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A Lower Semicontinuous Regularization for Set-Valued Mappings and its Applications

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Dedicated to the memory of Thomas Lachand-Robert.

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A basic fact in real analysis is that every real-valued function f admits a lower semicontinuous regularization $\underline{f}$, defined by means of the lower limit of f:

$$\underline{f}\,(x) := \liminf_{y \to x} f\,(y)\,.$$

This fact breaks down for set-valued mappings. In this note, we first provide some counterexamples. We try further to define a kind of lower semicontinuous regularization for a given set-valued mapping and we point out some general applications.

Keywords: Set-valued mappings, lower semicontinuity, regularization, approximate selections, fixed points, differential inclusions, variational inequalities

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1. Introduction

The lower semicontinuity property of functions and set-valued mappings is crucial in solving many problems arising in mathematical analysis and in particular the field of optimization theory and also in the study of existence of fixed points and continuous selections or the existence of solutions for differential inclusions and variational inequalities.

In the present work, our primary goal is to construct a lower semicontinuous set-valued

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mapping based on a given arbitrary one and we as well envision some theoretical applications in fixed points theory and its related areas.

Usually, a regularization process supposes to construct on the basis of a given mathematical object another object having better properties than the initial one and a good regularization technique supposes also to have the possibility to "measure" in a sense the "distance" between these objects. The regularization techniques for extended real-valued functions are among important tools in optimization (we mention here, among others, the lower semicontinuity regularization and various convexification procedures) and for this reason some regularization schemes were recently extended for the case of vector-valued functions in the works [2] and [3]. In the last years an important amount of attention was paid for regularization of sets as well: see [11], [8] and the references therein.

The aim of this paper is to look at a possible lower semicontinuous regularization for set-valued mappings. The lower semicontinuity (or inner semicontinuity) of set-valued mappings is a very important and useful property. But, in contrast with upper semicontinuity, this property is difficult to be achieved. In fact, this paper is basically motivated by the following assertion made by Rockafellar and Wets in their book [12] (pp. 155): "Inner semicontinuity of a given mapping is typically harder to verify than outer semicontinuity, and isn't constructive in such an easy sense". Our motivation is to present a construction of a kind of approximate lower semicontinuous regularization for a given set-valued mapping acting between two general normed vector spaces.

The paper is organized as follows. In Section 2 we present the notations and the basic concepts we need in the sequel. Section 3 is devoted to the analysis of some particular situations of set-valued mappings for which the lower semicontinuity of the lower limit mapping holds or not. The examples and counterexamples we present emphasize the fact that the same construction which in the single-valued case (even for vector-valued functions) give a lower semicontinuous regularization break down in the set-valued case. We also show that, due to the property of total ordering of $\mathbb{R}$, the case of real set-valued mappings with convex values is completely different from the case of set-valued mappings taking values in a higher dimensional normed vector space. All these facts lie in the heart of our motivation in this research topic. In Section 4 we present the main results of the paper together with some special cases. More precisely, starting from an arbitrary set-valued mapping we construct a regularization and we prove, thereafter, its semicontinuity. The fact that in some important cases this l.s.c. regularization is not too far (in the sense of inclusion) from the initial mapping is used in the final section where we apply the main results to obtain an approximate continuous selection theorem and an approximate (an exact, as well) fixed points result. Finally, we present two applications in differential inclusion theory and variational inequalities. A forthcoming paper will be dedicated to concrete optimization problems.

2. Preliminaries

Let X and Y be two normed vector spaces whose norms are denoted by $\|\cdot\|$. We denote by $\bar{B}(x,r)$ the closed ball centered at x with radius $r>0$ and by $\bar{B}_\rho$ (resp. B_ρ) the closed (resp. open) ball centered at 0 with radius ρ in any of the spaces X, Y. We use the notation $\mathcal{V}(x)$ for the filter of neighborhoods of x. We recall that if $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is an extended-real-valued function, then its lower and upper limits at $\bar{x}$ are defined respectively by

$$\liminf_{x\to\bar{x}} f(x) := \sup_{U\in\mathcal{V}(\bar{x})} \inf_{x\in U} f(x)$$

and

$$\limsup_{x\to\bar{x}} f(x) := \inf_{U\in\mathcal{V}(\bar{x})} \sup_{x\in U} f(x).$$

One says that f is lower semicontinuous (l.s.c. for short) at $\bar{x}$ if $f(\bar{x}) = \liminf_{x\to\bar{x}} f(x)$ and f is called upper semicontinuous (u.s.c. for short) at $\bar{x}$ if $f(\bar{x}) = \limsup_{x\to\bar{x}} f(x)$. It is well known that for an arbitrary extended-real-valued function f one can easily construct a l.s.c. regularization $\underline{f} : X \to \mathbb{R}\cup\{-\infty, +\infty\}$ by $\underline{f}(x) = \liminf_{u\to x} f(u)$ for every $u \in X$.

Let us consider a set-valued mapping $F : X \rightrightarrows Y$, that is a mapping which assigns to each $x \in X$ a subset $F(x)$ of Y. When $F(x) \neq \emptyset$ we say that $x \in \operatorname{Dom} F$.

For the lower limit and upper limit of F at a point $\bar{x} \in \operatorname{Dom} F$ defined as below, we prefer to use slightly different notations in order to distinguish these notions from their single-valued counterparts:

$$\operatorname*{Liminf}_{x\to\bar{x}} F(x) = \{y \in Y \mid \forall V \in \mathcal{V}(y),\, \exists U \in \mathcal{V}(\bar{x}),\, \forall x \in U, F(x) \cap V \neq \emptyset\},$$
$$\operatorname*{Limsup}_{x\to\bar{x}} F(x) = \{y \in Y \mid \forall V \in \mathcal{V}(y),\, \forall U \in \mathcal{V}(\bar{x}),\, \exists x \in U, F(x) \cap V \neq \emptyset\}.$$

Both sets are closed. We are mostly interested in the properties of the lower limit. In this sense, observe that if $\operatorname{Liminf}_{x\to\bar{x}} F(x) \neq \emptyset$, then $\bar{x} \in \operatorname{Int}(\operatorname{Dom} F)$. The converse is not true. One can easily observe also that for every $\bar{x} \in \operatorname{Dom} F$, $\operatorname{Liminf}_{x\to\bar{x}} F(x) \subset \operatorname{cl}(F(\bar{x}))$, the closure of $F(x)$, and this inclusion may be strict. Note that the above definitions have sense for any set-valued mapping F acting between two topological spaces. For the particular case of normed vector spaces one has the following sequential characterizations of the above defined notions:

$$\operatorname*{Liminf}_{x\to\bar{x}} F(x) = \{y \in Y \mid \forall x_n \to \bar{x},\, \exists y_n \to y,\ y_n \in F(x_n), \forall n \in \mathbb{N}\},$$
$$\operatorname*{Limsup}_{x\to\bar{x}} F(x) = \{y \in Y \mid \exists x_n \to \bar{x},\, \exists y_n \to y,\ y_n \in F(x_n), \forall n \in \mathbb{N}\}.$$

Definition 2.1. One says that a mapping $F : X \rightrightarrows Y$ is lower semicontinuous at $\bar{x} \in \operatorname{Dom} F$ if

$$F(\bar{x}) \subset \operatorname*{Liminf}_{x\to\bar{x}} F(x).$$

If $\bar{x} \notin \operatorname{Dom} F$, then one considers that F is automatically l.s.c. at $\bar{x}$.

Note that this lower semicontinuity concept was adopted by Aubin and Frankowska [1] and by Rockafellar and Wets [12], while in the more recent book [6], Göpfert, Riahi, Tammer and Zălinescu adopted a slightly different notion.

We present now an example for arbitrary single-valued functions viewed as set-valued mappings.

Example 2.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. Consider $F_f : \mathbb{R} \rightrightarrows \mathbb{R}$, defined, for every $x \in \mathbb{R}$, by $F_f(x) = \{f(x)\}$. Clearly, $\operatorname{Dom} F_f = \operatorname{Dom} f$. Let $\bar{x} \in \operatorname{Dom} f$. Since F_f is closed-valued, whenever $\operatorname{Liminf}_{x\to\bar{x}} F_f(x)$ is nonempty it must be reduced to $\{f(\bar{x})\}$ and in this case F_f is l.s.c. at a point $\bar{x}$ if and only if f is continuous at $\bar{x}$.

3. Examples and counterexamples

As we already saw from the above examples, in general, for a real-valued function f, we have two different objects if we consider the $\liminf$ in the sense of functions or Liminf in the sense of set-valued case (when we identify the function f with F_f). This is one of the reasons for which, in contrast to the scalar functions, the lower limit set-valued mapping is not necessarily l.s.c. In order to illustrate this assertion, we have the following counterexample. In the sequel if F is a set valued mapping we denote by G the lower limit set-valued mapping, i.e., $G(x) = \operatorname{Liminf}_{u\to x} F(u)$. Consider the mapping $F : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by:

$$F(x) = \begin{cases} \{0, 1 + x\}, & \text{if } x \in \mathbb{Q} \\ \{0, 1 - x\}, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

where, as usual, $\mathbb{Q}$ denotes the set of rational numbers. Clearly, $\operatorname{Dom} F = \mathbb{R}$. It is easy to show that $\operatorname{Liminf}_{x\to 0} F(x) = \{0, 1\}$ and, for $\bar{x} \neq 0$, $\operatorname{Liminf}_{x\to\bar{x}} F(x) = \{0\}$, i.e., $G(x) = \{0\}$ for $x \neq 0$ and $G(0) = \{0, 1\}$. Observe that $\operatorname{Liminf}_{x\to 0} G(x) = \{0\}$ and this set does not include $G(0) = \{0, 1\}$. Consequently, G is not lower semicontinuous at 0.

Let us now consider the case when the set-valued mapping has convex values. We start with the special case when $Y = \mathbb{R}$, because of the particular form of the convex sets in this setting. It is not difficult to see that, if for a function $f : X \to \mathbb{R}$ we construct the set-valued mapping $F : X \rightrightarrows \mathbb{R}$, $F(x) = f(x) + [0, +\infty)$, then F is l.s.c. if and only if f is u.s.c. More generally, we have the next proposition.

Proposition 3.1. *Let $F : X \rightrightarrows \mathbb{R}$ be a set-valued mapping with nonempty convex values and let $\bar{x} \in X$. If $G(x) \neq \emptyset$ for every x in a neighborhood of $\bar{x}$, then G is l.s.c. in $\bar{x}$. In particular, if G has nonempty values, then it is l.s.c. on $\mathbb{R}$. The nonemptiness assumption is essential.*

Proof. Since the convex sets in $\mathbb{R}$ are the intervals one can represent the value of F at $x \in X$ in the form $F(x) = [\alpha(x), \beta(x)]$, where $\alpha(x), \beta(x) \in \mathbb{R} \cup \{-\infty, \infty\}$ (we choose to work with closed intervals, but this is not essential in our setting). Accordingly, we have two extended-real-valued functions α, β. We claim that $G(x) = [\limsup_{u\to x} \alpha(u), \liminf_{u\to x} \beta(u)]$, with the convention that this interval is the empty set if $\limsup_{u\to x} \alpha(x) > \liminf_{u\to x} \beta(x)$. Indeed, let $y \in G(x)$. Then for every sequence $(x_n) \to x$ there exists a sequence $(y_n) \to y$ s.t., for n large enough, $y_n \in [\alpha(x_n), \beta(x_n)]$. Then we have that for every sequence $(x_n) \to x$, $\limsup_n \alpha(x_n) \leq y \leq \liminf_n \beta(x_n)$, whence $\limsup_{u\to x} \alpha(u) \leq y \leq \liminf_{u\to x} \beta(u)$. Let us prove the converse inclusion. If $\limsup_{u\to x} \alpha(u) = \liminf_{u\to x} \beta(u) := \theta$ then the fact that $G(x) \neq \emptyset$ implies $G(x) = \{\theta\}$. Otherwise, since $G(x)$ is closed, it is enough to prove that $(\limsup_{u\to x} \alpha(x), \liminf_{u\to x} \beta(x)) \subset G(x)$. Accordingly, take y s.t. $\limsup_{u\to x} \alpha(u) < y < \liminf_{u\to x} \beta(u)$. This means that there exists $\varepsilon > 0$ s.t. $\sup_{u\in\bar{B}(x,\varepsilon)} \alpha(u) < y < \inf_{u\in\bar{B}(x,\varepsilon)} \beta(u)$. Thus, for every sequence $(x_n) \to x$ one has $\alpha(x_n) < y < \beta(x_n)$, i.e., $y \in F(x_n)$ for all n large enough. The claim is proved. Now, if G has nonempty values on a neighborhood of $\bar{x}$, since the lower limit function associated to a function is l.s.c. and the upper limit function associated to a function is u.s.c., from the above claim one has that $\operatorname{Liminf}_{x\to\bar{x}} G(x) = [\limsup_{x\to\bar{x}} \alpha(x), \liminf_{x\to\bar{x}} \beta(x)] = G(\bar{x})$, whence G is l.s.c. at $\bar{x}$.

If the nonemptiness assumption is not fulfilled, the above conclusion is not longer true. In order to illustrate this situation we consider the following example: take $F : \mathbb{R} \rightrightarrows \mathbb{R}$

given by

$$F(x) = \begin{cases} \{1\}, & \text{if } x > 1 \\ \{n^{-1}\}, & \text{if } x \in ((n+1)^{-1}, n^{-1}],\ (\forall n \in \mathbb{N} \setminus \{0\}) \\ \{0\}, & \text{if } x \leq 0. \end{cases}$$

For this singleton-valued mapping one has

$$G(x) = \begin{cases} \{1\}, & \text{if } x > 1 \\ \{n^{-1}\}, & \text{if } x \in ((n+1)^{-1}, n^{-1}),\ (\forall n \in \mathbb{N} \setminus \{0\}) \\ \emptyset, & \text{if } x = (n+1)^{-1},\ (\forall n \in \mathbb{N} \setminus \{0\}) \\ \{0\}, & \text{if } x \leq 0, \end{cases}$$

and now it is clear that G is not l.s.c. at 0, because $\operatorname{Liminf}_{x\to 0} G(x) = \emptyset$. The proposition is proved. $\square$

Of course, the above result can be extended to any finite dimensional space $\mathbb{R}^m$ $(m > 1)$ and for set-valued mappings $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ having values from a special class of convex sets: $F(x) = \prod_{i=1}^{m} [\alpha_i(x), \beta_i(x)]$, where α_i, β_i $(i = 1, 2, .., m)$ are extended-real-valued functions. But, in this setting, for set-valued mappings with general convex values the lower limit set-valued mapping is not necessarily l.s.c. (even if it has nonempty values). To see this, let us consider $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$,

$$F(x) = \begin{cases} [(0,0); (x+1, -x)], & \text{if } x \in \mathbb{Q} \\ [(0,0); (1-x, x)], & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

where $[(a,b); (c,d)]$ is the closed segment joining the points (a,b) and $(c,d) \in \mathbb{R}^2$. It is not difficult to see that $G(0) = [(0,0); (1,0)]$, while for $x \neq 0$, $G(x) = \{(0,0)\}$, whence G is not l.s.c. at 0.

Having in mind all these examples, the aim of the next section is to introduce a l.s.c. set-valued mapping as a regularization for an arbitrarily set-valued mapping.

4. Main results

In this section, as above, X and Y are supposed to be normed vector spaces and the space Y will be partially ordered by a closed, convex (not necessarily pointed) cone C with nonempty interior.

Let $F : X \rightrightarrows Y$ be a set-valued mapping. For $\varepsilon \geq 0$, we consider $F^\varepsilon : X \rightrightarrows Y$ defined for each $x \in X$ by $F^\varepsilon(x) = F(x) - C_\varepsilon$, where $C_\varepsilon := C \cap \bar{B}_\varepsilon$. We use the convention $\emptyset - C_\varepsilon = \emptyset$, so $\operatorname{Dom} F = \operatorname{Dom} F^\varepsilon$. Let us introduce $L_F^\varepsilon : X \rightrightarrows Y$ defined for each $\bar{x} \in X$ by

$$L_F^\varepsilon(\bar{x}) = \operatorname*{Liminf}_{x \to \bar{x}} F^\varepsilon(x).$$

Equivalently,

$$L_F^\varepsilon(\bar{x}) = \{y \in Y \mid \forall x_n \to \bar{x},\ \exists y_n \to y,\ y_n \in F(x_n) - C_\varepsilon,\ \forall n \geq n_0\}. \tag{1}$$

Of course, if $\bar{x} \notin \operatorname{Dom} F$, $L_F^\varepsilon(\bar{x}) = \emptyset$.

Remark 4.1. Note that, in general,

$$\operatorname*{Liminf}_{x\to\bar{x}} F(x) \subset L_F^{\varepsilon}(\bar{x}) \subset \mathrm{cl}\,[F(\bar{x}) - C_{\varepsilon}].$$

The inclusion

$$L_F^{\varepsilon}(\bar{x}) \subset F(\bar{x}) - C_{\varepsilon}$$

holds if $F(\bar{x}) - C_{\varepsilon}$ is closed and, in particular, in each of the next situations:

- $F(\bar{x})$ is compact;
- $F(\bar{x})$ is weakly closed and Y is a reflexive Banach space;
- $F(\bar{x})$ is convex, closed, locally compact.

Example 4.2. We consider in $\mathbb{R}$ the positive cone $C = \mathbb{R}_+$. Let $k > 0$ and $F : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by $F(x) = \{0\}$ if $x \neq 0$ and $F(0) = [-k, 0]$. One can easily check that F is not lower semicontinuous at 0. Computing L_F^{ε}, for some $\varepsilon > 0$, we find that $L_F^{\varepsilon}(x) = [-\varepsilon, 0]$ for all $x \in \mathbb{R}$. L_F^{ε} is then lower semicontinuous at every point and for all $x \in \mathbb{R}$ we have $L_F^{\varepsilon}(x) \subset F(x) + [-\varepsilon, 0]$.

The crucial technical result for defining our regularization set-valued mapping is the following one.

Lemma 4.3. *For all $\bar{x} \in X$, for all $\varepsilon \geq 0$, and $\eta > 0$, one has $L_F^{\varepsilon}(\bar{x}) \subset \operatorname{Liminf}_{x\to\bar{x}} L_F^{\varepsilon+\eta}(x)$.*

Proof. Take $\varepsilon \geq 0$ and $\eta > 0$. Let us consider $c \in Y$ and $r > 0$ such that $\|c\| = 1$ and $\bar{B}(c, r) \subset C$. Note that, since C is a cone, for every $\varepsilon' > 0$ one has $C_{\varepsilon} + C_{\varepsilon'} = C_{\varepsilon+\varepsilon'}$. Let $y \in L_F^{\varepsilon}(\bar{x})$. We claim that:

$$\forall\, \nu \in (0, \eta), \exists\, \delta > 0, \forall\, x \in \bar{B}(\bar{x}, \delta) : y - \nu c \in F(x) - C_{\varepsilon+\eta}. \tag{2}$$

Indeed, fix $\nu \in (0, \eta)$ and consider $0 < \varepsilon' \leq \min\{r\nu, \eta - \nu\}$. Because $y \in L_F^{\varepsilon}(\bar{x})$,

$$\exists\, \delta > 0, \forall\, x \in \bar{B}(\bar{x}, \delta) : (y + \varepsilon' B_1) \cap (F(x) - C_{\varepsilon}) \neq \emptyset.$$

Hence, for $x \in \bar{B}(\bar{x}, \delta)$, there exists $u \in B_1$, $\omega \in F(x)$ and $c' \in C_{\varepsilon}$ such that $y + \varepsilon' u = \omega - c'$. Thus,

$$w - y + \nu c = c' + \nu c + \varepsilon' u = c' + \nu(c + (\varepsilon'/\nu)u).$$

Since $\varepsilon'/\nu \leq r$, we have $c + (\varepsilon'/\nu)u \in C$ and on the other hand, $\|\nu c + \varepsilon' u\| \leq \nu + \varepsilon'$. We deduce that

$$\nu c + \varepsilon' u \in C_{\nu+\varepsilon'},$$

whence

$$\omega - y + \nu c \in C_{\varepsilon} + C_{\varepsilon'+\nu} = C_{\varepsilon+\nu+\varepsilon'} \subset C_{\varepsilon+\eta},$$

and the claim is proved. Let now $\nu \in (0, \eta)$ and $\delta > 0$ provided by (2). Hence, for a fixed $x \in \bar{B}(\bar{x}, \delta/2)$ one has $y - \nu c \in F^{\varepsilon+\eta}(x)$. So, we have:

$$\forall\, \eta > 0, \forall\, \nu \in (0, \eta), \exists\, \delta > 0, \forall\, x \in \bar{B}(\bar{x}, \delta/2) : y - \nu c \in F^{\varepsilon+\eta}(x).$$

Consider $\varepsilon' > 0$, $\nu = \frac{1}{2}\min\{\eta, \varepsilon'\} \in (0, \eta)$ and $\delta > 0$ given by the above relation. Accordingly, for every $x \in \bar{B}(\bar{x}, \delta)$, we have $(y + \varepsilon' B_1) \cap F^{\varepsilon+\eta}(x) \neq \emptyset$. This shows that $y \in \operatorname{Liminf}_{x\to\bar{x}} L_F^{\varepsilon+\eta}(x)$, completing the proof. □

Notice that, as we have seen in the previous section, $L_F^0(\cdot) = \operatorname{Liminf}_{x\to\cdot} F(\cdot)$ is not necessarily lower semicontinuous, and using for $\varepsilon = +\infty$, as a convention, $\bar{B}_\infty = Y$, from Lemma 4.3, we deduce that $L_F^\infty(\cdot) = \operatorname{Liminf}_{x\to\bar{x}}(F(\cdot) - C)$ is lower semicontinuous but, of course, this mapping can be very far from the initial one.

It is worth pointing out that if C is pointed and F is identified with a single-valued vector mapping f, for any point $\bar{x} \in \operatorname{Dom} f$, $L_f^\infty(\bar{x})$ coincides with the lower level set $\{y \in Y \mid \forall V \in \mathcal{V}(y), \exists U \in \mathcal{V}(\bar{x}), f(U) \subset V - C\}$ introduced and studied in [3] (see also [2]). Moreover, if Y is a reflexive Banach space and a conditionally complete lattice w.r.t. to C, then for a given vector-valued function $f : X \to Y$, one can consider the set-valued mapping $F : X \rightrightarrows Y$ defined for every $x \in \operatorname{Dom} f$ by $F(x) = f(x) - C$ and subsequently the function $\sup L_F^\infty(\cdot)$, where the supremum is taken with respect to the order induced by C. It has been proved in [3] that the latter vector-valued function is l.s.c. whenever C is normal. Thus, it can be viewed as a lower semicontinuous selection (see also the next section) of L_F^∞, and in the meantime defines the greatest lower semicontinuous minorant of f.

We are now in a position to find a candidate which is expected to be a l.s.c. regularization for a given set-valued mapping F. To do that, consider a positive number α. We introduce the set-valued mapping $R_F^\alpha : X \rightrightarrows Y$, given by:

$$R_F^\alpha(\bar{x}) := \operatorname{cl}\left(\bigcup_{\mu\in[0,\alpha)} \operatorname*{Liminf}_{x\to\bar{x}} L_F^\mu(x)\right).$$

The next two results are essential in the sequel, being the main results of the paper.

Theorem 4.4. *Let $F : X \rightrightarrows Y$. Then for all $\alpha > 0$ the mapping $R_F^\alpha : X \rightrightarrows Y$ is lower semicontinuous.*

Proof. Obviously, R_F^α is l.s.c. at any $\bar{x}$ with $R_F^\alpha(\bar{x}) = \emptyset$. Let $\bar{x} \in X$ and take $y \in \bigcup_{\mu\in[0,\alpha)} \operatorname{Liminf}_{x\to\bar{x}} L_F^\mu(x)$. This means that there exists $\mu \in [0, \alpha)$ s.t. $y \in \operatorname{Liminf}_{x\to\bar{x}} L_F^\mu(x)$, i.e., there exists $\mu \in [0, \alpha)$ s.t. for every $x_n \to \bar{x}$, there exist $y_n \to y$ with $y_n \in L_F^\mu(x_n)$ for every n large enough. Consider $\delta \in (\mu, \alpha)$. Then, following Lemma 4.3 and the definition of R_F^α, one has that for n large enough:

$$y_n \in L_F^\mu(x_n) \subset \operatorname*{Liminf}_{u\to x_n} L_F^\delta(u) \subset R_F^\alpha(x_n).$$

Consequently, one concludes that $y \in \operatorname{Liminf}_{x\to\bar{x}} R_F^\alpha(x)$ and since the last set is closed, we conclude that $R_F^\alpha(\bar{x}) \subset \operatorname{Liminf}_{x\to\bar{x}} R_F^\alpha(x)$. The proof is complete. □

Proposition 4.5. *Assume that either $F(\bar{x})$ is compact or Y is reflexive and $F(\bar{x})$ is weakly closed or $F(\bar{x})$ is closed, convex, locally compact. Then for all $\alpha > 0$ we have $R_F^\alpha(\bar{x}) \subset F(\bar{x}) - C_\alpha$.*

Proof. Let $\alpha > 0$. If $R_F^\alpha(\bar{x}) = \emptyset$, then it is nothing to prove. Take $y \in R_F^\alpha(\bar{x})$. Since L_F^ε

is closed-valued for every positive ε, one can successively write:

$$\begin{aligned} y \in R_F^\alpha(\bar{x}) &= \operatorname{cl}\left(\bigcup_{\mu\in[0,\alpha)} \operatorname*{Liminf}_{x\to\bar{x}} L_F^\mu(x) \right) \\ &\subset \operatorname{cl}\left(\bigcup_{\mu\in[0,\alpha)} L_F^\mu(\bar{x}) \right) \\ &\subset \operatorname{cl}\left(\bigcup_{\mu\in[0,\alpha)} (F(\bar{x}) - C_\mu) \right) \\ &\subset \operatorname{cl}\,(F(\bar{x}) - C_\alpha) = F(\bar{x}) - C_\alpha, \end{aligned}$$

where for the last equality we have used the compactness assumptions we made (see Remark 4.1). □

Example 4.6. Coming back to the mapping of Example 4.2, we observe that for every $\alpha > 0$, and $x \in \mathbb{R}$, one has $R_F^\alpha(x) = [-\alpha, 0]$. Clearly, R_F^α is lower semicontinuous (even continuous) at every point.

In fact, since the process described in the previous results depends on the cone C we can call it "C-regularization". In order to justify this name we use for the above defined mapping, let us note that if F is l.s.c. with closed values and if we allow $\alpha = 0$ (with the convention $[0,0) = \{0\}$), then $F = R_F^0$, so in this case F coincides with the first mapping of the collection of set-valued mappings defined by $(R_F^\alpha)_{\alpha\geq 0}$.

5. Applications

The aim of this section is to present some (mostly) theoretical applications of the results presented above. Is it well known that the semicontinuity of the underlying set-valued mapping is an essential ingredient for proving many important results concerning the existence of continuous selections or existence of fixed points. In turn, these results are helpful to ensure the existence of solutions for a large number of mathematical models including differential inclusions and variational inequalities. In the case of missing lower semicontinuity, using the regularization we defined above, we give in what follows approximative counterparts of some classical results which traditionally impose the lower semicontinuity as main assumption.

Definition 5.1. Let A and B be two sets and $F : A \rightrightarrows B$ a set-valued mapping. A function $f : A \to B$ with the property $f(x) \in F(x)$ for every $x \in A$ is called a selection of F.

In general, the problem is to find selections (when $\operatorname{Dom} F = A$) with certain properties, e.g., continuity or measurability. The main result in the theory of continuous selections is the following theorem due to E. Michael (see [9], [10]).

Theorem 5.2 (Michael). *Let X be a metric space, Y be a Banach space and $F : X \rightrightarrows Y$ be a l.s.c. set-valued mapping with nonempty, closed, convex values. Then F admits a continuous selection.*

Note that the lower semicontinuity of F is essential in this result. Now we are able to present an approximate continuous selection result for set-valued mappings with no continuity assumptions.

Theorem 5.3. *Let X be a normed vector space, Y be a Banach space and $C \subset Y$ be a pointed closed convex cone with nonempty interior. Let $K \subset X$ be a nonempty set and $F : X \rightrightarrows Y$ be a set-valued mapping with closed, locally compact values on K and convex values on an open neighborhood of K and suppose that there exists $\alpha > 0$ s.t. for every $\bar{x} \in K$, $R_F^\alpha(\bar{x}) \neq \emptyset$. Then there exists a continuous function $f_\alpha : K \to Y$ s.t. for every $x \in K$, $f_\alpha(x) \in F(x) - C_\alpha$.*

Proof. Consider $\alpha > 0$. Observe that R_F^α has nonempty closed values for every $\bar{x} \in K$. Since F has convex values on an open neighborhood of K, then L_F^γ has convex values on the same neighborhood of K for every positive γ, and hence R_F^α has convex values on K (as a directed union of convex sets). Taking into account that R_F^α is l.s.c. on K and K is a metric space endowed with the distance given by the norm, we can apply Michael selection theorem in order to find a continuous selection f_α of R_F^α on K. We are in the conditions of Proposition 4.5, so we can conclude that for every $\bar{x} \in K$, $f_\alpha(\bar{x}) \in F(\bar{x}) - C_\alpha$. This completes the proof. □

Of course, in the case $Y = \mathbb{R}$, one can obtain a better result on the base of Proposition 3.1. Moreover, in the general setting, if the cone C is locally closed, then the corresponding condition for the values of F can be dropped.

Remark 5.4. The conclusion of the above theorem still holds if one supposes that Y is a reflexive Banach space and F has weak-closed values on K (see Proposition 4.5).

Roughly speaking, the number α measures the enlargement needed by F in order to have a continuous selection and similar remarks can be made on the next results where we assume that $R_F^\alpha(\bar{x}) \neq \emptyset$ for a positive α. This condition must be checked in every specific case. We consider the following example.

Example 5.5. Let $F : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by $F(x) = \{1\}$ if $x > 0$, $F(x) = \{-1\}$ if $x < 0$ and $F(0) = \{0\}$. Of course, F is not lower semicontinuous at 0 and F does not admit a continuous selection. Take $C = \mathbb{R}_+$. Computing R_F^α, for some $\alpha > 0$, we find that $R_F^\alpha(x) = [1 - \alpha, 1]$ for $x > 0$ and $R_F^\alpha(x) = [-1 - \alpha, -1]$ for $x < 0$. But, in order to have $R_F^\alpha(0) \neq \emptyset$ one needs $\alpha > 2$ and in this case $R_F^\alpha(0) = [1 - \alpha, -1]$.

Up to our knowledge, the problem of finding approximate continuous selections for a non l.s.c. set-valued mapping was considered for the first time by A. Cellina in [5] and the results of this kind were applied in order to derive fixed point theorems. In the next result we consider the same problematic. We recall two classical results in fixed point theory in both single-valued and set-valued cases. We start with the Schauder fixed point theorem which is an extension of the famous Brouwer topological fixed point theorem to infinite dimensional spaces. Both of them inspired Kakutani to construct his celebrated fixed point theorem for set-valued mappings.

Theorem 5.6 (Schauder). *Let K be a convex compact set in a normed vector space X and $f : K \to K$ be a continuous function. Then there exists $x \in K$ s.t. $f(x) = x$ (i.e., F has a fixed point in K).*

Theorem 5.7 (Kakutani). *Let K be a convex compact set in a normed vector space X and $F : K \rightrightarrows K$ be an upper semicontinuous set-valued mapping with nonempty closed convex values. Then there exists $x \in K$ s.t. $x \in F(x)$ (i.e., F has a fixed point in K).*

For upper semicontinuous set-valued mappings the Kakutani theorem is an adequate tool for proving the existence of fixed points. Our purpose is to suggest an alternative for the case where the continuity of the underlying mapping is missing. The result we present next does not impose neither upper semicontinuity nor lower semicontinuity of the mapping, but instead uses assumptions written for a "smaller" (in the sense of inclusion) mapping. So, the price to pay is that we work on finite dimension and we impose somehow stronger conditions corresponding to the other assumptions in Kakutani's theorem (see the remarks after the proof). In fact, the first conclusion of the result below speaks about approximate fixed points: for a positive ε, we say that F has an ε-fixed point x if $d(x, F(x)) \leq \varepsilon$, where $d(x, F(x))$ denotes the distance between the element x and the set $F(x)$. Note that approximate fixed point results for set-valued mappings can be useful, for instance, in game theory: see [4] where, on the basis of some approximate fixed point theorems, the authors have developed a method of finding ε-equilibrium results for non-cooperative games for which Nash equilibria do not exist.

Theorem 5.8. *Let X be a finite dimensional vector space and $C \subset X$ be a pointed closed convex cone with nonempty interior. Let $K \subset X$ be a compact convex set and $F : X \rightrightarrows X$ be a set-valued mapping with compact values on K and convex values on an open neighborhood of K and suppose that for every $\bar{x} \in K$, $\liminf_{x\to\bar{x}} F(x) \neq \emptyset$. If there exists $\mu > 0$ s.t. $F(K) - C_\mu \subset K$, then for every $\varepsilon > 0$, F has an ε-fixed point. If, moreover, the closure of the graph of the set valued mapping $F(\cdot) \cap (\cdot + C_\mu) \cap K$ is included in the graph of F, then F has a fixed point in K.*

Proof. Take $\alpha < \mu$. Then, using Theorem 5.3, the set valued mapping $F(\cdot) - C_\alpha$ has a continuous selection f_α on K. Moreover, since $F(K) - C_\mu \subset K$ we have that $f_\alpha(K) \subset K$, hence we can apply Schauder's fixed point theorem to obtain an element $x_\alpha \in K$ with $f_\alpha(x_\alpha) = x_\alpha$. Of course, x_α is in turn an α-approximate fixed point for F and since we have taken any $\alpha < \mu$, the first part of the conclusion follows. Take now, under the additional assumption, $\alpha = n^{-1}$ for n large enough and denote x_α by x_n. Then, the sequence (x_n) has a convergent subsequence to an element $x_0 \in K$. On the other hand, $x_n \in F(x_n) - C_{n^{-1}}$, so there exist $y_n \in F(x_n)$ and $c_n \in C_{n^{-1}}$ s.t. $y_n = x_n + c_n$. Since $c_n \to 0$, one obtains that $y_n \to x_0$ (on a subsequence). One can use the last assumption (because $y_n \in F(x_n) \cap (x_n + C_\mu)$ for n large enough), whence $x_0 \in F(x_0)$, i.e., F has a fixed point. □

Remark 5.9. The hypotheses that the closure of the graph of the set valued mapping $F(\cdot) \cap (\cdot + C_\mu) \cap K$ is included in the graph of F is weaker than the requirement that the graph of F is closed. On the other hand, the latter set-valued mapping can have empty values. Note that this is the main difference with respect to Kakutani theorem, since if the graph of F is closed, then the other assumptions imply the upper semicontinuity of F.

We present now two applications of Theorem 5.3 concerning the existence of approximate solutions for a differential inclusion and for a variational inequality problem.

For the first aim, let us consider an open interval $I \subset \mathbb{R}$, an open set $\Omega \subset \mathbb{R}^n$ and a set-valued mapping $F : I \times \Omega \rightrightarrows \mathbb{R}^n$. Fix $t_0 \in I$, $x_0 \in \Omega$ and consider the differential inclusion problem:

$$\begin{cases} x'(t) \in F(t, x(t)) \\ x(t_0) = x_0. \end{cases} \tag{3}$$

A solution of this problem is an absolutely continuous function $x : J \to \Omega$, (where $J \subset I$ is a nontrivial interval containing t_0) s.t. $x(t_0) = x_0$ and $x'(t) \in F(t, x(t))$ a.e. for $t \in J$. For a positive α, we call an α-approximate solution for the problem (3), an absolutely continuous function $x : J \to \Omega$ which satisfies $x(t_0) = x_0$ and $d(x'(t), F(t, x(t))) \leq \alpha$ a.e. for $t \in J$.

Theorem 5.10. *In the above notations, suppose that $C \subset \mathbb{R}^n$ is a pointed closed convex cone with nonempty interior, that the set-valued mapping F has closed convex values on $I \times \Omega$ and that there exists $\alpha > 0$ s.t. for every $\bar{u} \in I \times \Omega$, $R_F^\alpha(\bar{u}) \neq \emptyset$. Then there exists an α-approximate solution for the problem (3) defined on a neighborhood of t_0 and this solution can be continued up to a saturated one.*

Proof. Following Theorem 5.3 there exists a continuous function $f_\alpha : I \times \Omega \to \mathbb{R}^n$ s.t. $f_\alpha(u) \in F(u) - C_\alpha$ for all $u \in I \times \Omega$. Now, using the classical Peano theorem, the problem:

$$\begin{cases} x'(t) = f_\alpha(t, x(t)) \\ x(t_0) = x_0 \end{cases}$$

has a solution defined on a neighborhood of t_0 and this solution can be continued up to a saturated one. This solution is, in turn, an approximate solution for the differential inclusion problem we considered. □

One can apply the same pattern in the following setting: let K be a compact subset of $\mathbb{R}^n$ and $F : K \rightrightarrows \mathbb{R}^n$ be a set valued mapping. Consider the variational problem: find $x^* \in K$ s.t. there exists $u^* \in F(x^*)$ with $\langle u^*, x - x^* \rangle \geq 0$ for all $x \in K$ (where $\langle \cdot, \cdot \rangle$ denotes the inner product of $\mathbb{R}^n$). Using Theorem 5.3 one can give an existence result by means of weaker solutions.

Theorem 5.11. *Suppose that $C \subset \mathbb{R}^n$ is a pointed closed convex cone with nonempty interior, that the set-valued mapping F has closed convex values on a neighborhood of K and that there exists $\alpha > 0$ s.t. for every $x \in K$, $R_F^\alpha(u) \neq \emptyset$. Then there exists $x^* \in K$ and $u^* \in F(x^*)$ with $\langle u^*, x - x^* \rangle \geq -\alpha \|x - x^*\|$ for all $x \in K$.*

Proof. Following Theorem 5.3 there exists a continuous function $f_\alpha : K \to \mathbb{R}^n$ s.t. $f_\alpha(u) \in F(u) - C_\alpha$ for all $u \in K$. Now, using a classical existence result (see, e.g., [7]), one can find some $x^* \in K$ s.t. $\langle f_\alpha(x^*), x - x^* \rangle \geq 0$ for all $x \in K$. Since one can write $f_\alpha(x^*) = u^* - c^*$ with $u^* \in F(x^*)$ and $c^* \in C_\alpha$, one has $\langle u^*, x - x^* \rangle \geq \langle c^*, x - x^* \rangle \geq -\alpha \|x - x^*\|$ for all $x \in K$. □

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