such that

$$C \int_{\Omega\setminus\mathbb{D}E} |f|^2 \, d\mathfrak{L}^{2n} \;\leq\; \int_{\partial\Omega\setminus\partial\mathbb{D}E} \int_{|\lambda|\leq 1} |f(\lambda z)|^2 \, d\mathfrak{L}^2(\lambda) d\nu(z) \leq \int_{\partial\Omega\setminus\partial\mathbb{D}E} G d\nu \leq 2.$$

Moreover, since $E = G^{-1}(\infty)$ therefore $E^2_\Omega(f) \subset E$ and $\eta(E) = \eta(E^2_\Omega(f))$ which finishes the proof. □

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