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On Conditions for the Minimality of Exhausters

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A definition of the minimality of an upper (lower) exhauster is discussed; necessary conditions for the minimality of exhausters are formulated along with a sufficient condition for the minimality of sets in an exhauster.

1. Introduction

Upper and lower exhausters of a positively homogeneous function $h : R^n \to R$ were first introduced by Demyanov in [1]. The notions of exhausters are closely related to exhaustive families of upper convex and lower concave approximations of the function h. For an arbitrary function $f : R^n \to R$ exhausters can be employed to effectively study its (upper or lower) directional derivatives. For more details and historical background we refer the reader to [1], [2] and [6].

We say that a family of compact convex sets $E^*(h)$ is an *upper exhauster* of a p.h. function h, if h can be represented in the form

$$h(g) = \inf_{C\in E^*(h)} \max_{v\in C}(v, g) \quad \forall g \in R^n. \tag{1}$$

Analogously, if h can be represented as

$$h(g) = \sup_{C\in E_*(h)} \min_{v\in C}(v, g) \quad \forall g \in R^n, \tag{2}$$

where $E_*(h)$ is a family of compact convex sets from R^n, then $E_*(h)$ is called a *lower exhauster* of h. The pair $[E^*(h), E_*(h)]$ is called a *biexhauster* of h.

Exhausters can be employed to describe necessary optimality conditions and to find steepest ascent and descent directions. One needs to use an upper (lower) exhauster for the necessary conditions for a minimum (maximum) and to find steepest descent (ascent) directions. Moreover, it was shown recently (See [4, 5]) that an upper exhauster can be also employed to formulate conditions for the maximality and a lower one for the minimality, but in this case we can't find the steepest directions. An upper exhauster is called proper for minimality conditions and adjoint for the conditions of maximality. Analogously, a lower exhauster is a proper one for maximality and adjoint for minimality.

What is also important is that exhausters have a well-developed calculus – that is, if biexhausters of corresponding functions are known, one is able to calculate a biexhauster

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of a linear combination, product, quotient, superposition, max- and min-functions (e.g. see [1], Section 4 and [2], Section 12).

It is not difficult to observe that exhausters are not uniquely defined, hence, the problem of minimality naturally arises (see [2], Section 13). The problems of minimality and reduction of set-valued tools are quite common in nonsmooth optimization, since very often these tools are not uniquely defined. For example, see [3], [7] and [8]. An attempt for reduction of exhausters was undertaken in [10], but still there is no algorithm for reduction or minimization of exhausters.

In the current paper we provide a definition and state necessary conditions for the minimality of upper and lower exhausters. We also provide sufficient conditions for the minimality of sets in an exhauster. The paper contains numerous examples.

Exhausters in abstract spaces were studied in [11], but in this paper we limit ourselves to a finite-dimensional case only.

Throughout the paper we assume that h is positively homogeneous (p.h.) if

$$h(\lambda x) = \lambda h(x) \quad \forall \lambda \geq 0.$$

By S_1^n we denote a unit sphere in R^n, that is,

$$S_1^n = \{g \in R^n \mid ||g|| = 1\}.$$

Note that if the representations (1) or (2) are valid for every $g \in S_1^n$, then they are valid for all $g \in R^n$ since h is positively homogeneous. This would be implied throughout the paper.

The paper is organized as follows: in Section 2 we provide definitions of minimality of exhausters and state necessary conditions for the minimality; in Section 3 we discuss sufficient conditions for minimality of sets in exhausters. In Section 4 a short summary is given.

2. Necessary conditions for the minimality of exhausters

The definition of a *minimal biexhauster by inclusion* was given in [2], Section 13. Since we are not intended to operate with a biexhauster as a whole throughout this paper, we provide the definitions for the minimality of upper and lower exhausters separately.

Definition 2.1 (Minimal exhauster by inclusion). We say that an upper (lower) exhauster $E(h)$ of a p.h. function h is minimal by inclusion, if there exists no other upper (lower) exhauster $\tilde{E}(h)$ of h such that $\tilde{E}(h) \subset E(h)$ and $\tilde{E}(h) \neq E(h)$.

It is not difficult to observe that the above definition of minimality deals only with the quantity of sets in an exhauster, but has nothing to do with the shape of the sets. The following example demonstrates that in some cases the exhauster which is minimal in this definition, can be made much smaller by modifying the shape of the sets of which it consists.

Example 2.2. Consider a real-valued function from R^1: $h_1(x) = |x|$. It is not difficult to see that

$$E_1^*(h_1) = \{[-2, 1], [-1, 2]\}$$

is an upper exhauster of h_1. Note that if we remove one of the sets from $E_1^*(h_1)$, it will not be an upper exhauster of h_1 any more. However, we can modify the shape of the sets in the exhauster. Both of the sets in $E_1^*(h_1)$ can be reduced to $[-1,1]$, which is the subdifferential of the function h_1. Since the subdifferential of a convex function is uniquely defined (See [9]), there's exactly one upper exhauster consisting of a single set: $E_2^*(h_1) = \{[-1,1]\}$. Note that this exhauster also satisfies Definition 2.1, but is much "smaller" than $E_1^*(h_1)$. □

The example provided above demonstrates the need for another definition of a minimal exhauster, which covers not only minimality by the inclusion, but also the minimality of the shape of sets in an exhauster. We provide such definition below.

Definition 2.3. We say that an upper(lower) exhauster $E(h)$ of h is a minimal one, if there exists no other upper (lower) exhauster $\tilde{E}(h)$ of h such that

$$\forall \tilde{C} \in \tilde{E}(h)\ \exists C \in E(h) \mid \tilde{C} \subset C.$$

Note that if an upper (lower) exhauster is minimal in the sense of Definition 2.3, then it is also minimal by inclusion (Definition 2.1), but the converse is not true, as it was shown in Example 2.2.

Let C be any arbitrary set from an upper exhauster. Construct a set

$$M^*(C) = \operatorname{cl}\operatorname{co}\{v_0 \in C \mid \exists g \in S_1^n,\ v_0 = \arg\max_{v\in C}(v,g);\ (v_0,g) = h(g)\}.$$

Analogously, for any arbitrary set from a lower exhauster we put

$$M_*(C) = \operatorname{cl}\operatorname{co}\{v_0 \in C \mid \exists g \in S_1^n,\ v_0 = \arg\min_{v\in C}(v,g);\ (v_0,g) = h(g)\}.$$

Theorem 2.4. *Let E^* be an upper exhauster of a p.h. function h. If $M^*(C_0) = \emptyset$ for some $C_0 \in E^*$, then $\tilde{E}^* = E^* \setminus \{C_0\}$ is also an upper exhauster of h.*

Proof. Let E^* be an upper exhauster of a p.h. function h and $C_0 \in E^*$ be such that $M^*(C_0) = \emptyset$. Then for every $g \in S_1^n$ we have $\max_{v\in C_0}(v,g) > h(g)$, hence, for every $g \in S_1^n$ there exists $C_g \in E^*$ such that $\max_{v\in C_0}(v,g) > \max_{v\in C_g}(v,g)$ (obviously, $C_0 \neq C_g$). Hence, for every $g \in S_1^n$ we have

$$\max_{v\in C_0}(v,g) > \max_{v\in C_g}(v,g) \geq \inf_{C\in E^*\setminus\{C_0\}} \max_{v\in C}(v,g),$$

hence,

$$h(g) = \min\{\inf_{C\in E^*\setminus\{C_0\}} \max_{v\in C}(v,g), \max_{v\in C_0}(v,g)\} = \inf_{C\in E^*\setminus\{C_0\}} \max_{v\in C}(v,g) \quad \forall g \in S_1^n,$$

and the family of sets $\tilde{E}^* = E^* \setminus \{C_0\}$ is again an upper exhauster of h. □

The following theorem is a corollary of Theorem 2.4 and provides a necessary condition for the minimality of an upper exhauster by inclusion.

Theorem 2.5. *If E^* is a minimal by inclusion upper exhauster of a p.h. function h, then $\forall C \in E^*\ M^*(C) \neq \emptyset$.*

Proof. Assume the opposite, that is, let E^* be a minimal upper exhauster by inclusion and $C_0 \in E^*$ be such that $M^*(C_0) = \emptyset$. It follows from Theorem 2.4, that the family of sets $E^* \setminus \{C_0\}$ is also an upper exhauster of h and is smaller than E^*, which contradicts the assumption. □

The symmetric theorems are valid for a lower exhauster. We omit the proofs, since they are similar to the proofs of Theorems 2.4 and 2.5.

Theorem 2.6. *Let E_* be a lower exhauster of a p.h. function h. If $M_*(C_0) = \emptyset$ for some $C_0 \in E_*$, then $\tilde{E}_* = E_* \setminus \{C_0\}$ is also a lower exhauster of h.*

Theorem 2.7. *If E_* is a minimal by inclusion lower exhauster of a p.h. function h, then $\forall C \in E_*\ M_*(C) \neq \emptyset$.*

Remark 2.8. Note that since the minimality of an exhauster in the sense of Definition 2.3 implies the minimality by inclusion, the necessary conditions for the minimality by inclusion provided by Theorems 2.5 and 2.7 are also necessary for the minimality in the sense of Definition 2.3.

We will need the below lemma in what follows.

Lemma 2.9. *Let C be a compact convex set in R^n, $a \in S_1^n$. For any $d \in R^1$ denote*

$$C_d = \{v \in C \mid (a, v) \leq d\},\ d^* = \max_{v \in C}(a, v),\ d_* = \min_{v \in C}(a, v). \tag{3}$$

Then if $d_* < d^*$,

$$\max_{v \in C_d}(v, g) \xrightarrow[d_* < d < d^*]{d \to d^*,} \max_{v \in C}(v, g) \quad \forall g \in S_1^n. \tag{4}$$

Proof. Assume the opposite, that is, there exist $g_0 \in S_1^n$, a sequence $\{d_n\}$, $d_n \subset (d_*, d^*)$, $d_n \xrightarrow[n \to \infty]{} d^*$, and a constant $c \in R^1$ such that

$$\max_{v \in C}(v, g_0) - \max_{v \in C_{d_n}}(v, g_0) \geq c > 0 \quad \forall n \in 1 : \infty. \tag{5}$$

Let $v_0 \in C$ be such that $v_0 = \arg\max_{v \in C}(v, g_0)$ and let $w_0 \in C_{d_*}$. Note that $(a, v_0) = d^*$, otherwise $v_0 \in C_{d_n}$ for sufficiently large n and $\max_{v \in C}(v, g_0) = \max_{v \in C_{d_n}}(v, g_0)$, which makes (5) impossible. Put $v_n = (1 - \alpha_n)v_0 + \alpha_n w_0$, where $\alpha_n = \frac{d^* - d_n}{d^* - d_*}$. Note that for every $n \in \mathbb{N}$, $\alpha \in (0, 1)$, hence, $v_n \in C$ for all $n \in \mathbb{N}$. Moreover,

$$\begin{aligned}(a, v_n) &= (a, (1 - \alpha_n)v_0 + \alpha_n w_0) \\ &= (a, v_0) + \alpha_n[(a, w_0) - (a, v_0)] = d^* + \alpha_n(d_* - d^*) = d_n,\end{aligned}$$

hence, $v_n \in C_{d_n}$. Note that $(v_0, g_0) - (v_n, g_0) = \alpha_n(v_0 - w_0, g)$. We have

$$\max_{v \in C}(v, g) - \max_{v \in C_{d_n}}(v, g) = (v_0, g) - \max_{v \in C_{d_n}}(v, g) \leq (v_0, g) - (v_n, g) = \alpha_n(g, v_0 - w_0).$$

Noticing that α_n goes to zero as n goes to infinity, we get a contradiction with the initial assumption. □

Lemma 2.10. *Let E^* be an upper exhauster of a p.h. function h. If for some $C_0 \in E^*$*

$$M^*(C_0) \neq C_0 \quad \text{and} \quad M^*(C_0) \neq \emptyset,$$

then there exists a set $C_1 \subset C_0$, $C_1 \neq C_0$ such that $\tilde{E}^* = (E^* \setminus \{C_0\}) \cup \{C_1\}$ is an upper exhauster of h, that is, there exists a smaller upper exhauster.

Proof. Let E^* be an upper exhauster of a p.h. function h and $C \in E^*$ be such that $M^*(C) \neq C$ and $M^*(C) \neq \emptyset$. Take any $v_0 \in C \setminus M^*(C)$. Since $M^*(C)$ is a compact convex set, by the separation theorem (see [9]) there exist $d_0 \in R^1$ and $a \in S_1^n$, such that

$$(a, v) < d_0 \quad \forall v \in M^*(C) \quad \text{and} \quad (a, v_0) > d_0. \tag{6}$$

We shall prove that there exists $\tilde{d} < d^*$ such that for every $g \in S_1^n$

$$\max_{v \in C_{\tilde{d}}} (v, g) \geq h(g),$$

where $C_{\tilde{d}}$ and d^* are defined by (3), and, hence, $\tilde{E}^* = (E^* \setminus \{C\}) \cup \{C_{\tilde{d}}\}$ is an upper exhauster of h. Suppose that this is not true, then there exists a sequence $\{d_n\}$, $d_n \to d^*$, $d_n \in (d_*, d^*)$, such that for every $n \in \mathbb{N}$ there is a point $g_n \in S_1^n$ satisfying $\max_{v \in C_{d_n}} (v, g_n) < h(g_n)$. Denote $v_n = \arg\max_{v \in C_{d_n}} (v, g_n)$. Since the sequences $\{g_n\}$ and $\{v_n\}$ are bounded and C is compact, without loss of generality we may assume that

$$g_n \xrightarrow[n\to\infty]{} g^* \in S_1^n, \qquad v_n \xrightarrow[n\to\infty]{} v^* \in C.$$

There might be two cases: $(a, v^*) < d^*$ and $(a, v^*) = d^*$. Consider the first case. Then there exists $\bar{d} < d^*$, such that for sufficiently large N_1 we have

$$(a, v_n) < \bar{d} \quad \forall n > N_1. \tag{7}$$

Since for N_2 large enough for every $n > N_2$ we have $d_n > \bar{d}$, it follows from (7) that

$$(v_n, g_n) = \max_{v \in C_{\bar{d}}} (v, g_n) = \max_{v \in C_{d_n}} (v, g_n) = \max_{v \in C} (v, g_n) \geq h(g_n) \quad \forall n > N = \max\{N_1, N_2\}, \tag{8}$$

which contradicts the assumption. So the only possibility is that $(a, v^*) = d^*$.

Since $(v_n, g_n) < h(g_n)$ and $h(g)$ is upper semicontinuous as an infimum of continuous functions, we have

$$\lim_{n\to\infty} (v_n, g_n) = (v^*, g^*) \leq h(g^*). \tag{9}$$

Denote $u_n = \arg\max_{v \in C_{d_n}} (v, g^*)$. We have

$$(v_n, g_n) - \max_{v \in C} (v, g^*) = \Big((v_n, g_n) - (u_n, g^*)\Big) + \Big((u_n, g^*) - \max_{v \in C} (v, g^*)\Big).$$

The first summand goes to zero when $n \to \infty$ due to the continuity of the max-function, the second one goes to zero by Lemma 2.9. Hence,

$$(v^*, g^*) = \max_{v \in C} (v, g^*) \geq h(g^*). \tag{10}$$

It follows from (9) and (10), that $(v^*, g^*) = h(g^*)$, hence, $v^* \in M^*(C)$, which is impossible, since $(a, v^*) = d^* > d_0$. Then our assumption is wrong and there exists $\tilde{d} < d^*$ such that $\tilde{E}^* = (E^* \setminus \{C\}) \cup \{C_{\tilde{d}}\}$ is an upper exhauster of h. □

The symmetric lemma is true for a lower exhauster. The proof is omitted due to its similarity to the proof of Lemma 2.10.

Lemma 2.11. *Let E_* be a lower exhauster of a p.h. function h. If for some $C_0 \in E_*$*

$$M_*(C_0) \neq C_0 \quad \text{and} \quad M_*(C_0) \neq \emptyset,$$

then there exists a set $C_1 \subset C_0$, $C_1 \neq C_0$ such that $\tilde{E}_* = (E_* \setminus \{C_0\}) \cup \{C_1\}$ is a lower exhauster of h, that is, there exists a smaller lower exhauster.

Theorem 2.12. *Let E^* be an upper exhauster of a p.h. function h. If E^* is a minimal upper exhauster, then for every $C \in E^*$ we have $M^*(C) = C$.*

Proof. Assume the opposite, that is, the exhauster is minimal, but there exists a set $C_0 \in E^*$ such that $C_0 \setminus M^*(C_0) \neq \emptyset$. Since the exhauster is minimal, by Theorem 2.5 we have $M^*(C_0) \neq 0$. Then by Lemma 2.10 there exists an exhauster which is smaller than E^*, which contradicts the assumption. □

The symmetric result for a lower exhauster also holds.

Theorem 2.13. *Let E_* be a lower exhauster of a p.h. function h. If E_* is a minimal lower exhauster, then for every $C \in E_*$ we have $M_*(C) = C$.*

It could be expected that the conditions provided in Theorems 2.12 and 2.13 are also sufficient for the minimality, however, this is not true. The following example illustrates it.

Example 2.14. Consider a function $h_2 : R^2 \to R$,

$$h_2(x_1, x_2) = \max\{x_1, -x_1, x_2, -x_2\} \quad \forall\, (x_1, x_2) \in R^2.$$

This function is subdifferentiable as a maximum of linear functions and

$$\underline{\partial} h_2 = \operatorname{co}\{(-1, 0), (1, 0), (0, -1), (0, 1)\}.$$

Hence, $E_1^*(h_2) = \{C_1\}$, $C_1 = \underline{\partial} h_2$ is a minimal upper exhauster of h_2. In this case $M^*(C_1) = C_1$ and the necessary condition is satisfied. But what if we add one more set to this exhauster, for example, $C_2 = \operatorname{co}\{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$ (see Fig. 2.1)? Since $C_1 \subset C_2$, for every $g \in S_1^2$ we have $\max_{v \in C_2}(g, v) \geq \max_{v \in C_1}(g, v) = h_2(g)$, hence,

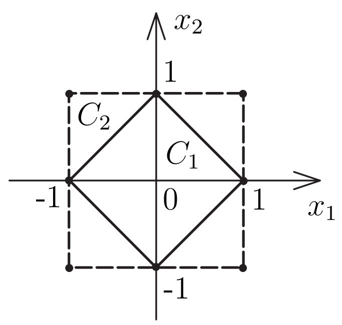

Figure 2.1: Example 2.14

$E_2^*(h_2) = \{C_1, C_2\}$ is also an upper exhauster of h. One can check that the necessary condition for the minimality of an upper exhauster is satisfied, but $E_2^*(h_2)$ is obviously not a minimal exhauster. □

Remark 2.15. When it comes to reduction of an exhauster, it seems quite tempting to replace C by $M^*(C)$ for every set in an exhauster and immediately get a smaller exhauster. However, this wouldn't work. The following example demonstrates it.

Example 2.16. Consider the same function h_2 as in Example 2.14. It is not difficult to see that $E_3^*(h_2) = \{C_1, C_3\}$, where

$$C_1 = \operatorname{co}\{(-1,0),(1,0),(0,-1),(0,1)\}, \qquad C_3 = \operatorname{co}\{(-1,0),(1,0),(0,-2),(0,2)\},$$

is an upper exhauster of h (Not a minimal one, since $M^*(C_3) = \operatorname{co}\{(-1,0),(1,0)\} \neq C_3$). It can be easily verified that the family of sets $\{M^*(C_1), M^*(C_3)\}$ is not an upper exhauster of h_2. □

3. Sufficient conditions for the minimality of sets in an exhauster

Let $h : R^n \to R$ be a p.h. function with $E^*(h)$ being its upper exhauster. For every $g \in S_1^n$ and $C \in E^*(h)$ put

$$m_C(g) = \{v_0 \in C \,|\, v_0 = \arg\max_{v \in C}(v, g); (v_0, g) = h(g)\}$$

and construct a set

$$\tilde{M}^*(C) = \operatorname{cl}\operatorname{co}\{m_C(g) \,|\, g \in S_1^n,\ m_C(g) \text{ is a singleton}\}.$$

There's an example shown on Figure 3.1 of the sets $M^*(C)$ and $\tilde{M}^*(C)$ for a simple upper exhauster consisting of two sets C_1 and C_2. Here $M^*(C_i) = M_i$ and $\tilde{M}^*(C_i) = \tilde{M}_i$, $i \in 1:2$.

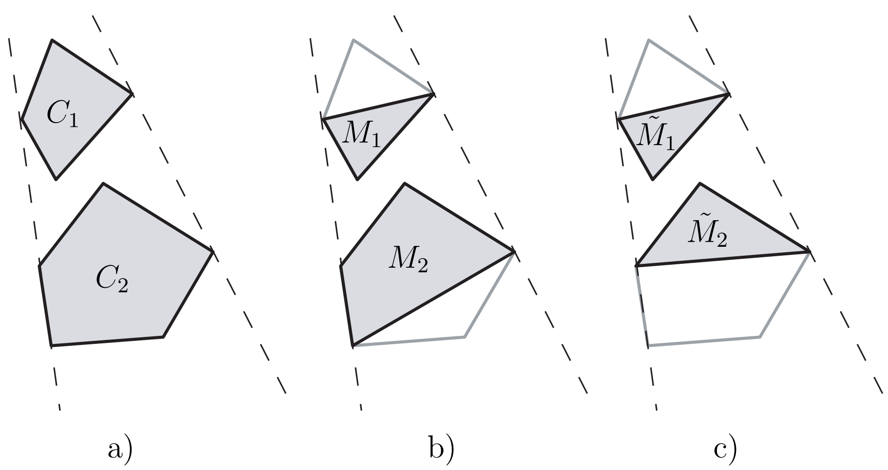

Figure 3.1: Difference between $M^*(C)$ and $\tilde{M}^*(C)$

Theorem 3.1. *Let E^* be an upper exhauster of a p.h. function h. If for some $C_0 \in E^*$ we have $\tilde{M}^*(C_0) = C_0$, then it is impossible to replace C_0 with a smaller set $C_1 \neq \emptyset$, such that $\tilde{E}^* = (E^* \setminus \{C_0\}) \cup \{C_1\}$ is again an upper exhauster of h.*

Proof. Consider the opposite, that is, there exists a set $C_0 \in E^*$ such that $\tilde{M}^*(C_0) = C_0$, but a smaller compact convex set $C_1 \subset C_0$ exists such that $\tilde{E}^* = (E^* \setminus \{C_0\}) \cup \{C_1\}$ is also an upper exhauster of h. Since C_1 is smaller than C_0, the set $C_0 \setminus C_1$ is nonempty. Choose any point $v_0 \in C_0 \setminus C_1$. Since C_1 is convex and compact, it can be strictly separated from v_0. That is, there exist $a \in R^n$, $d \in R^1$, such that

$$(a, v_0) > d, \quad (a, v) \leq d \quad \forall v \in C_1.$$

For the sake of convenience we denote

$$m(C) = \cup\{m_C(g) \,|\, g \in S_1^n,\ m_C(g) \text{ is a singleton}\}; \quad \tilde{m}(C) = \mathrm{co}\,\{m(C)\}, \quad C \in E^*.$$

Note that $\tilde{M}^*(C) = \mathrm{cl}\,\tilde{m}(C)$. Since $v_0 \in \tilde{M}^*(C_0)$, there exists a point $\bar{v} \in \tilde{m}(C)$ close enough to v_0 to satisfy

$$(a, \bar{v}) > d. \tag{11}$$

Recall that then by the Carathéodory theorem there exist $p \leq n+1$, $v_i \in m(C)$, $\alpha_i \in R^1$ for $i \in 1:p$ such that

$$\bar{v} = \sum_{i=1}^{p} \alpha_i v_i, \quad \sum_{i=1}^{p} \alpha_i = 1, \quad \alpha_i > 0 \quad \forall i \in 1:p.$$

It is not difficult to see that for at least one index $i_0 \in 1:p$ we have $(a, v_{i_0}) > d$ (otherwise we would have $(a, \bar{v}) \leq d$, which contradicts (11). Put $u = v_{i_0}$. Since $u \in m(C)$, there exists $g \in S_1^n$ such that $(u, g) = h(g)$ and $(g, v) < h(g)$ for all $v \in \{C_0\} \setminus \{u\}$. Since $C_1 \subset \{C_0\} \setminus \{u\}$ and C_1 is compact, we have

$$\max_{v \in C_1}(v, g) < h(g),$$

which contradicts our assumption that $\tilde{E}^*$ is an upper exhauster of h. □

Denote

$$n_C(g) = \{v_0 \in C \,|\, v_0 = \arg\min_{v \in C}(v, g); (v_0, g) = h(g)\} \quad \forall g \in S_1^n;$$

$$\tilde{M}_*(C) = \mathrm{cl}\,\mathrm{co}\,\{n_C(g) \,|\, g \in S_1^n,\ n_C(g) \text{ is a singleton}\}.$$

The result similar to Theorem 3.1 is true for a lower exhauster.

Theorem 3.2. *Let E_* be a lower exhauster of a p.h. function h. If for some $C_0 \in E_*$ we have $\tilde{M}_*(C_0) = C_0$, then it is impossible to replace C_0 with a smaller set $C_1 \neq \emptyset$, such that $\tilde{E}_* = (E_* \setminus \{C_0\}) \cup \{C_1\}$ is again a lower exhauster of h.*

Remark 3.3. The question of whether the conditions of Theorems 3.1 and 3.2 are necessary for the minimality of sets in an upper or lower exhauster or not is still open.

The following example shows that the conditions in Theorems 3.1 and 3.2 are not sufficient for the minimality of an exhauster, they only ensure the minimality of sets in an exhauster.

Example 3.4. Let a p.h. function $h_1 : R^2 \to R$ be defined by its upper exhauster $E_1^* = \{C_1, C_2, C_3\}$, where $C_1 = \mathrm{co}\,\{(-2,3), (1,0), (-2,-3)\}$, $C_2 = \mathrm{co}\,\{(2,3), (-1,0), (2,-3)\}$ and $C_3 = \mathrm{co}\,\{(1,0), (0,3), (-1,0), (0,-3)\}$ (See Fig. 3.2).

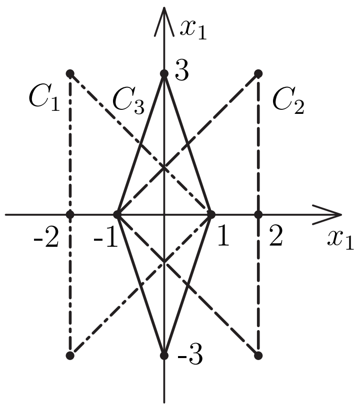

Figure 3.2: Example 3.4

It is not difficult to see that E_1^* satisfies the sufficient conditions for the minimality of sets in an exhauster, however, one can check that $E_2^* = \{C_1, C_2\}$ is also an upper exhauster of h. Hence, the sufficient conditions for the minimality of sets are not sufficient for the minimality of an exhauster itself. □

4. Summary

In this article we have provided certain conditions for the minimality of exhausters. However, there are still many open questions. For example, if a minimal exhauster exists, we do not know whether it is unique or not, and there is no algorithm for minimization or even reduction of exhausters. Moreover, it appears that in some cases a minimal exhauster doesn't exist (for example, consider the function $h(g) = -||g||$, $g \in R^2$, and try to find its minimal upper exhauster). All these questions need careful consideration and are subjects to further study.

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