Journal of Convex Analysis
Volume 15 (2008), No. 1, 55–62

The Weighted Fermat-Torricelli Problem and an "Inverse" Problem

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Received: May 23, 2006
Revised manuscript received: December 1, 2006

We study the weighted Fermat-Torricelli point of spherical, hyperbolic and plane triangles and an inverse weighted Fermat-Torricelli problem. We show that a fundamental application of the inverse weighted Fermat-Torricelli problem is the invariance property of the weighted Fermat-Torricelli point.

Keywords: Fermat-Torricelli point, inverse Fermat-Torricelli problem

1991 Mathematics Subject Classification: 51E10, 52A55

1. Introduction

The Fermat-Torricelli problem is to find the (unique) point that minimizes the sum of distances from three given points in $\mathbb{R}^2$. P. de Fermat (1601-1665) posed this problem to E. Torricelli (1608-1647) who solved it, and his student V. Viviani published the solution in 1659 (see [3] and Chapter II from [2]). R. Courant and H. Robbins have called this problem "Steiner problem", since J. Steiner solved it independently from Torricelli and Viviani in a more elegant and systematic way (see [6] and [8]). Following Y. S. Kupitz and H. Martini [9] (see also Chapter II of [2]), we will call this point the *Fermat-Torricelli point* of the given points, due to the first contributions of Fermat and Torricelli. The weighted Fermat-Torricelli point of a plane triangle and the inverse weighted Fermat-Torricelli problem in the plane have been studied in [7], where also a detailed historical exposition of the subject is given.
In this paper, a direct method is described to find the weighted Fermat-Torricelli point of a given spherical or hyperbolic triangle $\nabla A_1A_2A_3$ with non-negative weights B_i that correspond to each vertex A_i, respectively. Concerning the floating and absorbed case (see below) of the weighted Fermat-Torricelli point in $\mathbb{R}^N$, see [9] and Chapter II of [2]. For the solution of the Fermat-Torricelli problem concerning spherical triangles with unit weights we refer to [4] and [5]. For the solution of the problem in any regular surface of $\mathbb{R}^3$ we refer to the indirect method used in [11] by studying the problem in the tangent plane of the regular surface at the Fermat-Torricelli point. We also show the invariance property of the weighted Fermat-Torricelli point for a given spherical, hyperbolic and planar (Euclidean) triangle with non-negative weights that are given at the vertices. This property is derived by a fundamental condition which is obtained by the inverse weighted

ISSN 0944-6532 / $ 2.50

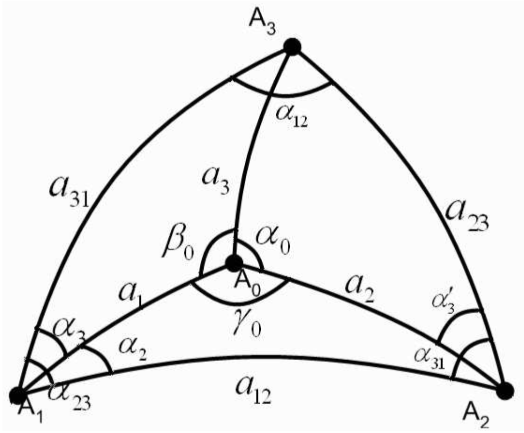

Figure 2.1.

Fermat-Torricelli problem in the two-dimensional sphere, hyperboloid and in $\mathbb{R}^2$. The inverse weighted Fermat-Torricelli problem in $\mathbb{R}^2$ was also studied in [7].

2. The weighted Fermat-Torricelli point

We start by stating the problem for the two-dimensional unit sphere S^2. For background material from spherical trigonometry we refer to [1], and the following notations are used in our paper: By a_i we denote the length of the geodesic A_0A_i, by a_{ij} the length of the geodesic A_iA_j, and by α_{ij} the spherical angle between the two geodesics A_iA_k, A_kA_j for $i,j,k = 1,2,3$, $i \neq j \neq k \neq i$. Moreover, a_i is seen on the left by the spherical angle α_i, and it is seen on the right by the spherical angle α_i'. Furthermore, let α_0 be the spherical angle between A_2A_0 and A_3A_0, β_0 be the spherical angle between A_1A_0 and A_3A_0, and γ_0 be the spherical angle between A_1A_0 and A_2A_0 (see Figure 2.1).

Problem 2.1. *Let $\nabla A_1A_2A_3$ be a spherical triangle. Suppose that a weight $B_i \in \mathbb{R}^+$ corresponds to each vertex A_i for $i = 1,2,3$, respectively. Find the weighted Fermat-Torricelli point A_0 of the spherical triangle $\nabla A_1A_2A_3$ which minimizes the sum of the lengths of the geodesics that connect every vertex with A_0 multiplied by the positive weight B_i:*

$$B_1a_1 + B_2a_2 + B_3a_3 = minimum. \tag{1}$$

Theorem 2.2. *The weighted Fermat-Torricelli point A_0 of the spherical triangle $\nabla A_1A_2A_3$ exists and is unique.*

- (i) *If $|B_i - B_j| < B_k < B_i + B_j$ for $i,j,k = 1,2,3$, then the weighted Fermat-Torricelli point is an interior point of the spherical triangle $\nabla A_1A_2A_3$ (Floating Case).*
- (ii) *If there is some i with $B_i \geq B_j + B_k$ for $i,j,k = 1,2,3$, then the weighted Fermat-Torricelli point is the vertex A_i (Absorbed Case).*

It is possible to show the existence and uniqueness of the Fermat-Torricelli point by reduction ad absurdum. We start with the description of the exponential map in [10]. Let $\mathfrak{D}$ be the set of vectors v that belong to the tangent space $\mathrm{T}(S^2)$ or $\mathrm{T}(H^2)$ such that 1 lies in the domain of β_v. The generalized inverse stereographic projection is the exponential map

$$\exp : \mathfrak{D} \rightarrow X$$

or

$$\exp(v) = \pi\beta_v(1).$$

If x∈ X (S^2 or H^2) and 0_x is the zero vector in T_x, then

$$\exp(0_x) = x.$$

Write $\exp_x$ for the restriction of exp to the tangent space T_x.

Theorem 2.3. *Let X be a manifold and ξ a spray on X. Then*

$$\exp_x : T_x \rightarrow X$$

induces a local isomorphism at 0_x, and

$$(\exp_x)_\star(0_x) = id.$$

Proof. See [10]. □

The generalized inverse stereographic projection exp: $\mathfrak{D} \rightarrow S^2$, or H^2, or any regular surface X, which is a conformal mapping, transfers the existence and uniqueness result of Kupitz-Martini in $\mathbb{R}^d$ (see Theorem 1.1 (I), pp. 58–61 from [9]) to the differentiable manifold S^2 or H^2, or to any differential submanifold of $\mathbb{R}^d$. For the case of S^2 we have the classical inverse stereographic projection from $\mathbb{R}^2$ to S^2. If the Fermat-Torricelli point does not exist for a spherical triangle, then the stereographic projection will not give any point for the corresponding plane triangle in $\mathbb{R}^2$. This is not true due to [9], since otherwise the Fermat-Torricelli point for plane triangles would not exist (Existence). If there were two Fermat-Torricelli points, the 1-1 correspondence of the stereographic projection would yield such points for the plane triangle, contradicting the uniqueness result of [9] (Uniqueness).

Proof of (*i*) of Theorem 2.2. The variables a_2, a_3 can be expressed as functions of a_1 and α_3:

$$a_2 = a_2(a_1, \alpha_3), \qquad a_3 = a_3(a_1, \alpha_3). \tag{2}$$

From (2) and (1) the following equation is obtained:

$$B_1 a_1 + B_2 a_2(a_1, \alpha_3) + B_3 a_3(a_1, \alpha_3) = \text{minimum}. \tag{3}$$

The “cosine law" regarding the spherical triangle $\nabla A_0 A_1 A_3$ is given by

$$\cos(a_3) = \cos(a_1)\cos(a_{31}) + \sin(a_1)\sin(a_{31})\cos(\alpha_3). \tag{4}$$

Similarly, the “cosine law" concerning the spherical triangle $\nabla A_0 A_1 A_2$ is given by

$$\cos(a_2) = \cos(a_1)\cos(a_{21}) + \sin(a_1)\sin(a_{21})\cos(\alpha_{23} - \alpha_3). \tag{5}$$

By differentiation of (3) with respect to the variables a_1 and α_3 we get

$$B_2 \frac{\partial a_2}{\partial a_1} + B_3 \frac{\partial a_3}{\partial a_1} = -B_1, \tag{6}$$

$$B_2\frac{\partial a_2}{\partial \alpha_3}+B_3\frac{\partial a_3}{\partial \alpha_3}=0. \tag{7}$$

Differentiating (4) and (5) with respect to α_3, we can replace $\frac{\partial a_2}{\partial \alpha_3}$, $\frac{\partial a_3}{\partial \alpha_3}$ in (7):

$$\frac{B_2}{B_3}=\frac{\sin(a_{31})\sin(\alpha_3)\sin(a_2)}{\sin(a_{12})\sin(\alpha_2)\sin(a_3)}. \tag{8}$$

Also one can consider the "sine law" for the spherical triangles $\nabla A_0A_1A_3$, $\nabla A_0A_1A_2$:

$$\frac{\sin(\alpha_3)}{\sin(a_3)}=\frac{\sin(\beta_0)}{\sin(a_{31})}, \tag{9}$$

$$\frac{\sin(\alpha_2)}{\sin(a_2)}=\frac{\sin(\gamma_0)}{\sin(a_{12})}. \tag{10}$$

The next equation is derived by combining equations (9) and (10) with (8):

$$\frac{B_2}{\sin(\beta_0)}=\frac{B_3}{\sin(\gamma_0)}=c. \tag{11}$$

Let a_1 and a_3 be expressed as functions of a_2 and α_3' (see Figure 2.1):

$$a_1=a_1(a_2,\alpha_3'), \qquad a_3=a_3(a_2,\alpha_3'). \tag{12}$$

From (12) and (1) the following equation is obtained:

$$B_1a_1(a_2,\alpha_3')+B_2a_2+B_3a_3(a_2,\alpha_3')=\text{minimum}. \tag{13}$$

Differentiating (13) with respect to the variable α_3', we deduce that

$$B_1\frac{\partial a_1}{\partial \alpha_3'}+B_3\frac{\partial a_3}{\partial \alpha_3'}=0. \tag{14}$$

The same procedure as described for (7) can be applied to the spherical triangles $\nabla A_0A_1A_2$ and $\nabla A_0A_2A_3$, and so we obtain the relation

$$\frac{B_1}{\sin(\alpha_0)}=\frac{B_3}{\sin(\gamma_0)}=c. \tag{15}$$

From (15) and (11) we get

$$\frac{B_1}{\sin(\alpha_0)}=\frac{B_2}{\sin(\beta_0)}=\frac{B_3}{\sin(\gamma_0)}=c. \tag{16}$$

The relationship between the spherical angles α_0, β_0, γ_0 (see Figure 2.1) can be used in order to clarify the value of c:

$$\alpha_0+\beta_0+\gamma_0=2\pi, \tag{17}$$

$$\sin(\alpha_0)=-\sin(\beta_0+\gamma_0). \tag{18}$$

From (16) and (18) we obtain that c depends only on B_1, B_2, B_3:

$$c = \frac{2B_1B_2B_3}{\sqrt{(B_1+B_2+B_3)(B_1+B_2-B_3)(B_2+B_3-B_1)(B_1+B_3-B_2)}}. \tag{19}$$

(16) and (19) give three important formulas:

$$\cos(\alpha_0) = \frac{B_1^2 - B_2^2 - B_3^2}{2B_2B_3}, \tag{20}$$

$$\cos(\beta_0) = \frac{B_2^2 - B_1^2 - B_3^2}{2B_1B_3}, \tag{21}$$

$$\cos(\gamma_0) = \frac{B_3^2 - B_1^2 - B_2^2}{2B_1B_2}. \tag{22}$$

From (20), (21), (22), the desired inequalities are obtained, namely

$$|B_i - B_j| < B_k < B_i + B_j$$

for $i, j, k = 1, 2, 3$, $i \neq j \neq k$. Three additional inequalities are needed to obtain A_0 inside $A_1A_2A_3$:

$$\alpha_0 > \alpha_{23}, \qquad \beta_0 > \alpha_{31}, \qquad \gamma_0 > \alpha_{12}$$

or

$$\frac{B_1^2 - B_2^2 - B_3^2}{2B_2B_3} < \frac{\cos(a_{23}) - \cos(a_{31})\cos(a_{12})}{\sin(a_{31})\sin(a_{12})},$$
$$\frac{B_2^2 - B_1^2 - B_3^2}{2B_1B_3} < \frac{\cos(a_{31}) - \cos(a_{23})\cos(a_{12})}{\sin(a_{23})\sin(a_{12})},$$
$$\frac{B_3^2 - B_1^2 - B_2^2}{2B_1B_2} < \frac{\cos(a_{12}) - \cos(a_{31})\cos(a_{23})}{\sin(a_{31})\sin(a_{23})}.$$

A_0 is the intersection point of α_0 and β_0 of $\nabla A_1A_2A_3$. This gives the geometrical construction of A_0. Another approach to construct A_0 is by calculating the angles α_3 and α_3'. The solutions of the two derived implicit equations referring to α_3 and α_3' , respectively, are unique. □

Proof of *(ii)* of Theorem 2.2. Suppose that $B_1 \geq B_2 + B_3$. Then

$$\begin{aligned} B_1a_1 + B_2a_2 + B_3a_3 &\geq (B_2+B_3)a_1 + B_2a_2 + B_3a_3 \\ &= B_2(a_1+a_2) + B_3(a_1+a_3) \geq B_2a_{12} + B_3a_{13}. \end{aligned}$$

By using the triangle inequality, we deduce that the minimum point is attained at the vertex A_1. □

Remark 2.4. The equation (20) can also be obtained by combining the two equations (6) and (7):

$$B_1 + B_2\cos(\gamma_0) + B_3\cos(\beta_0) = 0,$$
$$B_2\sin(\gamma_0) - B_3\sin(\beta_0) = 0$$

or, equivalently,

$$B_1 + B_2e^{i\gamma_0} + B_3e^{-i\beta_0} = 0.$$

The same process can be applied to (21) and (22).

Corollary 2.5. *Given the weights $B_1 = B_2 = B_3$, we have $\alpha_0 = \beta_0 = \gamma_0 = 120°$.*

Proposition 2.6. *Theorem 2.2 and Corollary 2.5 are also valid for*

(*a*) *a hyperbolic triangle $\nabla A_1A_2A_3$,*
(*b*) *a triangle $\nabla A_1A_2A_3$ in $\mathbb{R}^2$.*

Proof of (*a*). Similar identities are used for the hyperbolic triangle $\nabla A_1A_2A_3$ referring to (4) and (5):

$$\cosh(a_3) = \cosh(a_1)\cosh(a_{31}) - \sinh(a_1)\sinh(a_{31})\cos(\alpha_3), \tag{23}$$

$$\cosh(a_2) = \cosh(a_1)\cosh(a_{21}) - \sinh(a_1)\sinh(a_{21})\cos(\alpha_{23} - \alpha_3). \tag{24}$$

As analogue of the "sine law" for spherical triangles there is a "sine law" for the hyperbolic triangles $\nabla A_0A_1A_3$ and $\nabla A_0A_1A_2$, respectively:

$$\frac{\sin(\alpha_3)}{\sinh(a_3)} = \frac{\sin(\beta_0)}{\sinh(a_{31})}, \tag{25}$$

$$\frac{\sin(\alpha_2)}{\sinh(a_2)} = \frac{\sin(\gamma_0)}{\sinh(a_{12})}. \tag{26}$$

Differentiating (23) and (24) with respect to the hyperbolic angle α_3, the partial derivatives $\frac{\partial a_2}{\partial \alpha_3}$, $\frac{\partial a_3}{\partial \alpha_3}$ are replaced in (7) by taking into account (25) and (26):

$$\frac{B_2}{\sin(\beta_0)} = \frac{B_3}{\sin(\gamma_0)} = c. \tag{27}$$

The same can be done with the spherical triangle, and the same result is derived for the hyperbolic angles α_0, β_0 and γ_0. Thus (20), (21) and (22) are obtained. □

Proof of (*b*). Here our analytical approach uses differentiation with respect to the variable of the distance a_1 of the plane triangle $\nabla A_1A_2A_3$. We use the same symbols for the variables as in the case of spherical triangles. We start with the "cosine law" that is valid for the triangles $\nabla A_0A_1A_2$, and $\nabla A_0A_1A_3$, respectively:

$$a_2^2 = a_1^2 + a_{21}^2 - 2a_1a_{21}\cos(\alpha_{23} - \alpha_3), \tag{28}$$

$$a_3^2 = a_1^2 + a_{31}^2 - 2a_1a_{31}\cos(\alpha_3). \tag{29}$$

Furthermore, we apply the "sine law" to the triangles $\nabla A_0A_1A_2$ and $\nabla A_0A_1A_3$, respectively:

$$\frac{a_1}{a_2} = \frac{\sin(\alpha_{12} + \alpha_{23} - \alpha_3)}{\sin(\alpha_{23} - \alpha_3)}, \frac{a_{12}}{a_2} = \frac{\sin(\alpha_{12})}{\sin(\alpha_{23} - \alpha_3)}, \tag{30}$$

$$\frac{a_1}{a_3} = \frac{\sin(\alpha_{13} + \alpha_3)}{\sin(\alpha_3)}, \frac{a_{13}}{a_3} = \frac{\sin(\alpha_{13})}{\sin(\alpha_3)}. \tag{31}$$

We differentiate (28) and (29) with respect to the distance a_1, and we replace the new relations by combining (30) and (6). We square both parts of the derived equation and both parts of the equation (7) by applying (30) and (31). We add both of them, in order to obtain (20).
Similarly, by differentiating (1) with respect to a_2 and a_3 we get the relations (21) and (22). □

Remark 2.7. The relations (20), (21) and (22) are also obtained, since we derive (16) by differentiating (28) and (29) with respect to the angle α_3 using the same procedure for a spherical and hyperbolic triangle.

3. The inverse weighted Fermat-Torricelli problem

Problem 3.1. *Given the weighted Fermat-Torricelli point A_0 of the weighted spherical or hyperbolic triangle $\nabla A_1A_2A_3$ and the angles α_0, β_0, γ_0, find the ratios between the non-negative weights $\frac{B_i}{B_j}$, $i, j = 1, 2, 3$, such that*

$$B_1 + B_2 + B_3 = \textit{constant}.$$

This is the inverse weighted Fermat-Torricelli problem in the two-dimensional sphere S^2 or two-dimensional hyperboloid H^2.

The generalized inverse weighted Fermat-Torricelli problem in $\mathbb{R}^2$ is studied in [12] for $n > 3$ and in [7] for $n = 3$.

Proposition 3.2. *Given the angles α_0, β_0, γ_0 and the weighted Fermat-Torricelli point in S^2 or H^2 or $\mathbb{R}^2$, the ratio of the three weights B_1, B_2, B_3 is given by*

$$B_1 : B_2 : B_3 = \sin(\alpha_0) : \sin(\beta_0) : \sin(\gamma_0).$$

Proof. The ratio $B_1 : B_2 : B_3$ is obtained from (16). This equation also holds in H^2 and $\mathbb{R}^2$. □

Corollary 3.3. *Concerning the spherical, hyperbolic and plane triangle $\nabla A_1A_2A_3$, there are three common equations in a complex form (Remark 2.4) that provide the location of the weighted Fermat-Torricelli point:*

$$B_1 + B_2e^{i\gamma_0} + B_3e^{-i\beta_0} = 0,$$

$$B_1e^{i\gamma_0} + B_2 + B_3e^{-i\alpha_0} = 0,$$

$$B_1e^{i\beta_0} + B_2e^{-i\alpha_0} + B_3 = 0.$$

Corollary 3.4. *The inverse weighted Fermat-Torricelli problem gives the same ratio of the weights $B_1 : B_2 : B_3$ for a spherical, hyperbolic or plane triangle.*

Proposition 3.5. *The weighted Fermat-Torricelli point of a spherical, hyperbolic or plane triangle $\nabla A_1A_2A_3$ remains the same for any spherical, hyperbolic or plane triangle $\nabla A_1'A_2'A_3'$ if the floating case (Theorem 2.2) occurs for constant values of B_i that correspond to any vertex $A_i, i = 1, 2, 3$.*

Proof. The result follows from the fundamental condition

$$B_1 : B_2 : B_3 = \sin(\alpha_0) : \sin(\beta_0) : \sin(\gamma_0)$$

(see Proposition 3.2). □

Corollary 3.6. *The weighted Fermat-Torricelli point of a spherical, hyperbolic or plane triangle $\nabla A_1A_2A_3$ remains the same for any spherical, hyperbolic or plane triangle $\nabla A_1'A_2'A_3'$ with vertices on geodesic cycles that are defined by the segments of the geodesics A_0A_1, A_0A_2, A_0A_3.*

Proof. This corollary is a direct consequence of Proposition 3.2. □

In conclusion, we would like to mention that the notion of invariance which possesses the weighted Fermat-Torricelli point for given three points in the plane, two-dimensional sphere and two-dimensional hyperboloid, is stronger than the notion of similarity. This means that the invariance of the weighted Fermat-Torricelli point holds for similar and non-similar triangles. This fundamental result occurs because the angles $\alpha_0, \beta_0, \gamma_0$ depend only on the values of the weights B_1, B_2, B_3 and not on the side lengths of the triangles (see (20), (21), and (22)). Among known points (so-called centers) of Euclidean, spherical or hyperbolic triangles the weighted Fermat-Torricelli point is the only point that is connected with this invariance property (Proposition 3.5, Corollary 3.6). The invariance of the weighted Fermat-Torricelli point in Proposition 3.5 (strong invariance) is stronger than the invariance of the weighted Fermat-Torricelli point in Corollary 3.6 (weak invariance).

Acknowledgements. The first author acknowledges useful discussions with Lecturer Andreas Arvanitoyeorgos and Professor Athanasios Cotsiolis. Both authors are grateful to the referee for valuable comments.

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