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Calculus Rules for Maximal Monotone Operators in General Banach Spaces

M. D. Voisei
Department of Mathematics, Towson University, Towson, MD 21252, USA

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The goal of this article is to provide characterizations of monotonicity and maximality via new properties of the Fitzpatrick function associated with a multi-valued operator. Several calculus rules for maximal monotone operators in non-reflexive Banach space settings are presented. In particular positive answers to Rockafellar's conjecture on the maximality of the sum and the chain rule in the linear case are given.

Keywords: Maximal monotone operator, sum and chain rules

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1. Introduction

More than thirty years ago, Rockafellar [14, Theorem A, p. 210] proved that the subdifferential of a proper convex lower semicontinuous function is maximal monotone, followed by the following sum rule

Theorem ([15, Theorem 1, p. 76]). *Let X be reflexive, and let A and B be maximal monotone operators from X to X^*. Suppose that*

$$D(A) \cap \operatorname{int} D(B) \neq \emptyset. \tag{1}$$

Then $A + B$ is maximal monotone.

Since a sum formula for the subdifferential holds in a general Banach space under similar qualification constraints (see e.g. [19, Theorem 28.2]), and this could be seen as a maximality result for the sum of subdifferentials, a question was raised whether the previous theorem remains true without the reflexivity assumption on the Banach space X.

While the reflexivity of the Banach space remained unchanged, in time, the qualification constraint (1) evolved to the weaker forms

$$0 \in \operatorname{int}(D(A) - D(B)), \text{ Attouch [2]}, \tag{2}$$

$$\operatorname{co} D(A) - \operatorname{co} D(B) \text{ is absorbing, Penot [11]}, \tag{3}$$

$$\operatorname{co} D(A) - \operatorname{co} D(B) \text{ is a neighborhood of 0 in } \overline{\operatorname{lin}(D(A) - D(B))}, \tag{4}$$

Simons [17, 18], Chu [4], Attouch, Riahi & Théra [3], Zălinescu [27]. Here "co", "lin" stand for the convex, linear hull while the bar denotes the closure.

In the non-reflexive Banach space settings, the maximality of the sum problem continues to remain open. Progress has been made by Phelps & Simons [13] in the linear case, by

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Heisler for full-space domain operators (see e.g. [19]), and recently by Voisei [24, 25] for closed convex domain or linear operators.

Our aim in this article is to improve the best known results for basic sum and chain rules of maximal monotone operators.

Several approaches have been used in the attempt of solving this problem completely. We mention among others the saddle function approach of Krauss & Tiba [8], Krauss [9], Rockafellar [16], Gossez [6], the decomposition method of Asplund [1], and the study of monotonicity using tools of convex analysis (see e.g. Simons [19], Verona & Verona [22]).

A breakthrough in the study of maximal monotone operators was represented by the introduction of the Fitzpatrick function [5] and later, by its rediscovery in Martinez-Legaz & Théra [10]. Basically, the Fitzpatrick function characterizes maximal monotone operators in any Banach space possibly non-reflexive. Contrary to the general belief, the Fitzpatrick function is very useful for monotone operators which are not necessarily maximal and this is part of the goal of the present note. As a consequence, one of our main results, Proposition 3.2 contains more applicable characterizations of monotonicity and maximality in terms of the Fitzpatrick function and its convex conjugate.

The main difficulty in the study of monotone operators in a non-reflexive Banach space X is represented by the fact that $X \times X^*$ forms two distinct dual systems with $X^* \times X$ and $X^* \times X^{**}$. A monotone operator A in $X \times X^*$ can be seen via its inverse A^{-1} as monotone in $X^* \times X^{**}$, while the maximal monotonicity of A from $X \times X^*$ to $X^* \times X^{**}$ does not follow even if we consider closures.

This has led to several classification of maximal monotone operators in terms of their properties that resemble subdifferential features, such as the D-operators Gossez [7], locally maximal monotone and maximal monotone locally classes, and other classes (see e.g. Simons [19, 20, 21], Verona & Verona [23]). Subdifferentials and linear maximal monotone operators belong to all these classes.

Another difficulty is related to the fact that, for non-reflexive X, no topology on $X \times X^*$ compatible with the duality $(X \times X^*, X^* \times X)$ has good properties such as the lower semicontinuity of the dual product in $X \times X^*$. Some trials were attempted by Rockafellar [14], Gossez [7], Penot [12], Simons [20] in the introduction of a topology on $X \times X^*$ with better qualities.

It is worth mentioning that several useful properties such as the convexity of the closure of the domain of a maximal monotone operator are consequences of the maximality of the sum problem.

The plan of the paper is as follows. In Section 2 basic notions and notations are presented, while Section 3 deals with special properties of the Fitzpatrick function related to monotonicity and maximality. In Section 4 a chain and a sum rule for representable and maximal monotone operators are discussed together with some of their consequences.

This paper concludes with a result which gives a complete answer to Rockafellar's conjecture in the linear case.

2. Preliminaries

Let X be a real Banach space with topological dual X^* and bi-dual X^{**}. A multi-valued operator $A : X \rightrightarrows X^*$ is *monotone* if for every $x_1^* \in Ax_1$, $x_2^* \in Ax_2$,

$$\langle x_1 - x_2, x_1^* - x_2^* \rangle \geq 0. \tag{5}$$

Here

$$p(x, x^*) = \langle x, x^* \rangle = x^*(x), \ (x, x^*) \in X \times X^*,$$

denotes the dual product in $X \times X^*$.

Standard notations for a multi-valued operator A domain, range, and graph are

$$D(A) = \{x \in X;\ x^* \in Ax \text{ for some } x^* \in X^*\},$$

$$R(A) = \{x^* \in X^*;\ x^* \in Ax \text{ for some } x \in D(A)\},$$
$$\mathrm{Graph}A = \{(x, x^*) \in X \times X^*;\ x \in D(A),\ x^* \in Ax\}.$$

We identify operators with their graphs for the sake of notation simplicity. A monotone operator is called *maximal monotone* if it is maximal in the sense of inclusion in $X \times X^*$.

For a pair of linear spaces Z, W in duality, the convex conjugate of $f : Z \to \overline{\mathbb{R}}$ is defined by

$$f^*(w) = \sup\{w \cdot z - f(z);\ z \in Z\},\ w \in W, \tag{6}$$

where $w \cdot z$ denotes the dual product in $Z \times W$.

Let $Z = X \times X^*$. For $z = (x, x^*) \in Z$, $A \subset Z$, and $f : Z \to \bar{\mathbb{R}}$ define the transpose operations

$$z^T = (x^*, x) \in X^* \times X,$$
$$A^T = \{z^T;\ z \in A\} \subset X^* \times X,$$
$$f^T : Z^T \to \bar{\mathbb{R}},\ f^T(z^T) = f(z),\ z \in Z.$$

Similar transpose operations are considered in Z^T.

The set Z forms two possible different dual systems with Z^T and $Z^* = X^* \times X^{**}$. Correspondingly, for $f : Z \to \bar{\mathbb{R}}$ we have two convex conjugates $f^* : Z^* \to \bar{\mathbb{R}}$ given by (6) and $f^*_{/Z^T} : Z^T \to \bar{\mathbb{R}}$. In addition, we consider the transformation $f^\square : Z \to \bar{\mathbb{R}}$ given by

$$f^\square = (f^*_{/Z^T})^T, \tag{7}$$

or in expanded form

$$f^\square(x, x^*) = \sup\{\langle x, a^* \rangle + \langle a, x^* \rangle - f(a, a^*);\ (a, a^*) \in Z\}. \tag{8}$$

On Z we fix a topology compatible with the simple duality (Z, Z^T) such as the strong topology τ in X times the weakly star topology w^* in X^*.

A function $f : Z \to \mathbb{R} \cup \{\infty\}$ is called proper if f is not identically equal to $+\infty$ or equivalently its effective domain

$$D(f) = \{z \in Z;\ f(z) < \infty\}, \tag{9}$$

is non-empty.

We denote by $\Gamma(Z)$ the set of all proper convex functions $f : Z \to \mathbb{R} \cup \{\infty\}$ which are lower semicontinuous with respect to the established topology on Z. By the Biconjugate Theorem every proper convex lower semicontinuous function $f : Z \to \bar{\mathbb{R}}$ satisfies $f^{\Box\Box} = f$.

The following is a direct consequence of [26, Theorem 2.8.6 (v), p. 125].

Proposition 2.1. *If X, Y are Banach spaces, $g : Y \to \overline{\mathbb{R}}$ is proper convex lower semicontinuous, $C : X \rightrightarrows Y$ is a closed convex cone in $X \times Y$, and*

$$0 \in^{\mathrm{ic}} (D(g) - R(C)), \tag{10}$$

then the convex conjugate of $h : X \to \overline{\mathbb{R}}$ defined by

$$h(x) = \inf\{g(y);\ y \in Cx\},\ x \in X,$$

is given by

$$h^*(x^*) = \min\{g^*(y^*);\ x^* \in C^* y^*\},\ x^* \in X^*. \tag{11}$$

Moreover, if $x \in D(h) = C^{-1}(D(g))$, and $y \in Cx$ is such that $h(x) = g(y)$, then

$$\partial h(x) \subset C^* \partial g(y), \tag{12}$$

with equality when C is a linear subspace.

Here ${}^{\mathrm{ic}}S$ stands for the relative algebraic interior of S when the affine hull of S is closed, while ${}^{\mathrm{ic}}S$ is considered empty otherwise, "min" means that the infimum is attained when it is finite, and $C^* : Y^* \rightrightarrows X^*$ is the negative polar cone of C. By convention an infimum taken over an empty set is considered $+\infty$.

For other convex analysis notions used in this article we refer to the book of Zălinescu [26] and the references therein.

3. The Fitzpatrick function

Let $\emptyset \neq A \subset X \times X^*$. Define

$$p_A(z) = p(z),\ \text{if } z \in A,\ p_A(z) = +\infty,\ \text{otherwise}. \tag{13}$$

The Fitzpatrick function of A is given by $h_A = p_A^{\Box}$, or in extended form $h_A : X \times X^* \to \overline{\mathbb{R}}$,

$$h_A(x, x^*) = \sup\{\langle x, a^*\rangle + \langle a, x^*\rangle - \langle a, a^*\rangle;\ (a, a^*) \in A\},\ (x, x^*) \in X \times X^*. \tag{14}$$

The convex conjugate of the Fitzpatrick function, denoted by

$$\varphi_A = h_A^{\Box} = \operatorname{cl}\operatorname{co} p_A, \tag{15}$$

is the greatest convex lower semicontinuous function majorized by p_A.

For $f : Z \to \overline{\mathbb{R}}$ the following set notation will be frequently used

$$\{f \le p\} = \{z \in Z;\ f(z) \le p(z)\},$$
$$\{f = p\} = \{z \in Z;\ f(z) = p(z)\},$$
$$\{f \ge p\} = \{z \in Z;\ f(z) \ge p(z)\}.$$

Definition 3.1. $A \subset Z$ is called *representable* if there exists $h \in \Gamma(Z)$ such that $h \ge p$ in Z and $A = \{h = p\}$. A function h with these properties is called a *representative* of A. It is easily checked that every representative h of A satisfies

$$h_A \le h,\ h^\square \le \varphi_A \text{ in } Z. \tag{16}$$

and every representable operator is monotone (see e.g. [11]).

Consider the mapping,

$$E : \mathcal{P}(Z) \to \mathcal{P}(Z),\ E(A) := \{h_A \le p\}.$$

Here $\mathcal{P}(Z)$ stands for the class of subsets of Z. When A is monotone, $E(A)$ represents the union of all maximal monotone extensions of A.

Proposition 3.2.

(*i*) *For every* $A \subset X \times X^*$

$$A \subset D(A) \times X^* \cup X \times R(A) \subset \{h_A \ge p\}. \tag{17}$$

(*ii*) *A is monotone iff $A \subset \{h_A = p\}$.*

(*iii*) *A is maximal monotone iff $A = \{h_A = p\}$ and $h_A \ge p$ in Z.*

(*iv*) *A is maximal monotone iff $A = E(A)$.*

(*v*) *A is monotone iff $\varphi_A \ge p$ in Z.*

(*vi*) *If A is monotone then $A \subset \{\varphi_A = p\}$.*

(*vii*) *If A is monotone then*

$$p_A \ge \varphi_A \ge \max\{h_A, p\} \text{ in } Z. \tag{18}$$

(*viii*) *If A is monotone then*

$$A \subset \{\varphi_A = p\} \subset \{h_A = p\}. \tag{19}$$

(*ix*) *If A is monotone then*

$$\operatorname{co} D(A) \subset P_X D(h_A), \tag{20}$$

where $P_X : X \times X^* \to X$, $P_X(x, x^*) = x$, $(x, x^*) \in X \times X^*$ and "co" denotes the convex hull.

(*x*) *A is representable iff A is monotone and $A = \{\varphi_A = p\}$.*

(*xi*) *A is maximal monotone iff A is representable and $h_A \ge p$ in Z.*

(*xii*) *If A is maximal monotone and $D(A)$ is closed convex then*

$$P_X D(h_A) = D(A).$$

Proof. (*i*) Let $(x, x^*) \in D(A) \times X^*$. Take $a = x, a^* \in Aa$ in the definition of h_A to get

$$h_A(x, x^*) \ge \langle x - a, a^* \rangle + \langle a, x^* \rangle = \langle x, x^* \rangle, \tag{21}$$

that is $(x,x^*) \in \{h_A \geq p\}$. Hence $D(A) \times X^* \subset \{h_A \geq p\}$. We proceed similarly for $X \times R(A) \subset \{h_A \geq p\}$.

(ii) If A is monotone then for every $(x,x^*),(a,a^*) \in A$, $\langle x-a, x^*-a^*\rangle \geq 0$, that is $\langle x-a, a^*\rangle + \langle a, x^*\rangle \leq \langle x, x^*\rangle$. Pass to supremum over $(a,a^*) \in A$ to get $(x,x^*) \in \{h_A \leq p\}$, i.e, $A \subset \{h_A \leq p\}$. Combined with (i) this implies $A \subset \{h_A = p\}$ and $h_A \leq p_A$ in Z.

Conversely, if $A \subset \{h_A = p\}$ then every $(x,x^*) \in A$ satisfies

$$\inf_{(a,a^*)\in A} \langle x-a, x^*-a^*\rangle = (p-h_A)(x,x^*) = 0, \tag{22}$$

that is, A is monotone.

(iii)&(iv) If A is maximal monotone then for every $(x,x^*) \notin A$ there exists $(a,a^*) \in A$ such that $\langle x-a, x^*-a^*\rangle < 0$. This yields that $(x,x^*) \in \{h_A > p\}$ or $Z \setminus A \subset \{h_A > p\}$, or $E(A) = \{h_A \leq p\} \subset A$.

Taking (ii) into account we get $h_A \geq p$ in Z, $A = \{h_A = p\}$, and $E(A) = A$.

For the converse implications, if $h_A \geq p$ in Z and $A = \{h_A = p\}$ then $A = \{h_A \leq p\}$, A is monotone by (i) and (ii) and maximal since every (x,x^*) with $\langle x-a, x^*-a^*\rangle \geq 0$ for all $(a,a^*) \in A$ belongs to $\{h_A \leq p\} = A$.

(v) For every A monotone we have $h_A \leq \varphi_A$ in Z, because h_A is proper convex lower semicontinuous and $h_A \leq p_A$ in Z. Let $\tilde{A}$ be a maximal monotone extension of A. Since $A \subset \tilde{A}$ we have $p_A \geq p_{\tilde{A}}$ in Z and subsequently we find, from (iii), that

$$\varphi_A \geq \varphi_{\tilde{A}} \geq h_{\tilde{A}} \geq p \text{ in } Z. \tag{23}$$

Conversely in (v), if $\varphi_A \geq p$ in Z, then from $p_A \geq \varphi_A$ in Z, we get $A \subset \{\varphi_A = p\}$. But $\{\varphi_A = p\}$ is monotone because it is representable. The previous inclusion makes A monotone.

Implicitly we have proved (vi) and (vii).

For $(viii)$ it is sufficient to show $\{\varphi_A = p\} \subset \{h_A = p\}$. To this end let $z = (x,x^*) \in \{\varphi_A = p\}$. Then $z^T \in \partial\varphi_A(z)$, where the subdifferential "∂" is taken with respect to the dual system (Z, Z^T), because for arbitrary $v = (u,u^*) \in Z$, the directional derivative of φ_A at z in the direction of v satisfies

$$\begin{aligned} \varphi_A'(z;v) &= \lim_{t\downarrow 0} \frac{\varphi_A(z+tv)-\varphi_A(z)}{t} \\ &\geq \lim_{t\downarrow 0} \frac{p(z+tv)-p(z)}{t} = \langle x, u^*\rangle + \langle u, x^*\rangle = z^T \cdot v. \end{aligned} \tag{24}$$

Relation $z^T \in \partial\varphi_A(z)$ is equivalent to

$$h_A(z) + \varphi_A(z) = 2p(z), \tag{25}$$

and implies $h_A(z) = p(z)$ because $\varphi_A(z) = p(z)$, that is $\{\varphi_A = p\} \subset \{h_A = p\}$.

(ix) Let $x \in \operatorname{co} D(A)$, i.e., $x = \sum\limits_{i=1}^{n} \lambda_i a_i$, for some $\lambda_i \geq 0$, $(a_i, a_i^*) \in A$, $i = \overline{1,n}$ with $\sum\limits_{i=1}^{n} \lambda_i = 1$. Then $(x,x^*) \in D(h_A)$, where $x^* = \sum\limits_{i=1}^{n} \lambda_i a_i^*$.

Indeed, we have

$$h_A(x,x^*) = h_A(\sum_{i=1}^{n} \lambda_i(a_i,a_i^*)) \leq \sum_{i=1}^{n} \lambda_i h_A(a_i,a_i^*) = \sum_{i=1}^{n} \lambda_i p(a_i,a_i^*) < \infty. \tag{26}$$

Another argument could be performed as follows. Since $A \subset \{h_A = p\}$ we get $D(A) \subset P_X D(h_A)$ and $P_X D(h_A)$ is convex. Therefore, $\operatorname{co} D(A) \subset P_X D(h_A)$.

For the direct implication in (x) let A be representable, that is, $A = \{h = p\}$ for some $h \in \Gamma(Z)$ with $h \geq p$ in Z. Subsequently, A is monotone, $h \leq p_A$ in Z, and $p \leq h \leq \varphi_A$ in Z. Therefore $\{\varphi_A = p\} \subset \{h = p\} = A$. Taking (vi) into account, we get the desired set equality. The converse in (x) is straightforward because φ_A is a representative of A with all the required properties (see (v)).

The direct implication in (xi) is plain since, according to (iii), for a maximal monotone A we can take h_A as a representative. For the converse in (xi), if A is representable we know from (x) that A is monotone and $A = \{\varphi_A = p\}$.

According to (iii) and $(viii)$, it suffices to prove that $\{h_A = p\} \subset \{\varphi_A = p\}$. We repeat the argument in $(viii)$, namely, if $z = (x,x^*) \in \{h_A = p\}$ then $z^T \in \partial h_A(z)$, because for every $v = (u,u^*) \in Z$, the directional derivative of h_A at z in the direction of v satisfies

$$\begin{aligned} h'_A(z;v) &= \lim_{t\downarrow 0} \frac{h_A(z+tv) - h_A(z)}{t} \\ &\geq \lim_{t\downarrow 0} \frac{p(z+tv) - p(z)}{t} = \langle x,u^*\rangle + \langle u,x^*\rangle = z^T \cdot v. \end{aligned} \tag{27}$$

This is again equivalent to $h_A(z) + \varphi_A(z) = 2p(z)$, and implies $\varphi_A(z) = p(z)$ because $h_A(z) = p(z)$, i.e. $\{h_A = p\} \subset \{\varphi_A = p\}$.

(xii) For $K \subset X$, by $N_K = \partial i_K$ we understand the normal cone to K defined as the convex subdifferential of the indicator function of K, $i_K(x) = 0$, if $x \in K$, $i_K(x) = +\infty$, otherwise.

From (ix) we know $D(A) \subset P_X D(h_A)$. For the converse inclusion, let $x \in P_X D(h_A)$, i.e., there is $x^* \in X^*$, such that $(x,x^*) \in D(h_A)$. Hence

$$\langle x-a,a^*\rangle + \langle a,x^*\rangle \leq L < +\infty, \text{ for every } (a,a^*) \in A.$$

We have $A = A + N_{D(A)}$ because $N_{D(A)}$ is monotone, $0 \in N_{D(A)}(x)$, for every $x \in D(A)$, and A is maximal monotone. From this equality and previous inequality we get

$$\lambda\langle x-a,n^*\rangle + \langle x-a,a^*\rangle + \langle a,x^*\rangle \leq L < +\infty, \tag{28}$$

for every $\lambda > 0$, $a \in D(A)$, $a^* \in Aa$, $n^* \in N_{D(A)}(a)$, because $N_{D(A)}(a)$ is a cone for every $a \in D(A)$.

Relation (28) implies $\langle x-a,n^*\rangle \leq 0$, for every $a \in D(A)$, $n^* \in N_{D(A)}(a)$, i.e., $(x,0)$ is monotonically related to the graph of $N_{D(A)}$. Since $D(A)$ is closed convex, $i_{D(A)}$ is convex lower semicontinuous and $N_{D(A)}$ is maximal monotone. This yields that $(x,0) \in N_{D(A)}$, that is $x \in D(A)$. Therefore, $P_X D(h_A) \subset D(A)$. The proof is complete. □

4. Sum and chain rules

Theorem 4.1. *Let X, Y be two Banach spaces.*

(a) *If $L : X \to Y$ is linear bounded, $M : Y \rightrightarrows Y^*$ is representable, $D(M) \cap R(L) \neq \emptyset$, and*

$$0 \in^{\mathrm{ic}} (P_Y D(h_M) - R(L)), \tag{29}$$

then $T := L^* M L : X \rightrightarrows X^*$ is representable. Here h_M denotes the Fitzpatrick function of M, L^* denotes the adjoint of L, and $P_Y : Y \times Y^* \to Y$, $P_Y(y, y^*) = y$, $(y, y^*) \in Y \times Y^*$.

(b) *If $A, B : X \rightrightarrows X^*$ are representable with $D(A) \cap D(B) \neq \emptyset$ and*

$$0 \in^{\mathrm{ic}} (P_X D(h_A) - P_X D(h_B)), \tag{30}$$

then $A + B$ is representable. Here h_A, h_B stand for the Fitzpatrick functions of A, B.

Proof. (a) Consider $h : X \times X^* \to \overline{\mathbb{R}}$,

$$h(x, x^*) = \inf\{h_M(Lx, y^*);\ L^* y^* = x^*\},\ (x, x^*) \in X \times X^*. \tag{31}$$

Apply Proposition 2.1 for the Banach spaces $X \times X^*, Y \times Y^*$, $g = h_M$, and $C \subset X \times X^* \times Y \times Y^*$ given by

$$(x, x^*, y, y^*) \in C \text{ iff } y = Lx,\ x^* = L^* y^*. \tag{32}$$

We have $h(x, x^*) = \inf\{g(y, y^*);\ (y, y^*) \in C(x, x^*)\}$ and

$$(y^*, y^{**}, x^*, x^{**}) \in C^* \text{ iff } y^{**} = L^{**} x^{**},\ x^* = L^* y^*. \tag{33}$$

Since $0 \in^{\mathrm{ic}} (P_Y D(h_M) - R(L))$, $R(C) = R(L) \times Y^*$, and

$$\mathrm{aff}\ (D(h_M) - R(C)) = \mathrm{aff}\ (P_Y D(h_M) - R(L)) \times Y^*,$$

where "aff" denotes the affine hull, we know that $0 \in^{\mathrm{ic}} (D(h_M) - R(C))$.

According to Proposition 2.1 we get

$$h^*(x^*, x^{**}) = \min\{h_M^*(y^*, y^{**});\ (x^*, x^{**}) \in C^*(y^*, y^{**})\},\ (x^*, x^{**}) \in X^* \times X^{**}.$$

For $x^{**} = x \in X$ this reduces to

$$h^\square(x, x^*) = \min\{\varphi_M(Lx, y^*);\ L^* y^* = x^*\},\ (x, x^*) \in X \times X^*. \tag{34}$$

Since M is representable we know that M is monotone, $\varphi_M \geq p$ in $Y \times Y^*$, and $M = \{\varphi_M = p\}$. This yields $h^\square \geq p$ in $X \times X^*$.

If $(x, x^*) \in \{h^\square = p\}$ then there exists $y^* \in Y^*$ such that $L^* y^* = x^*$ and $\varphi_M(Lx, y^*) = p(x, x^*) = p(Lx, y^*)$, that is, $(Lx, y^*) \in \{\varphi_M = p\} = M$. Therefore $(x, x^*) \in T$ and this shows that $\{h^\square = p\} \subset T$. Conversely, if $(x, x^*) \in T$ then there exists $y^* \in Y^*$ such that $L^* y^* = x^*$ and $(Lx, y^*) \in M$. Hence $\varphi_M(Lx, y^*) = p(Lx, y^*)$, $h^\square(x, x^*) \leq p(Lx, y^*) = p(x, x^*)$, and $(x, x^*) \in \{h^\square = p\}$, i.e., $T \subset \{h^\square = p\}$.

We proved that $T = \{h^\square = p\}$ and $h^\square \geq p$ in $X \times X^*$.

Clearly, $h^\square$ is convex lower semicontinuous and a representative of T.

(b) Take $Y = X \times X$, $Y^* = X^* \times X^*$, $L : X \to Y$, $Lx = (x, x)$, $x \in X$, $L^* : Y^* \to X^*$, $L^*(x_1^*, x_2^*) = x_1^* + x_2^*$, $x_1^*, x_2^* \in X^*$, $M : Y \rightrightarrows Y^*$, $M(x_1, x_2) = Ax_1 \times Bx_2$, $(x_1, x_2) \in Y$. Then $L^*ML = A + B$.

Moreover,

$$h_M(x_1, x_2, x_1^*, x_2^*) = h_A(x_1, x_1^*) + h_B(x_2, x_2^*),\ x_1, x_2 \in X,\ x_1^*, x_2^* \in X^*, \tag{35}$$

$P_Y D(h_M) = P_X D(h_A) \times P_X D(h_B)$, and

$$F := \operatorname{aff}\,(P_Y D(h_M) - R(L)) = \operatorname{aff}\, P_X D(h_A) \times \operatorname{aff}\, P_X D(h_B) - R(L).$$

Consider $\mathcal{D} : X \times X \to X$,

$$\mathcal{D}(x_1, x_2) = x_1 - x_2,\ x_1, x_2 \in X. \tag{36}$$

We know that $\operatorname{Ker}\mathcal{D} = R(L)$ and

$$\mathcal{D}(F) = \operatorname{aff}\,(P_X D(h_A) - P_X D(h_B)) \text{ is closed.}$$

This is equivalent to $F = F + \operatorname{Ker}\mathcal{D}$ is closed (see e.g. [26, Corollary 1.3.15, p. 26]). From, $0 \in P_X D(h_A) - P_X D(h_B)$ we get $0 \in P_Y D(h_M) - R(L)$ and because $\mathbb{R}_+(P_Y D(h_M) - R(L)) = F$ is a closed subspace, we have $0 \in^{\mathrm{ic}} (P_Y D(h_M) - R(L))$ (see [26, (1.1), p. 3]). All the assumptions are verified and the conclusion follows from (a). The proof is complete. $\square$

Remark 4.2. The conclusions of Theorem 4.1 hold if we replace h_M in (29) by any function f_M whose conjugate $f_M^\square$ is a representative of M and h_A, h_B in (30) by any functions f_A, f_B whose conjugates $f_A^\square, f_B^\square$ are representatives of A, B. (We use f_M instead of h_M in the previous proof).

Corollary 4.3. *Let X, Y be two Banach spaces.*

(a) *If $L : X \to Y$ is linear bounded, $M : Y \rightrightarrows Y^*$ is representable, $D(M) \cap R(L) \neq \emptyset$ and*

$$\operatorname{co} D(M) - R(L) \textit{ is absorbing in } Y,$$

then $T := L^*ML : X \rightrightarrows X^*$ is representable.

(b) *If $A, B : X \rightrightarrows X^*$ are representable with $D(A) \cap D(B) \neq \emptyset$ and*

$$\operatorname{co} D(A) - \operatorname{co} D(B) \textit{ is absorbing in } X,$$

then $A + B$ is representable.

Proof. Since $\operatorname{co} D(M) \subset P_Y D(h_M)$ (see Proposition 3.2(ix)), the condition $\operatorname{co} D(M) - R(L)$ is absorbing in X implies $P_Y D(h_M) - R(L)$ absorbing and (29) follows . Similarly, from $\operatorname{co} D(A) \subset P_X D(h_A)$, $\operatorname{co} D(B) \subset P_X D(h_B)$ the condition $\operatorname{co} D(A) - \operatorname{co} D(B)$ is absorbing in X is stronger than (30) and can be used instead of (30). $\square$

Theorem 4.4. *If A, B are maximal monotone operators in a general Banach space X such that $D(A), D(B)$ are closed convex, and*

$$0 \in^{\mathrm{ic}} (D(A) - D(B)), \tag{37}$$

then $A + B$ is maximal monotone.

Proof. Under the given assumptions we know from Proposition 3.2(xii) and Theorem 4.1(b) that $A + B$ is representable. According to Proposition 3.2(xi) it suffices to show that $h_{A+B} \geq p$ in $X \times X^*$. We have,

$$A = A + \partial i_{D(A)}, \quad B = B + \partial i_{D(B)},$$

since A, B are maximal monotone $\partial i_{D(A)}, \partial i_{D(B)}$ are monotone and $0 \in \partial i_K(x)$, for every $K \subset X$, $x \in K$.

This implies

$$A + B = A + B + \partial i_{D(A)} + \partial i_{D(B)} = A + B + N_{D(A)\cap D(B)}, \tag{38}$$

since from (37) we have $\partial i_{D(A)} + \partial i_{D(B)} = \partial(i_{D(A)} + i_{D(B)}) = N_{D(A)\cap D(B)}$ (see [26, Theorem 2.8.7 (vii), p. 126]).

Let $(x, x^*) \in D(h_{A+B})$, that is

$$\sup\{\langle x - k, k^*\rangle + \langle k, x^*\rangle;\ (k, k^*) \in A + B\} \leq C < +\infty.$$

From (38) we obtain that, for every $k \in D(A) \cap D(B)$, $k^* \in (A+B)k$, $n^* \in N_{D(A)\cap D(B)}(k)$

$$\langle x - k, k^* + n^*\rangle + \langle k, x^*\rangle \leq C.$$

But $N_{D(A)\cap D(B)}(k)$ is a cone, i.e., for every $n^* \in N_{D(A)\cap D(B)}(k)$, $\lambda > 0$, $\lambda n^* \in N_{D(A)\cap D(B)}(k)$. We find

$$\lambda\langle x - k, n^*\rangle + \langle x - k, k^*\rangle + \langle k, x^*\rangle \leq C < +\infty,$$

for every $\lambda > 0$, $k \in D(A) \cap D(B)$, $k^* \in (A + B)k$, $n^* \in N_{D(A)\cap D(B)}(k)$. Therefore

$$\langle x - k, n^*\rangle \leq 0, \tag{39}$$

for every $k \in D(A) \cap D(B)$, $n^* \in N_{D(A)\cap D(B)}(k)$.

Relation (39) shows that $(x, 0)$ is monotonically related to the graph of $N_{D(A)\cap D(B)}$ and since $N_{D(A)\cap D(B)}$ is maximal monotone we get that $(x, 0) \in N_{D(A)\cap D(B)}$, that is $x \in D(A) \cap D(B)$. According to Proposition 3.2(i) we conclude that $(x, x^*) \in \{h_{A+B} \geq p\}$ or in other words $D(h_{A+B}) \subset \{h_{A+B} \geq p\}$. This proves that $h_{A+B} \geq p$ in $X \times X^*$. The proof is complete. □

Corollary 4.5. *If A, B are maximal monotone operators in a general Banach space X such that $D(A), D(B)$ are closed convex, and*

$$D(A) - D(B) \textit{ is absorbing in } X,$$

then $A + B$ is maximal monotone.

Theorem 4.6. *Let X, Y be two Banach spaces. If $L : X \to Y$ is linear bounded, $M : Y \rightrightarrows Y^*$ is maximal monotone with $D(M)$ closed convex and*

$$0 \in^{\mathrm{ic}} (D(M) - R(L)), \tag{40}$$

then $T := L^*ML : X \rightrightarrows X^*$ is maximal monotone.

Proof. From Proposition 3.2(xii) and Theorem 4.1(a) we get that T is representable. Therefore, it is sufficient to show that $h_T \geq p$ in $X \times X^*$. We know

$$D(T) = \{x \in X;\ Lx \in D(M)\}$$

is closed convex because so is $D(M)$ and L is linear bounded. We write

$$i_{D(T)}(x) = \inf\{i_{D(M)}(y);\ y = Lx\}, \tag{41}$$

and apply the second part in Proposition 2.1 to get, for every $x \in D(T)$

$$N_{D(T)}(x) = L^* N_{D(M)}(Lx). \tag{42}$$

From $M = M + N_{D(M)}$ we find $M(Lx) = M(Lx) + N_{D(M)}(Lx)$ and

$$Tx = Tx + N_{D(T)}(x), \tag{43}$$

for every $x \in D(T)$. As previously seen, (43) implies $h_T \geq p$ in $X \times X^*$ and the proof is complete. □

Corollary 4.7. *If X, Y are two Banach spaces, $L : X \to Y$ is linear bounded, $M : Y \rightrightarrows Y^*$ is maximal monotone with $D(M)$ closed convex and*

$$D(M) - R(L) \text{ is absorbing in } Y,$$

then $T := L^*ML : X \rightrightarrows X^*$ is maximal monotone.

We conclude this section with a surprising result which proves that, in the linear case, Rockafellar's conjecture holds.

Theorem 4.8. *Let X, Y be two Banach spaces.*

(a) *If $L : X \to Y$ is linear bounded, $M : Y \rightrightarrows Y^*$ is linear maximal monotone, and $D(M) - R(L)$ is a closed subspace of Y, then $T := L^*ML$ is maximal monotone.*

(b) *If A, B are linear maximal monotone operators in X such that $D(A) - D(B)$ is a closed subspace of X, then $A + B$ is maximal monotone.*

Proof. (a) Notice that for every $(x, x^*) \in X \times X^*$

$$\begin{aligned} p_T(x, x^*) &= \inf\{p_M(Lx, y^*);\ L^*y^* = x^*\} \\ &= \inf\{p_M(y, y^*);\ (y, y^*) \in C(x, x^*)\}, \end{aligned} \tag{44}$$

where C is defined in (32).

We verify the conditions of the chain rule contained in Proposition 2.1. First, p_M is convex lower semicontinuous, since M is linear monotone and strongly closed, while condition

$D(M)-R(L)$ closed is equivalent to $M-R(C)=(D(M)-R(L))\times Y^*$ closed. Because we are dealing with subspaces this yields $0\in^{\text{ic}}(D(p_M)-R(C))$. Therefore, by the chain rule, we find

$$h_T(x,x^*)=p_T^{\square}(x,x^*)=\min\{h_M(Lx,y^*);\ L^*y^*=x^*\}. \tag{45}$$

This is sufficient in order to conclude that T is maximal monotone.

Subpoint (b) follows from (a) for $Y=X\times X$, $Y^*=X^*\times X^*$, $L:X\to Y$, $Lx=(x,x)$, $x\in X$, $L^*:Y^*\to X^*$, $L^*(x_1^*,x_2^*)=x_1^*+x_2^*$, $x_1^*,x_2^*\in X^*$, $M:Y\rightrightarrows Y^*$, $M(x_1,x_2)=Ax_1\times Bx_2$, $(x_1,x_2)\in D(M)=D(A)\times D(B)$, $L^*ML=A+B$ while $\mathcal{D}(D(M))=D(A)-D(B)$ closed is equivalent to $D(M)-R(L)$ is closed, where $\mathcal{D}$ is defined in (36). For an alternative proof of (b) see [25]. □

Remark 4.9. In the linear case, conditions (40), (37) are sharp in the sense that there is a counter-example with A,B linear maximal monotone such that $D(A)-D(B)$ is dense and $A+B$ is not maximal monotone (see e.g. [19, Problem 34.2] or [13, Remark 7.3]).

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