The Weighted Fermat-Torricelli Problem and an “Inverse” Problem

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We study the weighted Fermat-Torricelli point of spherical, hyperbolic and plane triangles and an inverse weighted Fermat-Torricelli problem. We show that a fundamental application of the inverse weighted Fermat-Torricelli problem is the invariance property of the weighted Fermat-Torricelli point.

Keywords: Fermat-Torricelli point, inverse Fermat-Torricelli problem

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1. Introduction

The Fermat-Torricelli problem is to find the (unique) point that minimizes the sum of distances from three given points in $\mathbb{R}^2$. P. de Fermat (1601-1665) posed this problem to E. Torricelli (1608-1647) who solved it, and his student V. Viviani published the solution in 1659 (see [3] and Chapter II from [2]). R. Courant and H. Robbins have called this problem "Steiner problem", since J. Steiner solved it independently from Torricelli and Viviani in a more elegant and systematic way (see [6] and [8]). Following Y. S. Kupitz and H. Martini [9] (see also Chapter II of [2]), we will call this point the Fermat-Torricelli point of the given points, due to the first contributions of Fermat and Torricelli. The weighted Fermat-Torricelli point of a plane triangle and the inverse weighted Fermat-Torricelli problem in the plane have been studied in [7], where also a detailed historical exposition of the subject is given.

In this paper, a direct method is described to find the weighted Fermat-Torricelli point of a given spherical or hyperbolic triangle $\nabla A_1A_2A_3$ with non-negative weights $B_i$ that correspond to each vertex $A_i$, respectively. Concerning the floating and absorbed case (see below) of the weighted Fermat-Torricelli point in $\mathbb{R}^N$, see [9] and Chapter II of [2]. For the solution of the Fermat-Torricelli problem concerning spherical triangles with unit weights we refer to [4] and [5]. For the solution of the problem in any regular surface of $\mathbb{R}^3$ we refer to the indirect method used in [11] by studying the problem in the tangent plane of the regular surface at the Fermat-Torricelli point. We also show the invariance property of the weighted Fermat-Torricelli point for a given spherical, hyperbolic and planar (Euclidean) triangle with non-negative weights that are given at the vertices. This property is derived by a fundamental condition which is obtained by the inverse weighted
Fermat-Torricelli problem in the two-dimensional sphere, hyperboloid and in $\mathbb{R}^2$. The inverse weighted Fermat-Torricelli problem in $\mathbb{R}^2$ was also studied in [7].

2. The weighted Fermat-Torricelli point

We start by stating the problem for the two-dimensional unit sphere $S^2$. For background material from spherical trigonometry we refer to [1], and the following notations are used in our paper: By $a_i$ we denote the length of the geodesic $A_0 A_i$, by $a_{ij}$ the length of the geodesic $A_i A_j$, and by $\alpha_{ij}$ the spherical angle between the two geodesics $A_i A_k$, $A_k A_j$ for $i, j, k = 1, 2, 3$, $i \neq j \neq k \neq i$. Moreover, $a_i$ is seen on the left by the spherical angle $\alpha_i$, and it is seen on the right by the spherical angle $\alpha'_i$. Furthermore, let $\alpha_0$ be the spherical angle between $A_2 A_0$ and $A_3 A_0$, $\beta_0$ be the spherical angle between $A_1 A_0$ and $A_3 A_0$, and $\gamma_0$ be the spherical angle between $A_1 A_0$ and $A_2 A_0$ (see Figure 2.1).

**Problem 2.1.** Let $\nabla A_1 A_2 A_3$ be a spherical triangle. Suppose that a weight $B_i \in \mathbb{R}^+$ corresponds to each vertex $A_i$ for $i = 1, 2, 3$, respectively. Find the weighted Fermat-Torricelli point $A_0$ of the spherical triangle $\nabla A_1 A_2 A_3$ which minimizes the sum of the lengths of the geodesics that connect every vertex with $A_0$ multiplied by the positive weight $B_i$:

$$B_1 a_1 + B_2 a_2 + B_3 a_3 = \text{minimum}. \quad (1)$$

**Theorem 2.2.** The weighted Fermat-Torricelli point $A_0$ of the spherical triangle $\nabla A_1 A_2 A_3$ exists and is unique.

(i) If $|B_i - B_j| < B_k < B_i + B_j$ for $i, j, k = 1, 2, 3$, then the weighted Fermat-Torricelli point is an interior point of the spherical triangle $\nabla A_1 A_2 A_3$ (Floating Case).

(ii) If there is some $i$ with $B_i \geq B_j + B_k$ for $i, j, k = 1, 2, 3$, then the weighted Fermat-Torricelli point is the vertex $A_i$ (Absorbed Case).

It is possible to show the existence and uniqueness of the Fermat-Torricelli point by reduction ad absurdum. We start with the description of the exponential map in [10]. Let $\mathcal{D}$ be the set of vectors $v$ that belong to the tangent space $T(S^2)$ or $T(H^2)$ such that 1 lies in the domain of $\beta_v$. The generalized inverse stereographic projection is the
exponential map
\[ \exp : \mathcal{D} \to X \]
or
\[ \exp(v) = \pi \beta_x(1). \]
If \( x \in X \) (\( S^2 \) or \( H^2 \)) and \( 0_x \) is the zero vector in \( T_x \), then
\[ \exp(0_x) = x. \]
Write \( \exp_x \) for the restriction of \( \exp \) to the tangent space \( T_x \).

**Theorem 2.3.** Let \( X \) be a manifold and \( \xi \) a spray on \( X \). Then
\[ \exp_x : T_x \to X \]
induces a local isomorphism at \( 0_x \), and
\[ (\exp_x)_*(0_x) = id. \]

**Proof.** See [10].

The generalized inverse stereographic projection \( \exp : \mathcal{D} \to S^2 \), or \( H^2 \), or any regular surface \( X \), which is a conformal mapping, transfers the existence and uniqueness result of Kupitz-Martini in \( \mathbb{R}^d \) (see Theorem 1.1 (I), pp. 58–61 from [9]) to the differentiable manifold \( S^2 \) or \( H^2 \), or to any differential submanifold of \( \mathbb{R}^d \). For the case of \( S^2 \) we have the classical inverse stereographic projection from \( \mathbb{R}^2 \) to \( S^2 \). If the Fermat-Torricelli point does not exist for a spherical triangle, then the stereographic projection will not give any point for the corresponding plane triangle in \( \mathbb{R}^2 \). This is not true due to [9], since otherwise the Fermat-Torricelli point for plane triangles would not exist (Existence). If there were two Fermat-Torricelli points, the 1-1 correspondence of the stereographic projection would yield such points for the plane triangle, contradicting the uniqueness result of [9] (Uniqueness).

**Proof of (i) of Theorem 2.2.** The variables \( a_2, a_3 \) can be expressed as functions of \( a_1 \) and \( \alpha_3 \):
\[ a_2 = a_2(a_1, \alpha_3), \quad a_3 = a_3(a_1, \alpha_3). \] (2)
From (2) and (1) the following equation is obtained:
\[ B_1 a_1 + B_2 a_2(a_1, \alpha_3) + B_3 a_3(a_1, \alpha_3) = \text{minimum}. \] (3)
The "cosine law" regarding the spherical triangle \( \nabla A_0 A_1 A_3 \) is given by
\[ \cos(a_3) = \cos(a_1) \cos(a_{31}) + \sin(a_1) \sin(a_{31}) \cos(\alpha_3). \] (4)
Similarly, the "cosine law" concerning the spherical triangle \( \nabla A_0 A_1 A_2 \) is given by
\[ \cos(a_2) = \cos(a_1) \cos(a_{21}) + \sin(a_1) \sin(a_{21}) \cos(\alpha_{23} - \alpha_3). \] (5)
By differentiation of (3) with respect to the variables \( a_1 \) and \( \alpha_3 \) we get
\[ B_2 \frac{\partial a_2}{\partial a_1} + B_3 \frac{\partial a_3}{\partial a_1} = -B_1, \] (6)
\[ B_2 \frac{\partial a_2}{\partial \alpha_3} + B_3 \frac{\partial a_3}{\partial \alpha_3} = 0. \]  

Differentiating (4) and (5) with respect to \( \alpha_3 \), we can replace \( \frac{\partial a_2}{\partial \alpha_3}, \frac{\partial a_3}{\partial \alpha_3} \) in (7):

\[ \frac{B_2}{B_3} \frac{\sin(a_{31}) \sin(a_3) \sin(a_2)}{\sin(a_{12}) \sin(a_2) \sin(a_3)} = \sin(a_{31}) \sin(a_3) \sin(a_2). \]

Also one can consider the “sine law” for the spherical triangles \( \nabla A_0A_1A_3 \), \( \nabla A_0A_1A_2 \):

\[ \frac{\sin(a_3)}{\sin(a_3)} = \frac{\sin(\beta_0)}{\sin(a_{31})}, \quad \frac{\sin(a_2)}{\sin(a_2)} = \frac{\sin(\gamma_0)}{\sin(a_{12})}. \]

The next equation is derived by combining equations (9) and (10) with (8):

\[ \frac{B_2}{\sin(\beta_0)} = \frac{B_3}{\sin(\gamma_0)} = c. \]

Let \( a_1 \) and \( a_3 \) be expressed as functions of \( a_2 \) and \( \alpha'_3 \) (see Figure 2.1):

\[ a_1 = a_1(a_2, \alpha'_3), \quad a_3 = a_3(a_2, \alpha'_3). \]

From (12) and (1) the following equation is obtained:

\[ B_1a_1(a_2, \alpha'_3) + B_2a_2 + B_3a_3(a_2, \alpha'_3) = \text{minimum.} \]

Differentiating (13) with respect to the variable \( \alpha'_3 \), we deduce that

\[ B_1 \frac{\partial a_1}{\partial \alpha'_3} + B_3 \frac{\partial a_3}{\partial \alpha'_3} = 0. \]

The same procedure as described for (7) can be applied to the spherical triangles \( \nabla A_0A_1A_2 \) and \( \nabla A_0A_2A_3 \), and so we obtain the relation

\[ \frac{B_1}{\sin(a_0)} = \frac{B_2}{\sin(\beta_0)} = \frac{B_3}{\sin(\gamma_0)} = c. \]

From (15) and (11) we get

\[ \frac{B_1}{\sin(a_0)} = \frac{B_2}{\sin(\beta_0)} = \frac{B_3}{\sin(\gamma_0)} = c. \]

The relationship between the spherical angles \( a_0, \beta_0, \gamma_0 \) (see Figure 2.1) can be used in order to clarify the value of \( c \):

\[ a_0 + \beta_0 + \gamma_0 = 2\pi, \]

\[ \sin(a_0) = -\sin(\beta_0 + \gamma_0). \]
From (16) and (18) we obtain that \( c \) depends only on \( B_1, B_2, B_3 \):

\[
c = \frac{2B_1B_2B_3}{\sqrt{(B_1 + B_2 + B_3)(B_1 + B_2 - B_3)(B_2 + B_3 - B_1)(B_1 + B_3 - B_2)}}.
\]  

(16) and (19) give three important formulas:

\[
\cos(\alpha_0) = \frac{B_2^2 - B_3^2 - B_4^2}{2B_2B_3},
\]

(20)

\[
\cos(\beta_0) = \frac{B_3^2 - B_1^2 - B_4^2}{2B_3B_1},
\]

(21)

\[
\cos(\gamma_0) = \frac{B_4^2 - B_1^2 - B_2^2}{2B_2B_1}.
\]

(22)

From (20), (21), (22), the desired inequalities are obtained, namely

\[ |B_i - B_j| < B_k < B_i + B_j \]

for \( i, j, k = 1, 2, 3, \ i \neq j \neq k \). Three additional inequalities are needed to obtain \( A_0 \) inside \( A_1A_2A_3 \):

\[ \alpha_0 > \alpha_{23}, \quad \beta_0 > \alpha_{31}, \quad \gamma_0 > \alpha_{12} \]

or

\[
\frac{B_2^2 - B_3^2 - B_4^2}{2B_2B_3} < \frac{\cos(a_{23}) - \cos(a_{31}) \cos(a_{12})}{\sin(a_{31}) \sin(a_{12})},
\]

\[
\frac{B_3^2 - B_1^2 - B_4^2}{2B_3B_1} < \frac{\cos(a_{31}) - \cos(a_{23}) \cos(a_{12})}{\sin(a_{23}) \sin(a_{12})},
\]

\[
\frac{B_4^2 - B_1^2 - B_2^2}{2B_2B_1} < \frac{\cos(a_{12}) - \cos(a_{31}) \cos(a_{23})}{\sin(a_{31}) \sin(a_{23})}.
\]

\( A_0 \) is the intersection point of \( \alpha_0 \) and \( \beta_0 \) of \( \nabla A_1A_2A_3 \). This gives the geometrical construction of \( A_0 \). Another approach to construct \( A_0 \) is by calculating the angles \( \alpha_3 \) and \( \alpha_3' \), respectively, are unique.

**Proof of (ii) of Theorem 2.2.** Suppose that \( B_1 \geq B_2 + B_3 \). Then

\[
B_1a_1 + B_2a_2 + B_3a_3 \geq (B_2 + B_3)a_1 + B_2a_2 + B_3a_3
\]

\[
= B_2(a_1 + a_2) + B_3(a_1 + a_3) \geq B_2a_{12} + B_3a_{13}.
\]

By using the triangle inequality, we deduce that the minimum point is attained at the vertex \( A_1 \).

**Remark 2.4.** The equation (20) can also be obtained by combining the two equations (6) and (7):

\[
B_1 + B_2 \cos(\gamma_0) + B_3 \cos(\beta_0) = 0,
\]

\[
B_2 \sin(\gamma_0) - B_3 \sin(\beta_0) = 0
\]

or, equivalently,

\[
B_1 + B_2e^{i\gamma_0} + B_3e^{-i\beta_0} = 0.
\]

The same process can be applied to (21) and (22).
Corollary 2.5. Given the weights \( B_1 = B_2 = B_3 \), we have \( \alpha_0 = \beta_0 = \gamma_0 = 120^\circ \).

Proposition 2.6. Theorem 2.2 and Corollary 2.5 are also valid for

(a) a hyperbolic triangle \( \nabla A_1 A_2 A_3 \),
(b) a triangle \( \nabla A_1 A_2 A_3 \) in \( \mathbb{R}^2 \).

Proof of (a). Similar identities are used for the hyperbolic triangle \( \nabla A_1 A_2 A_3 \) referring to (4) and (5):

\[
\cosh(a_3) = \cosh(a_1) \cosh(a_{31}) - \sinh(a_1) \sinh(a_{31}) \cos(\alpha_3), \quad (23)
\]

\[
\cosh(a_2) = \cosh(a_1) \cosh(a_{21}) - \sinh(a_1) \sinh(a_{21}) \cos(\alpha_2 - \alpha_3). \quad (24)
\]

As an analogue of the “sine law” for spherical triangles there is a "sine law" for the hyperbolic triangles \( \nabla A_0 A_1 A_3 \) and \( \nabla A_0 A_1 A_2 \), respectively:

\[
\frac{\sin(\alpha_3)}{\sinh(a_3)} = \frac{\sin(\beta_0)}{\sinh(a_{31})}, \quad (25)
\]

\[
\frac{\sin(\alpha_2)}{\sinh(a_2)} = \frac{\sin(\gamma_0)}{\sinh(a_{12})}. \quad (26)
\]

Differentiating (23) and (24) with respect to the hyperbolic angle \( \alpha_3 \), the partial derivatives \( \frac{\partial a_2}{\partial \alpha_3}, \frac{\partial a_3}{\partial \alpha_3} \) are replaced in (7) by taking into account (25) and (26):

\[
\frac{B_2}{\sin(\beta_0)} = \frac{B_3}{\sin(\gamma_0)} = c. \quad (27)
\]

The same can be done with the spherical triangle, and the same result is derived for the hyperbolic angles \( \alpha_0, \beta_0 \) and \( \gamma_0 \). Thus (20), (21) and (22) are obtained.

Proof of (b). Here our analytical approach uses differentiation with respect to the variable of the distance \( a_1 \) of the plane triangle \( \nabla A_1 A_2 A_3 \). We use the same symbols for the variables as in the case of spherical triangles. We start with the "cosine law" that is valid for the triangles \( \nabla A_0 A_1 A_2 \) and \( \nabla A_0 A_1 A_3 \), respectively:

\[
a_2^2 = a_1^2 + a_{21}^2 - 2a_1a_{21} \cos(\alpha_{23} - \alpha_3), \quad (28)
\]

\[
a_3^2 = a_1^2 + a_{31}^2 - 2a_1a_{31} \cos(\alpha_3). \quad (29)
\]

Furthermore, we apply the "sine law" to the triangles \( \nabla A_0 A_1 A_2 \) and \( \nabla A_0 A_1 A_3 \), respectively:

\[
\frac{a_1}{a_2} = \frac{\sin(\alpha_{12} + \alpha_{23} - \alpha_3)}{\sin(\alpha_{23} - \alpha_3)}, \quad \frac{a_{12}}{a_2} = \frac{\sin(\alpha_{12})}{\sin(\alpha_{23} - \alpha_3)}, \quad (30)
\]

\[
\frac{a_1}{a_3} = \frac{\sin(\alpha_{13} + \alpha_3)}{\sin(\alpha_3)}, \quad \frac{a_{13}}{a_3} = \frac{\sin(\alpha_{13})}{\sin(\alpha_3)} \quad (31)
\]

We differentiate (28) and (29) with respect to the distance \( a_1 \), and we replace the new relations by combining (30) and (6). We square both parts of the derived equation and both parts of the equation (7) by applying (30) and (31). We add both of them, in order to obtain (20).

Similarly, by differentiating (1) with respect to \( a_2 \) and \( a_3 \) we get the relations (21) and (22).
Remark 2.7. The relations (20), (21) and (22) are also obtained, since we derive (16) by differentiating (28) and (29) with respect to the angle $\alpha_3$ using the same procedure for a spherical and hyperbolic triangle.

3. The inverse weighted Fermat-Torricelli problem

Problem 3.1. Given the weighted Fermat-Torricelli point $A_0$ of the weighted spherical or hyperbolic triangle $\nabla A_1A_2A_3$ and the angles $\alpha_0$, $\beta_0$, $\gamma_0$, find the ratios between the non-negative weights $B_i/B_j$, $i, j = 1, 2, 3$, such that

\[ B_1 + B_2 + B_3 = \text{constant}. \]

This is the inverse weighted Fermat-Torricelli problem in the two-dimensional sphere $S^2$ or two-dimensional hyperboloid $H^2$.

The generalized inverse weighted Fermat-Torricelli problem in $\mathbb{R}^2$ is studied in [12] for $n > 3$ and in [7] for $n = 3$.

Proposition 3.2. Given the angles $\alpha_0$, $\beta_0$, $\gamma_0$ and the weighted Fermat-Torricelli point in $S^2$ or $H^2$ or $\mathbb{R}^2$, the ratio of the three weights $B_1, B_2, B_3$ is given by

\[ B_1 : B_2 : B_3 = \sin(\alpha_0) : \sin(\beta_0) : \sin(\gamma_0). \]

Proof. The ratio $B_1 : B_2 : B_3$ is obtained from (16). This equation also holds in $H^2$ and $\mathbb{R}^2$. \hfill \Box

Corollary 3.3. Concerning the spherical, hyperbolic and plane triangle $\nabla A_1A_2A_3$, there are three common equations in a complex form (Remark 2.4) that provide the location of the weighted Fermat-Torricelli point:

\[ B_1 + B_2e^{i\gamma_0} + B_3e^{-i\beta_0} = 0, \]
\[ B_1e^{i\gamma_0} + B_2 + B_3e^{-i\alpha_0} = 0, \]
\[ B_1e^{i\beta_0} + B_2e^{-i\alpha_0} + B_3 = 0. \]

Corollary 3.4. The inverse weighted Fermat-Torricelli problem gives the same ratio of the weights $B_1 : B_2 : B_3$ for a spherical, hyperbolic or plane triangle.

Proposition 3.5. The weighted Fermat-Torricelli point of a spherical, hyperbolic or plane triangle $\nabla A_1A_2A_3$ remains the same for any spherical, hyperbolic or plane triangle $\nabla A'_1A'_2A'_3$ if the floating case (Theorem 2.2) occurs for constant values of $B_i$ that correspond to any vertex $A_i$, $i = 1, 2, 3$.

Proof. The result follows from the fundamental condition

\[ B_1 : B_2 : B_3 = \sin(\alpha_0) : \sin(\beta_0) : \sin(\gamma_0) \]

(see Proposition 3.2). \hfill \Box
Corollary 3.6. The weighted Fermat-Torricelli point of a spherical, hyperbolic or plane triangle $\nabla A_1A_2A_3$ remains the same for any spherical, hyperbolic or plane triangle $\nabla A'_1A'_2A'_3$ with vertices on geodesic cycles that are defined by the segments of the geodesics $A_0A_1$, $A_0A_2$, $A_0A_3$.

Proof. This corollary is a direct consequence of Proposition 3.2.

In conclusion, we would like to mention that the notion of invariance which possesses the weighted Fermat-Torricelli point for given three points in the plane, two-dimensional sphere and two-dimensional hyperboloid, is stronger than the notion of similarity. This means that the invariance of the weighted Fermat-Torricelli point holds for similar and non-similar triangles. This fundamental result occurs because the angles $\alpha_0, \beta_0, \gamma_0$ depend only on the values of the weights $B_1, B_2, B_3$ and not on the side lengths of the triangles (see (20), (21), and (22)). Among known points (so-called centers) of Euclidean, spherical or hyperbolic triangles the weighted Fermat-Torricelli point is the only point that is connected with this invariance property (Proposition 3.5, Corollary 3.6). The invariance of the weighted Fermat-Torricelli point in Proposition 3.5 (strong invariance) is stronger than the invariance of the weighted Fermat-Torricelli point in Corollary 3.6 (weak invariance).

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