Journal of Convex Analysis
Volume 15 (2008), No. 4, 891–903

Heat Flow for Closed Geodesics on Finsler Manifolds

Mamadou Sango
*Department of Mathematics, University of Pretoria,
Pretoria 0002, South Africa*
mamadou.sango@up.ac.za, sango177@yahoo.com

Received: April 1, 2006

In the present paper we use the celebrated heat flow method of Eells and Sampson to the question of deformation of a smooth loop $M \in \mathbf{R}^2$ on a Finsler manifold (N, h) to a closed geodesic in N. This leads to the investigation of the corresponding heat equation which is the parabolic initial value problem

$$\frac{\partial u^i}{\partial t} - \frac{\partial^2 u^i}{\partial x^2} = \Gamma^i_{hk}\left(u, \frac{\partial u}{\partial x}\right)\frac{\partial u^h}{\partial x}\frac{\partial u^k}{\partial x} \quad \text{in } M \times [0, T),$$
$$u(x, 0) = f(x);$$

$i = 1, \ldots, n$. The existence of a global in time solution $u(x, t)$ and its subsequent convergence to a closed geodesic $u_\infty : M \to N$ as $t \to \infty$, are dealt with. Appropriate concepts arising from the Finslerian nature of the problem are introduced.

1. Introduction

Let (M, g) and (N, h) be some smooth compact Finsler (or Riemannian) manifolds with the metrics g and h respectively; $\dim M = m$, $\dim N = n$. Given a C^1-map $f : M \to N$, an important question in geometric analysis is to determine whether or not f can be deformed into a harmonic map $u : M \to N$. For Riemannian manifolds this question has been successfully settled in the pioneering work of Eells and Sampson [14]. However the corresponding issue in the framework of Finsler geometry seems to be far more complicated when $m \geq 2$. While the powerful tools developed in the Riemannian case have a potential of being suitable devices in order to solve the problem, an appropriate notion of energy functional for the purpose is still to be found; we refer to [13] and [25] for some attempts; in fact Mo considers the case of a Riemannian target which through a suitable change of metric reduces to [14].

In the present paper we consider the case when the source manifold M is one-dimensional (the circle $S^1 \subset \mathbf{R}^2$ for instance) but the target is a genuine multi-dimensional Finsler manifold; this is related to the very rich area of closed geodesics. The problem is a classical one in global differential geometry and has been extensively dealt with in the Riemannian case via the direct method of Calculus of Variations (see for instance [26], p. 74-75) as well as through the celebrated heat flow method of Eells and Sampson ([4], [14], [16], [17]). Closed geodesic on Finsler manifolds have also been the object of interesting research since the deep work of Fet [15] in the fifties. Important contributions were later made by Anosov [5], Katok [19] (this paper has now become a milestone in Finsler geometry), [23]

ISSN 0944-6532 / $ 2.50

and [24]. The papers mainly derive the existence of closed geodesics through the method of critical points.

The aim of the present paper is to study the existence, uniqueness and large time behavior of the solution of the Cauchy problem for a semi-linear parabolic system which arises from the use of the celebrated Eells and Sampson's heat flow method in the corresponding Finslerian setting.

After a dormant period, Finsler geometry is generating a lot of interest in recent years. While purely geometric questions are being vigorously tackled, results in Finslerian geometric analysis are scarced; notable among the few contributions in this direction are works by members of the famous Italian School of Calculus of Variations [1], [10], [11], [6], [8], [9]; the first three papers for instance establish that Finsler metrics can arise as Γ-limits of some Riemannian metrics. This is a very powerful statement which can deeply influence the study of Finsler manifolds in years to come, in a way similar to the revolution brought by Gromov's groundbreaking result on convergence of Riemannian manifolds which gave rise to deep studies of Alexandrov spaces, graph manifolds, and many other geometries.

The paper is structured as follows. In Section 2, we introduce some definitions and formulate our main results. Section 3 is devoted to the short-time existence of the heat flow. In Section 4, we introduce some kind of "normal coordinates" in which a contraction of the Finslerian Christofell's symbols vanish (normal coordinates in which the Christofell's symbols vanish can be introduced only on Berwald spaces), we obtain a Weitzenboch-Bochner type estimate for the energy density and subsequently use it to derive appropriate a priori estimates that enable us to establish the theorem on the global existence of the heat flow. The closing Section 5 is devoted to the proof the Corollary 2.4 on the existence of closed geodesics.

In the sequel Einstein's summation rule is used throughout.

2. Finslerian geodesics and the heat flow

Basic informations about Finsler manifolds can be found in [7], [12] or [29].

On the tangent bundle TN of the smooth compact n-dimensional ($n \geq 1$) manifold N with local coordinates (v, w), we consider the function $H : TN \to [0, \infty)$, satisfying the following conditions:

(i) $H \in C^\infty (TN \backslash 0)$, 0 being the zero section.

(ii)

$$H(v, \lambda w) = |\lambda| \, H(v, w), \quad \forall \lambda \in \mathbf{R}, \ \forall (v, w) \in TN \tag{1}$$

(iii) the $(n \times n)$ hessian matrix

$$h_{ij} = \left(\left[\frac{1}{2} H^2 \right]_{w^i w^j} \right), \quad i, j = 1, \ldots, n \tag{2}$$

is positive definite at every point of $TN \backslash 0$. We have $H^2(v, w) = h_{ij}(v, w) \, w^i w^j$. The matrix $h = (h_{ij})$ is called the fundamental matrix.

A function H satisfying the conditions (i)-(iii) is called a Finsler structure and the pair (N, h) a Finsler manifold.

Let

$$A_{ijk}(v,w) = \frac{\partial h_{ij}(v,w)}{\partial w^k}, \quad i,j,k = 1,\dots,n. \tag{3}$$

(A_{ijk}) is known as the Cartan Tensor. The homogeneity condition (1) leads to the relations

$$A_{ijk}w^k = A_{ijk}w^i = A_{ijk}w^j = 0, \tag{4}$$

known as Euler's formulae. Christofell symbols for the metric h are defined by

$$\Gamma^i_{jk} = \frac{1}{2}\sum_{l=1}^{n} h^{il}\left(\frac{\partial h_{lj}}{\partial v^k} + \frac{\partial h_{lk}}{\partial v^j} - \frac{\partial h_{jk}}{\partial v^l}\right), \tag{5}$$

where (h^{ij}) is the inverse matrix of h.

Let M be a the one dimensional unit sphere S^1 in $\mathbf{R}^2$ parametrized by the central angle that we denote by $x \in [0, 2\pi]$, or any smooth loop in $\mathbf{R}^2$. We consider the map $u : M \to N$, $u(x) = (u^1(x), \dots, u^n(x))$ which describes a closed curve in N. The density energy of u is given by

$$e(u) = \frac{1}{2}h_{ij}\left(u^1,\dots,u^n,\frac{du^1}{dx},\dots,\frac{du^n}{dx}\right)\frac{du^i}{dx}\frac{du^i}{dx} \tag{6}$$

and the energy reads

$$E(u) = \int_0^{2\pi} e(u)\,dx. \tag{7}$$

We shall for simplicity write the integral as

$$E(u) = \int_M e(u)\,dx, \tag{8}$$

where $x \in M$ will mean $x \in [0, 2\pi]$.

Remark 2.1. The energy E arises also as the Gamma-Limit of some sequences of Riemannian energies (see [1], [10], [11]). We also refer to [27] for the corresponding complex variables version.

We assume that there exists a constant C independent of v, w such that

$$h_{ij}(v,w)\,w^i w^j \geq C\,|w|^2, \tag{9}$$

$$\left|\Gamma^i_{hk}(v,w)\right| + \left|\frac{\partial}{\partial v^l}\Gamma^i_{hk}(v,w)\right| + \frac{\partial}{\partial w^l}\Gamma^i_{hk}(v,w) \leq C, \tag{10}$$

for all v, $w \in \mathbf{R}^n$ and $l = 1,\dots,n$.

The Euler-Lagrange equations for E obtained by evaluating the quantity $\frac{dE(u+\varepsilon\varphi)}{d\varepsilon}|_{\varepsilon=0}$, for any test function $\varphi : M \to N$, are

$$\frac{d^2u^i}{dx^2} = \Gamma^i_{hk}\left(u, \frac{du}{dx}\right)\frac{du^h}{dx}\frac{du^k}{dx}, \quad i = 1,\dots,n. \tag{11}$$

These are the differential equations governing the geodesics on (N, h).

As previously mentioned we make use of the celebrated heat flow method of Eells and Sampson. It consists at looking for a map $u(x,t) : M \times [0,T) \to N$ which is a solution of the initial value problem for the system of quasilinear parabolic equations

$$\frac{\partial u^i}{\partial t} - \frac{\partial^2 u^i}{\partial x^2} = \Gamma^i_{hk}\left(u, \frac{\partial u}{\partial x}\right)\frac{\partial u^h}{\partial x}\frac{\partial u^k}{\partial x}, \quad (x,t) \in M \times [0,T), \tag{12}$$

$$u(x,0) = f(x); \tag{13}$$

here $T > 0$, and $f : M \to N$ is a smooth loop playing the role of initial condition.

The following issues are to be dealt with.

1) For any initial value f, does the problem (12)-(13) have a unique solution in $M \times [0,T)$?
2) Set $u_t(x) = u(x,t)$. As $t \to \infty$, does u_t converge to a closed geodesic $u_\infty : M \to N$ which is free-homotopic to $f = u_0$?

Remark 2.2. It is worth mentioning that unlike the Riemannian case the Christofell symbols Γ^i_{hk} have a dependence on $\partial u/\partial x$; this makes the study of the system (12) more involved.

(12) is a local system of equations since it is expressed in local coordinates. It needs to be replaced by a more convenient global system in global coordinates. In the Riemannian case this is achieved generally by isometrically embedding the target manifold in a Euclidean manifold, through the celebrated Nash's embedding theorem. A similar result for Finsler manifolds is still out of reach and its quest is one of the most outstanding open problems in Finsler geometry. Lukily an approach due originally to Eells and Sampson [14] and Hamilton [16] which uses instead Whitney's embedding theorem is adaptable to our needs.

N being a differentiable manifold, Whitney's theorem enables us to embed it in a Euclidean space $\mathbf{R}^k$. Let $i : N \to \mathbf{R}^k$ be such an embedding. Its differential induces an embedding $i_* : TN \to \mathbf{TR}^k =: \mathbf{R}^q$. We construct in $\mathbf{R}^q$ a tubular neighbourhood $\mathcal{U}_{2\delta} = \{x \in \mathbf{R}^q : dist(x, TN) < 2\delta\}$, $dist$ denotes the usual Euclidean distance and extend the metric h on N to a metric $h' = \left(h'_{ij}\right)_{i,j=1,\dots,q}$ on $\mathcal{U}_{2\delta}$ by keeping h on TN and by taking the induced Euclidean metric on the normal bundle of TN in $\mathcal{U}_{2\delta}$. Locally $h'_{ij} = (h_{ij}) \otimes (\delta_{ij})$ on $TN \times B^{q-n}_{2\delta}$ where $B^{q-n}_{2\delta}$ is the ball in $\mathbf{R}^{q-n}$ of radius 2δ with center on TN, δ_{ij} denotes the Kroneker symbols. We extend h' smoothly to $\mathbf{R}^q$ by $h'' = \left(h''_{ij}\right)_{i,j=1,\dots,q}$ with $h''_{ij} = \varphi h'_{ij} + (1-\varphi)\delta_{ij}$ where $\varphi \in C^\infty_0(\mathbf{R}^q, \mathbf{R})$, $\varphi = 1$ on $\mathcal{U}_\delta$ and $\varphi = 0$ outside $\mathcal{U}_{2\delta}$. It is clear that h' and h'' are both Finsler metrics. This embedding induces an involutive isometry π from the tubular neighbourhood $\mathcal{U}_\delta(N)$ of N into itself having N as its fixed points.

Arguing as in [16], we get that a deformation $u_t : M \to N$ satisfies the problem (12)-(13) if and only if the function $W_t = \pi \circ u_t : M \to \mathbf{R}^k$ satisfies the problem

$$\frac{\partial W^c_t}{\partial t} - \frac{\partial^2 W^c_t}{\partial x^2} = \Gamma^c_{ab}\left(W_t, \frac{\partial W_t}{\partial x}\right)\frac{\partial W^a_t}{\partial x}\frac{\partial W^b_t}{\partial x}, \quad (x,t) \in M \times [0,T), \tag{14}$$

$$W_0(x) = f(x), \quad x \in M. \tag{15}$$

Here Γ_{ab}^{c} are the Christofell symbols induced by h'' in $\mathbf{R}^q$. This problem is the needed globalized version of (12)-(13). Since W_0 maps M into N any solution W_t of (14)-(15) for $t \in [0,T)$ maps M into N.

From now on, we assume that all conditions stated above are always valid. We are now in the position to state our main results. The first deals with the existence of a global in time solution of (14)-(15).

Theorem 2.3. *For any $f \in C^{2+\alpha}(M,N)$, there exists a solution*

$$W \in C^{2+\alpha,1+\alpha/2}(M \times [0,\infty), N) \tag{16}$$

of problem (14)-(15) *in $M \times [0,\infty)$.*

As a consequence of this theorem we recover the well-known result on the existence of closed geodesic on Finsler manifolds

Corollary 2.4. *Let $f \in C^{2+\alpha}(M,N)$ and let*

$$u \in C^{2+\alpha,1+\alpha/2}(M \times [0,\infty), N) \tag{17}$$

be the global in time solution of (12)-(13). *Then there exists a sequence $\{t_m\} \to \infty$ such that $u(x,t_m)$ converges to a closed geodesic $u_\infty(x) \in C^{2+\alpha}(M,N)$ free homotopic to f.*

Remark 2.5. f and u_∞ are free homotopic if there exists a continuous map $u(x,t)$: $M \times [0,T] \to N$ satisfying the relations: $u(x,0) = f(x)$ and $u(x,T) = u_\infty(x)$ for all $x \in M$.

Remark 2.6. The result of the corollary can be improved by showing that the convergence holds for all $t \to \infty$. This can be done by adapting the arguments of Hartman [17] to our case.

3. Short-time existence

We start with some notations. Given $T > 0$, set $Q = M \times [0,T]$, $0 < \alpha < 1$. For $u : Q \to \mathbf{R}^q$, set

$$|u|_Q = \sup_{(x,t)\in Q} |u(x,t)|, \tag{18}$$

$$\langle u\rangle_x^{(\alpha)} = \sup\left\{\frac{|u(x,t)-u(x',t)|}{|x-x'|^\alpha};\ (x,t),(x',t) \in Q,\ x \neq x'\right\}, \tag{19}$$

$$\langle u\rangle_t^{(\alpha)} = \left\{\sup\frac{|u(x,t)-u(x',t)|}{|t-t'|^\alpha};\ (x,t),(x',t) \in Q, t \neq t'\right\}; \tag{20}$$

$|\ |$ stands for the Euclidean norm.

$$|u|_Q^{(\alpha,\alpha/2)} = |u|_Q + \langle u\rangle_x^{(\alpha)} + \langle u\rangle_t^{(\alpha/2)}, \tag{21}$$

$$
\begin{aligned}
&|u|_Q^{(2+\alpha,1+\alpha/2)} \\
= \;& |u|_Q + \left|\frac{\partial u}{\partial t}\right|_Q + \left|\frac{\partial u}{\partial x}\right|_Q + \left|\frac{\partial^2 u}{\partial x^2}\right|_Q + \left\langle \frac{\partial u}{\partial t}\right\rangle_t^{(\alpha/2)} \\
&+ \left\langle \frac{\partial u}{\partial x}\right\rangle_t^{(1/2+\alpha/2)} + \left\langle \frac{\partial^2 u}{\partial x^2}\right\rangle_t^{(\alpha/2)} + \left\langle \frac{\partial u}{\partial x}\right\rangle_x^{(\alpha)} + \left\langle \frac{\partial^2 u}{\partial x^2}\right\rangle_x^{(\alpha)} .
\end{aligned}
$$

We introduce Hölder's spaces

$$
C^{2+\alpha}(M,N) = \left\{ u \in C^2(M) : \left\langle \frac{\partial^2 u}{\partial x^2}\right\rangle_x^{(\alpha)} < \infty \right\}, \tag{22}
$$

$$
C^{2+\alpha,1+\alpha/2}(Q,N) = \left\{ u \in C^{2,1}(Q) : |u|_Q^{(2+\alpha,1+\alpha/2)} < \infty \right\}, \tag{23}
$$

$$
C^{\alpha,\alpha/2}(Q,N) = \left\{ u \in C^{0}(Q) : |u|_Q^{(\alpha,\alpha/2)} < \infty \right\}. \tag{24}
$$

We have

Theorem 3.1. *For any map $f \in C^{2+\alpha}(M,N)$, there exists a positive number $\varepsilon = \varepsilon(M,N,f,\alpha)$ and a map $W \in C^{2+\alpha,1+\alpha/2}(M\times[0,\varepsilon],\mathbf{R}^q)$ such that W is a solution of (14)-(15) in $M\times[0,\varepsilon]$.*

Proof. We use the method of linearization. Let us denote the right-hand side of (14) by $H^c\left(W,\frac{\partial W}{\partial x}\right)$. Let $k \in C^{2+\alpha,1+\alpha/2}(M\times[0,\varepsilon],\mathbf{R}^q)$. We have

$$
\begin{aligned}
&\lim_{\theta\to 0}\frac{H^c\left(W+\theta k,\frac{\partial(W+\theta k)}{\partial x}\right) - H^c\left(W,\frac{\partial W}{\partial x}\right)}{\theta} \\
= \;& \frac{\partial \Gamma^c_{ab}}{\partial z^\alpha}\frac{\partial W^a}{\partial x}\frac{\partial W^b}{\partial x}k^\alpha + \frac{\partial \Gamma^c_{ab}}{\partial Z^\alpha}\frac{\partial W^a}{\partial x}\frac{\partial W^b}{\partial x}\frac{\partial k^\alpha}{\partial x} + 2\Gamma^c_{ab}\frac{\partial W^a}{\partial x}\frac{\partial k^b}{\partial x},
\end{aligned}
$$

and

$$
\lim_{\theta\to 0}\frac{\left(\frac{\partial (W^c+\theta k^c)}{\partial t} - \frac{\partial^2 (W^c+\theta k^c)}{\partial x^2}\right) - \left(\frac{\partial W^c}{\partial t} - \frac{\partial^2 W^c}{\partial x^2}\right)}{\theta} = \frac{\partial k^c}{\partial t} - \frac{\partial^2 k^c}{\partial x^2}.
$$

Thus k satisfies a system of linear parabolic equations of the form

$$
\frac{\partial k^c}{\partial t} - \frac{\partial^2 k^c}{\partial x^2} = A^c_\alpha k^\alpha + B^c_\alpha \frac{\partial k^\alpha}{\partial x}. \tag{25}
$$

The local existence of a solution for (14)-(15) follows from well-known arguments involving the regularity theory of linear parabolic systems [21] and the inverse function theorem in Banach spaces [22]. □

4. Global existence

We shall need on N some kind of "normal coordinates" which can be introduced on Finsler manifolds using the results of [3] or [28]. It can be shown that there exist some coordinates (z, Z) such that

$$\Gamma^c_{ab}(z, Z)\, Z^a Z^b = 0 \tag{26}$$

along the geodesic emanating from a point $z_0 \in N$ and this relation is valid at the point z_0 for all Z. This implies ([3], p. 42, equation (10.5)) after differentiation with respect to Z^a that

$$\Gamma^c_{ab}(z_0, Z)\, Z^b = 0, \quad \forall Z. \tag{27}$$

By "normal coordinates" we shall mean the pairs (z_0, Z) such that the above equations hold. Though we do not have the vanishing of the Γ^c_{ab}, the relations (27) will suffice for our purposes.

We now derive a Weitzenboch-Bochner's type formula in the following

Lemma 4.1. *Let W be a solution of* (14)-(15). *Then the energy density*

$$e(W) = \frac{1}{2} h''_{ab}\left(W^1, \ldots, W^q, \frac{\partial W^1}{\partial x}, \ldots, \frac{\partial W^q}{\partial x}\right) \frac{\partial W^a}{\partial x} \frac{\partial W^b}{\partial x} \tag{28}$$

satisfies the parabolic equation

$$\frac{\partial e(W)}{\partial t} - \frac{\partial^2 e(W)}{\partial x^2} = -h''_{ab} \frac{\partial^2 W^a}{\partial x^2} \frac{\partial^2 W^b}{\partial x^2}. \tag{29}$$

Proof. We have

$$\frac{\partial e(W)}{\partial t} = \frac{1}{2}\left[\frac{\partial h''_{ab}}{\partial z^k}\frac{\partial W^k}{\partial t} + \frac{\partial h''_{ab}}{\partial Z^k}\frac{\partial^2 W^k}{\partial t \partial x}\right]\frac{\partial W^a}{\partial x}\frac{\partial W^b}{\partial x} + h''_{ab}\frac{\partial W^a}{\partial x}\frac{\partial^2 W^b}{\partial t \partial x}. \tag{30}$$

Since h'' is a Finsler metric, by (4), we get

$$\frac{\partial h''_{ab}}{\partial Z^k}\frac{\partial W^a}{\partial x} = 0. \tag{31}$$

In view of (14), we have

$$\begin{aligned} \frac{\partial^2 W^b}{\partial t \partial x} \;=\; & \frac{\partial^3 W^b}{\partial x^3} + \frac{\partial \Gamma^b_{de}}{\partial z^l}\frac{\partial W^l}{\partial x}\frac{\partial W^d}{\partial x}\frac{\partial W^e}{\partial x} \\ & + \frac{\partial \Gamma^b_{de}}{\partial Z^l}\frac{\partial^2 W^l}{\partial x^2}\frac{\partial W^d}{\partial x}\frac{\partial W^e}{\partial x} + 2\Gamma^b_{de}\frac{\partial^2 W^d}{\partial x^2}\frac{\partial W^e}{\partial x}. \end{aligned} \tag{32}$$

Thanks to (27), (31), (30), (32) and the relation

$$\frac{\partial h''_{ab}}{\partial z^k} = h''_{sb}\Gamma^s_{ak} + h''_{as}\Gamma^s_{bk} \tag{33}$$

we get in normal coordinates

$$\frac{\partial e(W)}{\partial t} = h''_{ab}\left(\frac{\partial^3 W^b}{\partial x^3} + \frac{\partial \Gamma^b_{pk}}{\partial z^l}\frac{\partial W^l}{\partial x}\frac{\partial W^p}{\partial x}\frac{\partial W^k}{\partial x}\right)\frac{\partial W^a}{\partial x}. \tag{34}$$

Similarly we have

$$\frac{\partial^2 e(W)}{\partial x^2} = h''_{ab}\left(\frac{\partial^3 W^b}{\partial x^3} + \frac{\partial \Gamma^b_{pk}}{\partial z^l}\frac{\partial W^l}{\partial x}\frac{\partial W^p}{\partial x}\frac{\partial W^k}{\partial x}\right)\frac{\partial W^a}{\partial x} + h''_{ab}\frac{\partial^2 W^a}{\partial x^2}\frac{\partial^2 W^b}{\partial x^2}.$$

(29) readily follows from these last two relations. □

Remark 4.2. Unlike the multidimensional case ($\dim M > 1$), the curvature term is missing in (29). Hence no curvature sign condition is needed.

Lemma 4.3. *Let W be a solution of* (14)-(15). *Then the energy density $e(W)$ satisfies*

$$e(W)(x) \leq \max_{x\in M} e(f)(x). \tag{35}$$

Proof. Since h'' is positive definite we have by (29),

$$\frac{\partial e(W)}{\partial t} - \frac{\partial^2 e(W)}{\partial x^2} \leq 0, \quad (x,t) \in M \times (0,T). \tag{36}$$

Then by the Maximum Principle [21] for the Heat equation we get

$$\max_{M\times[0,T]} e(W)(x) = \max_{M\times\{0\}} e(W)(x) = \max_{M} e(f). \tag{37}$$

□

Next we have

Lemma 4.4. *Let W be a sufficiently smooth solution of* (14)-(15). *Then there exists a constant $C = C(M,N,f) > 0$, such that*

$$\left|\frac{\partial W}{\partial t}\right|_{L^\infty(M,N)} + |W|_{L^\infty(M,N)} \leq C. \tag{38}$$

Proof. We differentiate the system (14) with respect to t (this a formal process which can be made rigorous by considering the difference quotient of (14) with respect to t) and get

$$\frac{\partial}{\partial t}\left(\frac{\partial W^c}{\partial t}\right) - \frac{\partial^2}{\partial x^2}\frac{\partial W^c}{\partial t} = \frac{\partial}{\partial z^l}\Gamma^c_{ab}\frac{\partial W^l}{\partial t}\frac{\partial W^a}{\partial x}\frac{\partial W^b}{\partial x} + 2\Gamma^c_{ab}\frac{\partial}{\partial x}\left(\frac{\partial W^a}{\partial t}\right)\frac{\partial W^b}{\partial x},$$

where we have used (4).

By (9), (35) and the compactness of M, it follows that

$$\left|\frac{\partial W}{\partial x}\right|_{L^\infty(M,N)} \leq C \tag{39}$$

Thus the function $v_t = \partial W/\partial t$ satisfies the equation

$$\frac{\partial v^c_t}{\partial t} - \frac{\partial^2 v^c_t}{\partial x^2} = A^c_\alpha v^\alpha + B^c_\alpha \frac{\partial v^\alpha}{\partial x}, \quad \forall (x,t) \in M \times [0,T) \tag{40}$$

with bounded coefficients thanks to (10); also $v_t(x,0) = 0$ in M. Applying the results of [21](Chap. III, § 7) on the bounds for maximum of solutions of linear parabolic equations, we get that

$$\left| \frac{\partial W}{\partial t} \right|_{L^\infty(M,N)} \leq C. \tag{41}$$

Since N is compact we have that the image of W_t is contained in a bounded set. Thus

$$|W_t|_{L^\infty(M,N)} \leq C. \tag{42}$$

The lemma is proved. □

Thanks to the estimates in Lemmas 4.3 and 4.4 and the Schauder's estimates for linear elliptic and parabolic equations [20], [21], we obtain

Lemma 4.5. *Let* $W \in C^{2,1}(M \times [0,T], N)$ *be a solution of* (14)-(15). *Then there for any* $\alpha \in (0,1)$

$$|W(,t)|_{C^{2+\alpha}(M,N)} + \left| \frac{\partial W}{\partial t} \right|_{C^\alpha(M,N)} \leq C \tag{43}$$

for any $t \in [0,T)$; *where* $C = C(M,N,f,\alpha)$.

Now we are in the position to prove the existence of global in time solution.

Proof of Theorem 2.3. By Theorem 3.1 there exists a $\varepsilon = \varepsilon(M,N,f,\alpha)$ such that a solution

$$W \in C^{2+\alpha,1+\alpha/2}(M \times [0,\infty), N) \tag{44}$$

of (14)-(15) exists in $M \times [0,\epsilon]$. Let us show that this solution can be extended to $M \times [0,\infty)$. Set

$$T_0 = \sup\{t \in [0,\infty) : \text{(14)-(15) has a solution in } [0,t]\}. \tag{45}$$

We show that $T_0 = \infty$. Indeed assume the contrary, that is $T_0 < \infty$ and let $\{t_m\}$ be a sequence converging to T_0. Let $1 > \alpha' > \alpha > 0$. We have by Lemma 4.5, that

$$|W(.,t_m)|_{C^{2+\alpha'}(M,N)}, \qquad \left| \frac{\partial W(.,t_m)}{\partial t} \right|_{C^{\alpha'}(M,N)} \tag{46}$$

are uniformly bounded. Thus since $C^{2+\alpha'}(M,N)$ is compactly embedded into $C^{2+\alpha}(M,N)$ (see for instance [2, p. 11], modulo the extraction of a subsequence from $\{t_m\}$ it follows

$$W(.,t_m) \to W(.,T_0), \qquad \frac{\partial W(.,t_m)}{\partial t} \to \frac{\partial W(.,T_0)}{\partial t} \tag{47}$$

uniformly, $W(.,T_0) \in C^{2+\alpha}(M,N)$, $\frac{\partial W(.,T_0)}{\partial t} \in C^\alpha(M,N)$ and (14)-(15) is satisfied on $[0,T_0]$. Applying Theorem 3.1 to (14)-(15) with initial condition $W(x,T_0)$ we get the existence of a solution

$$W \in C^{2+\alpha,1+\alpha/2}(M \times [T_0, T_0+\varepsilon], N) \tag{48}$$

of an $\varepsilon' = \varepsilon'(M, N, W(x, T_0), \alpha)$. Since $W\left(x, T_0^-\right) = W\left(x, T_0^+\right)$, we have thus obtained a solution

$$W \in C^{2+\alpha, 1+\alpha/2}(M \times [0, T_0 + \varepsilon'], N). \tag{49}$$

A bootstrap argument yields a solution in $M \times [0, \infty)$.

We embark now on the proof of the uniqueness of the solution. Let

$$W_1, W_2 \in C^{2+\alpha, 1+\alpha/2}(M \times [0, \infty), N) \tag{50}$$

be solutions of (14)-(15). Set $\Psi = W_1 - W_2$. We have

$$\frac{\partial \Psi}{\partial t} - \frac{\partial^2 \Psi}{\partial x^2} = \Phi(W_1, W_2), \tag{51}$$

with

$$\Phi(W_1, W_2) = \Gamma_{ab}^c\left(W_1, \frac{\partial W_1}{\partial x}\right) \frac{\partial W_1^a}{\partial x} \frac{\partial W_1^b}{\partial x} - \Gamma_{ab}^c\left(W_2, \frac{\partial W_2}{\partial x}\right) \frac{\partial W_2^a}{\partial x} \frac{\partial W_2^b}{\partial x}.$$

We write $\Phi(W_1, W_2)$ as

$$\begin{aligned} \Phi &= 2\left[\Gamma_{ab}^c\left(W_1, \frac{\partial W_1}{\partial x}\right) - \Gamma_{ab}^c\left(W_2, \frac{\partial W_2}{\partial x}\right)\right] \times \frac{\partial W_1^a}{\partial x} \frac{\partial W_1^b}{\partial x} \\ &\quad + \Gamma_{ab}^c\left(W_2, \frac{\partial W_2}{\partial x}\right) \frac{\partial \Psi^a}{\partial x} \frac{\partial W_1^b}{\partial x} + \Gamma_{ab}^c\left(W_2, \frac{\partial W_2}{\partial x}\right) \frac{\partial W_2^a}{\partial x} \frac{\partial \Psi^b}{\partial x}. \end{aligned}$$

We have

$$\begin{aligned} &\Gamma_{ab}^c\left(W_2, \frac{\partial W_2}{\partial x}\right) - \Gamma_{ab}^c\left(W_1, \frac{\partial W_1}{\partial x}\right) \\ = \; &\Gamma_{ab}^c\left(W_2, \frac{\partial W_2}{\partial x}\right) - \Gamma_{ab}^c\left(W_1, \frac{\partial W_2}{\partial x}\right) + \Gamma_{ab}^c\left(W_1, \frac{\partial W_2}{\partial x}\right) - \Gamma_{ab}^c\left(W_1, \frac{\partial W_1}{\partial x}\right). \end{aligned}$$

Applying the mean value theorem we get

$$\begin{aligned} &\Gamma_{ab}^c\left(W_2, \frac{\partial W_2}{\partial x}\right) - \Gamma_{ab}^c\left(W_1, \frac{\partial W_1}{\partial x}\right) \\ = \; &\frac{\partial \Gamma_{ab}^c}{\partial z^l}\left(\omega, \frac{\partial W_2}{\partial x}\right) \Psi^l + \frac{\partial \Gamma_{ab}^c}{\partial Z^l}(W_1, \Omega) \frac{\partial \Psi^l}{\partial x}, \end{aligned}$$

with $\omega, \ \Omega \in \mathbf{R}^n$. Thus thanks to (10) and Cauchy-Schwarz's inequality we get

$$\Phi \leq C\left(\left|\frac{\partial \Psi}{\partial x}\right|_{\mathbf{R}^n} + |\Psi|_{\mathbf{R}^n}\right). \tag{52}$$

Taking the inner product of (51) with Ψ, and integrating the resulting equation over $M \times [0, t]$ and making use of (52) and Young's inequality we get

$$\int_M |\Psi(x, t)|_{\mathbf{R}^n}^2 dx \leq C \int_0^t \int_M |\Psi(x, \tau)|_{\mathbf{R}^n}^2 dx d\tau, \quad \forall t > 0. \tag{53}$$

This implies that $\Psi \equiv 0$ on $[0, t]$, by Gronwall's inequality. The uniqueness is proved.

Remark 4.6. It can further be proved that the global solution

$$W \in C^\infty(M \times (0, \infty), N). \tag{54}$$

5. Proof of Corollary 2.4

From Lemma 4.5 and Theorem 2.3 $u(x,t)$ and $\partial u(x,t)/\partial t$ are uniformly bounded in $C^{2+\alpha}(M,N)$ and $C^{\alpha}(M,N)$ respectively for any $t>0$. Thus there exists a sequence $\{t_m\}\to\infty$ and a function $u_\infty \in C^{2+\alpha}(M,N)$ such that

$$u(x,t)\to u_\infty \tag{55}$$

uniformly.

We have

$$\frac{d}{dt}E(u(x,t)) = \int_M h_{ij}\frac{\partial}{\partial x}\left(\frac{\partial u^i}{\partial t}\right)\frac{\partial u^j}{\partial x}dx + \frac{1}{2}\int_M \frac{\partial h_{ij}}{\partial z^k}\frac{\partial u^k}{\partial t}\frac{\partial u^j}{\partial x}\frac{\partial u^i}{\partial x}dx;$$

the term containing $\partial h_{ij}/\partial Z^k$ vanished in account of (4). After integration by parts and using the relation $\partial h_{ij}/\partial z^k = h_{sj}\Gamma^s_{ik} + h_{is}\Gamma^s_{jk}$ together with the symmetry of h_{ij} we get

$$\frac{d}{dt}E(u(x,t)) = \int_M h_{ij}\frac{\partial u^i}{\partial t}\left(\frac{\partial^2 u^j}{\partial x^2} + \Gamma^j_{hk}\left(u,\frac{\partial u}{\partial x}\right)\frac{\partial u^h}{\partial x}\frac{\partial u^k}{\partial x}\right). \tag{56}$$

Thanks to (12), it follows that

$$\frac{d}{dt}E(u(x,t)) = -\int_M h_{ij}\frac{\partial u^i}{\partial t}\frac{\partial u^j}{\partial t}dx. \tag{57}$$

This shows that $E(u(x,t))$ is decreasing. Integrating this relation over $[\tau,T]$, we get

$$E(u(x,T)) - E(u(x,\tau)) = -\int_\tau^T\int_M h_{ij}\frac{\partial u^i}{\partial t}\frac{\partial u^j}{\partial t}dx. \tag{58}$$

As $\tau\to\infty$, $T\to\infty$ the left-hand side approaches zero. Therefore from the uniform boundedness of $\partial u/\partial t$ we get

$$\int_M h_{ij}\frac{\partial u^i}{\partial t}\frac{\partial u^j}{\partial t}dx \to 0 \quad \text{as } t\to\infty. \tag{59}$$

Thus by (9), we get that $\partial u(x,t_m)/\partial t\to 0$ as $t_m\to\infty$. Passing to the limit in

$$\begin{aligned} &\frac{\partial u^i(x,t_m)}{\partial t} - \frac{\partial^2 u^i(x,t_m)}{\partial x^2} \\ = \ &\Gamma^i_{hk}\left(u(x,t_m),\frac{\partial u(x,t_m)}{\partial x}\right)\frac{\partial u^h(x,t_m)}{\partial x}\frac{\partial u^k(x,t_m)}{\partial x}, \end{aligned}$$

and arguing as we did in the proof of the uniqueness of the solution we get that

$$\frac{\partial^2 u^i_\infty}{\partial x^2} = \Gamma^i_{hk}\left(u_\infty,\frac{\partial u_\infty}{\partial x}\right)\frac{\partial u^h_\infty}{\partial x}\frac{\partial u^k_\infty}{\partial x}. \tag{60}$$

Thus u_∞ is a closed geodesic.

Clearly $f(x) = u(0,x)$ and $u(x,t)$ are free homotopic in view of the continuity of u. Further $u(x,t)\to u_\infty(x)$ uniformly. Hence f and u_∞ are free homotopic.

References

[1] E. Acerbi, G. Buttazzo: On the limits of periodic Riemannian metrics, J. Anal. Math. 43 (1984) 183–201.

[2] R. A. Adams: Sobolev Spaces, Pure and Applied Mathematics 65, Academic Press, New York (1975).

[3] H. Akbar-Zadeh: Les espaces de Finsler et certaines de leurs généralisations, Ann. Sci. Éc. Norm. Supér., III. Sér. 80 (1963) 1–79.

[4] S. I. Alber: The topology of functional manifolds, and the calculus of variations in the large, Usp. Mat. Nauk 25(1970) 57–122 (in Russian); Engl. transl.: Russ. Math. Surv. 25(4) (1970) 51–117.

[5] D. V. Anosov: Geodesic flows that satisfy the U-condition (Y), in: Current Problems in Mathematics. Mathematical Analysis, Algebra, Topology, A. A. Logunov (ed.), Tr. Mat. Inst. Steklova 167, Nauka, Moscow (1985) 3–24 (in Russian); Engl. transl.: Proc. Steklov Inst. Math. 167 (1986) 1–22.

[6] M. Amar, E. Vitali: Homogenization of periodic Finsler metrics, J. Convex Analysis 5(1) (1998) 171–186.

[7] D. Bao, S.-S. Chern, Z. Shen: An Introduction to Riemann-Finsler Geometry, Graduate Texts in Mathematics 200, Springer, New York (2000).

[8] G. Bellettini, M. Paolini: Anisotropic motion by mean curvature in the context of Finsler geometry, Hokkaido Math. J. 25(3) (1996) 537–566.

[9] G. Bellettini, I. Fragalà: Elliptic approximations of prescribed mean curvature surfaces in Finsler geometry, Asymptotic Anal. 22(2) (2000) 87–111.

[10] A. Braides, G. Buttazzo, I. Fragalà: Riemannian approximation of Finsler metrics, Asymptotic Anal. 31(2) (2002) 177–187.

[11] A. Braides, A. Defranceschi: Homogenization of Multiple Integrals, Oxford Lecture Series in Mathematics and its Applications 12, Clarendon Press, Oxford (1998).

[12] E. Cartan: Les Espaces de Finsler, Hermann, Paris (1934).

[13] P. Centore: Finsler Laplacians and minimal-energy maps, Int. J. Math. 11(1) (2000) 1–13.

[14] J. Eells, Jr., J. H. Sampson: Harmonic mappings of Riemannian manifolds, Amer. J. Math. 86 (1964) 109–160.

[15] A. I. Fet: Variational problems on closed manifolds, Mat. Sb., Nov. Ser. 30(72) (1952) 271–316 (in Russian); Engl. transl.: Amer. Math. Soc. Translation 1953(90) (1953) 15–41.

[16] R. S. Hamilton: Harmonic Maps of Manifolds with Boundary, Lecture Notes in Mathematics 471, Springer, Berlin (1975).

[17] P. Hartman: On homotopic harmonic maps, Canad. J. Math. 19 (1967) 673–687.

[18] J. Jost: Nonpositive Curvature: Geometric and Analytic Aspects, Birkhäuser, Basel (1997).

[19] A. B. Katok: Ergodic perturbations of degenerate integrable Hamiltonian systems, Izv. Akad. Nauk SSSR, Ser. Mat. 37 (1973) 539–576 (in Russian); Engl. transl.: Math. USSR, Izv. 7(1973) (1974) 535–571.

[20] O. A. Ladyzhenskaya, N. N. Uraltseva: Linear and Quasilinear Elliptic Equations, Academic Press, New York (1968).

[21] O. A. Ladyzhenskaja, V. A. Solonnikov, N. N. Uraltseva: Linear and Quasilinear Equations of Parabolic Type, Translations of Mathematical Monographs 23, AMS, Providence (1967).

[22] S. Lang: Introduction to Differentiable Manifolds, Interscience, New York (1962).

[23] L. Kozma, A. Kristály, C. Varga: Critical point theorems on Finsler manifolds, Beitr. Algebra Geom. 45(1) (2004) 47–59.

[24] F. Mercuri: The critical points theory for the closed geodesics problem, Math. Z. 156 (1977) 231–245.

[25] Mo. Xiaohuan: Harmonic maps from Finsler manifolds, Ill. J. Math. 45(4) (2001) 1331–1345.

[26] S. Nishikawa: Variational Problems in Geometry, Translations of Mathematical Monographs 205, AMS, Providence (2002).

[27] S. Nishikawa: Harmonic maps in complex Finsler geometry, in: Variational Problems in Riemannian Geometry, P. Baird et al. (ed.), Progr. Nonlinear Differential Equations Appl. 59, Birkhäuser, Basel (2004) 113–132.

[28] T. Otsuki: On geodesic coordinates in Finsler spaces, Math. J. Okayama Univ. 6(2) (1957) 135–145.

[29] H. Rund: The Differential Geometry of Finsler Spaces, Springer, Berlin (1959).