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Monotonic Analysis over Cones: III*

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Dedicated to the memory of Thomas Lachand-Robert.

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This paper studies the class of increasing and co-radiant (ICR) functions over a cone equipped with an order relation which agrees with the conic structure. In particular, a representation of ICR functions as abstract convex functions is provided. This representation suggests the introduction of some polarity notions between sets. The relationship between ICR functions and increasing positively homogeneous functions is also shown.

Keywords: Monotonic analysis, ICR functions, Abstract convexity, Radiant sets, Co-radiant sets, Normal sets, Co-normal sets

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1. Introduction

It is well known that convex analysis plays a fundamental role in optimization theory. There are many approaches to generalize the notion of convexity in order to tackle non-convex optimization problems. One of the major approaches is to generalize the local aspect of a convex function based on the subdifferential. This gave rise to the subject of nonsmooth analysis. Another approach, which we follow in this paper, is to generalize a very important global aspect of a convex function. It is well known that every proper and lower semicontinuous convex function can be expressed as a pointwise supremum of the family of affine functions majorized by it [6]. Thus affine functions can be considered as

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elementary functions with respect to convex functions since they are the basic building blocks for convex functions. It is natural to see what happens if we replace affine functions with some other class of elementary functions. This gave rise to the subject of *Abstract Convexity*; for details see [5, 9, 7]. Abstract convexity studies *abstract convex functions*, that is, functions that can be realized in a pointwise fashion through the supremum of a class of elementary functions. It has been observed that some classes of increasing functions are abstract convex. This gave rise to a special part of abstract convexity called *Monotonic Analysis*, which studies increasing functions with some additional properties. Notable among them are *Increasing and Positively Homogeneous functions* (IPH) and *Increasing and Convex-Along-Rays functions* (ICAR). The first studies of these functions were carried out over the interior of the cone $\mathbb{R}^n_+$; one can find the details in Rubinov [7]. This was generalized by Dutta, Martínez-Legaz and Rubinov [1, 2], where IPH and ICAR functions defined over cones were studied. This study helped to provide a broader perspective on these classes of functions and threw new light on their properties, which cannot be easily visualized if one restricts oneself to $\mathbb{R}^n_+$. Some other contributions to Monotonic Analysis are [4] and [8].

In this paper we study another class of increasing functions, called *Increasing and Co-radiant* (ICR), over cones. These functions arise, for instance, in mathematical economics; see, e.g., [3], where quasiconcave ICR functions have been studied. Unlike in our previous papers [1] and [2], in this article we do not assume that the ordering is induced by a closed, convex, solid and pointed cone; we rather impose some weaker properties on the order relation, which are still sufficient for the study of ICR functions. Thus this approach differs considerably from the approach in [1] and [2]. We introduce a class of elementary functions, with respect to which ICR functions are abstract convex. The paper is organized as follows. In Section 2 we study the required properties of the order relation and provide the definition of ICR functions and their basic properties. Sections 3 and 4 are devoted to the representation of ICR functions as abstract convex functions and to the study of the properties of their support sets. In Section 5 we study polarity of sets and, finally, in Section 6 we show the relations between IPH and ICR functions.

Before we end this section let us mention the following conventions, which we shall use throughout the article: $\sup\emptyset = 0$, $\inf\emptyset = +\infty$. Some other conventions will be introduced when needed, sometimes in an apparently inconsistent way. For instance, in (19) we use $\frac{0}{0} := 0$, whereas in (20) we take $\frac{0}{0} := +\infty$. In fact, in the given context this second choice is necessary for the sake of consistency with the first one; indeed, if $\frac{a}{b} = 0$ then we should consistently have $\frac{b}{a} = \left(\frac{a}{b}\right)^{-1} = 0^{-1} = +\infty$, even though $\frac{a}{b}$ and $\frac{b}{a}$ yield the same expression $\frac{0}{0}$ when both a and b are equal to 0. We could solve this apparent inconsistency regarding notation by introducing two different operations, a "lower" division and an "upper" division, similarly to the lower addition and upper addition often used in Abstract Convex Analysis, but we prefer to avoid this in order to keep our notation as simple as possible.

2. Preliminaries: ICR functions and their properties

A nonempty set $C \neq \{0\}$ in a vector space X is called a cone if $(x \in C, \lambda > 0) \implies \lambda x \in C$. We consider a cone C equipped with an order relation $\leq$. It is assumed in this paper that this relation agrees with the conic structure of C. This means that the following assumption holds:

Assumption 2.1. $(x, y \in C,\ x \le y;\ \lambda > 0) \Longrightarrow \lambda x \le \lambda y$.

We also make the following assumptions:

Assumption 2.2. $(x \in C,\ 0 < \lambda \le 1) \implies \lambda x \le x$.

Assumption 2.3. $(x, y \in C,\ \lambda_k y \le x,\ \lambda_k \to 1) \implies y \le x$.

It follows from Assumption 2.2 and the transitivity of $\le$ that

$$(x, y \in C,\ y \le x;\ \lambda \le 1) \implies \lambda y \le x. \tag{1}$$

From Assumption 2.2 it also follows that

$$(x \in C,\ \lambda \ge 1) \implies x \le \lambda x. \tag{2}$$

Further using Assumption 2.1 and (2) one has $(x, y \in C,\ x \le y;\ \lambda \ge 1) \implies x \le \lambda y$. As a consequence of Assumptions 2.1 and 2.3, for each $\lambda > 0$ one has:

$$(\lambda_k y \le x,\ \lambda_k \to \lambda) \implies \lambda y \le x. \tag{3}$$

If $0 \in C$ then it makes sense to consider Assumption 2.2 also for $\lambda = 0$. This condition holds if and only if $0 \le x$ for all $x \in C$, that is, 0 is the least element of C. This property will also be assumed throughout the paper:

Assumption 2.4. If $0 \in C$ then $0 \le x$ for all $x \in C$.

A function $f : C \to \mathbb{R} \cup \{+\infty\}$ is called increasing if $x \le y \implies f(x) \le f(y)$; f is called co-radiant if $f(\lambda x) \ge \lambda f(x)$ for all $x \in C$ and $\lambda \in (0, 1]$. In this paper we study increasing co-radiant (ICR) functions.

We shall now define two important types of sets associated with our study, namely radiant and co-radiant sets. A set $B \subseteq C$ is called *radiant* if $x \in B$ and $0 < \lambda \le 1$ imply that $\lambda x \in B$. A set $B \subseteq C$ is called *co-radiant* if $x \in B$ and $\lambda \ge 1$ imply that $\lambda x \in B$. The Minkowski gauge $\mu_B : C \to \mathbb{R} \cup \{+\infty\}$ of a radiant set B is defined by

$$\mu_B(x) := \inf\left\{\lambda > 0 : \frac{x}{\lambda} \in B\right\}.$$

Two other types of sets that arise in Monotonic Analysis are the normal and the co-normal sets. Assume as before that $B \subset C$. The set B is said to be *normal* if $y \in C$, $x \in B$ and $y \le x$ imply that $y \in B$. The set B is said to be *co-normal* if $y \in C$, $x \in B$ and $x \le y$ imply that $y \in B$. Notice that, by Assumption 2.2, every normal (co-normal) set is radiant (respectively, co-radiant). The *normal hull* of a set B, denoted $Nh(B)$, is defined as the smallest normal set in C containing B:

$$Nh(B) := \{z \in C : z \le x \text{ for some } x \in B\}.$$

Proposition 2.5.

1) f *is co-radiant if and only if* $f(\mu x) \le \mu f(x)$ *for all* $x \in C$ *and* $\mu > 1$.

2) *If f is ICR and $f(x) = +\infty$ then $f(\lambda x) = +\infty$ for all $\lambda > 0$.*

3) *An ICR function f is nonnegative.*

4) *If f is increasing and $0 \in C$ then $f(0) = \min_{x\in C} f(x)$.*

Proof. *1)* Let $\mu > 1$, $\lambda = \mu^{-1}$ and $\mu x = y$. Then the inequality $f(\lambda y) \geq \lambda f(y)$ is equivalent to $f(\mu x) \leq \mu f(x)$;

2) Let f is ICR and $f(x) = +\infty$. It follows from Assumptions 2.1 and 2.2 and the monotonicity of f that $f(\lambda x) \geq f(x)$ for $\lambda > 1$. We also have $f(\lambda x) \geq \lambda f(x)$ for $\lambda \in (0, 1)$. So $f(\lambda x) = +\infty$ for all $\lambda > 0$.

3) Let $x \in C$. Assume without loss of generality that $f(x) < +\infty$. The function $f_x(\lambda) = f(\lambda x)$ $(\lambda > 0)$ is increasing so there exists $\lim_{\lambda\to 0+} f_x(\lambda)$. Since $f(\lambda x) \geq \lambda f(x)$ for $\lambda \in (0, 1)$ it follows that $\lim_{\lambda\to 0+} f_x(\lambda) \geq 0$, therefore $f(x) = f_x(1) \geq 0$.

4) It follows from Assumption 2.4. □

The set of all ICR functions is closed under pointwise convergence, this set is a convex cone (if f and g are ICR and $\alpha, \beta > 0$ then $\alpha f + \beta g$ is ICR) and a complete lattice (if $(f_t)_{t\in T}$ is a family of ICR functions then $\bar{f}(x) := \sup_{t\in T} f_t(x)$ and $\underline{f}(x) := \inf_{t\in T} f_t(x)$ are ICR functions).

A function $f : C \to \mathbb{R}_{+\infty}$ is called positively homogeneous of the first degree if $f(\lambda x) = \lambda f(x)$ for all $\lambda > 0$. Increasing positively homogeneous of the first degree (IPH) functions form an important subclass of the class of ICR functions.

For each $x, y \in C$ consider the set $\Lambda_{x,y} := \{0\} \cup \{\lambda > 0 : \lambda y \leq x\}$. It follows from (1) that this set is a segment with the left endpoint equal to zero. Let us show that $\Lambda_{x,y}$ is a closed set. Without loss of generality assume that $\overline{\lambda} := \sup \Lambda_{x,y} < +\infty$. If $\overline{\lambda} > 0$ then in view of (3) we get $\overline{\lambda} \in \Lambda_{x,y}$. If $\overline{\lambda} = 0$ then $\Lambda_{x,y} = \{0\}$, hence $\Lambda_{x,y}$ is closed.

Define $l : C \times C \to [0, +\infty]$ by

$$l(x, y) := \sup\{\lambda > 0 : \lambda y \leq x\}. \tag{4}$$

Since the set $\Lambda_{x,y}$ is closed it follows that either $l(x, y) = 0$, or $l(x, y) = +\infty$, or $l(x, y) = \max\{\lambda > 0 : \lambda y \leq x\}$. It is easy to see that for each $x, y \in C$ and $\lambda > 0$ it holds:

$$l(\lambda x, y) = \lambda l(x, y), \qquad l(x, \lambda y) = \lambda^{-1} l(x, y), \tag{5}$$

$$x \leq x' \implies l(x, y) \leq l(x', y), \qquad y \leq y' \implies l(x, y) \geq l(x, y'). \tag{6}$$

Since $y \leq x$ is valid if and only if $\lambda = 1$ belongs to the closed segment $\Lambda_{x,y}$ it follows that

$$y \leq x \iff l(x, y) \geq 1 \tag{7}$$

We now present some examples of co-radiant and ICR functions.

Example 2.6. Let C be a conic set and $f : C \to [0, +\infty]$ be a concave-along-rays function, that is, the function f_x defined on $(0, +\infty)$ by $f_x(t) = f(tx)$ is concave for all $x \in C$. Then f is co-radiant; indeed, for every $x \in C$, $\lambda \in (0, 1]$ and $\epsilon \in (0, \lambda)$ one has

$$f(\lambda x) = f_x(\lambda) = f_x\left(\frac{1-\lambda}{1-\epsilon}\epsilon + \frac{\lambda-\epsilon}{1-\epsilon}1\right) \geq \frac{1-\lambda}{1-\epsilon} f_x(\epsilon) + \frac{\lambda-\epsilon}{1-\epsilon} f_x(1) \geq \frac{\lambda-\epsilon}{1-\epsilon} f_x(1),$$

and by setting ϵ go to 0 in this inequality one obtains $f(\lambda x) \geq \lambda f(x)$. In particular, a positively homogeneous function of degree $\delta \in (0,1]$ is co-radiant.

Example 2.7. Let X be the space $C(S)$ of continuous functions defined on a compact subset S of the Euclidean space $\mathbb{R}^n$. Consider the cone $C \subset C(S)$ of nonnegative functions with the order relation $\geq$ generated by this cone: $x \leq y \implies y - x \in C$. Let

$$f(x) := \int_S x(s)^{\delta(s)} d\mu,$$

where $x \in C$, $\delta(s) \in (0,1]$ for all $s \in S$ and μ is a nonnegative measure on S. It is easy to check that f is an ICR function on C.

Example 2.8. Let $X = \mathbb{R}^2$ and let C be the cone $\mathbb{R}^2_{++}$ of vectors with positive coordinates. Consider the order relation $\leq$ defined on C by

$$(x_1, x_2) \leq (y_1, y_2) \iff x_1 \leq y_1 \quad \text{and} \quad \frac{y_2}{x_2} \leq \frac{y_1}{x_1}.$$

It is easy to check that Assumptions 2.1, 2.2 and 2.3 are valid for $\leq$. Consider the function $f : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$, where $f(x_1, x_2) = \frac{x_1}{x_2}$. The restriction of f to the ray $R_x = \{\lambda x : \lambda > 0\}$ is a constant positive function for each $x \in \mathbb{R}^n_{++}$, hence f is co-radiant. If $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2_{++}$ and $(x_1, x_2) \leq (y_1, y_2)$ then $\frac{x_1}{x_2} \leq \frac{y_1}{y_2}$.This means that f is increasing, hence f is ICR.

3. The supremal generator approach

Let $C \neq \{0\}$ be a nonempty cone in a vector space X equipped with an order relation $\leq$ satisfying Assumptions 2.1, 2.2, 2.3 and 2.4, and define $c : C \times C \to [0,1]$ by

$$c(x, y) := \max\{\lambda \in [0,1] : \lambda y \leq x\} = \min\{l(x, y), 1\}, \tag{8}$$

where l is defined by (4).

Proposition 3.1. *For every $x, x', y, y' \in C$ and $\lambda \in (0,1]$, one has*

$$c(\lambda x, y) \geq \lambda c(x, y), \tag{9}$$
$$c(x, \lambda y) \leq \lambda^{-1} c(x, y), \tag{10}$$
$$x \leq x' \Longrightarrow c(x, y) \leq c(x', y) \tag{11}$$
$$y \leq y' \Longrightarrow c(x, y) \geq c(x, y'), \tag{12}$$
$$y \leq x \Longleftrightarrow c(x, y) = 1. \tag{13}$$

The proof easily follows from the properties of function l (see (5), (6) and (7)).

Proposition 3.2. *Let $x, y \in C$ be such that $c(x, y) > 0$. Then $c(x, y)y \leq x$.*

Proof. Since $c(x, y) > 0$ it follows that the set $\{\lambda > 0 : \lambda y \leq x\}$ is not empty. Let $\overline{\lambda} = \sup\{\lambda > 0 : \lambda y \leq x\}$. If $\overline{\lambda} > 1$ then $c(x, y) = 1$ and $c(x, y)y = y \leq x$. Let $\overline{\lambda} \leq 1$. Since the segment $\Lambda_{x,y} = \{0\} \cup \{\lambda > 0 : \lambda y \leq x\}$ is closed it follows that $\overline{\lambda} \in \Lambda_{x,y}$ and $\overline{\lambda} \neq 0$. Then $\overline{\lambda} y \leq x$ and $\overline{\lambda} = c(x, y)$. □

Theorem 3.3. *Let $f : C \to [0, +\infty]$. Then the following statements are equivalent:*

(*i*) f *is ICR.*
(*ii*) $f(x) \geq \lambda f(y)$ *for all* $x, y \in C$ *and* $\lambda \in (0, 1]$ *such that* $\lambda y \leq x$.
(*iii*) $f(x) \geq c(x, y) f(y)$ *for all* $x, y \in C$.

In (*iii*) *we use the convention* $0 \cdot (+\infty) := 0$.

Proof. $(i) \Longrightarrow (ii)$. Obvious.

$(ii) \Longrightarrow (iii)$. Let $c(x, y) > 0$. Then in view of Proposition 3.2 we get $c(x, y)y \leq x$, so $c(x, y)f(y) \leq f(x)$. Since f is nonnegative, this inequality also holds if $c(x, y) = 0$ using the convention that $0(+\infty) = 0$.

$(iii) \Longrightarrow (i)$. Let (iii) hold. Then f is increasing; indeed, if $x, y \in C$ are such that $y \leq x$ then, by (8), $c(x, y) = 1$ and hence (iii) yields $f(x) \geq f(y)$. Finally, if $y \in C$ and $\lambda \in (0, 1]$ then, by (iii), (9) and (13), $f(\lambda y) \geq c(\lambda y, y) f(y) \geq \lambda c(y, y) f(y) = \lambda f(y)$; therefore f is co-radiant. □

Given $y \in C$, let us define $c_y : C \to [0, 1]$ by $c_y(x) := c(x, y)$ for all $x \in C$. It follows from (9) that c_y is an ICR function. We introduce the following class of functions:

$$\mathcal{L} := \{\alpha c_y : y \in C,\ \alpha \in \mathbb{R}_{++}\}$$

and the mapping $\psi : C \times \mathbb{R}_{++} \to \mathcal{L}$ defined by

$$\psi(y, \alpha) := \alpha c_y.$$

Let us now consider the pointwise ordering in $\mathcal{L}$: For any pair of points $(y, \alpha), (y', \alpha')$ in $C \times \mathbb{R}_{++}$, $\alpha c_y \geq \alpha' c_{y'}$ if and only if $\alpha c_y(x) \geq \alpha' c_{y'}(x)$ for all $x \in C$.

Proposition 3.4. *The mapping* ψ *is a bijection from* $C \times \mathbb{R}_{++}$ *onto* $\mathcal{L}$. *Moreover, it is increasing in* α *and decreasing in* y:

$$y \leq y' \Longleftrightarrow \psi(y, \alpha) \geq \psi(y', \alpha) \quad (y, y' \in C,\ \alpha \in \mathbb{R}_{++}). \tag{14}$$

$$\alpha \leq \alpha' \Longleftrightarrow \psi(y, \alpha) \leq \psi(y, \alpha') \quad (y \in C,\ \alpha, \alpha' \in \mathbb{R}_{++}). \tag{15}$$

Proof. We will first prove the equivalences. The implication $\Longrightarrow$ in (14) follows from (12) and the nonnegativity of the functions c_y and $c_{y'}$. To prove $\Longleftarrow$, assume that $y, y' \in C$ and $\alpha \in \mathbb{R}_{++}$ are such that $\psi(y, \alpha) \geq \psi(y', \alpha)$. Since $\alpha > 0$, one has $c_y \geq c_{y'}$; hence, using (13), we obtain $c(y', y) = c_y(y') \geq c_{y'}(y') = 1$. As $c(y', y) \in [0, 1]$, we deduce that $c(y', y) = 1$; therefore, by (13), $y \leq y'$. The implication $\Longrightarrow$ in (15) is obvious, since the function c_y is nonnegative. To prove the converse implication, notice that, by (13),

$$\psi(y, \alpha)(y) = \alpha c_y(y) = \alpha c(y, y) = \alpha;$$

hence from $\psi(y, \alpha) \leq \psi(y, \alpha')$ we get $\alpha = \psi(y, \alpha)(y) \leq \psi(y, \alpha')(y) = \alpha'$.

Since, by the definition of $\mathcal{L}$, ψ is onto, to prove that it is a bijection it only remains to prove that it is one-to-one. Let $\alpha, \alpha' \in \mathbb{R}_{++}$ and $y, y' \in C$ be such that $\psi(y, \alpha) = \psi(y', \alpha')$. We have

$$\alpha = \psi(y, \alpha)(y) = \psi(y', \alpha')(y) = \alpha' c_{y'}(y) \leq \alpha';$$

by symmetry, we also obtain $\alpha' \leq \alpha$, so that $\alpha = \alpha'$ and hence $\psi(y, \alpha) = \psi(y', \alpha)$. Using (14), we conclude that $y = y'$. □

For a function $f : C \to [0, +\infty]$ define the positive hypograph $hypo^+ f$ by

$$hypo^+ f := \{(y, \alpha) \in C \times \mathbb{R}_{++} : f(y) \geq \alpha\} \tag{16}$$

(recall that that the hypograph of f is $hypo\,(f) = \{(y, \alpha) \in C \times \mathbb{R} : f(y) \geq \alpha\}$). The notion of positive hypograph fits well to nonnegative functions, as one can recover f from $hypo^+ f$ by

$$f(y) = \sup \left\{\alpha : (y, \alpha) \in hypo^+ f\right\} \quad (y \in C) \tag{17}$$

in view of our convention $\sup \emptyset = 0$. It follows directly from the definition of a co-radiant function and the definition of the positive hypograph that $hypo^+ f$ is a radiant set if and only if f is co-radiant. It is easy to check that f is increasing if and only if the section $\{x \in C : (x, \alpha) \in hypo^+ f\}$ of the set $hypo^+ f$ is a co-normal set for every $\alpha \in \mathbb{R}_{++}$.

Theorem 3.5. *A function $f : C \to [0, +\infty]$ is ICR if and only if there exists a set $S \subseteq C \times \mathbb{R}_{++}$ such that*

$$f(x) = \sup_{(y,\alpha) \in S} \alpha c_y(x) \quad (x \in C) \tag{18}$$

Hence, $f : C \to [0, +\infty]$ is ICR if and only if it is $\mathcal{L}$-convex.

Proof. It is easy to prove that $\mathcal{L}$ consists of ICR functions; hence the pointwise supremum of such functions is also ICR. We will now prove that (18) holds by taking S equal to the positive hypograph $hypo^+\ f$ of f. This is obviously true if $\text{hypo}^+\ f = \emptyset$, as this means that f is identically 0. Assume then that $\text{hypo}^+\ f \neq \emptyset$. For any $(y, \alpha) \in \text{hypo}^+\ f$, one has $\alpha c_y(x) \leq f(y)\, c(x, y) \leq f(x)$, due to (*iii*) of Theorem 3.3. Hence the inequality $\geq$ holds in (18). We next prove that for every $x \in C$ with $f(x) < +\infty$ there exists $(y, \alpha) \in \text{hypo}^+\ f$ such that $\alpha c_y(x) = f(x)$. If $f(x) > 0$, take $(y, \alpha) = (x, f(x))$; one then has $\alpha c_y(x) = f(x)\, c_x(x) = f(x)\, c(x, x) = f(x)$, in view of (13). If $f(x) = 0$ then take any $(y, \alpha) \in \text{hypo}^+\ f$ and observe that $f(x) \geq c(x, y) f(y) \geq \alpha c_y(x) \geq 0 = f(x)$. Thus, to prove (18) it only remains to consider the case when $f(x) = +\infty$. In this case $(x, \alpha) \in \text{hypo}^+\ f$ for any $\alpha > 0$ and so one has $\sup_{(y,\alpha) \in S} \alpha c_y(x) \geq \sup_{\alpha > 0} \alpha c_x(x) = \sup_{\alpha > 0} \alpha c(x, x) = \sup_{\alpha > 0} \alpha = +\infty = f(x)$. □

Definition 3.6. The sup-polar function of $f : C \to [0, +\infty]$ is the function $f^\nabla : C \to [0, +\infty]$ defined by

$$f^\nabla(y) := \sup_{x \in C} \frac{c(x, y)}{f(x)} \tag{19}$$

(with the convention $\frac{0}{0} := 0$).

We will mainly consider the function $\frac{1}{f^\nabla}$. It follows from the definition of f^∇ that

$$\frac{1}{f^\nabla}(y) = \inf_{x \in C} \frac{f(x)}{c(x, y)} \tag{20}$$

(for this formula to be consistent with the convention adopted in the definition of f^∇, we need here the opposite convention $\frac{0}{0} := +\infty$).

Theorem 3.7. *Let $f : C \to [0, +\infty]$. Then $\frac{1}{f^\nabla}$ is the largest ICR minorant of f. Hence*

$$f^\nabla \geq \frac{1}{f},$$

and f is ICR if and only if

$$f^\nabla = \frac{1}{f}.$$

Proof. By (20), (12) and (10), $\frac{1}{f^\nabla}$ is ICR. Moreover, for every $y \in C$ one has $f^\nabla(y) \geq \frac{c(y,y)}{f(y)} = \frac{1}{f(y)}$; hence $\frac{1}{f^\nabla}$ is an ICR minorant of f. Let g be any ICR minorant of f and let $y \in C$. By Theorem 3.3, for every $x \in C$ we have $g(y) \leq \frac{g(x)}{c(x,y)} \leq \frac{f(x)}{c(x,y)}$ (with the convention $\frac{0}{0} := +\infty$); therefore $g(y) \leq \inf_{x \in C} \frac{f(x)}{c(x,y)} = \frac{1}{f^\nabla(y)}$. This shows that $\frac{1}{f^\nabla}$ is the largest ICR minorant of f. □

Consider the lower support set of a function $f : C \to [0, +\infty]$ with respect to $\mathcal{L}$:

$$\operatorname{supp}_l f := \{(y, \alpha) \in C \times \mathbb{R}_{++} : \alpha c_y \leq f\}.$$

The next proposition shows that, in the case of an ICR function f, this set coincides with its positive hypograph as defined in (16).

Proposition 3.8. *Let $f : C \to [0, +\infty]$. Then*

$$\operatorname{supp}_l f = hypo^+ \frac{1}{f^\nabla}. \tag{21}$$

Hence, f is ICR if and only if

$$\operatorname{supp}_l f = hypo^+ f. \tag{22}$$

Proof.

$$\begin{aligned} \operatorname{supp}_l f &= \{(y, \alpha) \in C \times \mathbb{R}_{++} : \alpha c_y \leq f\} \\ &= \{(y, \alpha) \in C \times \mathbb{R}_{++} : \alpha c(x, y) \leq f(x) \quad \forall\, x \in C\} \\ &= \left\{(y, \alpha) \in C \times \mathbb{R}_{++} : \alpha f^\nabla(y) \leq 1\right\} = \text{hypo}^+ \frac{1}{f^\nabla}. \end{aligned}$$

By Theorem 3.7, f is ICR if and only if $\frac{1}{f^\nabla} = f$, which is equivalent to the equality $\text{hypo}^+ \frac{1}{f^\nabla} = \text{hypo}^+ f$; in view of (21) this is in turn equivalent to (22). □

The following definition will be useful to characterize lower supports.

Definition 3.9. A subset U of $C \times \mathbb{R}_{++}$ is called hypographical if $U = hypo^+ f$ for some function $f : C \to [0, +\infty]$.

The following proposition is immediate.

Proposition 3.10. *A subset U of $C \times \mathbb{R}_{++}$ is hypographical if and only if for every $y \in C$ the set $\{\alpha \in \mathbb{R}_{++} : (y, \alpha) \in U\}$ is either of the form $(0, \overline{\alpha}]$, with $\overline{\alpha} \in [0, +\infty)$, or $\mathbb{R}_{++}$.*

Proposition 3.11. *For any set $U \subseteq C \times \mathbb{R}_{++}$, the following statements are equivalent:*

(*i*) *There exists a function $f : C \to [0, +\infty]$ such that* $\operatorname{supp}_l f = U$.

(*ii*) *There exists an ICR function $f : C \to [0, +\infty]$ such that* $\operatorname{supp}_l f = U$.

(*iii*) *U is hypographical and radiant and for every $\alpha \in \mathbb{R}_{++}$ the section $U_\alpha := \{y \in C : (y, \alpha) \in U\}$ is co-normal.*

Furthermore, the function f of (ii) is unique, namely, it is determined by the equality $hypo^+ f = U$.

Proof. (*i*) $\Longrightarrow$ (*iii*). If (*i*) holds then, by Proposition 3.8, U is hypographical; moreover, it is easy to check that it is radiant and has co-normal sections U_α, using (10) and (12), respectively.

(*iii*) $\Longrightarrow$ (*ii*). Assume that $U = \operatorname{hypo}^+ f$ for some function $f : C \to [0, +\infty]$. If the sections U_α are co-normal then f is increasing, and if U is radiant then f is co-radiant. So f is ICR and hence, by (22), $\operatorname{supp}_l f = U$.

Implication (*ii*) $\Longrightarrow$ (*i*) is obvious.

The last assertion in the statement follows from (17) and (22). □

Corollary 3.12. *The mapping $f \longmapsto hypo^+ f$ is a bijection from the set of all ICR functions $f : C \to [0, +\infty]$ onto the set of all hypographical radiant sets $U \subseteq C \times \mathbb{R}_{++}$ with co-normal sections U_α.*

Associated with the duality mapping $f \mapsto f^\nabla$, we introduce the ∇-subdifferential of $f : C \to [0, +\infty]$ at a point $x_0 \in f^{-1}((0, +\infty))$ as follows:

$$\partial^\nabla f(x_0) := \left\{ y \in C : \frac{f(x)}{f(x_0)} \geq \frac{c(x, y)}{c(x_0, y)} \quad \forall\, x \in C \right\} \tag{23}$$

(with the convention $\frac{0}{0} := 0$). The next assertion gives a reformulation of the definition of ∇-subdifferential in terms of the sup-polar function f^∇.

Proposition 3.13. *Let $f : C \to [0, +\infty]$, $x_0 \in f^{-1}((0, +\infty))$ and $y \in C$. Then $y \in \partial^\nabla f(x_0)$ if and only if $f^\nabla(y) = \frac{c(x_0, y)}{f(x_0)}$.*

The next proposition, whose proof is immediate, shows the relationship existing between the ∇-subdifferentials of a function and its support set.

Proposition 3.14. *Let $f : C \to [0, +\infty]$, $x_0 \in f^{-1}((0, +\infty))$ and $y \in C$ be such that $c(x_0, y) > 0$. Then $y \in \partial^\nabla f(x_0)$ if and only if $\left(y, \frac{f(x_0)}{c(x_0, y)}\right) \in \operatorname{supp}_l f$.*

Combining Proposition 3.13 with Theorem 3.7, one easily obtains the following result:

Theorem 3.15. *Let $f : C \to [0, +\infty]$ and $x_0 \in f^{-1}((0, +\infty))$. Then*

$$\partial^\nabla f(x_0) \subseteq \left\{ y \in C : f(y) \geq \frac{f(x_0)}{c(x_0, y)} \right\}. \tag{24}$$

If f is ICR then equality holds in (24) and one also has

$$\partial^{\nabla} f(x_0) = \left\{ y \in C : f(y) = \frac{f(x_0)}{c(x_0, y)} \right\}.$$

4. The infimal generator approach

We define $d : C \times C \to [1, +\infty]$ by

$$d(x, y) := \frac{1}{c(y, x)} = \min\{\mu \geq 1 : x \leq \mu y\} \tag{25}$$

(here and in the sequel we use the convention $\min \emptyset := +\infty$). One has

$$d(x, y) = \max\{u(x, y), 1\}, \tag{26}$$

with

$$u(x, y) := \frac{1}{l(y, x)} = \min\{\mu \geq 0 : x \leq \mu y\}.$$

Using (25) we reformulate Proposition 3.1 in the following form:

Proposition 4.1. *For every $x, x', y, y' \in C$ and $\mu \geq 1$, one has*

$$\begin{aligned}
&d(\mu x, y) \leq \mu d(x, y), \\
&d(x, \mu y) \geq \mu^{-1} d(x, y), \\
&x \leq x' \Longrightarrow d(x, y) \leq d(x', y), \\
&y \leq y' \Longrightarrow d(x, y) \geq d(x, y'), \\
&y \leq x \Longleftrightarrow d(x, y) = 1,
\end{aligned}$$

Proposition 3.2 can be presented as

Proposition 4.2. *Let $x, y \in C$ be such that $d(x, y) < +\infty$. Then $x \leq d(x, y)y$.*

We also reformulate Theorem 3.3:

Theorem 4.3. *Let $f : C \to [0, +\infty]$. Then the following statements are equivalent:*

(*i*) *f is ICR.*
(*ii*) *$f(x) \leq \mu f(y)$ for all $x, y \in C$ and $\mu \geq 1$ such that $x \leq \mu y$.*
(*iii*) *$f(x) \leq d(x, y) f(y)$ for all $x, y \in C$ (here and in the sequel we use the convention $(+\infty) 0 := +\infty$).*

Given $y \in C$, let us define $d_y : C \to [1, +\infty)$ by $d_y(x) := d(x, y)$ for all $x \in C$. We introduce the following class of functions:

$$\mathcal{U} := \{\alpha d_y : y \in C,\ \alpha \in \mathbb{R}_{++}\}$$

and the mapping $\eta : C \times \mathbb{R}_{++} \to \mathcal{U}$ defined by

$$\eta(y, \alpha) := \alpha d_y.$$

The statements below are a reformulation of the corresponding statements from Section 3 and we omit their proofs.

Proposition 4.4. *The mapping η is a bijection from $C \times \mathbb{R}_{++}$ onto $\mathcal{U}$. Moreover, it is increasing in α and decreasing in y :*

$$y \leq y' \Longleftrightarrow \eta(y,\alpha) \geq \eta(y',\alpha) \quad (y, y' \in C,\ \alpha \in \mathbb{R}_{++}).$$
$$\alpha \leq \alpha' \Longleftrightarrow \eta(y,\alpha) \leq \eta(y,\alpha') \quad (y \in C,\ \alpha, \alpha' \in \mathbb{R}_{++}).$$

Theorem 4.5. *A function $f : C \to [0,+\infty]$ is ICR if and only if there exists a set $T \subseteq C \times \mathbb{R}_{++}$ such that*

$$f(x) = \inf_{(y,\alpha)\in T} \alpha d_y(x) \quad (x \in C).$$

Hence, $f : C \to [0,+\infty]$ is ICR if and only if it is $\mathcal{U}$-concave.

Definition 4.6. The inf-polar function of $f : C \to [0,+\infty]$ is the function $f_\nabla : C \to [0,+\infty]$ defined by

$$f_\nabla(y) := \inf_{x\in C} \frac{d(x,y)}{f(x)}$$

(with the convention $\frac{+\infty}{+\infty} := +\infty$).

Theorem 4.7. *Let $f : C \to [0,+\infty]$. Then $\frac{1}{f_\nabla}$ is the smallest ICR majorant of f. Hence*

$$f_\nabla \leq \frac{1}{f},$$

and f is ICR if and only if

$$f_\nabla = \frac{1}{f}.$$

On combining Theorems 3.7 and 4.7, one gets

Theorem 4.8. *Let $f : C \to [0,+\infty]$. Then $f_\nabla \leq f^\nabla$, and f is ICR if and only if $f_\nabla = f^\nabla$.*

The set

$$\operatorname{supp}_u f := \{(y,\alpha) \in C \times \mathbb{R}_{++} : \alpha d_y \geq f\}$$

is called the upper support of $f : C \to [0,+\infty]$. The next proposition shows that, in the case of an ICR function f, this set coincides with its positive epigraph, defined as follows:

$$epi^+ f := \{(y,\alpha) \in C \times \mathbb{R}_{++} : f(y) \leq \alpha\}.$$

Similarly to the case of the positive hypograph introduced in the preceding section, the definition of positive epigraph fits well to nonnegative functions, as one can recover f from $epi^+ f$ by

$$f(y) = \inf\{\alpha : (y,\alpha) \in epi^+ f\} \quad (y \in C). \tag{27}$$

Proposition 4.9. *Let $f : C \to [0,+\infty]$. Then*

$$\operatorname{supp}_u f = epi^+ \frac{1}{f_\nabla}.$$

Hence, f is ICR if and only if

$$\operatorname{supp}_u f = epi^+ f.$$

The following definition will be useful to characterize upper supports.

Definition 4.10. A subset U of $C \times \mathbb{R}_{++}$ is called epigraphical if $U = epi^{+} f$ for some function $f : C \to [0, +\infty]$.

Proposition 4.11. *A subset U of $C \times \mathbb{R}_{++}$ is epigraphical if and only if for every $y \in C$ the set $\{\alpha \in \mathbb{R}_{++} : (y, \alpha) \in U\}$ is either $\mathbb{R}_{++}$ or of the form $[\overline{\alpha}, +\infty)$, with $\overline{\alpha} \in (0, +\infty]$.*

Upper support sets are characterized next.

Proposition 4.12. *For any set $U \subseteq C \times \mathbb{R}_{++}$, the following statements are equivalent:*

(i) *There exists a function $f : C \to [0, +\infty]$ such that* $\operatorname{supp}_u f = U$.

(ii) *There exists an ICR function $f : C \to [0, +\infty]$ such that* $\operatorname{supp}_u f = U$.

(iii) *U is epigraphical and co-radiant and for every $\alpha \in \mathbb{R}_{++}$ the section U_α is normal.*

Furthermore, the function f of (ii) is unique.

Corollary 4.13. *The mapping $f \longmapsto epi^{+} f$ is a bijection from the set of all ICR functions $f : C \to [0, +\infty]$ onto the set of all epigraphical co-radiant sets $U \subseteq C \times \mathbb{R}_{++}$ with normal sections U_α.*

Associated with the duality mapping $f \mapsto f_\nabla$, we introduce the ∇-superdifferential of $f : C \to [0, +\infty]$ at a point $x_0 \in f^{-1}((0, +\infty))$ as follows:

$$\partial_\nabla f(x_0) := \left\{ y \in C : \frac{f(x)}{f(x_0)} \leq \frac{d(x, y)}{d(x_0, y)} \quad \forall\, x \in C \right\}.$$

Proposition 4.14. *Let $f : C \to [0, +\infty]$, $x_0 \in f^{-1}((0, +\infty))$ and $y \in C$. Then $y \in \partial_\nabla f(x_0)$ if and only if $f_\nabla(y) = \frac{d(x_0, y)}{f(x_0)}$.*

Proposition 4.15. *Let $f : C \to [0, +\infty]$, $x_0 \in f^{-1}((0, +\infty))$ and $y \in C$ be such that $d(x_0, y) < +\infty$. Then $y \in \partial_\nabla f(x_0)$ if and only if $\left(y, \frac{f(x_0)}{d(x_0, y)}\right) \in \operatorname{supp}_u f$.*

Proposition 4.16. *Let $f : C \to [0, +\infty]$ and $x_0 \in f^{-1}((0, +\infty))$. Then*

$$\partial_\nabla f(x_0) \subseteq \left\{ y \in C : f(y) \leq \frac{f(x_0)}{d(x_0, y)} \right\}. \tag{28}$$

If f is ICR then equality holds in (28) and one also has

$$\partial_\nabla f(x_0) = \left\{ y \in C : f(y) = \frac{f(x_0)}{d(x_0, y)} \right\}.$$

5. Polarity between sets

Motivated by the abstract convexity representations of ICR functions given in the preceding sections, we will now introduce and study several related polarities between sets.

Definition 5.1. Let $U \subseteq C \times \mathbb{R}_{++}$. The left polar set of U is

$$U^{l} := \{(x, \beta) \in C \times \mathbb{R}_{++} : \alpha c(x, y) \leq \beta \;\; \forall\; (y, \alpha) \in U\}.$$

Proposition 5.2. *Let $U \subseteq C \times \mathbb{R}_{++}$. Then*

$$U^l = \mathrm{supp}_u \; h_U,$$

$h_U : C \to [0, +\infty]$ *being the function defined by*

$$h_U(y) := \sup\{\alpha : (y, \alpha) \in U\} \quad (y \in C).$$

Proof. The proof is straightforward:

$$\begin{aligned} U^l &= \{(x,\beta) \in C \times \mathbb{R}_{++} : \alpha c(x,y) \leq \beta \;\; \forall \; (y,\alpha) \in U\} \\ &= \{(x,\beta) \in C \times \mathbb{R}_{++} : \alpha \leq \beta d(y,x) \;\; \forall \; (y,\alpha) \in U\} \\ &= \{(x,\beta) \in C \times \mathbb{R}_{++} : h_U(y) \leq \beta d(y,x) \;\; \forall \; (y,\alpha) \in U\} \\ &= \{(x,\beta) \in C \times \mathbb{R}_{++} : h_U \leq \beta d_x\} = \mathrm{supp}_u \; h_U. \end{aligned}$$

□

Definition 5.3. Let $U \subseteq C \times \mathbb{R}_{++}$. The right polar set of U is

$$U^r := \{(y,\alpha) \in C \times \mathbb{R}_{++} : \alpha c(x,y) \leq \beta \;\; \forall \; (x,\beta) \in V\}.$$

Similarly to Proposition 5.2, we have:

Proposition 5.4. *Let $U \subseteq C \times \mathbb{R}_{++}$. Then*

$$U^r = \mathrm{supp}_l \; e_U,$$

$e_U : C \to [0, +\infty]$ *being the function defined by*

$$e_U(x) := \inf\{\beta : (x, \beta) \in U\} \quad (x \in C).$$

The sets that are closed under the closure operators $U \longmapsto U^{lr}$ and $U \longmapsto U^{rl}$ are identified in the next theorem.

Theorem 5.5. *Let $U \subseteq C \times \mathbb{R}_{++}$. Then the following statements hold:*

(*1*) *One has $U = U^{lr}$ if and only if U is hypographical and radiant and for every $\alpha \in \mathbb{R}_{++}$ the section $U_\alpha := \{y \in C : (y,\alpha) \in U\}$ is co-normal.*

(*2*) *One has $U = U^{rl}$ if and only if U is epigraphical and co-radiant and for every $\alpha \in \mathbb{R}_{++}$ the section U_α is normal.*

Proof. We will only prove (*1*), since the proof of (*2*) is very similar.

If $U = U^{lr}$ then, by Proposition 5.4, $U = \mathrm{supp}_l \; e_{U^l}$; hence, by Proposition 4.12, U is hypographical and radiant and its sections U_α are co-normal.

Conversely, assume that U is hypographical and radiant and has co-normal sections. Then, by Propostions 3.11 and 3.8, there exists an ICR function $f : C \to [0, +\infty]$ such that $U = \mathrm{supp}_l \; f = \mathrm{hypo}^+ \; f$. In view of (17), $f = h_U$; hence, by Propositions 5.2 and 4.9, $U^l = \mathrm{supp}_u \; f = \mathrm{epi}^+ \; f$. Using (27), we see that $f = e_{U^l}$. Therefore, by Proposition 5.4, $U^{lr} = \mathrm{supp}_l \; f = U$. □

Remark 5.6. Theorem 5.5 combined with Propositions 4.12 and 3.11 shows that the relevant sets to consider in connection with the polarities $U \longmapsto U^l$ and $U \longmapsto U^r$ are of the type $U = hypo^+ f$ and $U = epi^+ f$, respectively, $f : C \to [0, +\infty]$ being an ICR function. According to the preceding proof, for such sets the polarity relations can be simply written as $(hypo^+ f)^l = epi^+ f$ and $(epi^+ f)^r = hypo^+ f$.

We will now consider another polarity relations between sets.

Definition 5.7. Let $B \subseteq C$. The polar set of B is

$$B^o := \{(y, \alpha) \in C \times \mathbb{R}_{++} : \alpha c(x, y) \leq 1 \ \ \forall\, x \in B\}.$$

Proposition 5.8. *Let $B \subseteq C$. Then*

$$B^o = hypo^+ \mu_{Nh(B)} \cup (C \times (0, 1]), \tag{29}$$

$\mu_{Nh(B)} : C \to [0, +\infty]$ *denoting the Minkowski gauge of* $Nh(B)$.

Proof. Clearly, $B^o = (B \times \{1\})^r$, so by Propositions 5.4 and 3.8 we have $B^o = \mathrm{supp}_l\, e_{B\times\{1\}} = \mathrm{hypo}^+ \frac{1}{\left(e_{B\times\{1\}}\right)^{\triangledown}}$. Let us compute $\frac{1}{\left(e_{B\times\{1\}}\right)^{\triangledown}}$. For every $y \in C$,

$$\begin{aligned}\left(e_{B\times\{1\}}\right)^{\triangledown}(y) &= \sup_{x\in C} \frac{c(x, y)}{e_{B\times\{1\}}(x)} = \sup_{x\in B} c(x, y) = \sup_{x\in B} \max_{\lambda\in[0,1]} \{\lambda : \lambda y \leq x\} \\ &= \sup_{\lambda\in[0,1]} \sup_{x\in B} \{\lambda : \lambda y \leq x\} = \sup_{\lambda\in[0,1]} \{\lambda : \lambda y \in \underline{B}\}.\end{aligned}$$

By taking reciprocals, after a straightforward computation one gets the equality $\frac{1}{\left(e_{B\times\{1\}}\right)^{\triangledown}} = \max\{\mu_{\underline{B}}, 1\}$, which yields (29). □

Definition 5.9. Let $U \subseteq C \times \mathbb{R}_{++}$. The polar set of U is

$$U^0 := \{x \in C : \alpha c(x, y) \leq 1 \ \ \forall\ (y, \alpha) \in U\}.$$

Proposition 5.10. *Let $U \subseteq C \times \mathbb{R}_{++}$. Then*

$$U^o = \{x \in C : (h_U)_{\triangledown}(x) \geq 1\}.$$

Proof. One has $U^0 = \left\{x \in C : (x, 1) \in U^l\right\}$; since, by Propositions 5.2 and 4.9, $U^0 = \mathrm{epi}^+ \frac{1}{(h_U)_{\triangledown}}$, the result immediately follows. □

The following lemma will be useful to characterize the sets $B \subseteq C$ that coincide with their second polars.

Lemma 5.11. *For every ICR function $f : C \to [0, +\infty]$ and $x \in C$, the function $\mathbb{R}_{++} \ni t \longmapsto f(tx)$ is continuous.*

Proof. By Theorem 3.5, f is the pointwise supremum of a collection of functions of the type αc_y, with $(y, \alpha) \in C \times \mathbb{R}_{++}$. Each of these functions is continuous along rays since, for every $x \in C$ and $t \in \mathbb{R}_+$, in view of (8) one has

$$\alpha c_y(tx) = \alpha c(tx, y) = \alpha \min\{l(tx, y), 1\} = \alpha \min\{tl(x, y), 1\}.$$

So it follows that f is lower semicontinuous along rays. Similarly, using Theorem 4.5 and (26) one can easily prove that the restriction of f to $C\setminus\{0\}$ is upper semicontinuous along rays. $\square$

The following definition will be used in the characterization of the sets $U \subseteq C \times \mathbb{R}_{++}$ that coincide with their second polars.

Definition 5.12. Let S and E be two subsets of a common vector space. The set S is said to be an E-enlarged cone if it is the union of E with a cone.

The next proposition, which has an immediate proof, provides two characterizations of E-enlarged cones.

Proposition 5.13. *Let S and E be two subsets of a common vector space. Then the following statements are equivalent:*

(i) S is an E-enlarged cone.
(ii) $S \supseteq E$ and satisfies the following property: for every $x \in S\setminus E$ and every $\lambda > 0$ one has $\lambda x \in S$.
(iii) $S = cone\,(S\setminus E) \cup E$, $cone\,(S\setminus E)$ denoting the smallest cone containing $S\setminus E$.

Theorem 5.14. *Let $B \subseteq C$ and $U \subseteq C \times \mathbb{R}_{++}$. Then the following statements hold:*

(1) One has $B = B^{oo}$ if and only if B is normal and closed along rays (in the sense that for every $x \in C$ the set $\{t \in \mathbb{R}_{++} : tx \in C\}$ is closed in $\mathbb{R}_{++}$).
(2) One has $U = U^{oo}$ if and only if U is a hypographical $C \times (0,1]$-enlarged cone and for every $\alpha \in \mathbb{R}_{++}$ the section U_α is co-normal.

Proof. (1) Let $B \subseteq C$. If $B = B^{oo}$ then, by Proposition 5.10, $B = \{x \in C : (h_{B^o})_\nabla (x) \geq 1\}$. Since, according to Theorem 4.7, the function $\frac{1}{(h_{B^o})_\nabla}$ is ICR, it follows that B is normal and, by Lemma 5.11, it is closed along rays.

Assume now that B is normal and closed along rays. The function μ_B is positively homogeneous and, since B is normal, it is also increasing. Hence it is ICR. Moreover, as B is closed along rays, $B = \{x \in C : \mu_B(x) \leq 1\}$. In view of Theorem 4.7, we can equivalently write $B = \{x \in C : (\mu_B)_\nabla (x) \geq 1\}$. On the other hand, by (17) with $f := \mu_B$, we have $\mu_B = h_{\text{hypo}^+\mu_B}$; therefore, using Propositions 5.10, the normality of B and the equalities $(C \times (0,1])^o = C$ and (29), we get

$$\begin{aligned} B &= \left\{x \in C : \left(h_{\text{hypo}^+\mu_B}\right)_\nabla (x) \geq 1\right\} = \left(\text{hypo}^+\mu_B\right)^o = \left(\text{hypo}^+\mu_{Nh(B)}\right)^o \cap (C \times (0,1])^o \\ &= \left(\text{hypo}^+\mu_{Nh(B)}\right)^o \cap (C \times (0,1])^o = \left(\text{hypo}^+\mu_{Nh(B)} \cup (C \times (0,1])\right)^o = B^{oo}. \end{aligned}$$

(2) Let $U \subseteq C \times \mathbb{R}_{++}$. If $U = U^{oo}$ then, by Proposition 5.8, $U = \text{hypo}^+\ \mu_{Nh(U^o)} \cup (C \times (0,1]) = \text{hypo}^+\ \max\left\{\mu_{Nh(U^o)}, 1\right\}$. Given that $\mu_{Nh(U^o)}$ is positively homogeneous, $\text{hypo}^+\ \mu_{Nh(U^o)}$ is a cone; therefore U is an $C \times (0,1]$-enlarged cone. Since the function $\max\left\{\mu_{Nh(U^o)}, 1\right\}$ is ICR, by Propositions 3.8 and 3.11 U is hypographical and its sections U_α are co-normal.

Assume now that U is a hypographical $C\times(0,1]$-enlarged cone and has co-normal sections. Then there exists a cone $L \subseteq C \times \mathbb{R}_{++}$ such that $U = L \cup (C \times (0,1])$. Since, as one can easily check, the smallest hypographical set that contains a given cone is also a cone, without loss of generality we assume that L is hypographical, so $L = \mathrm{hypo}^+ f$ for some function $f : C \to [0, +\infty]$. This function f must be positively homogeneous. Moreover the sections L_α are co-normal, since so are the sections U_α (indeed, since L is a cone it suffices to check that one of its sections is co-normal, and so is the case because $L_2 = U_2$). Therefore f is increasing and hence it is an IPH function. Let $B := \{y \in C : f(y) \leq 1\}$. Then B is normal, that is, $B = Nh(B)$, and $f = \mu_B = \mu_{Nh(B)}$. Moreover, by Theorem 4.7,

$$\begin{aligned} B &= \{y \in C : f_\nabla(y) \geq 1\} = \{y \in C : (h_L)_\nabla(y) \geq 1\} = L^o = L^o \cap (C \times (0,1])^o \\ &= (L \cup (C \times (0,1]))^o = U^0. \end{aligned}$$

So, using (29) we conclude that

$$\begin{aligned} U &= L \cup (C \times (0,1]) = \mathrm{hypo}^+ f \cup (C \times (0,1]) = \mathrm{hypo}^+ \mu_{Nh(B)} \cup (C \times (0,1]) = B^o \\ &= U^{oo}. \end{aligned}$$

□

Corollary 5.15. *The mapping $B \longmapsto B^o$ is a bijection from the set of all normal and closed along rays subsets of C onto the set of all hypographical $C \times (0,1]$-enlarged cones $U \subseteq C \times \mathbb{R}_{++}$ that have co-normal sections U_α; the inverse mapping is $U \longmapsto U^o$.*

Proof. Let $B \subseteq C$ be normal and closed along rays. Using (29), one can easily see that B^o is an hypographical $C \times (0,1]$-enlarged cone with co-normal sections, so the mapping $B \longmapsto B^o$ is onto. By Theorem 5.5, $B = B^{oo}$, which shows that this mapping is also one-to-one and has $U \longmapsto U^o$ as its inverse. □

6. IPH functions

In this section we will study relations between duality for IPH functions and ICR functions under an additional Assumption 6.1, which we impose on the order relation $\geq$ under consideration. (Recall that we consider order relations for which Assumptions 2.1, 2.2, 2.3 and 2.4 are valid.)

Assumption 6.1. For each $x, y \in C, y \neq 0$ there exists $\lambda > 0$ such that the inequality $\lambda y \leq x$ does not hold.

Assumption 6.1 is equivalent to the finiteness of $l(x, y)$ for all $x \in C$ and $y \neq 0$. Recall that an order relation $\leq$ is called Archimedean if for all $x, y \in C$, $y \neq 0$ there exists a natural number n such that $x < ny$. Clearly Assumption 6.1 holds for Archimedean order relations. Assumption 6.1 does not hold for the lexicographic order relation on the cone $\mathbb{R}^n_+$.

Let $f : C \to [0, +\infty)$ be a finite IPH function. Consider, as in [1], the lower polar function $f^\circ : C \to [0, +\infty]$ of f defined by

$$f^\circ(y) = \sup_{x \in C} \frac{l(x, y)}{f(x)}$$

(with the convention $\frac{0}{0} := 0$) and the upper polar function $f_\circ : C \to [0, +\infty]$ of f defined by

$$f_\circ(y) = \inf_{x \in C} \frac{u(x, y)}{f(x)}.$$

(with the convention $\frac{0}{0} := +\infty$). Both polar functions were defined for IPH functions in [1].

Proposition 6.2. *Let f be an IPH function. Then $f^\nabla = f^\circ$ and $f_\nabla = f_\circ$.*

Proof. First we present the proof for the lower case. Assume that $0 \in C$ and $y = 0$. Since $0 \leq x$ for all $x \in C$ it follows that $l(x, y) = +\infty$ for all $x \in C$ so $c(x, y) = 1$ for all $x \in C$. Since $f(0) = 0$ for a finite IPH function f, we have

$$f^\nabla(0) = \sup_{x \in C} \frac{1}{f(x)} \geq \frac{1}{0} = +\infty; \qquad f^\circ(0) = \sup_{x \in C} \frac{l(x, 0)}{f(x)} \geq \frac{l(0, 0)}{f(0)} = +\infty.$$

Hence $f^\nabla(0) = f^\circ(0)$.

Let $y \in C, y \neq 0$. Since $c(x, y) = \min(l(x, y), 1)$ it follows that

$$f^\nabla(y) = \sup_{x \in C} \frac{c(x, y)}{f(x)} = \max\left(\sup_{x \in C, l(x,y) \leq 1} \frac{l(x, y)}{f(x)}, \sup_{x \in C, l(x,y) \geq 1} \frac{1}{f(x)} \right). \tag{30}$$

Due to Assumptions 6.1 and 2.2, $l(x, y) < +\infty$ for all $x \in X$. Let $l(x, y) := \mu \geq 1$ and $x' = x/\mu$. Then $l(x', y) = 1$ and $f(x) = f(\mu x') = \mu f(x') \geq f(x')$. Therefore

$$\sup_{x \in C, l(x,y) \geq 1} \frac{1}{f(x)} = \sup_{x' \in C, l(x',y) = 1} \frac{1}{f(x')}. \tag{31}$$

Since both f and l_y (recall that $l_y(x) = l(x, y)$) are positively homogeneous of degree one and finite it follows that

$$\sup_{x \in C} \frac{l(x, y)}{f(x)} = \sup_{x \in C, l(x,y) \leq 1} \frac{l(x, y)}{f(x)} = \sup_{x \in C, l(x,y) = 1} \frac{l(x, y)}{f(x)} = \sup_{x \in C, l(x,y) = 1} \frac{1}{f(x)} \tag{32}$$

Hence

$$f^\circ(y) = \sup_{x \in C} \frac{l(x, y)}{f(x)} = \sup_{x \in C, l(x,y) = 1} \frac{1}{f(x)} \tag{33}$$

The result in the lower case follows from (30), (31) and (32).

Consider now the upper case. Let $0 \in C$ and $y = 0$. Due to Assumption 2.4, the set $\{\mu \geq 0 : x \leq \mu y\}$ is empty for all $x \neq 0$, so $u(x, y) = +\infty$ and $d(x, y) = +\infty$ for all $x \neq 0$. At the same time $u(0, 0) = 0$ and $d(0, 0) = 1$. Since $0/0 = +\infty$ it follows that $\frac{u(x,y)}{f(x)} = +\infty$ for all $x \in C$. We also have $\frac{d(x,y)}{f(x)} = +\infty$ for all $x \in C$. Hence $f_\circ(0) = +\infty = f_\nabla(0)$.

Assume now that $y \neq 0$. Then

$$f_\nabla(y) = \inf_{x \in C} \frac{d(x, y)}{f(x)} = \min\left(\inf_{x \in C, u(x,y) \geq 1} \frac{u(x, y)}{f(x)}, \inf_{x \in C, u(x,y) \leq 1} \frac{1}{f(x)} \right). \tag{34}$$

Let $u_y(x) = u(x, y)$. It is easy to check that u_y is positively homogeneous of degree one. Since f is also positively homogeneous of degree one, we have

$$\inf_{x\in C, u(x,y)\geq 1} \frac{u(x, y)}{f(x)} = \inf_{x\in C, u(x,y)=1} \frac{u(x, y)}{f(x)}. \tag{35}$$

We now show that $u(x, y) > 0$ for all $x \neq 0$. Indeed, if $x \neq 0$ then due to Assumption 2.1 and Assumption 6.1 there exists $\mu > 0$ such that the inequality $x \leq \mu y$ does not hold, hence, by Assumption 2.2, $u(x, y) > 0$. At the same time $u(0, y) = 0$ due to Assumption 2.4.

Let $x \neq 0$ and $u(x, y) := \mu \leq 1$. Let $x' = x/\mu$. Since u_y is positively homogeneous we have $u(x', y) = 1$. We also have $f(x) = f(\mu x') = \mu f(x') \leq f(x')$. If $0 \in C$ and $x = 0$ then $f(x) = 0$ which implies $\frac{1}{f(x)} = +\infty$ and $\frac{0}{f(x)} = +\infty$, so we can omit $x = 0$ calculating $\inf \frac{1}{f(x)}$ or $\inf \frac{0}{f(x)}$ to compute $f_0(y)$. Then we have

$$\inf_{x\in C, u(x,y)\leq 1} \frac{1}{f(x)} = \inf_{x\in C, u(x,y)=1} \frac{1}{f(x)}. \tag{36}$$

Using positive homogeneity of u_y and f we get

$$f_\circ(y) = \inf_{x\in C} \frac{u_y(x)}{f(x)} = \inf_{x\in C, u(x,y)=1} \frac{u_y(x)}{f(x)} = \inf_{x\in C, u(x,y)=1} \frac{1}{f(x)}. \tag{37}$$

The result in the upper case follows from (34), (35), (36) and (37). □

The proof of the following proposition is immediate and we omit it.

Proposition 6.3. *Let $f : C \to [0, +\infty]$ be a function. The following assertions are equivalent:*

(*i*) *f is positive homogeneous of the first degree;*
(*ii*) *the positive hypograph hypo$^+ f$ is a cone;*
(*iii*) *the positive epigraph epi$^+ f$ is a cone.*

Using results from Sections 3 and 4 and Propositions 6.3 and 6.2 we can give a description of IPH functions in terms of the lower and upper polar functions and their positive hypographs and positive epigraphs.

We now consider the ∇-subdifferential $\partial^\nabla f(x_0)$ of an IPH function f at a point $x_0 \in C$ such that $0 < f(x_0) < +\infty$:

Proposition 6.4. *Let f be an IPH function and $x_0 \in f^{-1}((0, +\infty))$. Then $y \in \partial^\nabla f(x_0)$ if and only if*

$$f(x) \geq f(x_0) \frac{l(x, y)}{l(x_0, y)}, \quad \textit{if } l(x_0, y) \leq 1 \tag{38}$$

$$f(x) \geq f(x_0) l(x, y), \quad \textit{if } l(x_0, y) \geq 1. \tag{39}$$

Proof. By definition $y \in \partial^{\nabla} f(x_0)$ if $f(x) \geq f(x_0)\dfrac{c(x,y)}{c(x_0,y)}$ for all $x \in C$. Replacing x with λx, where $\lambda > 0$ and dividing into λ we get that $y \in \partial^{\nabla} f(x_0)$ implies

$$f(x) \geq f(x_0)\frac{1}{\lambda}\frac{c(\lambda x,y)}{c(x_0,y)}, \quad \lambda > 0.$$

We have $l(\lambda x, y) = \lambda l(x,y) \leq 1$ for sufficiently small numbers λ. So $c(\lambda x, y) = l(\lambda x, y) = \lambda l(x,y)$ for these λ and we obtain (38) and (39). Obviously (38) and (39) imply that $y \in \partial^{\nabla} f(x_0)$. □

A similar characterization can be given for the ∇-superdifferential $\partial_{\nabla} f(x_0)$.

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