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Differentiability of Approximately Convex, Semiconcave and Strongly Paraconvex Functions

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It is shown that continuous approximately convex, semiconcave and strongly $\alpha(\cdot)$-paraconvex functions on Banach spaces have almost all (but not all) known first order differentiability properties of continuous convex functions. The main results easily follow from known (or essentially known) results on single-valuedness and continuity of submonotone operators.

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1. Introduction

A number of notions of generalized convex functions (surfaces, sets) was considered (frequently independently) in the literature.

Functions f defined on $\mathbf{R}^n$ (or corresponding surfaces in $\mathbf{R}^{n+1}$) which are locally representable in the form $f(x) = g(x) + c\|x\|^2$, where g is concave and $c > 0$, were treated in the literature many times under different names and different (equivalent) definitions (see e.g. [25]). Now these functions f are most frequently called semiconcave functions (with linear modulus, see [4]), and the functions $-f$ are called semiconvex or lower-C^2 functions (see [24]).

More general are locally (γ)-paraconvex functions considered in Banach spaces in 1979 by Rolewicz [26].

On $\mathbf{R}^n$, a more general class of lower-C^1 functions was considered in 1981 by Spingarn [31].

A class of "uniformly almost superdifferentiable" functions (defined on an open subset of a Banach space) was introduced and applied in the abstract approximation theory [38] in 1984. It is not difficult to show (but we will not use this fact) that this class of functions coincides with the class of continuous (general) semiconcave functions considered [2] in 1992 in $\mathbf{R}^n$ and in general Banach spaces [1] in 1998.

The corresponding class of continuous (general) semiconvex functions (i.e. those, for which $-f$ is semiconcave) essentially coincides with the class of continuous strongly $\alpha(\cdot)$-paraconvex functions (with $\lim_{t\to 0+} \alpha(t)/t = 0$) defined [27] in 2000. Note that strongly t^γ-paraconvex functions ($1 < \gamma \leq 2$) coincide [27] with (γ)-paraconvex functions, and also

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[6] with lower-$C^{1,\gamma-1}$ functions.

The class of approximately convex functions on a Banach space was considered [19] in 2000. Continuous approximately convex functions on $\mathbf{R}^n$ coincide with lower-C^1 functions [5], and also with continuous locally (general) semiconvex functions (see Remark 2.6). Each locally semiconvex function (or each locally strongly $\alpha(\cdot)$-paraconvex function) is approximately convex. (Indeed, strongly $\alpha(\cdot)$-paraconvex functions coincide with uniformly approximately convex functions [28].)

In Section 3 we show that an approximately convex function which is Fréchet differentiable at a point is even strictly differentiable at this point.

The differentiability properties of some non-convex functions were infered from generalized monotonicity of its subdifferential in [38] and [16]. We show in Sections 4 and 5 that this method easily gives that continuous approximately convex functions (and so also continuous general semiconcave functions and continuous strongly paraconvex functions) have almost all known differentiability properties (of the first order) of continuous convex functions (but not all; see Section 6). In particular, the result of [1] is generalized and the results of [29] and [30] are generalized and refined.

Note that we work only with functions which are finite and continuous on an open subset of a Banach space. Approximately convex functions which generalize extended real valued convex functions $f : X \to \mathbf{R} \cup \{\infty\}$ are not considered.

2. Preliminaries

2.1. Basic notation

In the following, X will be always a (real) Banach space and $B(x,r)$ will denote the open ball with center x and radius r.

In the following definitions, f is a real function defined on an open subset G of X.

Recall that $A \in X^*$ is called a (Fréchet; uniform) strict derivative of f at $a \in X$ if

$$\lim_{x,y\to a, x\neq y} \frac{f(y)-f(x)-\langle A, y-x\rangle}{\|y-x\|} = 0,$$

(where we allow that $x = a$ or $y = a$).

The one-sided directional derivative of f at x in the direction v is defined by

$$f'_+(x,v) := \lim_{t\to 0+} \frac{f(x+tv)-f(x)}{t}.$$

The Fréchet subdifferential of f at a is defined by

$$\partial^F f(a) := \{x^* \in X^* : \ \liminf_{h\to 0} \frac{f(a+h)-f(a)-\langle x^*, h\rangle}{\|h\|} \geq 0\}.$$

If f is locally Lipschitz on G, then

$$f^0(a,v) := \limsup_{z\to a, t\to 0+} \frac{f(z+tv)-f(z)}{t}$$

is the Clarke derivative of f at a in the direction v and

$$\partial^C f(a) := \{x^* \in X^* : \langle x^*, v\rangle \le f^0(a,v) \text{ for all } v \in X\}$$

is the Clarke subdifferential of f at a (which is always non-empty). We say that f is (Clarke) regular if $f^0(x,v) = f'_+(x,v)$ for each $x \in G$ and $v \in X$.

Recall that (see e.g. [11], p. 5)

$$f \text{ is Gâteaux differentiable at } a \text{ whenever } \partial^C f(a) \text{ is a singleton.} \tag{1}$$

(Note that, in [11], strict differentiability is called "uniform strict differentiability", and "strict differentiability" is a weaker notion; see [11, pp. 5, 7].)

We will denote by $N_F(f)$ (resp. $N_G(f)$) the set of points of G at which f is not Fréchet (resp. Gâteaux) differentiable.

Now we recall the notion of a submonotone mapping (operator) which is basic for this article.

Definition 2.1. Let $T : X \to X^*$ be a multivalued mapping (operator). We will say that T is submonotone at $a \in X$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle x_1^* - x_2^*, x_1 - x_2\rangle \ge -\varepsilon\|x_1 - x_2\|,$$

whenever $x_1, x_2 \in B(a,\delta)$, $x_1^* \in T(x_1)$ and $x_2^* \in T(x_2)$.

We say that T is submonotone on $G \subset X$ if it is submonotone at each $a \in G$.

This definition is taken from [5]. The notion was introduced by Spingarn [31] in $\mathbf{R}^n$ under the name "strictly submonotone mapping". (An essentially same notion was used in a Banach space in [38] and [16], where the notion of "locally almost nonincreasing" mappings was defined.)

2.2. Three classes of generalized convex functions

We will give definitions only for the case of real functions defined on open (convex) sets, since we will work with such functions only.

Denote by $\mathcal{M}$ the set of all functions $\omega : [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$ which are non-decreasing and right continuous at 0.

The following definition is taken from [4]; the definitions in [2] and [1] are slightly different but essentially equivalent.

Definition 2.2. A continuous real function f on an open convex set $\Omega \subset X$ is called semiconcave with modulus $\omega \in \mathcal{M}$ if

$$\lambda f(x) + (1-\lambda)f(y) - f(\lambda x + (1-\lambda)y) \le \lambda(1-\lambda)\omega(\|x-y\|)\|x-y\| \tag{2}$$

whenever $\lambda \in [0,1]$ and $x, y \in \Omega$.

A function is called semiconcave on Ω if it is semiconcave on Ω with some modulus $\omega \in \mathcal{M}$. A function f is called semiconvex if $-f$ is semiconcave.

Definition 2.3 ([27], [29]). Let $\alpha \in \mathcal{M}$ be such that $\lim_{t\to 0+} \alpha(t)/t = 0$ and $\Omega \subset X$ be a convex open set. A function $f : \Omega \to \mathbf{R}$ is called strongly $\alpha(\cdot)$-paraconvex if there exists $C > 0$ such that

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) + C\min(\lambda, 1-\lambda)\alpha(\|x-y\|) \tag{3}$$

whenever $\lambda \in [0,1]$ and $x, y \in \Omega$.

We will say that f is strongly paraconvex if it is strongly $\alpha(\cdot)$-paraconvex for some $\alpha \in \mathcal{M}$ with $\lim_{t\to 0+} \alpha(t)/t = 0$.

Definition 2.4 ([19]). A real function f on an open set $\Omega \subset X$ is called approximately convex at $x_0 \in \Omega$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) + \varepsilon\lambda(1-\lambda)\|x-y\| \tag{4}$$

whenever $\lambda \in [0,1]$ and $x, y \in B(x_0, \delta)$.

We say that f is approximately convex on Ω if it is approximately convex at each $x_0 \in \Omega$. We say that f is uniformly approximately convex on Ω if for every $\varepsilon > 0$ there exists $\delta > 0$ such that (4) holds whenever $\lambda \in [0,1]$, $x, y \in \Omega$, and $\|x-y\| < \delta$.

(Note that the term "approximately convex functions" is used for a long time for another type of functions, namely for ε-convex functions in the sense of Hyers and Ulam.)

It is easy to see that f is semiconvex on an open convex $\Omega \subset X$ if and only if f is continuous and strongly paraconvex on Ω. (It is sufficient to observe that $\lambda(1-\lambda) \leq \min(\lambda, 1-\lambda) \leq 2\lambda(1-\lambda)$ for $\lambda \in [0,1]$ and that for each $\omega^* : [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$ which is right continuous at 0 there exists $\omega \in \mathcal{M}$ such that $\omega^* \leq \omega$.)

However, for us it is sufficient to observe that each semiconvex function on Ω is approximately convex on Ω and that each strongly paraconvex function on Ω is approximately convex on Ω. Indeed, all results below hold for the (largest) class of approximately convex functions.

We will need the following well-known properties of approximately convex functions.

Lemma 2.5. *Let f be a lower semicontinuous real approximately convex function on an open subset G of a Banach space X. Then the following hold.*

(*i*) *f is locally Lipschitz on G and the one-sided directional derivative $f'_+(x, v)$ exists for each $x \in G$ and $v \in X$.*
(*ii*) *$\partial^F f(x) = \partial^C f(x)$ for each $x \in G$.*
(*iii*) *The multi-valued mapping $T(x) := \partial^C f(x)$ is submonotone on G.*
(*iv*) *f is (Clarke) regular.*

Proof. The property (*i*) is proved in [19, Proposition 3.2 and Corollary 3.5]. The property (*ii*) follows from [19, Theorem 3.6], because f is locally Lipschitz and so $\partial^C f(x) = \partial^{C-R} f(x)$ for $x \in G$. The property (*iii*) follows from [5, Theorem 2]. The property (*iv*) follows from [19, Corollary 3.5. and Theorem 3.6] (see also [5, Remark 6]). □

The following remark was added in the revised version as a response to a question formulated by an anonymous referee.

Remark 2.6. It is natural to ask whether results on differentiability of approximately convex functions are essential generalizations of corresponding results about strongly paraconvex (equivalently: general semiconcave, or locally uniformly approximately convex) functions.

It is not the case if we deal with functions on $\mathbf{R}^n$ (which is locally compact), since each approximately convex function f on an open $\Omega \subset \mathbf{R}^n$ is locally uniformly approximately convex. It follows easily from results of [31] and [5] (via [5, Theorem 2] and [31, Proposition 3.8]; local compactness of $\mathbf{R}^n$ is used in [31, Lemma 3.6]).

However, the following simple direct proof is preferable:

Let $a \in \Omega$, $\omega > 0$ and $B := \overline{B(a, \omega)} \subset \Omega$ be given. Choose an arbitrary $\varepsilon > 0$. For each $x_0 \in B$, choose $\delta = \delta_{x_0}$ by Definition 2.4. Let $L > 0$ be the Lebesgue number of the covering $\{B(x_0, \delta_{x_0}) \cap B : \ x_0 \in B\}$ of the compact B. Let now $x, y \in B(a, \omega)$ with $|x - y| < L$ be given. Then, for each $\lambda \in [0, 1]$, the inequality (4) holds, since (by the definition of the Lebesgue number) there is $x_0 \in B$ with $x, y \in B(x_0, \delta_{x_0})$.

On the other hand, it is not difficult to construct an approximately convex function f on ℓ^2, which is uniformly approximately convex on no ball. Since this example does not concern directly the topic of the present article, and the detailed proof is not short, I present here only a construction of a weaker example of an approximately convex function f which is uniformly approximately convex on no neighbourhood of $0 \in \ell^2$. I will publish the complete example elsewhere, if I will not find it in the literature.

For each $n \in \mathbf{N}$, let $\varphi_n : [-1, 1] \to \mathbf{R}$ be an even concave continuous function such that $\varphi_n(-1) = 0$, $\varphi_n(0) = 1 - 1/2n$, φ_n is C^2 on $(-1, 1)$, and $\varphi_n(x) = x + 1$ for $x \in [-1, -1/n]$. For $k, l \in \mathbf{N}$, let $B_{k,l} := B(2^{-l} e_k, 2^{-(l+4)})$, where e_k is the k-th member of the canonical basis of ℓ^2. Obviously, the family $\{B_{k,l}\}$ is disjoint. Now set $f(x) := 2^{-2l} \varphi_k(2^{l+4} \|x - 2^{-l} e_k\|)$ if $x \in B_{k,l}$ and $f(x) := 0$ for $x \in \ell^2 \setminus \bigcup B_{k,l}$. It is not difficult to prove that f is an approximately convex function on ℓ^2, but f is uniformly approximately convex on no neighbourhood of $0 \in \ell^2$. (Note that, after estimating Lipschitz constants of f on balls $B_{k,l}$, it is easy to show that f is strictly differentiable at 0.)

2.3. Definitions of some systems of small sets

We recall here definitions of some systems of small sets in Banach spaces. For more information see [42].

Definition 2.7. Let $M \subset X$, $x \in X$ and $R > 0$. Then we define $\gamma(x, R, M)$ as the supremum of all $r \geq 0$ for which there exists $z \in X$ such that $B(z, r) \subset B(x, R) \setminus M$. Further define the upper porosity of M at x as

$$\overline{p}(M, x) := 2 \limsup_{R \to 0+} \frac{\gamma(x, R, M)}{R}$$

and the lower porosity of M at x as

$$\underline{p}(M, x) := 2 \liminf_{R \to 0+} \frac{\gamma(x, R, M)}{R}.$$

We say that M is upper porous (lower porous) at x if $\overline{p}(M, x) > 0$ ($\underline{p}(M, x) > 0$).

We say that M is *upper porous (lower porous)* if M is upper porous (lower porous) at each point $y \in M$. We say that M is *σ-upper porous (σ-lower porous)* if it is a countable union of upper porous (lower porous) sets.

It is clear that each σ-lower porous set is σ-upper porous and each σ-upper porous set is a first category set.

Definition 2.8. Let X be a Banach space. We say that $A \subset X$ is a *Lipschitz hypersurface* (or a Lipschitz surface of codimension 1) if we can write $A = \{x + \varphi(x)v : \ x \in E\}$, where $X = E \oplus \mathbf{R}v$ and φ is a real Lipschitz function on E.

Recall the important fact that, in a separable Banach space X, each Lipschitz hypersurface (and so also each Borel set which can be covered by countably many Lipschitz hypersurfaces) is Gaussian null (and therefore also Haar null) and Γ-null. (For information on Gaussian and Haar null sets see [3], the notion of Γ-null sets was defined in [14].)

Definition 2.9. If X is a Banach space, $v \in X$, $\|v\| = 1$ and $0 < c < 1$, then we define the cone $A(v, c) := \bigcup_{\lambda>0} \lambda \cdot B(v, c)$. We say that $M \subset X$ is *cone supported* if for each $x \in M$ there exist $r > 0$ and a cone $A(v, c)$ such that $M \cap (x + A(v, c)) \cap B(x, r) = \emptyset$. The notion of a *$\sigma$-cone supported* set is defined in the usual way.

If X is separable, it is easy to show (see Lemma 1 of [34]) that the system of all σ-cone supported sets coincides with sets which can be covered by countably many Lipschitz hypersurfaces. Each σ-cone supported set is clearly σ-lower porous.

Definition 2.10. Let X be a Banach space. If $x^* \in X^*$, $x^* \neq 0$ and $0 \leq \alpha < 1$, define (the α-cone)

$$C(x^*, \alpha) := \{x \in X : \ \alpha\|x\| \cdot \|x^*\| < (x, x^*)\}.$$

A set $M \subset X$ is said to be α-cone porous at $x \in X$ if there exists $R > 0$ such that for each $\varepsilon > 0$ there exists $z \in B(x, \varepsilon)$ and $0 \neq x^* \in X^*$ such that

$$M \cap B(x, R) \cap (z + C(x^*, \alpha)) = \emptyset. \tag{5}$$

A subset of X is said to be α-cone porous if it is α-cone porous at all its points; σ-α-cone porous sets are defined in the obvious way. A set is said to be *cone-small* if it is σ-α-cone porous for each $0 < \alpha < 1$.

If we write in (5) $M \cap (z + C(x^*, \alpha)) = \emptyset$ then we obtain (instead of the notion of a cone small set) the notion of an *angle small set* (cf. [23], [22]). If X is separable then it is easy to see that the notions of cone smallness and angle smallness coincide. It is not true in non-separable Hilbert spaces ([13]). Clearly each cone small set is σ-lower porous.

3. Connections between approximate convexity and strict differentiability

The following proposition is an immediate consequence of [33, Proposition 3.7] (cf. [40, Theorem A]).

Proposition 3.1. *Let f be a mapping from a normed linear space X to a Banach space Y. Then f is strictly differentiable at a point $a \in X$ if and only if f is continuous at a*

and for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left\| \frac{f(y+kv)-f(y)}{k} - \frac{f(y)-f(y-hv)}{h} \right\| < \varepsilon, \tag{6}$$

whenever $\|v\| = 1$, $k > 0$, $h > 0$, $y - hv \in B(a,\delta)$, $y + kv \in B(a,\delta)$.

So the following easy (geometrically more understandable) reformulation of the definition of approximate convexity shows that approximate convexity at a point has the meaning of "semi-strict differentiability".

Lemma 3.2. *Let X be a Banach space and f be a real function defined on a neighbourhood of a point $a \in X$. Then f is approximately convex at a if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\frac{f(y+kv)-f(y)}{k} - \frac{f(y)-f(y-hv)}{h} > -\varepsilon, \tag{7}$$

whenever $\|v\| = 1$, $k > 0$, $h > 0$, $y - hv \in B(a,\delta)$, $y + kv \in B(a,\delta)$.

Proof. Denote $x_1 := y + kv$, $x_2 := y - hv$, $t := h/(h+k)$. Then $1 - t = k/(k+h)$, $y = tx_1 + (1-t)x_2$ and an easy computation shows that

$$\begin{aligned} &\frac{f(y+kv)-f(y)}{k} - \frac{f(y)-f(y-hv)}{h} \\ &= \frac{1}{t(1-t)\|x_1 - x_2\|} \left(tf(x_1) + (1-t)f(x_2) - f(tx_1 + (1-t)x_2) \right). \end{aligned}$$

The rest of the proof is clear. □

Proposition 3.1 and Lemma 3.2 have the following immediate consequence.

Corollary 3.3. *Let X be a Banach space and f be a real function defined on an open neighbourhood of a point $a \in X$. Then f is strictly differentiable at a if and only if f is continuous at a and both f and $-f$ are approximately convex at a.*

Using Lemma 3.2, we give a short proof of the following easy result which generalizes the well-known property of convex functions.

Proposition 3.4. *Let X be a Banach space and f be approximately convex and Fréchet differentiable at $a \in X$. Then f is strictly differentiable at a.*

Proof. We can clearly suppose (adding an affine function to f, if necessary) that $f(a) = 0$ and $f'(a) = 0$. Let $\varepsilon > 0$ be given. Let $\delta > 0$ be a number which corresponds to $\varepsilon^* := \varepsilon/2$ by Lemma 3.2. Further choose $\omega \in (0,\delta)$ such that

$$|f(x)| < (\varepsilon/8)\|x - a\|, \text{ whenever } x \in B(a,\omega). \tag{8}$$

Put $\rho := \omega/2$ and consider arbitrary different points $y, z \in B(a,\rho)$. Denote $v := (z - y)/\|z - y\|$. Since the points y, z, $z + \rho v$, $y - \rho v$ belong to $B(a,\omega) \subset B(a,\delta)$, by the choice of δ and (8) we obtain that

$$\frac{f(z+\rho v)-f(z)}{\rho} - \frac{f(z)-f(y)}{\|z-y\|} > -\varepsilon/2, \tag{9}$$

$$\frac{f(z)-f(y)}{\|z-y\|}-\frac{f(y)-f(y-\rho v)}{\rho}>-\varepsilon/2, \tag{10}$$

and the numbers $|f(y)|$, $|f(z)|$, $|f(z+\rho v)|$, $|f(y-\rho v)|$ are less than $(\varepsilon/8)2\rho=\varepsilon\rho/4$. Therefore $|(f(z+\rho v)-f(z))/\rho|<\varepsilon/2$ and $|(f(y)-f(y-\rho v))/\rho|<\varepsilon/2$. So (9) implies $(f(z)-f(y))/\|z-y\|<\varepsilon$ and (10) gives $(f(z)-f(y))/\|z-y\|>-\varepsilon$. We obtain that 0 is the strict derivative of f at a. □

4. Gâteaux differentiability

4.1. Gâteaux differentiability on separable spaces

Theorem 4.1. *Let X be a separable Banach space, $G\subset X$ be an open set and $f: G\to\mathbf{R}$ be a continuous approximate convex function on G. Then the set $N_G(f)$ of all points at which f is not Gâteaux differentiable can be covered by countably many Lipschitz hypersurfaces. In particular, $N_G(f)$ is Gaussian null, Haar null and Γ-null.*

Proof. Lemma 2.5(i) says that

$$f \text{ is locally Lipschitz and } f \text{ has all one-sided derivatives at all points.} \tag{11}$$

By [18, Theorem 3.3], condition (11) implies that $N_G(f)$ can be covered by countably many Lipschitz hypersurfaces. □

Remark 4.2.

1. Theorem 4.1 is also an almost immediate consequence of Lemma 2.5(i), (ii) and [36, Lemma 2] (and also of a related [15, Theorem 3]). In fact, [18, Theorem 3.3] implies slightly more: $N_G(f)$ can be covered by countably many of "special parts" of Lipschitz hypersurfaces.
2. A complete characterization of smallness of sets $N_G(f)$ for continuous convex functions on a separable space was given in [35] (see also [3, Theorem 4.20]). A complete characterization of sets $N_G(f)$ for continuous convex functions on $\mathbf{R}^n$ was given in [21].

Theorem 4.1 for semiconcave functions (and thus also for continuous strongly paraconvex functions) was essentially proved in [1]. Although the definition of σ-$(\infty-1)$-dimensional rectifiable sets used in [1] is too weak (since the whole space can be such a set), the proof in fact gives the result.

4.2. Gâteaux differentiability on non-separable spaces

It was proved in [39] that, if $f: X\to\mathbf{R}$ is a continuous convex function and X is Asplund or X^* is strictly convex (i.e. rotund), then $N_G(f)$ is σ-cone supported. If X is Asplund, this result was generalized in [16] to the case when f is "uniformly almost superdifferentiable", and thus essentially also to the case of a semiconcave function (or a continuous strongly paraconvex function).

The proof of [39] was given via single-valuedness of monotone operators and the proof of [16] via single-valuedness of submonotone operators. Further generalizations of the above mentioned result of [39] were given in [12] (where monotone operators are considered) and in [10] (where directionally submonotone operators are considered). The above mentioned articles contain (or essentially contain) the following result on submonotone operators. Recall that X is called an Asplund generated space (or a GSG space, see [7]) if there exists an Asplund space V and a continuous linear mapping $F : V \to X$ such that $F(V)$ is dense in X.

Proposition 4.3. *Let $T : X \to X^*$ be a locally bounded (multivalued) submonotone operator with an arbitrary domain $D(T) = \{x \in X : T(x) \neq \emptyset\}$. Let X be Gâteaux smooth or X be a subspace of an Asplund generated (i.e. GSG) space. Then there exists a σ-cone supported set $A \subset D(T)$ such that T is single-valued at each point of $D(T) \setminus A$.*

Proof. The case when X is Gâteaux smooth is a special case of [10, Theorem 5.1] (where directionally submonotone operators are called "submonotone"; so a more general result is proved). Note that σ-cone supported sets are called "σ-cone porous" in [10].

If X is a subspace of an Asplund generated (i.e. GSG) space, then the result is essentially proved in [12]. Indeed, Theorems 3.3 and 3.4 of [12] work with monotone operators T, but the monotonicity of T is used only via [12, Lemma 3.1] (which coincides with [38, Lemma 2]). But [16, Lemma 2.2] (formulated for locally almost nonincreasing T) easily implies (since T is locally almost nonincreasing if and only if $-T$ is submonotone) that [12, Lemma 3.1] holds for submonotone operators as well. □

Remark 4.4. The argument above and [12, Theorem 3.3] show that Proposition 4.3 (and so also Theorem 4.5 below) holds also for "countably dentable spaces" which form a (possibly strictly) larger class than subspaces of Asplund generated (i.e. GSG) spaces. We do not repeat here the (slightly technical) definition of countably dentable spaces.

Theorem 4.5. *Let X be Gâteaux smooth or X be a subspace of an Asplund generated (i.e. GSG) space. Let $G \subset X$ be an open set and $f : G \to \mathbf{R}$ be a continuous approximately convex function on G. Then the set $N_G(f)$ of all points $x \in G$ at which f is not Gâteaux differentiable is σ-cone supported.*

Proof. Lemma 2.5(i), (iii) implies that $T := \partial^C f$ is a locally bounded submonotone mapping on G. So Proposition 4.3 (we put $T(x) := \emptyset$ for $x \notin G$) and (1) immediately give the assertion. □

Since each WCG space is Asplund generated (i.e. GSG) space (see [7], p. 16), Theorem 4.5 gives a positive answer to [30, Problem 5].

Using Theorem 4.5 we obtain in the same way as [41, Proposition 6] the following corollary (which cannot be infered from the fact that $N_G(f)$ is a first category set).

Corollary 4.6. *Let Γ be an uncountable set, $p > 1$ and f be a continuous approximately convex function on $\ell^p(\Gamma)$. Then the set of all points $x \in \ell^1(\Gamma)$ at which f is Gâteaux differentiable is uncountable and dense in $\ell^1(\Gamma)$.*

5. Fréchet differentiability

5.1. Results which follow from Fréchet subdifferentiability of continuous approximatively convex functions

It was observed already in [37] that some differentiability properties of Lipschitz nonconvex functions can be easily obtained from their Fréchet subdifferentiability. [37, Theorem 2] immediately gives:

If X^* is separable and f is Lipschitz function on X, then the set of all points $x \in X$ at which $\partial^F f(x) \neq \emptyset$ and f is not Fréchet differentiable is σ-upper porous.

So, using Lemma 2.5(i), (ii) and separability of X, we easily obtain:

If X^* is separable, $G \subset X$ is an open set and $f : G \to \mathbf{R}$ is a continuous approximately convex function on G, then the set $N_F(f)$ of all points $x \in G$ at which f is not Fréchet differentiable is σ-upper porous.

So the general result [40, Theorem 8] on separable reduction (which concerns generic Fréchet differentiablity) immediately gives:

If X is Asplund, $G \subset X$ is an open set and $f : G \to \mathbf{R}$ is a continuous approximately convex function on G, then the set $N_F(f)$ of all points $x \in G$ at which f is not Fréchet differentiable is a first category set.

Remark 5.1. This result also immediately follows from Lemma 2.5(ii) and [40, Theorem 10] (or [37, Theorem 3]).

So we obtain a generalization of the result [29] on Fréchet differentiability of strongly paraconvex functions on Asplund spaces.

If X is not an Asplund space but $f : X \to \mathbf{R}$ is Fréchet subdifferentiable at all points and we have some information on the set of subdifferentials, then we can sometimes also obtain that f is generically Fréchet differentiable. The following two propositions are immediate consequences of [40, Theorem 9 and Theorem 10*]. (For related results on Fréchet differentiability of convex functions on non-Asplund spaces see [8], [32] and [43].)

Proposition 5.2. *Let X be a Banach space, $G \subset X$ be an open set and $f : G \to \mathbf{R}$ be a continuous approximately convex function on G. Let P be a separable subset of X^*. Then the set A of all points $x \in G$ at which $\partial^F f(x) \cap P \neq \emptyset$ but f is not Fréchet differentiable at x is a first category set.*

To formulate the second proposition, we need the following notion of [8]. A subset K of a dual space X^* is called *separably related to a subset A of X* provided, for every separable bounded subset S of A, the set K is separable for the topology of uniform convergence on S.

Proposition 5.3. *Let X be a Banach space, $G \subset X$ be an open set and $f : G \to \mathbf{R}$ be a continuous approximately convex function on G. Let $K \subset X^*$ be separably related to X and $\partial^F f(x) \cap K \neq \emptyset$ for each $x \in G$. Then the set $N_F(f)$ of all points $x \in G$ at which f is not Fréchet differentiable is a first category set.*

5.2. Results obtained via submonotonicity of the subdifferential mapping

We will need the following lemma which is an easy generalization of Lemma 2(ii) of [38].

Lemma 5.4. *Let X be a Banach space and f be a continuous approximately convex function on an open set $G \subset X$. Let $g : G \to X^*$ be a selection of $\partial^C f$ (i.e., $g(x) \in \partial^C f(x)$ for $x \in G$) and g be continuous at $a \in G$. Then f is Fréchet differentiable at a.*

Proof. Let $\varepsilon > 0$ be given. Choose $\delta_1 > 0$ such that $\|g(x) - g(a)\| < \varepsilon$ for all $x \in B(a, \delta_1)$. Further, by [5, Theorem 2(ii)] we can choose $\delta_2 > 0$ such that $f(x+u) - f(x) \geq \langle x^*, u \rangle - \varepsilon\|u\|$ whenever $x \in B(a, \delta_2)$, $x^* \in \partial^C f(x)$, $\|u\| < \delta_2$ and $x + u \in B(a, \delta_2)$. Put $\delta := \min(\delta_1, \delta_2)$. Then, for each $x \in B(a, \delta)$, we have

$$f(x) - f(a) \geq \langle g(a), x - a \rangle - \varepsilon\|x - a\| \quad \text{and} \tag{12}$$
$$f(a) - f(x) \geq \langle g(x), a - x \rangle - \varepsilon\|x - a\|.$$

Since $\|g(x) - g(a)\| < \varepsilon$, the last inequality implies

$$f(x) - f(a) \leq \langle g(a), x - a \rangle + \langle g(x) - g(a), x - a \rangle + \varepsilon\|x - a\| \leq \langle g(a), x - a \rangle + 2\varepsilon\|x - a\|. \tag{13}$$

The inequalities (12) and (13) immediately imply that $g(a)$ is the Fréchet derivative of f at a. □

Now we can easily prove the following generalization of [38, Proposition 1] (on uniformly almost superdifferentiable functions).

Theorem 5.5. *If X^* is separable, $G \subset X$ is an open set and $f : G \to \mathbf{R}$ is a continuous approximately convex function on G, then the set $N_F(f)$ of all points $x \in G$ at which f is not Fréchet differentiable is angle small.*

Proof. Choose a selection g of $\partial^C f$ on G. Since g is submonotone by Lemma 2.5(*iii*), [38, Lemma 3] implies that there exists an angle small set A such that g is continuous at all points $x \in G \setminus A$. Now Lemma 5.4 implies that f is Fréchet differentiable at all points $x \in G \setminus A$. □

Even if X is a separable Hilbert space, there is known no complete characterization of the smallest σ-ideal containing all sets $N_F(f)$, where f is a continuous convex function on X.

If f is a continuous convex function on a separable superreflexive infinite dimensional Banach space, the set $N_F(f)$ of Fréchet non-differentiable points need not be Haar null (in particular, it need not be Gaussian, resp. Aronszajn null), see [17].

However, if X is an infinite dimensional space with separable X^*, then $N_F(f)$ is Γ-null for each continuous convex f on X [14, Corollary 3.11]. In fact, the proof in [14] gives that this result holds also for continuous approximately convex functions:

Proposition 5.6. *If X^* is separable, $G \subset X$ is an open set and $f : G \to \mathbf{R}$ is a continuous approximately convex function on G, then the set $N_F(f)$ of all points $x \in G$ at which f is not Fréchet differentiable is Γ-null.*

Proof. In [14], the authors work with a notion of "regularity" ([14, Definition 3.1]) which is clearly weaker than Clarke regularity. Therefore f is "regular" at all points by Lemma 2.5(*iv*). So the assertion of the proposition follows (since X is separable and a countable union of Γ-null sets is Γ-null) from [14, Theorem 2.5 and Theorem 3.10]. □

Using Proposition 5.6 and [14, Theorem 2.5], we immediately obtain:

Corollary 5.7. *Let X^* be separable and Y have RNP. Let f be a continuous approximately convex function on X and $g : X \to Y$ be locally Lipschitz. Then the set $N_F(f) \cup N_G(g)$ is Γ-null. In particular, the set of points at which f is Fréchet differentiable and g is Gâteaux differentiable is uncountable and dense in X.*

Remark 5.8. Of course, the proofs give that both Proposition 5.6 and Corollary 5.7 hold for each (Clarke) regular locally Lipschitz f.

The following result is an easy generalization of [16, Theorem 2.6].

Theorem 5.9. *Let X be an Asplund space, $G \subset X$ be an open set and $f : G \to \mathbf{R}$ be a continuous approximately convex function on G. Then there exists a σ-cone supported set $A \subset G$ and a cone-small set $B \subset G$ such that f is Fréchet differentiable at all points of $G \setminus (A \cup B)$.*

Proof. Denote $T(x) := \partial^C f(x)$. Then T is submonotone by Lemma 2.5(iii) and $T(x) \neq \emptyset$ for each $x \in G$. Since f is locally Lipschitz, T is clearly locally bounded and T is norm-to-weak* upper semicontinuous (see e.g. [11, Theorem 1.2.2]). So [16, Proposition 2.3 and Proposition 2.4] imply that there exist a σ-cone supported set $A \subset G$ and a cone-small set $B \subset G$ such that $T = \partial^C f$ is single-valued and norm-to-norm upper semicontinuous at all points of $G \setminus (A \cup B)$. So [11, Theorem 1.2.3] implies that f is Fréchet differentiable at all points of $G \setminus (A \cup B)$. □

6. An example

Recall that a function on an open convex subset of a Banach space is called a d.c. function if it is the difference of two continuous convex functions.

Proposition 6.1. *Let $\mathcal{I}$ be the smallest σ-ideal of subsets of $\mathbf{R}^2$ which contains all sets of the form $N_F(g)$, where g is a continuous convex function on an open convex subset of $\mathbf{R}^2$. Let $\omega \in \mathcal{M}$ be a modulus with $\lim_{t\to 0+} t/\omega(t) = 0$. Then there exists an open convex set $G \subset \mathbf{R}^2$ and a function f on G which is semiconcave with modulus ω and $N_F(f) \notin \mathcal{I}$.*

Proof. It is well-known (see [20]) that there exists a function φ on $[0,1]$ such that $|\varphi(y) - \varphi(x)| \leq \omega(|y-x|)$ and φ has finite derivative at no point $x \in (0,1)$. We can clearly suppose that $1 \leq \varphi(x) \leq 2$ for each $x \in [0,1]$. Define $\psi(x) := \int_0^x \varphi$ for $x \in (0,1)$, and $p(x,y) := \psi(x) - y$ for $(x,y) \in G := (0,1) \times \mathbf{R}$. Since $\psi' = \varphi$ is continuous with modulus of continuity ω, we easily see that p' is continuous with modulus of continuity ω as well. So the proof of [4, Proposition 2.1.2] clearly gives that p is semiconcave on G with modulus ω. Consequently (see [4, Proposition 2.1.5]) the function $f := \min\{p, 0\}$ is semiconcave on G with modulus ω as well. Clearly $f'_y(x,y)$ does not exist for $(x,y) \in M := \{(x,y) : y = \psi(x), x \in (0,1)\}$; so $M \subset N_F(f)$.

Suppose on the contrary that $N_F(f) \in \mathcal{I}$. Then [35, Theorem 1] implies that there exist sequences α_n, β_n of d.c. functions on $\mathbf{R}$ such that $M \subset \bigcup_1^\infty A_n \cup \bigcup_1^\infty B_n$, where $A_n := \{(x,y) : \ y = \alpha_n(x)\}$ and $B_n := \{(x,y) : \ x = \beta_n(y)\}$. The Baire theorem easily implies that there exists an open interval $I \subset (0,1)$ and $n \in \mathbf{N}$ such that

$$\{(x,y) : y = \psi(x), x \in I\} \subset A_n \quad \text{or} \quad \{(x,y) : y = \psi(x), x \in I\} \subset B_n.$$

In the first case clearly $\psi = \alpha_n$ and so $\varphi = \alpha_n'$ on I, which is impossible, since α_n' has locally bounded variation and therefore is a.e. differentiable.

In the second case denote $J := \psi(I)$. Then $\beta_n = \psi^{-1}$ on J and so $\psi|_I = (\beta_n|_J)^{-1}$. Consequently ψ is locally d.c. on I (it follows immediately from [33, Theorem 5.2], since ψ is bilipschitz). Thus we obtain a contradiction as in the first case. □

Recall that (by [4, Proposition 2.1.4]), for each $\alpha \in (0, 1)$, there exists a function u on an open interval $I \subset \mathbf{R}$ which is semiconcave with modulus $\omega(r) = r^\alpha$ and that cannot be written in the form $u = u_1 + u_2$ with u_1 concave and $u_2 \in C^1(I)$. (So, a conjecture in [6, Remark 3.6] concerning local decomposability of lower-$C^{1,\alpha}$ functions does not hold.)

The following easy consequence of Proposition 6.1 give in $\mathbf{R}^2$ a slightly more general and slightly more precise result.

Proposition 6.2. *Let $\omega \in \mathcal{M}$ be a modulus with $\lim_{t \to 0+} t/\omega(t) = 0$. Then there exists an open convex set $G \subset \mathbf{R}^2$ and a function f on G which is semiconcave with modulus ω and which cannot be written in the form $f = g + h$, where g is concave (or d.c.) and h is (Fréchet) differentiable on G.*

Proof. Let f be as in Proposition 6.1. If f can be written in the above form, then $N_F(f) \in \mathcal{I}$, a contradiction. □

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