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Convex Coradiant Sets with a Continuous Concave Cogauge

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Dedicated to the memory of Alex Rubinov.

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The paper studies convex coradiant sets and their cogauges. While the concave gauge of a convex coradiant set is superlinear but discontinuous and its Minkowski cogauge is (possibly) continuous but is not concave, we are interested in those convex coradiant sets which admit a continuous concave cogauge. These sets are characterized in primal terms using their outer kernel and in dual terms using their reverse polar set. It is shown that a continuous concave cogauge, if it exists, is not unique; we prove that the class of continuous concave cogauges of some set C admits a greatest element and characterize its support set as the intersection of the reverse polar of C and the polar of its outer kernel.

Keywords: Convex sets, concave gauge, cogauge, radiant sets, coradiant sets, reverse polar

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1. Introduction

This paper is devoted to the study of some properties of a particular class of convex sets, namely convex coradiant sets. A proper subset C of a vector space X is called *coradiant* if it satisfies $0 \notin C$ and $tc \in C$ for all $c \in C$ and $t \geq 1$ or equivalently if it is the complement of a radiant set, where $A \subseteq X$ is said to be *radiant* (or star-shaped at the origin) if $a \in A$, $t \in [0, 1]$ imply $ta \in A$.

While the importance of convex radiant sets, i.e. convex sets containing the origin, has always been recognized, and their study in connection with nonnegative sublinear functionals is at the basis of fundamental results of Functional Analysis and of the theory of normed spaces, on the contrary the study of convex coradiant sets raised less systematic interest, mainly motivated by applications in different fields. For instance Tind [21] studies convex coradiant sets as an intermediate step to analyze blocking and antiblocking sets, which are used in some extremal problems of combinatorics; Ruys and Weddepohl [16] underline the relevance of convex and aureoled sets (their name for convex coradiant sets) in Mathematical Economics; more recently Cornuejols and Lemarechal [3], though not explicitly interested in convex coradiant sets, analyze extensively the use of the reverse polar and of the reverse gauge of a set (see the definitions below) in their study on disjunctive cuts in combinatorial optimization. Theoretical interest seems to be more recent; Barbara and Crouzeix [2] study the concave gauge of a convex coradiant set; Levin [7, 6] extensively analyzes semiconic duality for convex sets and functions; Marechal [8]

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introduces a particular analytic operation which associates a subset of $\mathbb{R}^{n+m}$ to a convex set C of $\mathbb{R}^n$ and a convex function on $\mathbb{R}^m$ and exploits the assumption that C is radiant or coradiant in order that the resulting set be convex. Penot [9] uses the reverse polar to study the conjugate of a quasiconvex function.

Our interest in convex coradiant sets arose in connection with the study of separation properties of radiant and coradiant sets. It has been noted in [24] that the separation of some point x from a set A of a normed space X can be described, in a geometric fashion, by saying that convex coradiant sets separate points from a radiant set and that convex radiant sets separate points from a coradiant set. To be more precise a set A is closed and radiant (respectively closed and coradiant) if and only if for every point $x \notin A$ there exists an open convex, coradiant (resp. radiant) set G which contains x and is disjoint from A.

To obtain an analytic form of separation one needs to characterize the convex separating sets G by means of some simple functional form, which is possibly sub- or super-linear and continuous. This is very easily done for (open or closed) convex sets containing the origin, while no such description seems to be available for convex coradiant sets.

Indeed a set $C \subseteq X$ is closed, convex with $0 \in C$ if and only if its Minkowski gauge μ_C is sublinear with $C = [\mu_C \leq 1]$ and if and only if $C = C^{\circ\circ}$, where C° is the polar set of C. Moreover it is easy to describe those closed, convex, radiant sets whose Minkowski gauge is continuous: they have the origin as an interior point and this happens if and only if the polar set C° is bounded.

The situation is more intricate if we want to give an analogous functional description of a convex coradiant set. For this purpose, we will say that a function $p : X \to \overline{\mathbb{R}}$ is a *cogauge* of the set $C \subseteq X$ if p is positively homogeneous and $C = [p \geq 1]$. Two different instances are known in the literature, neither of which fits our purposes. Barbara and Crouzeix [2] introduced the concept of *concave gauge* (called reverse gauge in [3]) of a convex coradiant set, that is the function

$$\varphi_C(x) = \inf\{\ell(x) : \ell \in C^\oplus\},$$

where the set $C^\oplus = \{\ell \in X' : \ell(c) \geq 1, \forall c \in C\}$ is the reverse polar of C. The function φ_C is superlinear and upper semicontinuous with $\varphi_C(x) \geq 0$ for all $x \in \operatorname{cl}\operatorname{cone} C$ and $\varphi_C(x) = -\infty$ otherwise, and hence it not continuous whatever the set C is.

Rubinov [14] studied the *Minkowski cogauge* of a coradiant set C, that is the positively homogeneous function

$$\nu_C(x) = \sup\{\lambda > 0 : x \in \lambda C\},$$

and proved its continuity for a particular class of coradiant sets, called coradiative. But ν_C is not concave, in that it holds $\nu_C(x) = 0$ for all $x \notin \operatorname{cone} C$. The two functions are related by the equalities $C = [\varphi_C \geq 1] = [\nu_C \geq 1]$, which hold whenever C is closed, convex and coradiant. More generally ν_C and φ_C coincide on the set $\overline{K} = \operatorname{cl}\operatorname{cone} C$. These results are summarized in Section 2.

The main purpose of this paper, as discussed in Section 3, is to single out the class of convex coradiant sets for which the functions φ_C and ν_C can be extended from the set $K = \operatorname{cl}\operatorname{cone} C$ to a superlinear, continuous function defined on X. We prove that this is possible provided the origin is an interior point for the *outer kernel* of C (see Definition 3.2 below). This is quite analogous to the condition used to obtain the continuity of the Minkowski gauge of a convex radiant set C, which can be equivalently expressed as $0 \in \operatorname{int} C$ or $0 \in \operatorname{int} \ker C$, where $\ker A$ is the kernel of the set $A \subseteq X$.

The extension we obtain is not uniquely defined. There is rather an infinite choice of extensions and the concave gauge φ_C can be seen as the least element of this set. We show that also the greatest element exists, that is a superlinear continuous function γ_C which represents C and is greater than any other continuous concave representation of C. This is treated in Section 4, together with other side results on the outer kernel of a set.

The section is completed with a result which characterizes the support sets of the greatest superlinear gauge γ_C in terms of polar sets. This characterization is very helpful for the determination of γ_C and it can be used to show that, in the case where C is a closed coradiant halfspace (that is a closed halfspace for which $0 \notin C$), then the support set of γ_C reduces to a single point and hence γ_C is linear. Thus the rule for computing the greatest superlinear gauge gives a way to associate linear functionals to closed, coradiant halfspaces. Note that both the concave gauge φ_C and the cogauge ν_C fail to be linear in this case.

This study underlines the importance in Convex Analysis of some concepts, like the cogauge ν_C or the kernel of a radiant set, which initially found their motivation in star-shaped analysis.

We consider a normed space X, in which the closed ball of radius δ centered in x is denoted by $B_\delta(x) = B(x, \delta)$; the closure, interior, boundary of some set $S \subseteq X$ are denoted by $\operatorname{cl} S$, $\operatorname{int} S$ and $\operatorname{bd} S$ respectively; the convex hull and the conic hull of S are denoted, respectively, as $\operatorname{conv} S$ and $\operatorname{cone} S = \{y = \lambda x : x \in S, \lambda > 0\}$. Moreover we use the set $\operatorname{cone}_0 S = \operatorname{cone} S \cup \{0\} = \{y = \lambda x : x \in S, \lambda \geq 0\}$. Let X' be the topological dual space of X and denote by $\langle x, \ell \rangle$ or equivalently $\ell(c)$ the usual bilinear pairing between $x \in X$ and $\ell \in X'$. For a function $f : X \to \overline{\mathbb{R}} = [-\infty, +\infty]$ and $k \in \mathbb{R}$ we denote by $[f \leq k]$ the sublevel set $\{x \in X : f(x) \leq k\}$ and by $[f \geq k]$ the superlevel set $\{x \in X : f(x) \geq k\}$, while $\operatorname{epi} f$ and $\operatorname{hyp} f$ stand for the epigraph and, respectively, the hypograph of f.

2. Preliminaries on gauges and polarities

We recall in this section some preliminary concepts, as the ones of radiant and coradiant sets, and some known results about the description of closed convex radiant sets and closed, convex, coradiant sets by means of their gauges and of polarity relations.

Definition 2.1. The set $A \subseteq X$ is called *radiant* if $x \in A$, $t \in [0, 1]$ imply that $tx \in A$. It is called *coradiant* if its complement $A^C = X \backslash A$ is radiant, that is if either $A = X$ or $0 \notin A$ and $x \in A$, $t \geq 1$ imply that $tx \in A$.

We deduce that the empty set $\emptyset$ and the set X are both radiant and coradiant. We underline that the terms radiant and coradiant have been used previously, with a slightly different meaning. Rubinov [14] uses $t \in (0, 1]$ in the definition of a radiant set, so that the origin can either belong or not belong to a radiant or to a coradiant set. Penot [9] includes convexity in the definition of a radiant set. Levin [7] calls a set $A \subseteq X$ semiconic when $x \in A$, $t \geq 1$ imply that $tx \in A$ and strictly semiconic if moreover it holds $0 \notin \operatorname{cl} \operatorname{conv} A$.

Of particular importance are those radiant or coradiant sets which are also convex. It is easy to see that a convex set is radiant if and only if it contains the origin. Let $\mathcal{C}$

denote the class of nonempty closed convex sets of X containing the origin. Various characterizations of elements of $\mathcal{C}$ are well-known, starting from the following separation property: a set $C \subseteq X$ belongs to $\mathcal{C}$ if and only if for every point $x \notin C$ there exists some linear continuous functional $\ell \in X'$ such that

$$\langle x, \ell \rangle > 1 \quad \text{and} \quad \langle c, \ell \rangle \leq 1, \quad \forall c \in C.$$

This property can equivalently be described by means of a polarity relation. Let

$$C^\circ = \{\ell \in X' : \ell(c) \leq 1, \forall c \in C\}$$

be the polar of the set $C \subseteq X$ and

$$C^{\circ\circ} = (C^\circ)^\circ = \{x \in X : \ell(c) \leq 1, \forall \ell \in C^\circ\} \subseteq X$$

be the bipolar of C. Then $C \in \mathcal{C}$ if and only if $C = C^{\circ\circ}$. A different characterization, in primal terms, of elements of $\mathcal{C}$ can be given by means of their Minkowski gauge.

Definition 2.2. Given a radiant set $A \subseteq X$, its *Minkowski gauge* is the function $\mu_A : X \to \overline{\mathbb{R}}$ given by

$$\mu_A(x) = \inf\{\lambda > 0 : x \in \lambda A\}.$$

For a detailed study of the properties of the gauge of a radiant set, see [12, 14]. A classical result of Functional Analysis sets a one to one correspondence between sets in $\mathcal{C}$ and nonnegative sublinear functions. For our purposes it can be stated as follows.

Proposition 2.3. *Let C be a radiant subset of X. Then the following are equivalent:*

a) C *is closed and convex;*
b) $C = C^{\circ\circ}$*;*
c) μ_C *is sublinear and lower semicontinuous, with* $C = [\mu_C \leq 1]$.

The two mentioned characterizations are closely related in that, for any set $C \in \mathcal{C}$ it holds that C° is the subdifferential of μ_C at the origin and conversely μ_C is the support function of C°, i.e.

$$\mu_C(x) = \sigma_{C^\circ}(x), \tag{1}$$

where $\sigma_B(x) = \sup_{\ell \in B}\langle x, \ell \rangle$ is the support function of a subset $B \subseteq X'$. Further conditions on C are required in order that its gauge be continuous (and hence finite valued on X). They are expressed in the following proposition whose proof can be given by standard arguments.

Proposition 2.4. *Let C be a closed, convex, radiant subset of X. The following are equivalent:*

a) $0 \in \mathrm{int}\, C$*;*
b) C° *is bounded;*
c) μ_C *is continuous.*

We turn now our attention to convex coradiant sets, which we will call *shady*. This terminology follows Penot [9], but observe that the term shady in [10] does not include convexity. Given a set $A \subseteq X$, we call *shadow* of A the set

$$\operatorname{shw} A = \{x \in X : x = ta,\ a \in A,\ t \geq 1\}.$$

If $0 \notin A$ then the set $\operatorname{shw} A$ is coradiant; it is indeed the smallest coradiant set containing A, that is the coradiant hull of A. It follows from Definition 2.1 that, if $0 \in A$, then the coradiant hull of A always coincides with X, while its shadow does not. Our main interest is in the class $\mathcal{K}$ of closed, convex, coradiant sets of X; for their description a different notion of polarity is needed. Given a nonempty set $C \subseteq X$ we call *reverse polar* of C the set

$$C^{\oplus} = \{\ell \in X' : \langle c, \ell \rangle \geq 1, \forall c \in C\}.$$

The reverse polar was considered, for instance, in [1, 2, 3, 7, 9, 10, 16, 20, 21]. Different authors used different names for the same notion and our choice seems to be the one most frequently used, while the notation comes from [2]. Note that Penot [9] also considers the analogues of $\circ$-polarity and of $\oplus$-polarity with strict inequalities.

We adopt the convention that $C^{\oplus} = X'$ if $C = \emptyset$. It is easy to see that $C^{\oplus}$ is always closed, convex and coradiant in X' and that, for any nonempty set $C \subseteq X$, it holds $C^{\oplus} = (\operatorname{cl} \operatorname{conv} \operatorname{shw} C)^{\oplus}$. More precisely, since $C \subseteq X$ is closed and shady if and only if it is the intersection of all the halfspaces $[\ell \geq 1]$, with $\ell \in X'$, which contain C (see, e.g., [9]), it is easy to verify that $C^{\oplus\oplus} = C$ if and only if $C \in \mathcal{K}$. Actually the authors who studied the reverse polarity gave different characterizations of the class of sets $C \subseteq X$ for which $C = C^{\oplus\oplus}$, in which the relation between C and its conic hull or its recession cone is emphasized. We collect them in the following proposition. Recall that, the recession cone of a nonempty convex set C is

$$\operatorname{Rec} C = \{d \in X : x + td \in C, \forall x \in C, \forall t > 0\}.$$

The recession cone of a closed set C is closed and, in this case, it holds (see [11])

$$\operatorname{Rec} C = \{d \in X : \exists x \in C, \text{ such that } x + td \in C, \forall t > 0\}. \tag{2}$$

Proposition 2.5. *For a nonempty, closed, convex set $C \subseteq X$, with $0 \notin C$, the following are equivalent:*

a) C *is coradiant;*
b) $C \subseteq \mathit{Rec}\, C$;
c) $\mathit{Rec}\, C = \operatorname{cl} \operatorname{cone} C$*;*
d) $C + \mathit{cone}_0\, C = C$*;*
e) $C = C^{\oplus\oplus}$.

Proof. *a)* $\Rightarrow$ *b)*. If $c \in C$ and C is coradiant, then

$$C \ni \alpha c = c + (\alpha - 1)c, \quad \forall \alpha \geq 1,$$

which shows that $c \in \operatorname{Rec} C$, using (2).

b) $\Rightarrow$ *c)*. The relation $\operatorname{Rec} C \subseteq \operatorname{cl} \operatorname{cone} C$ is true for all convex set C; the opposite inclusion follows from $C \subseteq \operatorname{Rec} C$, noting that the recession cone of a closed set is closed.

c) $\Rightarrow$ *d)*. The inclusion $C \subseteq C + \operatorname{cone}_0 C$ is true for all sets C; to prove the opposite inclusion, note that $\operatorname{cone}_0 C \subseteq \operatorname{cl} \operatorname{cone} C$ and that the definition of recession cone implies that $C + \operatorname{Rec} C \subseteq C$.

d) $\Rightarrow$ *e)*. It was proved by Tind [21].

e) $\Rightarrow$ *a)*. It was proved for instance by Penot [9] or by Levin [6]. □

Condition $b)$ was used by Marechal [8] to define one of the classes of convex sets in which the paper is interested. Tind [21] proves that, for any nonempty set C with $0 \notin \operatorname{cl}\operatorname{conv} C$, it holds

$$C^{\oplus\oplus} = \operatorname{cl}\operatorname{conv} C + \operatorname{cone}_0(\operatorname{cl}\operatorname{conv} C).$$

The implication $a) \Rightarrow c)$ was proved in [7] and in [8]. In [18] one can find the proof of a result quite analogue to $c)$ for nonconvex coradiant sets, that is the equality

$$\operatorname{cl}\operatorname{cone} A = \operatorname{As} A$$

for a nonempty, closed, coradiant set $A \subseteq X$, where

$$\operatorname{As} A = \{y \in X : \, y = \lim t_i x_i, \, t_i \searrow 0, \, x_i \in A\}$$

is the cone of asymptotic directions of a set A and it holds $\operatorname{As} C = \operatorname{Rec} C$ for any closed convex set $C \subseteq X$.

We pass now to the functional characterization of convex coradiant sets, which follows the same geometric construction on which the radiant case is based. Indeed for a closed, convex, radiant set C, it holds $\operatorname{epi} \mu_C = \operatorname{cl}\operatorname{cone}(C \times \{1\})$; if we start from a closed, convex, coradiant set C, the set $\operatorname{cl}\operatorname{cone}(C \times \{1\})$, which is obviously a convex cone in $X \times \mathbb{R}_+$, can be seen as the positive part of the hypograph of a superlinear function, whose precise definition depends on the way the hypograph is completed in $X \times \mathbb{R}$. Two instances are known in the literature and we will propose another one. To ease the comparisons among these notions we will refer to the concepts of "Concave Analysis", instead of the more common convex ones.

Barbara and Crouzeix [2] give the following definition of concave gauge of a set C. Although the definition is meaningful for a larger class of sets, we restrict our interest, here and in the sequel, to elements of the class $\mathcal{K}$.

Definition 2.6 ([2]). Given a set $C \in \mathcal{K}$, the function $\varphi_C : X \to \overline{\mathbb{R}}$ given by

$$\varphi_C(x) = \inf\{\ell(x) : \, \ell \in C^{\oplus}\}$$

is called the *concave gauge* of C.

The concave gauge is clearly related to the support function of $C^{\oplus}$. If we let, for any nonempty set $B \subseteq X'$,

$$i_B(x) = \inf_{\ell \in B} \langle x, \ell \rangle$$

we obtain an upper semicontinuous superlinear function $i_B : X \to \mathbb{R} \cup \{-\infty\}$, which is continuous if and only if B is bounded. Moreover it holds, for all $x \in X$,

$$i_B(x) = i_{cl\,convB}(x) = -\sigma_{-B}(x).$$

Without fear of confusion with the 'convex' analogue, we will say that i_B is the *support function* of B and that $\operatorname{cl}\operatorname{conv} B$ is the *support set* of i_B.

Some special features of the concave gauge φ_C are consequences of $C^{\oplus}$ being coradiant. Indeed φ_C is nonnegative on its *effective domain*, $\operatorname{dom} \varphi_C = \{x : \, \varphi_C(x) > -\infty\}$, which is the set $\overline{K} = \operatorname{cl}\operatorname{cone} C$ and positive on $K = \operatorname{cone} C$, while $\varphi_C(x) = -\infty$ for all $x \notin \overline{K}$. Moreover $C = \{x \in X : \, \varphi_C(x) \geq 1\}$. Thus φ_C is an u.s.c. superlinear cogauge of C.

Clearly the superdifferential of φ_C coincides with $C^{\oplus}$. Here and in the sequel, by *superdifferential*, denoted $\partial\varphi$, of an upper semicontinuous superlinear function $\varphi : X \to \overline{\mathbb{R}}$, we mean its support set, or superdifferential at the origin

$$\partial\varphi(0) = \{\ell \in X' : \ell(x) \geq \varphi(x), \forall x \in X\}.$$

Moreover, for all $x \in \operatorname{dom}\varphi$, it holds

$$\partial\varphi(x) = \{\ell \in \partial\varphi : \ell(x) = \varphi(x)\}$$

and $\partial\varphi(x) \neq \emptyset$ if φ is continuous at x.

A different functional characterization of closed, convex, coradiant sets comes from the following concept (see [14] for details).

Definition 2.7. Given a coradiant set A, its *Minkowski cogauge* is the function $\nu_A : X \to \mathbb{R} \cup \{+\infty\}$ given by

$$\nu_A(x) = \sup\{\lambda > 0 : x \in \lambda A\}.$$

Since in Definition 2.7 we consider only positive values of λ, it is natural to impose that $\sup\emptyset = 0$. Using the above convention one can see that, for every coradiant set $A \subseteq X$, it holds $\nu_A(0) = 0$ (unless $A = X$, in which case $\nu_A(0) = +\infty$) and that, for $x \neq 0$, it holds $\nu_A(x) = 0$ if and only if $A \cap R_x = \emptyset$, where $R_x = \{z \in X : \lambda x, \lambda > 0\}$ is the open ray defined by x. Moreover $\nu_A(x) = +\infty$ if and only if $R_x \subseteq A$.

The equality $A = [\nu_A \geq 1]$ holds for all closed coradiant sets. It is easy to see that, for a radiant set A and its complement $A^C = B$, which is coradiant, it holds $\mu_A = \nu_B$.

Both Barbara and Crouzeix [2] and Penot and Zalinescu [10] set $\nu_A(x) = -\infty$ when the set $\{\lambda > 0 : x \in \lambda A\}$ is empty. In this case ν_A lacks upper semicontinuity even for A closed and shady. To overcome this drawback, in [10] the authors give a modified definition of cogauge, which is proved to be u.s.c. for all closed, shady sets. Moreover it is proved in [2, Prop 2.1] that the concave gauge φ_C of a set $C \in \mathcal{K}$ is the closure (that is the least u.s.c. majorant) of ν_C on $\overline{K}$. This amounts to say that φ_C coincide with the modified definition of cogauge in [10] and that φ_C and ν_C actually coincide on $\overline{K}$, since ν_A is upper semicontinuous for every closed coradiant set A (see [14] for the proof of this statement and see also [3] for an independent proof of the equivalence between ν_C and φ_C on the set $K = \operatorname{cone} C$).

On the other hand the two functions have a very different behaviour at points $x \notin \overline{K}$. Indeed it holds $\varphi_C(x) = -\infty$ and $\nu_C(x) = 0$ there. Thus we have

$$\operatorname{hyp}\varphi_C = \operatorname{cl}\operatorname{cone}(C \times \{1\}) \cup \{(\operatorname{cl}\operatorname{cone} C) \times \mathbb{R}_-\}$$

and

$$\operatorname{hyp}\nu_C = \operatorname{cl}\operatorname{cone}(C \times \{1\}) \cup \{X \times \mathbb{R}_-\},$$

and, unlike the convex case (illustrated by equality (1)), the two representations of a closed, shady set C, which we obtain following the support idea, namely φ_C, or the Minkowski idea, ν_C, do not coincide on X.

We note in passing that the functions φ_C and ν_C can be defined for nonclosed sets and may differ in this case even at points belonging to the boundary of K. This happens, for instance, for the set $C = \{(x_1, x_2) : x_1 > 0, x_2 > 0, x_1 + x_2 > 1\}$, for which we have $\varphi_C(0,1) = 1$ and $\nu_C(0,1) = 0$.

3. Continuous superlinear cogauges

We noted above that both the concave gauge φ_C and the Minkowski co-gauge ν_C represent a closed, convex, coradiant sets $C \subseteq X$ since it holds $C = [\nu_C \geq 1] = [\varphi_C \geq 1]$.

We are interested in the following problem: under what further conditions on the set C can we extend φ_C (and ν_C) from $K = \operatorname{cl}\operatorname{cone} C$ to a superlinear continuous function F defined on all of X, with $[F \geq 1] = C$. When a positively homogeneous function $F : X \to \overline{\mathbb{R}}$ satisfy $C = [F \geq 1]$ we will say that F is a *cogauge* of C or, equivalently, that F *represents* C.

The problem of extending a convex function from a convex set $C \subseteq \mathbb{R}^n$ to all $\mathbb{R}^n$ is studied in [19]. For us the problem is more complicated since we also want that the set C coincides with the level set $[F \geq 1]$ of its cogauge F. To illustrate the difference, let $C = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 + x_2 \geq 1\}$. Then it holds $\varphi_C(x_1, x_2) = x_1 + x_2$ if $(x_1, x_2) \in \mathbb{R}^2_+$ and $\varphi_C(x_1, x_2) = -\infty$ else. It is easy to see that the linear function $\ell(x_1, x_2) = x_1 + x_2$ is a continuous extension of φ_C to $\mathbb{R}^2$, but it is not a continuous cogauge of C in that $[\ell \geq 1] \neq C$.

Barbara and Crouzeix [2] showed partial interest in this issue; they name concave barrier functions those functions φ_C which vanish on the boundary of $\operatorname{cone} C$. We deduce from what has been said above, that this property coincides with ν_C being continuous and hence one can find in [14] a convenient necessary and sufficient condition for a closed convex coradiant set to admit a concave barrier gauge, namely that it is a coradiative set.

Definition 3.1. A proper coradiant set A is said to be coradiative if every ray from the origin has at most one intersection with the boundary of A.

It is proved in [14] that a set $A \subseteq X$ is *coradiative* if and only if its Minkowski cogauge ν_A is continuous. Moreover, for a coradiative set A, it holds $\operatorname{bd} A = [\nu_A = 1]$ and hence, for a convex coradiative set $C \subseteq X$, it holds

$$\operatorname{bd} C = [\nu_C = 1] = [\varphi_C = 1] \quad \text{and} \quad \operatorname{bd} K = [\varphi_C = 0].$$

Thus the requirement that C be coradiative is necessary if we want that φ_C be extended to a continuous function, since we need that F vanishes on the boundary of K. It is not sufficient though, as shown by the example of the set

$$C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\, x_1 x_2 \geq 1\},$$

whose concave gauge is $\varphi_C(x_1, x_2) = \sqrt{x_1 \cdot x_2}$ for $(x_1, x_2) \in \mathbb{R}^2_+$ and $-\infty$ elsewhere. This function cannot be extended to a continuous superlinear function defined on $\mathbb{R}^2$ since its superdifferential is empty at points $(0, x_2)$, with $x_2 \geq 0$ or $(x_1, 0)$, with $x_1 \geq 0$.

To reach our goal we need to remind some concepts from star-shaped analysis.

Definition 3.2. The *kernel* of a set $A \subseteq X$ is the set of points

$$\ker A = \{z \in A : z + t(x - z) \in A,\, \forall x \in A,\, \forall t \in (0, 1]\}.$$

The *outer kernel* of a set $A \subseteq X$, $\operatorname{oker} A$, is the kernel of its complement A^C, that is the set

$$\operatorname{oker} A = \{z \in X : z + t(x - z) \notin A,\, \forall x \notin A,\, \forall t \in (0, 1]\}.$$

Definition 3.2 considers only positive values of t and hence does not imply that $\ker A \subseteq A$, nor that $A \cap \operatorname{oker} A = \emptyset$. For instance if $A = \{x \in \mathbb{R}^2 : x_1 + x_2 < 1\}$, then it holds

$$\ker A = \operatorname{oker}(A^C) = \{x \in \mathbb{R}^2 : x_1 + x_2 \leq 1\},$$

which yields $A^C \cap \operatorname{oker}(A^C) = \{x \in \mathbb{R}^2 : x_1 + x_2 = 1\}$. On the other hand it holds $\ker A \subseteq \operatorname{cl} A$. A modified definition of kernel, with $t \in [0, 1]$, is called stage in [4].

It is obvious that a set $A \subseteq X$ including the origin is radiant if and only if $0 \in \ker A$ and that a proper set A excluding the origin is coradiant if and only if $0 \in \operatorname{oker} A$. It is easy to see that both the kernel and the outer kernel of a set $A \subseteq X$ are convex sets (if nonempty) and that a set C is convex if and only if $C \subseteq \ker C$. This means that a closed nonempty set is convex if and only if $C = \ker C$. The following equivalent description of the outer kernel of a set is often used in the sequel and straightforward to verify:

$$\operatorname{oker} A = \{z \in X : z + t(x - z) \in A, \forall t \geq 1, \forall x \in A\}.$$

We will see in Theorem 3.6 that the condition to be required to a convex coradiant set in order that its concave gauge may be extended to a continuous superlinear function is that the origin be an interior point of its outer kernel.

Definition 3.3. A proper, closed, convex, coradiant set C is called *strongly shady* if $0 \in \operatorname{int} \operatorname{oker} C$.

Since our main interest in this section is to study the sets which can be obtained as level sets $[\varphi \geq 1]$, for a continuous superlinear function $\varphi : X \to \mathbb{R}$, we will always consider closed sets. This entails some lack of generality in that some results do not require closedness. On the other hand we remind that, for a coradiative set A, it holds $\nu_A = \nu_{cl\, A}$.

The assumption that the origin belongs to the interior of the kernel of some radiant set A (which is the complement of C in the present case) was used in [14] to characterize those radiant sets whose Minkowski gauge is Lipschitz continuous. It is indeed equivalent to the requirement that A is the union of convex sets which contain the same open ball around the origin.

It is possible to prove that every strongly shady set is coradiative. It is indeed a consequence of the following result, for which convexity is not required.

Proposition 3.4. *If the set A is coradiant and $0 \in \operatorname{int} \operatorname{oker} A$, then A is coradiative.*

Proof. Reasoning by contradiction, suppose that the points x and $x' = \alpha x$, with $0 < \alpha < 1$, belong to the boundary of A, and let B_ε, with $\varepsilon > 0$, be contained in $\operatorname{oker} A$. The ball $B(x, s)$ with $s = \alpha^{-1}(1-\alpha)\varepsilon/2$, contains a point $y \in X \backslash A$ and the ball $B(x', \alpha s)$ contains a point $y' \in A$. Then $z = (1-\alpha)^{-1}(y' - \alpha y) \in B(0, \varepsilon)$ and $y' = (1-\alpha)z + \alpha y \in A$, a contradiction with the assumption that $z \in \operatorname{oker} A$. □

We also need the following further concept.

Definition 3.5. A closed shady set $C \subseteq X$ is said to be *reducible* if there exists some $M > 0$ such that $C = \operatorname{shw}(C \cap B_M(0))$.

Definition 3.5 can be rephrased as: for every $y \in C$ there exist $t \geq 1$ and $c \in C$ with $\|c\| \leq M$ and $y = tc$, so that C can be seen as the shadow of a bounded set. Moreover it holds $\operatorname{cone} C = \operatorname{cone}(C \cap B_M(0))$ so that $C \cap B_M(0)$ is a bounded base for $\operatorname{cone} C$. We observe that, for a convex coradiant set C, the specifications that C is coradiative and that C is reducible are mutually exclusive. We are now ready to present the companion to Proposition 2.4, which is the main result of this section.

Theorem 3.6. *Let C be a nonempty, closed, shady set. Then the following are equivalent:*

(a) *There exists a continuous superlinear function $F : X \to \mathbb{R}$ such that $[F \geq 1] = C$;*

(b) $0 \in \mathit{int}\, \mathit{oker}\, C$*;*

(c) $C^{\oplus}$ *is reducible.*

Proof. $(a) \Rightarrow (b)$. Let F be continuous and superlinear with $C = [F \geq 1]$; then $F(x) \leq 0$ for $x \notin \operatorname{cone} C$. Since F is superlinear and continuous, its superdifferential ∂F is norm bounded. Let $M > 0$ be such that $\|\ell\| \leq M$ for all $\ell \in \partial F$ and let $\varepsilon = \min(1/M, \eta)$, with $B_\eta(0) \cap C = \emptyset$. We want to show that for all $z \in B_\varepsilon(0)$ and all $t \in [0,1]$ and for $x \notin C$, we have $z + t(x - z) \notin C$, i.e. $z \in \operatorname{oker} C$. Suppose on the contrary that there exist $\tilde{x} \notin C$, $\tilde{z} \in B_\varepsilon(0)$, $\tilde{\lambda} \in (0,1)$ such that

$$\tilde{y} = \tilde{z} + \tilde{\lambda}(\tilde{x} - \tilde{z}) \in C.$$

Then $F(\tilde{y}) \geq 1$. On the other hand, it holds

$$\ell(\tilde{z}) \leq \|\ell\| \|\tilde{z}\| \leq M\varepsilon \leq 1$$

for all $\ell \in \partial F$ and, since $\tilde{x} \notin C$, there exists some $\ell \in \partial F(0)$ such that $\ell(\tilde{x}) < 1$. Thus $F(\tilde{y}) \leq \ell(\tilde{y}) = (1 - \tilde{\lambda})\ell(\tilde{z}) + \tilde{\lambda}\ell(\tilde{x}) < 1$, which is a contradiction.

$(b) \Rightarrow (c)$. Let $B_\varepsilon(0) \subseteq \operatorname{oker} C$ and $M = 1/\varepsilon$. If $\ell \in C^{\oplus}$ and $\|\ell\| > M$ then set $\alpha = M/\|\ell\|$ and $\ell' = \alpha\ell$ so that $\|\ell'\| = M$ and $\sup\{\ell'(z) | \, z \in B_\varepsilon(0)\} = 1$. We have to show that $\ell' \in C^{\oplus}$. Reasoning by contradiction, suppose that $\ell'(\bar{c}) = \bar{b} < 1$ for some $\bar{c} \in C$; we can find $\bar{z} \in B_\varepsilon(0)$ such that $\ell'(\bar{z})$ is as close to 1 as desired, so let $1 \geq \ell'(\bar{z}) = k > \bar{b}$. Since $\bar{z} \in \operatorname{oker} C$, it holds, for all $t > 1$, $\bar{z} + t(\bar{c} - \bar{z}) \in C$. But also:

$$\ell'(\bar{z} + t(\bar{c} - \bar{z})) = (1 - t)\ell'(\bar{z}) + t\ell'(\bar{c}) = (1 - t)k + t\bar{b} = k - t(k - \bar{b}) < 1.$$

Since t can be taken arbitrarily large and $k - \bar{b} > 0$, we have that ℓ' can take negative values on C. Hence ℓ can also take negative values on C, in contrast to $\ell \in C^{\oplus}$.

$(c) \Rightarrow (a)$. Let $C^{\oplus}$ be reducible and take $M > 0$ such that $D = C^{\oplus} \cap B_M(0)$ satisfies $\operatorname{shw} D = C^{\oplus}$. Since D is weak*-compact, the function

$$F(x) = \inf\{\ell(x) : \, \ell \in D\}$$

is superlinear and continuous on X. To show that F represents C, that is $C = [F \geq 1]$, we will first prove that $F(x) = \varphi_C(x) := \inf\{\ell(x) : \, \ell \in C^{\oplus}\}$ for all $x \in \overline{K}$ and, second, that $F(x) < 0$ for all $x \notin \overline{K}$.

Since $x \in \overline{K}$ is equivalent to $\varphi_C(x) \geq 0$ one has, in this case, that

$$\varphi_C(x) = \inf\{td(x), \, t \geq 1, \, d \in D\} = \inf\{d(x) : \, d \in D\} = F(x).$$

To complete the proof, we recall that, since D is weak*-compact and $0 \notin D$, then $\operatorname{cone} D \cup \{0\}$ is closed. Moreover we have that

$$\operatorname{cl} \operatorname{cone} C^{\oplus} = K^{+} := \{\ell \in X' : \, \ell(k) \geq 0, \, \forall k \in K\},$$

the positive polar cone of K. But as $\operatorname{cone} D \cup \{0\} = \operatorname{cone} C^{\oplus} \cup \{0\}$ is closed, we have $K^{+} = \operatorname{cone} D \cup \{0\}$.

If $x \notin \overline{K}$ there exists some $\ell \in X'$ such that $\ell(x) < 0$ and $\ell(k) \geq 0$ for all $k \in \overline{K}$, whence $\ell \in K^{+}$ and $\ell' = \alpha\ell \in D$ for some $\alpha > 0$. Since it holds $D = \partial F$ and $\ell'(x) < 0$ then it is $F(x) \leq \ell'(x) < 0$. □

We wish to underline the analogy between Theorem 3.6 and Proposition 2.4: since for a convex set C it holds $C \subseteq \ker C$, statement $a)$ in Proposition 2.4 corresponds to $0 \in \operatorname{int}\ker C$; the reverse polar cannot be required to be bounded, as it is done for C° in $c)$ of Proposition 2.4, but in our case it is the shadow of a bounded set.

It can be seen that a strongly shady set $C \subseteq X$ has nonempty interior. Indeed by Theorem 3.6 there exists a continuous representation φ such that $C = [\varphi \geq 1]$ and it is easy to see that the set $[\varphi \geq 1]$ has nonempty interior. Indeed this is true for all coradiative sets, since their co-gauge is continuous.

We obtain important instances of strongly shady sets, by considering particular shifts of convex solid cones. The possibility to obtain a continuous superlinear representation for these sets was used in [22] to characterize functions with radiant sublevel sets in terms of Abstract Convexity.

Proposition 3.7. *If $K \neq X$ is a closed, convex cone with nonempty interior and $x \in \operatorname{int} K$, then $C = x + K$ is strongly shady.*

Proof. To see that C is coradiant, let y be some element in C and $\alpha \geq 1$. Then $y = x + k$, for some $k \in K$ and we have

$$\alpha y = x + (\alpha - 1)x + \alpha k \in x + K + K \subseteq x + K.$$

Now we want to show that $x - K \subseteq \operatorname{oker} C$.

To this purpose, take $z \in x - K$, $t \geq 1$ and $c \in x + K$. Thus $z = x - k$ and $c = x + k'$ for some $k, k' \in K$ and

$$z + t(c - z) = x - k + t(x + k' - x + k) = x + (t-1)k + tk' \in x + K = C, \quad \forall t \geq 1$$

and therefore $z \in \operatorname{oker} C$. To finish note that $x \in \operatorname{int} K$ implies

$$0 \in x - \operatorname{int} K = \operatorname{int}(x - K) \subseteq \operatorname{int}\operatorname{oker} C$$

and $C = x + K$ is strongly shady. □

4. The greatest superlinear continuous cogauge

Simple examples show that the continuous representation of a convex coradiant set C (if it exists) is not unique. Indeed any of the functions given by

$$F_{\alpha}(x) = \begin{cases} x & \text{for } x \geq 0 \\ \alpha x & \text{for } x < 0 \end{cases} \qquad \alpha \geq 1$$

is a continuous superlinear representation of the set $C = [1, +\infty)$.

More precisely, we can refine Theorem 3.6 and obtain the following result.

Theorem 4.1. *Let $C \subseteq X$ be strongly shady and $\varphi : X \to \mathbb{R}$ be a superlinear function. Then φ is a continuous cogauge of C if and only if the superdifferential $\partial\varphi$ is bounded and satisfies*

$$\textit{shw}\, \partial\varphi = C^{\oplus}.$$

Proof. Let's prove necessity first. Since φ is continuous, $\partial\varphi$ is bounded, and since

$$[\varphi_C \geq 1] = C \subseteq [\varphi \geq 1],$$

we have

$$\varphi_C(x) \leq \varphi(x), \quad \forall x \in C. \tag{3}$$

Relation (3) can be extended to all $x \in \operatorname{cone} C$ due to positive homogeneity and to all $x \in \operatorname{cl}\operatorname{cone} C = \overline{K}$ due to upper semicontinuity. Moreover $\varphi_C(x) = -\infty$ for all $x \notin \overline{K}$ and hence we have $\varphi_C \leq \varphi$. This implies $\partial\varphi \subseteq C^{\oplus}$ and $\operatorname{shw}\partial\varphi \subseteq C^{\oplus}$.

To prove the opposite inclusion, let $\ell \notin \operatorname{shw}\partial\varphi$. Since $[\varphi \geq 1] \neq \emptyset$, it holds $0 \notin \partial\varphi$ and hence we have that $[0,1] \cdot \ell \cap \partial\varphi = \emptyset$. As both sets are convex and w^*-compact, there exists $x \in X$ such that

$$\alpha\ell(x) < 1 < \varphi(x), \quad \forall \alpha \in [0,1]. \tag{4}$$

Since $\varphi(x) > 1$, then $x \in C$. Since, in particular, (4) yields $\ell(x) < 1$, it holds $\ell \notin C^{\oplus}$ and necessity is proved.

Suppose now that $\partial\varphi$ is bounded with $\operatorname{shw}\partial\varphi = C^{\oplus}$. Boundedness of $\partial\varphi$ implies continuity of φ and $\partial\varphi \subseteq C^{\oplus}$ implies that $\varphi_C = i_{C^{\oplus}} \leq i_{\partial\varphi} = \varphi$, whence

$$C = [\varphi_C \geq 1] \subseteq [\varphi \geq 1].$$

To finish the proof, take $\varphi(x) \geq 1$ to prove the opposite inclusion. Indeed we have

$$1 \leq \varphi(x) = \inf\{\ell(x),\, \ell \in \partial\varphi\} = \inf\{t\ell(x),\, t \geq 1, \ell \in \partial\varphi\} = i_{\textit{shw}\,\partial\varphi}(x) = i_{C^{\oplus}}(x) = \varphi_C(x),$$

so that $x \in C$ and the proof is finished. □

If we compare all u.s.c. concave representations of some set $C \in \mathcal{K}$, we see that they all agree on $\overline{K}$ and differ outside $\overline{K}$. The concave gauge φ_C is the least element of the family, as it satisfies $\varphi_C(x) = -\infty$ for all $x \notin \overline{K}$.

We will show in this section that a greatest element also exists, that is a continuous superlinear representation which is greater than any other continuous concave representation. We will also give several descriptions of its superdifferential. To this purpose, we introduce the set of unitary support of a convex coradiant set C, which plays a key role in what follows.

Definition 4.2. Let the set $C \subseteq X$ be nonempty, closed and shady. The set of *unitary supports* of C is the set

$$\Lambda_C = \{\ell \in X' : \ell \in C^{\oplus}, \ell(c) = 1, \text{ for some } c \in C\}. \tag{5}$$

It is easy to verify that Λ_C is nonempty provided $\operatorname{int} C \neq \emptyset$ (hence for all strongly shady sets) and that any support point c belongs to $C \backslash \operatorname{int} C$.

The set Λ_C can be characterized by means of supergradients of the concave gauge of C.

Proposition 4.3. *Let $C \subseteq X$ be closed, convex and coradiative. Then it holds*

$$\Lambda_C = \Lambda'_C = \Lambda''_C,$$

where

$$\Lambda'_C = \{\ell \in X' : \ell \in \partial\varphi_C(x) \text{ for some } x \in C\} \tag{6}$$

and

$$\Lambda''_C = \{\ell \in X' : \ell \in \partial\varphi_C(x) \text{ for some } x \in K = \operatorname{cone} C\}. \tag{7}$$

Proof. The following implications hold:

$$\begin{aligned} \ell \in \Lambda_C \quad &\Leftrightarrow \quad \ell \in C^{\oplus},\, \exists c \in C : \ell(c) = 1 \\ &\Leftrightarrow \quad \ell \in C^{\oplus},\, \exists c \in \operatorname{bd} C : \ell(c) = 1 \\ &\Leftrightarrow \quad \ell \in \partial\varphi_C,\, \exists c \in C : \varphi_C(c) = 1 = \ell(c) \\ &\Leftrightarrow \quad \exists c \in C : \ell \in \partial\varphi_C(c) \Leftrightarrow \ell \in \Lambda'_C. \end{aligned}$$

The assumption that C be coradiative is used here to guarantee (see Rubinov [14]) that any boundary point c of C satisfies $\varphi_C(c) = 1$. Morever we have

$$\begin{aligned} \ell \in \Lambda''_C \quad &\Leftrightarrow \quad \ell \in \partial\varphi_C(x),\, x = \alpha c,\, \alpha > 0,\, c \in C \\ &\Leftrightarrow \quad \ell \in \partial\varphi_C,\, \ell(x) = \varphi_C(x),\, x = \alpha c,\, \alpha > 0,\, c \in C \\ &\Leftrightarrow \quad \ell \in \partial\varphi_C,\, \ell(c) = \varphi_C(c),\, c \in C \Leftrightarrow \ell \in \Lambda'_C. \end{aligned}$$

□

The following theorem is useful in the proof of the main result of this section, but it also has some interest in itself.

Theorem 4.4. *Let $C \subseteq X$ be proper, convex, closed and coradiant. Then it holds*

$$\mathit{oker}\, C = \bigcap_{c \in C} (c - \overline{K}) \subseteq (\Lambda_C)^{\circ},$$

where $\overline{K} = \operatorname{cl}\operatorname{cone} C$. If C is coradiative then it holds

$$\mathit{oker}\, C = \bigcap_{c \in C} (c - \overline{K}) = (\Lambda_C)^{\circ}.$$

Proof. 1. We prove first that $\operatorname{oker} C \subseteq \bigcap_{c\in C}(c - \overline{K})$. If $z \notin \bigcap_{c\in C}(c - \overline{K})$, then there exists some $\bar{c} \in C$ such that $z \notin \bar{c} - \overline{K}$, that is

$$\bar{c} - z \notin \overline{K}.$$

Consequently there exists some $\ell \in K^{+}$ such that $\ell(\bar{c} - z) < 0$.

Now suppose, ab absurdo, that $z \in \operatorname{oker} C$. Then it must be

$$z + t(\bar{c} - z) \in C, \quad \forall t \geq 1$$

and

$$\ell(z + t(\bar{c} - z)) = \ell(z) + t\ell(\bar{c} - z) \geq 0, \quad \forall t \geq 1. \tag{8}$$

But since $\ell(\bar{c} - z) < 0$ and t can be taken arbitrarily large, we obtain a contradiction to (8).

2. To prove the opposite inclusion, let $z \in \cap_{c \in C}(c - \overline{K})$, that is

$$c - z \in \overline{K}, \quad \forall c \in C.$$

If $z \notin \operatorname{oker} C$, there exist $\bar{c} \in C$ and $\tau > 1$ such that

$$y = z + \tau(\bar{c} - z) \notin C. \tag{9}$$

Since C is convex, this yields

$$z + t(\bar{c} - z) \notin C, \quad \forall t \geq \tau.$$

We obtain from (9) that there exists $\bar{\ell} \in C^{\oplus}$ such that $\bar{\ell}(y) < 1$. The hyperplane $H = \{z \in X : \bar{\ell}(z) = 1\}$ cuts the halfline $L = \{z + \alpha(\bar{c} - z), \alpha \geq 0\}$ in two parts. If we take $\alpha = 1$, we obtain the point $\bar{c}$, which satisfies $\bar{\ell}(\bar{c}) \geq 1$ and hence stays in the upper halfspace H^+, and if we take $\alpha = \tau$ we obtain the point y, which satisfies $\bar{\ell}(y) < 1$ and stays in the lower halfspace H^-. The points in L which lie besides y must also stay in H^- and hence

$$1 > \bar{\ell}(z + t(\bar{c} - z)) = \bar{\ell}(z) + t\bar{\ell}(\bar{c} - z) \quad \forall t \geq \tau. \tag{10}$$

For the same reason, we have $\bar{\ell}(z) \geq \bar{\ell}(\bar{c})$ and $\bar{\ell}(\bar{c} - z) \geq 0$. If it is $\bar{\ell}(\bar{c} - z) > 0$, we obtain a contradiction to (10), since t can be taken arbitrarily large. If it is $\bar{\ell}(\bar{c} - z) = 0$, then $\bar{\ell}(z) = \bar{\ell}(y) < 1$ and $\bar{\ell}(\bar{c}) = \bar{\ell}(z)$ against $\bar{\ell} \in C^{\oplus}$.

3. Now suppose that $z \notin (\Lambda_C)^{\circ}$. Thus there exists $\ell \in \Lambda_C$ such that $\ell(z) > 1$. Let $c \in C$ be such that $\ell(c) = 1$ and suppose that $z \in (c - \overline{K})$. Since $\ell \in C^{\oplus} \subseteq K^+$, we have

$$\ell(z) \leq \ell(c) = 1,$$

which is a contradiction. Hence $\bigcap_{c \in C}(c - \overline{K}) \subseteq (\Lambda_C)^{\circ}$.

4. To conclude the proof we have to show that $(\Lambda_C)^{\circ} \subseteq \operatorname{oker} C$, when C is radiative. To this purpose, take $z \notin \operatorname{oker} C$. Hence there exist $c \in C$ and $t > 1$ such that

$$z + t(c - z) \notin C.$$

Consider the halfline $L = \{z + \alpha(c - z), \alpha \geq 0\}$. Since C is closed and convex, the intersection between C and L is a nonempty, closed segment, containing the point c.

Suppose first that c is an interior point of C and let $\bar{c} \in \operatorname{bd} C$ be the extreme point of the segment which is closest to $z + t(c - z)$. Then find a linear functional $\ell \in X'$ which separate $\bar{c}$ from C, with $\ell(\bar{c}) = 1$ and $\ell \in C^{\oplus}$, whence $\ell \in \Lambda_C$. It follows that $\ell(c) > 1$ which implies $\ell(z) > \ell(c) > 1$ and hence $z \notin \Lambda_C^{\circ}$.

If $c \in \operatorname{bd} C$, then, since C is coradiative, $c' = \alpha c \in \operatorname{int} C$ for all $\alpha > 1$. We want to show that there exists some (sufficiently small) $\alpha > 1$ and some $\beta \geq 0$ such that $c' + \beta(c' - z) \notin C$.

Suppose, on the contrary that, for all $\alpha > 1$ and all $\beta \geq 0$ it holds $\alpha c + \beta(\alpha c - z) \in C$. Then the vector $\alpha c - z$ belongs to the recession cone of C, for all $\alpha > 1$; since α is arbitrary and $\operatorname{Rec} C$ is closed, it holds $c - z \in \operatorname{Rec} C$ and $z + t(c - z) \in C$ for all $t \geq 1$ which is false.

Hence, for some $\alpha > 1$ and some $\beta > 0$, we have $\alpha c \in \operatorname{int} C$ and $y = \alpha c + \beta(\alpha c - z) \notin C$. As before, the intersection between C and the halfline starting from z and going through $c' = \alpha x$ is a nonempty closed interval. Let $\bar{c} \in \operatorname{bd} C$ be the extreme point of the segment which is closest to y and take $\ell \in X'$ such that $\ell \in C^{\oplus}$ and $\ell(\bar{c}) = 1$. By evaluating ℓ along the halfline, we have $\ell(\alpha c) > 1$ and $\ell(z) > 1$ so that $z \notin (\Lambda_C)^{\circ}$ and the proof is finished. □

We observe, as a consequence of Theorem 4.4, that the outer kernel of a closed, convex, coradiant set is closed, besides being convex and radiant.

In view of Proposition 4.3 we denote in the sequel by Λ the set which can be characterized by any of the conditions (5) to (7). This will cause no mistake since the assumption that C is strongly shady (and hence coradiative) will always be standing.

We will see that among all continuous superlinear representation of the strongly shady set C, the greatest is the function $\gamma_C : X \to \mathbb{R}$ given by

$$\gamma_C(x) = \inf\{\lambda(x) : \lambda \in \Lambda\} = i_{\Lambda}(x).$$

Theorem 4.5. *If C is strongly shady, the function γ_C is a continuous superlinear cogauge of C.*

Proof. Theorem 4.4 yields $\operatorname{oker} C = \Lambda^{\circ}$ and hence $0 \in \operatorname{int} \Lambda^{\circ}$, since C is strongly shady. This shows that Λ is bounded in X'. Therefore the support set $\partial\gamma_C = \operatorname{cl}\operatorname{conv}\Lambda$ is weak*-compact and γ_C is continuous. It remains to show that $C = [\gamma_C \geq 1]$.

Since $\Lambda \subseteq C^{\oplus} = \partial\varphi_C$, we have, for all $x \in X$

$$\gamma_C(x) = \inf\{\lambda(x) : \lambda \in \Lambda\} \geq \inf\{\lambda(x) : \lambda \in C^{\oplus}\} = \varphi_C(x)$$

and this yields

$$\gamma_C(x) \geq \varphi_C(x) \geq 1,$$

for all $x \in C$, whence $C \subseteq [\gamma_C \geq 1]$.

To prove that $C = [\gamma_C \geq 1]$ we will prove that $\gamma_C(x) = \varphi_C(x)$ for all $x \in C$ and that for all $x \notin C$ it holds $\gamma_C(x) < 1$.

Thus let $x \in C$. If $\ell \in \partial\varphi_C(x)$ (which is nonempty, because φ_C is continuous at x), then it holds $\ell \in \partial\varphi_C$ and $\ell(x) = \varphi_C(x)$, which yields

$$\varphi_C(x) = \ell(x) \geq \inf\{\lambda(x), \lambda \in \partial\varphi_C(c), c \in C\} = \gamma_C(x)$$

and

$$\varphi_C(x) = \gamma_C(x) \tag{11}$$

for all $x \in C$. Using positive homogeneity and upper semicontinuity of γ_C and φ_C, we can extend equality (11) to $\overline{K} = \operatorname{cl}\operatorname{cone} C$. Remind that $\varphi_C(x) \in (0, 1)$ for all $x \in K \backslash C$ and vanishes on the boundary of K. So let now $x \notin \overline{K}$ and suppose it holds $\gamma_C(x) \geq 1$.

To find a contradiction, take any $c \in C$ and consider the line segment $tx + (1-t)c$ with $t \in [0,1]$. It holds $\gamma_C(x) \geq 1$, $\gamma_C(c) \geq 1$ and the segment touches the boundary of K, where γ_C vanishes, against the concavity of γ_C. □

We show next that γ_C is indeed the greatest continuous superlinear representation of C.

Theorem 4.6. *If $C \subseteq X$ is strongly shady and $\varphi : X \to \mathbb{R}$ is a continuous superlinear cogauge of C, then we have*

$$\varphi(x) \leq \gamma_C(x), \quad \forall x \in X.$$

Proof. It is enough to prove that the support set of φ satisfies $\Lambda \subseteq \partial\varphi$.

Take $\ell \in \Lambda$ and $x \in C$ such that

$$\ell(x) = 1 \leq \ell(c), \quad \forall c \in C. \tag{12}$$

We have to show that $\ell(z) \geq \varphi(z)$ for all $z \in X$, that is $\ell \in \partial\varphi$. We know from (12) that $\{x : \varphi(x) \geq 1\} \subseteq \{x : \ell(x) \geq 1\}$ and this implies, thanks to positive homogeneity of both ℓ and φ, that the inequality $\ell \geq \varphi$ holds on $K = \{x : \varphi > 0\}$. By continuity, the same inequality can be extended to $\overline{K}$ and further to $\{x : \ell \geq 0\}$. Hence $\varphi(y) \leq 0$ when $y \in H = \{x \in X : \ell(x) = 0\}$.

Take now $z \in X$ such that $\ell(z) < 0$. Since $x \notin H$ there exist $y \in H$ and $0 \neq \beta \in \mathbb{R}$ such that $z = y + \beta x$. Since $\ell(z) < 0$ and $\ell(x) = 1$, it follows $\beta < 0$.

Since φ is superadditive we have

$$\varphi(y) = \varphi(y + \beta x - \beta x) \geq \varphi(y + \beta x) + \varphi(-\beta x)$$

whence

$$\varphi(z) = \varphi(y + \beta x) \leq \varphi(y) - \varphi(-\beta x) \leq \beta\varphi(x) = \beta. \tag{13}$$

Since $\ell(z) = \beta\ell(x) = \beta$, we deduce that $\ell(z) \geq \varphi(z)$.

Thus $\ell \geq p$ and $\ell \in \partial\varphi$. □

The set Λ, used to define the greatest superlinear cogauge of C is not convex and the equalities proved in Proposition 4.3 are not much helpful in describing the function γ_C or its superdifferential. To this purpose, the next results offer a useful characterization of the support set of γ_C in terms of the reverse polar of C and the polar of its outer kernel.

Theorem 4.7. *Let $C \subseteq X$ be a strongly shady set. Then*

$$\partial\gamma_C = \mathrm{cl}\,\mathrm{conv}\,\Lambda = (oker\,C)^\circ \cap C^\oplus,$$

where the closure is taken in the weak topology of X'.*

Proof. The inclusion $\Lambda \subseteq C^\oplus$ comes from (5). To see that $\Lambda \subseteq (\mathrm{oker}\,C)^\circ$ take $\ell \in \Lambda$ and suppose that some $z \in \mathrm{oker}\,C$ exists such that $\ell(z) > 1$. Then choose $x \in C$ such that $\ell(x) = 1$. We have

$$z + t(x - z) \in C, \quad \forall t \geq 1$$

and, for all $t > \ell(z)/(\ell(z)-1)$ (which is greater than 1), we have

$$1 \leq \ell(z + t(x-z)) = \ell(z) + t(1-\ell(z)) < 0,$$

which is a contradiction.

To prove the inverse inclusion, let $Q = (\operatorname{oker} C)^\circ \cap C^\oplus$ and take $q \notin \operatorname{cl}\operatorname{conv}\Lambda$. The set Λ is bounded since it is contained in $(\operatorname{oker} C)^\circ$, which, being $0 \in \operatorname{int}\operatorname{oker} C$, is itself bounded in X'. Then there exists $x \in X$ and $\alpha \in \mathbb{R}$ such that

$$q(x) > \alpha = \sup\{\ell(x),\, \ell \in \Lambda\}.$$

We have to study three cases:

- if $\alpha < 0$: take $y = -x$ to find (with $\beta = -\alpha$):

  $$q(y) < \beta \leq \ell(y), \quad \forall \ell \in \Lambda,$$

  which yields

  $$y/\beta \in \Lambda^\oplus = (\operatorname{cl}\operatorname{shw}\operatorname{conv}\Lambda)^\oplus = (\operatorname{shw}\operatorname{cl}\operatorname{conv}\Lambda)^\oplus = (C^\oplus)^\oplus = C \tag{14}$$

  and $q(y/\beta) < 1$, whence $q \notin C^\oplus$, which implies $q \notin Q$. The first equality in (14) comes from the fact that the set Λ and its closed, convex, coradiant hull have the same reverse polar; the second equality is proved in [7]. For the third, note that γ_C is a superlinear representation of C and hence, by Proposition 4.1, it follows that $\operatorname{shw}\operatorname{cl}\operatorname{conv}\Lambda = C^\oplus$.
- if $\alpha = 0$, then $q(x) > 0$ and $\ell(x) \leq 0$ for all $\ell \in \Lambda$. Since $K^+ = \operatorname{cl}\operatorname{cone}\Lambda$, then $\ell(x) \leq 0$ for all $\ell \in K^+$ and $x \in -\overline{K}$. Since $q(x) > 0$ we deduce that $q \notin K^+$ and $q \notin Q$.
- if $\alpha > 0$, then for $y = x/\alpha$ we have $\sup_\Lambda \ell(y) = 1$ and $\ell(y) \leq 1$ for all $\ell \in \Lambda$, whence $y \in \Lambda^\circ$. From Proposition 4.4 we have $y \in (\operatorname{oker} C)^{\circ\circ} = \operatorname{oker} C$. In view of the inequality $q(y) > 1$ we have $q \notin (\operatorname{oker} C)^\circ$, which concludes the proof. □

Taking $Q = C^\oplus \cap (\operatorname{oker} C)^\circ$, we know from Theorem 4.7, that the function $i_Q : X \to \mathbb{R}$ given by

$$i_Q(x) = \inf\{\ell(x) : \ell \in Q\}$$

is superlinear and continuous and coincides with γ_C.

We can exploit the calculus rules for reverse polar sets developed in [7] to obtain the greatest superlinear cogauge of the set $C = x + K$, where K is a closed, convex cone, with $x \in \operatorname{int} K$. It holds

$$C^\oplus = (x+K)^\oplus = K^+ \cap H^+_{X'}(x,1),$$

where $K^+ = \{x^* \in X' : x^*(k) \geq 0, \forall k \in K\}$ is the positive polar cone of K and $H^+_{X'}(x,1) = \{x^* \in X' : x^*(x) \geq 1\}$ is the upper 1-halfspace defined in X' by the vector $x \in X$. Moreover

$$(\operatorname{oker} C)^\circ = (x-K)^\circ = K^+ \cap H^-_{X'}(x,1),$$

which yields

$$\partial\gamma_C = (x+K)^\oplus \cap (\operatorname{oker}(x+K))^\circ = K^+ \cap H_{X'}(x,1).$$

In the particular case in which K is a halfspace, $K = H_X^+(y^*, 0) = \{z \in X : y^*(z) \geq 0\}$ and $x \in \operatorname{int} H_X^+(y^*, 0)$, that is $y^*(x) > 0$, we have $K^+ = \operatorname{cone}_0(y^*)$ and $\partial\gamma_C$ reduces to a singleton, which means that γ_C is indeed linear. This observation is relevant if referred to the original purpose of this research, that is the continuous representation of convex, coradiant sets within the framework of superlinear separation of radiant sets, as discussed in [22, 23, 24]. It implies that if the point x can be separated from the set $A \subseteq X$ by means of an hyperplane, then the superlinear separation reduces to the classical linear separation.

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