Journal of Convex Analysis
Volume 15 (2008), No. 1, 87–104

Existence and Construction of Bipotentials for Graphs of Multivalued Laws

Marius Buliga
"Simion Stoilow" Institute of Mathematics of the Romanian Academy, PO BOX 1-764, 014700 Bucharest, Romania
Marius.Buliga@imar.ro

Géry de Saxcé
Laboratoire de Mécanique de Lille, UMR CNRS 8107, Université des Sciences et Technologies de Lille, Cité Scientifique, 59655 Villeneuve d'Ascq cedex, France
gery.desaxce@univ-lille1.fr

Claude Vallée
Laboratoire de Mécanique des Solides, UMR 6610, UFR SFA-SP2MI, bd M. et P. Curie, téléport 2, BP 30179, 86962 Futuroscope-Chasseneuil cedex, France
vallee@lms.univ-poitiers.fr

Received: October 3, 2006
Revised manuscript received: January 17, 2007

Based on an extension of Fenchel inequality, bipotentials are non smooth mechanics tools, used to model various non associative multivalued constitutive laws of dissipative materials (friction contact, soils, cyclic plasticity of metals, damage).

Let X, Y be dual locally convex spaces, with duality product $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{R}$. Given the graph $M \subset X \times Y$ of a multivalued law $T : X \to 2^Y$, we state a simple necessary and sufficient condition for the existence of a bipotential b for which M is the set of (x, y) such that $b(x, y) = \langle x, y \rangle$.

If this condition is fulfilled, we use convex lagrangian covers in order to construct such a bipotential, generalizing a theorem due to Rockafellar, which states that a multivalued constitutive law admits a superpotential if and only if its graph is cyclically monotone.

1991 Mathematics Subject Classification: 49J53, 49J52, 26B25

1. Introduction

The basic tools of the mechanics of continua are the kinematical compatibility and equilibrium local equations but they are not sufficient to describe the deformation and motion of the continuous media. Additional information must be given through the constitutive laws traducing the material behaviour. In its simplest form, a constitutive law is given by a graph collecting couples of dual variables resulting from experimental testing.

For many physically relevant situations, the constitutive laws are multivalued, but also associated. The graph of the constitutive law is included in the graph of the subdifferential of a convex (and lower semi continuous) superpotential ϕ. The constitutive law takes the form of a differential inclusion, $y \in \partial\phi(x)$. Any superpotential ϕ has a polar function ϕ^* satisfying a fundamental relation, Fenchel's inequality, $\forall x, y,\ \phi(x) + \phi^*(y) \geq \langle x, y \rangle$.

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

The constitutive law may be also written as $x \in \partial\phi^*(y)$. In the literature, this kind of materials are often called standard materials or generalized standard materials [12].

From the viewpoint of applications it is important to know whether there exists a superpotential for a given non smooth graph and how to construct it. The answer to this question is provided by a famous theorem due to Rockafellar [20] that ensures a graph admits a superpotential if and only if it is maximal cyclically monotone.

However, some of the constitutive laws are non-associated. They cannot be cast in the mould of the standard materials. To skirt this pitfall, a possible response, proposed first in [21], consists in constructing a function b of two variables, bi-convex and satisfying an inequality generalizing Fenchel's one, $\forall x, y, \;\; b(x,y) \geq \langle x, y\rangle$. We call it a bipotential. Physically, it represents the dissipation. In the case of associated constitutive laws the bipotential has the expression $b(x,y) = \phi(x) + \phi^*(y)$.

As for the non associated constitutive laws which can be expressed with the help of bipotentials, they have the form of an implicit relation between dual variables, $y \in \partial b(\cdot, y)(x)$. In Mechanics they are called implicit, or weak, normality rules. The applications of bipotentials to Solid Mechanics are various: Coulomb's friction law [22], non-associated Drücker-Prager [23] and Cam-Clay models [24] in Soil Mechanics, cyclic Plasticity ([22], [3]) and Viscoplasticity [16] of metals with non linear kinematical hardening rule, Lemaitre's damage law [2], the coaxial laws ([8], [30]). Such kind of materials are called implicit standard materials. A synthetic review of these laws expressed in terms of bipotentials can be found in [8] and [30].

The use of bipotentials in applications is particularly attractive in numerical simulations when using the finite element method, but the interest is not limited to this aspects. For instance, the bound theorems of the limit analysis ([26], [6]) and the plastic shakedown theory ([28], [8], [7], [4]) can be reformulated in a broader framework, precisely by means of weak normality rules. From an applied numerical viewpoint, the bipotential method suggests new algorithms, fast but robust, as well as variational error estimators assessing the accurateness of the finite element mesh ([14], [15], [25], [27], [5], [17], [18]). Applications to the contact Mechanics [9], the Dynamics of granular materials ([10], [11], [13], [29]), the cyclic Plasticity of metals [25] and the Plasticity of soils ([1], [17]) illustrate the relevancy of this approach.

In all the papers already mentioned about the mechanical applications, bipotentials for certain multivalued constitutive laws were constructed. Nevertheless, in order to better understand the bipotential approach, one has to solve the following problems:

1) (existence) what are the conditions to be satisfied by a multivalued law such that it can be expressed with the help of a bipotential?
2) is there a procedure to construct a class of bipotentials for a multivalued law? We expect that generically the law does not uniquely determine the bipotential.

We give a first mathematical treatment of these problems and we prove results of existence (Theorem 3.2) and construction (Theorem 6.7) of bipotentials for a class of graphs of multivaluate laws.

One of the key ideas is constructing the bipotential as an inferior envelope. That could be considered as paradoxal because, in general, it is strongly unprobable that an inferior envelope, even of convex functions, would be convex. Nevertheless, we were convinced of the relevancy of this approach by examples inspired from mechanics and we wished to understand the reason. That led us to introduce the main tool of convex lagrangian covers (Definition 4.1) satisfying an implicit convexity condition.

The recipe that we give in this paper applies only to BB-graphs (Definition 3.1) admitting at least one convex lagrangian cover by maximal cyclically monotone graphs. This is an interesting class of graph of multivalued laws for the following two reasons:

(a) it contains the class of graphs of subdifferentials of convex lsc superpotentials,
(b) any of the graphs of non associated laws from the mentioned mechanical applications of bipotentials is a BB-graph and it admits a physically relevant convex lagrangian cover by cyclically monotone graphs.

Relating to point (b), it is important to know that the results from this paper don't apply to some BB-graphs of mechanical interest, such as the graph of the bipotential associated to contact with friction [21]. This is because we use in this paper only convex lagrangian covers with *maximal* cyclically monotone graphs, see also Remark 5.1.

This paper is only a first step into the subject of constructions of bipotentials. Our aim is to explain a general method of construction in a reasonably simple situation, interesting in itself, leaving aside for the moment certain difficulties appearing in the general method. Another article, in preparation, is dedicated to the extension of the method presented here to a more general class of BB-graphs, by relaxing the notion of convex lagrangian cover. In this way we shall be able to construct bipotentials even for some of the BB-graphs described in Remark 5.1.

2. Notations and Definitions

X and Y are topological, locally convex, real vector spaces of dual variables $x \in X$ and $y \in Y$, with the duality product $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{R}$. We shall suppose that X, Y have topologies compatible with the duality product, that is: any continuous linear functional on X (resp. Y) has the form $x \mapsto \langle x, y \rangle$, for some $y \in Y$ (resp. $y \mapsto \langle x, y \rangle$, for some $x \in X$).

For any convex and closed set $A \subset X$, its indicator function, χ_A, is defined by

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \in A \\ +\infty & \text{otherwise} \end{cases}$$

The indicator function is convex and lower semi continuous. If the set A contains only one element $A = \{a\}$ then we shall use the notation χ_a for the indicator function of A.

We use the notation: $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$.

Given a function $\phi : X \to \bar{\mathbb{R}}$, the polar $\phi^* : Y \to \bar{\mathbb{R}}$ is defined by:

$$\phi^*(y) = \sup \{\langle y, x \rangle - \phi(x) \mid x \in X\}.$$

The polar is always convex and lower semi continuous.

We denote by $\Gamma(X)$ the class of convex and lower semicontinuous functions $\phi : X \to \bar{\mathbb{R}}$. The class of convex and lower semicontinuous functions $\phi : X \to \mathbb{R}$ is denoted by $\Gamma_0(X)$.

The subgradient of a function $\phi : X \to \bar{\mathbb{R}}$ in a point $x \in X$ is the (possibly empty) set:

$$\partial\phi(x) = \{u \in Y \mid \forall z \in X \ \langle z - x, u\rangle \leq \phi(z) - \phi(x)\} .$$

In a similar way is defined the subgradient of a function $\psi : Y \to \bar{\mathbb{R}}$ in a point $y \in Y$, as the set:

$$\partial\psi(y) = \{v \in X \mid \forall w \in Y \ \langle v, w - y\rangle \leq \psi(w) - \psi(y)\} .$$

With these notations we have the Fenchel inequality: let $\phi : X \to \bar{\mathbb{R}}$ be a convex lower semicontinuous function. Then:

(i) for any $x \in X, y \in Y$ we have $\phi(x) + \phi^*(y) \geq \langle x, y\rangle$;
(ii) for any $(x, y) \in X \times Y$ we have the equivalences:

$$y \in \partial\phi(x) \iff x \in \partial\phi^*(y) \iff \phi(x) + \phi^*(y) = \langle x, y\rangle.$$

Definition 2.1. We model **the graph of a constitutive law** by a set $M \subset X \times Y$. Equivalently, the law is given by the multivalued application

$$X \ni x \mapsto m(x) = \{y \in Y \mid (x, y) \in M\} .$$

The **dual** law is the multivalued application

$$Y \ni y \mapsto m^*(y) = \{x \in X \mid (x, y) \in M\} .$$

The **domain** of the law is the set $dom(M) = \{x \in X \mid m(x) \neq \emptyset\}$. The **image** of the law is the set $im(M) = \{y \in Y \mid m^*(y) \neq \emptyset\}$.

For example, if $\phi : X \to \mathbb{R}$ is a convex lower semi continuous function, the associated law is the multivalued application $\partial\phi$, the subdifferential of ϕ, [19], Def. 10.1, that is the set of subgradients. The dual law is $\partial\phi^*$ (the subdifferential of the Legendre-Fenchel dual of ϕ) and the graph of the law is the set

$$M(\phi) = \{(x, y) \in X \times Y \ \mid \ \phi(x) + \phi^*(y) = \langle x, y\rangle\} . \tag{1}$$

For any convex lower semi continuous function ϕ the graph $M(\phi)$ is *maximal cyclically monotone* ([20], Theorem 24.8 or [19], Proposition 12.2). Conversely, if M is closed and maximal cyclically monotone then there is a convex, lower semicontinuous ϕ such that $M = M(\phi)$.

Definition 2.2. A **bipotential** is a function $b : X \times Y \to \bar{\mathbb{R}}$, with the properties:

(a) b is convex and lower semicontinuous in each argument;
(b) for any $x \in X, y \in Y$ we have $b(x, y) \geq \langle x, y\rangle$;
(c) for any $(x, y) \in X \times Y$ we have the equivalences:

$$y \in \partial b(\cdot, y)(x) \iff x \in \partial b(x, \cdot)(y) \iff b(x, y) = \langle x, y\rangle. \tag{2}$$

The **graph** of b is

$$M(b) = \{(x, y) \in X \times Y \ \mid \ b(x, y) = \langle x, y\rangle\} . \tag{3}$$

Examples. 1. (Separable bipotential) To any convex lower semicontinuous function ϕ we can associate the **separable bipotential**

$$b(x, y) = \phi(x) + \phi^*(y).$$

The bipotential b and the potential ϕ define the same law: $M(b) = M(\phi)$.

2. (Cauchy bipotential) Let $X = Y$ be a Hilbert space and let the duality product be equal to the scalar product. Then we define the **Cauchy bipotential** by the formula

$$b(x, y) = \|x\| \; \|y\|.$$

Let us check the Definition 2.2. The point (a) is obviously satisfied. The point (b) is true by the Cauchy-Schwarz-Bunyakovsky inequality. We have equality in the Cauchy-Schwarz-Bunyakovsky inequality $b(x, y) = \langle x, y\rangle$ if and only if there is $\lambda > 0$ such that $y = \lambda x$ or one of x and y vanishes. This is exactly the statement from the point (c), for the function b under study.

Remark 2.3. The Cauchy bipotential is an ingredient in the construction of many bipotentials of mechanical interest, because the (graph of the) law associated to b is the set of pairs of collinear and with same orientation vectors. It can not be expressed by a separable potential because $M(b)$ is not a cyclically monotone graph. We shall apply the results of this paper to the Cauchy bipotential, in order to show that we are able to recover the expression of this bipotential from the graph of its associated law.

3. Existence of a bipotential

Given a non empty set $M \subset X \times Y$, Theorem 3.2 provides a necessary and sufficient condition on M for the existence of a bipotential b with $M = M(b)$. In order to shorten the notation we shall give a name to this condition:

Definition 3.1. The non empty set $M \subset X \times Y$ is a **BB-graph** (bi-convex, bi-closed) if for all $x \in dom(M)$ and for all $y \in im(M)$ the sets $m(x)$ and $m^*(y)$ are convex and closed.

The existence problem is easily settled by the following result.

Theorem 3.2. *Given a non empty set $M \subset X \times Y$, there is a bipotential b such that $M = M(b)$ if and only if M is a BB-graph.*

Proof. Let b be a bipotential such that $M(b)$ is not void. We first want to prove that for any $x \in X$ and $y \in Y$ the sets $m(x)$ and $m^*(y)$ are convex and closed.

Indeed, if $m(x)$ or $m(y)$ are empty or they contain only one element then there is nothing to prove. Let us suppose, for example, that $m(x)$ has more than one element. From the convexity and lower semi continuity hypothesis on b from Definition 2.2, it follows that $m(x)$ is closed and convex. Indeed, remark that $m(x)$ is a sub-level set for a convex and lower semi continuous mapping:

$$m(x) = \{y \in Y : b(x, y) - \langle x, y\rangle \leq 0\},$$

thus a closed and convex set.

Let us consider now a non empty set $M \subset X \times Y$ such that for any $x \in X$ and $y \in Y$ the sets $m(x)$ and $m^*(y)$ are convex and closed. We define then the function $b_\infty : X \times Y \to \overline{\mathbb{R}}$ by:

$$b_\infty(x,y) = \begin{cases} \langle x, y \rangle & \text{if } (x,y) \in M \\ +\infty & \text{otherwise.} \end{cases}$$

We have to prove that b_∞ is a bipotential and that $M = M(b_\infty)$. This last claim is trivial, so let us check the points from the Definition 2.2. For the point (a) notice that for any fixed $x \in X$ the function $b_\infty(x, \cdot)$ is the sum of a linear continuous function with the indicator function of $m(x)$. By hypothesis the set $m(x)$ is closed and convex, therefore its indicator function is convex and lower semicontinuous. It follows that the function $b_\infty(x, \cdot)$ is convex and lower semi continuous. In the same way we prove that for any fixed $y \in Y$ the function $b_\infty(\cdot, y)$ is convex and lower semi continuous. The points (b) and (c) are trivial by the Definition of the function b_∞. □

Remark 3.3. The uniqueness of b is *not true*. For example, in the case of the Cauchy bipotential we have two different bipotentials b and b_∞ with the same graph. Therefore the graph of the law alone is not sufficient to uniquely define the bipotential.

4. Construction of a bipotential

Theorem 3.2 does not give a satisfying bipotential for a given multivalued constitutive law, because the bipotential b_∞ is definitely not interesting for applications.

The most important conclusion of preceding section is contained in the Remark 3.3: in the hypothesis of Theorem 3.2, the graph of the law is not sufficient to uniquely construct an associated bipotential. This is in contrast with the case of a maximal cyclically monotone graph M, when by Rockafellar theorem ([20], Theorem 24.8) we have a method to reconstruct unambigously the associated separable bipotential (see point (a) below).

In our opinion this is the main reason why the bipotentials are not more often used in applications. Without a recipe for constructing the bipotential associated with (the experimental data contained in) the graph of a non associated mechanical law, there is little chance that one may guess a correct expression for this bipotential.

We are looking for a method of construction of a bipotential with the following properties:

(a) if the graph $M \subset X \times Y$ is maximal cyclically monotone then the constructed bipotential is separable (see Example 1.),
(b) the method applied to the graph associated to the Cauchy bipotential allows to reconstruct the named bipotential (as mentioned in Remark 2.3, this bipotential appears in many applications),
(c) the method should use only hypothesis related to the graph $M \subset X \times Y$.

Relating to point (c), we noticed that in all applications we were able to reconstruct the bipotentials by knowing a little more than the graph $M \subset X \times Y$, namely a decomposition:

$$M = \bigcup_{\lambda \in \Lambda} M_\lambda.$$

We have to mention that in all applications this decomposition stems out from physical considerations.

Thus we were led to the introduction of convex lagrangian covers.

Definition 4.1. Let $M \subset X \times Y$ be a non empty set. A **convex lagrangian cover** of M is a function $\lambda \in \Lambda \mapsto \phi_\lambda$ from Λ with values in the set $\Gamma(X)$, with the properties:

(a) The set Λ is a non empty compact topological space,

(b) Let $f : \Lambda \times X \times Y \to \bar{\mathbb{R}}$ be the function defined by

$$f(\lambda, x, y) = \phi_\lambda(x) + \phi^*_\lambda(y).$$

Then for any $x \in X$ and for any $y \in Y$ the functions $f(\cdot, x, \cdot) : \Lambda \times Y \to \bar{\mathbb{R}}$ and $f(\cdot, \cdot, y) : \Lambda \times X \to \bar{\mathbb{R}}$ are lower semi continuous on the product spaces $\Lambda \times Y$ and respectively $\Lambda \times X$ endowed with the standard topology,

(c) We have

$$M = \bigcup_{\lambda \in \Lambda} M(\phi_\lambda).$$

5. On the existence and uniqueness of convex lagrangian covers

Not any BB-graph admits a convex lagrangian cover. There are at least two sources of examples of such BB-graphs, described further. For more considerations along this line see the last section of the paper.

Remark 5.1. Let M be a BB-graph with the property: for any ϕ, convex, lower semi-continuous function defined on X, we have $M(\phi) \setminus M \neq \emptyset$. Then M does not admit any convex lagrangian cover.

As an example take any convex, lower semicontinuous $\phi : X \to \bar{\mathbb{R}}$ and consider $M \subset M(\phi)$, BB-graph, such that $M \neq M(\phi)$. Then M has the property described previously, therefore it does not admit any convex lagrangian cover.

Remark 5.2. If M is a BB-graph and A is any linear, continuous transformation of $X \times Y$ into itself, such that $A(X \times \{0\}) \subset X \times \{0\}$ and $A(\{0\} \times Y) \subset \{0\} \times Y$, then $A(M)$ is also a BB-graph. However, it may happen that M admits convex lagrangian covers, but not $A(M)$.

Indeed, we consider $X = Y = \mathbb{R}$ with natural duality and a $\mathcal{C}^2$ function $\phi : X \to \mathbb{R}$ with derivative ϕ' strictly increasing. Let us define $M = M(\phi)$ and $A(x, y) = (x, -y)$. The set $A(M)$ has a simple description as the graph of $-\phi'$. As ϕ' is stricly increasing, for any two different $x_1, x_2 \in \mathbb{R}$ and $y_i = -\phi'(x_i)$ $(i = 1, 2)$, we have

$$\langle x_1 - x_2, y_1 - y_2 \rangle = (x_1 - x_2)(y_1 - y_2) < 0.$$

This implies that $A(M)$ has the property described in Remark 5.1. For if there is a convex, lower semicontinuous $\psi : X \to \bar{\mathbb{R}}$ such that $M(\psi) \subset A(M)$ then for any two different $x_1, x_2 \in \mathbb{R}$ and $y_i \in \mathbb{R}$, $i = 1, 2$, such that $(x_i, y_i) \in M(\psi)$ we would have

$$\langle x_1 - x_2, y_1 - y_2 \rangle = (x_1 - x_2)(y_1 - y_2) \geq 0,$$

which leads to contradiction.

The bipotential b_∞ from the proof of Theorem 3.2 does not come from a convex lagrangian cover. There exist BB-graphs admitting only one convex lagrangian cover (up to reparametrization), as well as BB-graphs which have infinitely many lagrangian covers.

In conclusion, we think it is a hard and challenging mathematical problem to describe all convex lagrangian covers of a BB-graph.

6. Implicit convexity and the main result

The main result of this paper is Theorem 6.7, which gives a recipe for constructing a bipotential not from the graph M of a multivalued law, but from a convex lagrangian cover. Therefore the results in this section apply only to BB-graphs admitting at least one convex lagrangian cover.

In the next section we shall apply this recipe for two convex lagrangian covers of $M(b)$, with b equal to the Cauchy bipotential.

Remark 6.1. We give here a justification for the name "convex lagrangian cover". Suppose that for any $\lambda \in \Lambda$ the function ϕ_λ is smooth. Then it is well known that the graph (of the subdifferential of ϕ_λ) $M(\phi_\lambda)$ is a lagrangian manifold in the symplectic manifold $X \times Y$ with the canonical symplectic form

$$\omega\left((x, y), (x', y')\right) = \langle x, y' \rangle - \langle y, x' \rangle.$$

Therefore the set M is covered by the family of lagrangian manifolds $M(\phi_\lambda)$, $\lambda \in \Lambda$.

With the help of a convex lagrangian cover we shall define a function b. We intend to prove that (under a certain condition explained further) the function b is a bipotential and that $M = M(b)$.

Definition 6.2. Let $\lambda \mapsto \phi_\lambda$ be a convex lagrangian cover of the BB-graph M. To the cover we associate the function $b : X \times Y \to \mathbb{R} \cup \{+\infty\}$ by the formula

$$b(x, y) = \inf \{\phi_\lambda(x) + \phi^*_\lambda(y) \,:\, \lambda \in \Lambda\} = \inf_{\lambda \in \Lambda} f(\lambda, x, y).$$

We have to check if the function b has the properties (a), (b), (c) from the Definition 2.2 of a bipotential.

Proposition 6.3. *Let $\lambda \mapsto \phi_\lambda$ be a convex lagrangian cover of the BB-graph M and b given by Definition 6.2. Then:*

(a) *for all $(x, y) \in M$ we have $b(x, y) = \langle x, y \rangle$.*
(b) *for all $(x, y) \in X \times Y$ we have $b(x, y) \geq \langle x, y \rangle$.*

Proof. For all $\lambda \in \Lambda$ and $(x, y) \in X \times Y$ we have the inequality:

$$\phi_\lambda(x) + \phi^*_\lambda(y) \geq \langle x, y \rangle.$$

As a consequence of this inequality and Definition 6.2 of the function b we obtain the point (b).

For proving the point (a) it is enough to show that if $(x, y) \in M$ then $b(x, y) \leq \langle x, y \rangle$. But this is true. Indeed, if $(x, y) \in M$ then there is a $\lambda \in \Lambda$ such that $(x, y) \in M(\phi_\lambda)$ and thus

$$\phi_\lambda(x) + \phi_\lambda^*(y) = \langle x, y \rangle.$$

From the Definition 6.2 it follows that for any $\lambda \in \Lambda$ we have

$$b(x, y) \leq \phi_\lambda(x) + \phi_\lambda^*(y)$$

therefore $b(x, y) \leq \langle x, y \rangle$, which finishes the proof . □

Proposition 6.4. *Let $\lambda \mapsto \phi_\lambda$ be a convex lagrangian cover of the BB-graph M and b given by Definition* 6.2.

(a) *Suppose that $x \in X$ is given and that $y \in Y$ has the minimum property*

$$b(x, y) - \langle x, y \rangle \leq b(x, z) - \langle x, z \rangle$$

for any $z \in Y$. Then $b(x, y) = \langle x, y \rangle$.

(b) *If $b(x, y) = \langle x, y \rangle$ then $(x, y) \in M$.*

Proof. (a) We start from the Definition of b. We have

$$b(x, y) = \inf \{ \phi_\lambda(x) + \phi_\lambda^*(z) \, : \, \lambda \in \Lambda \} \, .$$

We use the compactness of Λ (point (a) from Definition 4.1) to obtain a net $(\lambda_n)_n$ in Λ, which converges to $\bar{\lambda} \in \Lambda$, such that $b(x, y)$ is the limit of the net $\left(\phi_{\lambda_n}(x) + \phi_{\lambda_n}^*(y) \right)_n$.

From the lower semicontinuity of the cover (point (b) from Definition 4.1) we infer that

$$b(x, y) = \phi_{\overline{\lambda}}(x) + \phi_{\overline{\lambda}}^*(y).$$

Remark that the value of the limit $\bar{\lambda}$ of the net $(\lambda_n)_n$ depends on (x, y).

The hypothesis from point (a) and the definition of the function b implies that for any $z \in Y$ and any $\lambda \in \Lambda$ we have

$$\phi_{\overline{\lambda}}(x) + \phi_{\overline{\lambda}}^*(y) - \langle x, y \rangle \leq \phi_\lambda(x) + \phi_\lambda^*(z) - \langle x, z \rangle.$$

In particular, for $\lambda = \overline{\lambda}$ we get that for all $z \in Y$

$$\phi_{\overline{\lambda}}^*(y) - \phi_{\overline{\lambda}}^*(z) \leq \langle x, y - z \rangle.$$

This means that $x \in \partial \phi_{\overline{\lambda}}^*(y)$, which implies that

$$b(x, y) = \phi_{\overline{\lambda}}(x) + \phi_{\overline{\lambda}}^*(y) = \langle x, y \rangle.$$

For the point (b), suppose that $b(x, y) = \langle x, y \rangle$. As we remarked before, there is $\overline{\lambda} \in \Lambda$ such that

$$b(x, y) = \phi_{\overline{\lambda}}(x) + \phi_{\overline{\lambda}}^*(y).$$

Putting all together we see that

$$\phi_{\overline{\lambda}}(x) + \phi_{\overline{\lambda}}^*(y) = \langle x, y \rangle,$$

therefore $(x, y) \in M(\phi_{\overline{\lambda}}) \subset M$. □

We shall give now a sufficient hypothesis for the separate convexity of b. This is the last ingredient that we need in order to prove that b is a bipotential.

We shall use the following notion of implicit convexity.

Definition 6.5. Let Λ be an arbitrary non empty set and V a real vector space. The function $f : \Lambda \times V \to \bar{\mathbb{R}}$ is **implicitly convex** if for any two elements $(\lambda_1, z_1), (\lambda_2, z_2) \in \Lambda \times V$ and for any two numbers $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$ there exists $\lambda \in \Lambda$ such that

$$f(\lambda, \alpha z_1 + \beta z_2) \leq \alpha f(\lambda_1, z_1) + \beta f(\lambda_2, z_2).$$

Let us state the last hypothesis from our construction, as a definition.

Definition 6.6. Let $\lambda \mapsto \phi_\lambda$ be a convex lagrangian cover of the BB-graph M and $f : \Lambda \times X \times Y \to \mathbb{R}$ the associated function introduced in Definition 4.1, that is the function defined by

$$f(\lambda, z, y) = \phi_\lambda(z) + \phi^*_\lambda(y).$$

The cover is bi-implicitly convex (or a **BIC-cover**) if for any $y \in Y$ and $x \in X$ the functions $f(\cdot, \cdot, y)$ and $f(\cdot, x, \cdot)$ are implicitly convex in the sense of Definition 6.5.

In the case of $M = M(\phi)$, with ϕ convex and lower semi continuous (this corresponds to separable bipotentials), the set Λ has only one element $\Lambda = \{\lambda\}$ and we have only one potential ϕ. The associated bipotential from Definition 6.2 is obviously

$$b(x, y) = \phi(x) + \phi^*(y).$$

This is a BIC-cover in a trivial way: the implicit convexity conditions are equivalent with the convexity of ϕ, ϕ^* respectively.

Therefore, in the case of separable bipotentials the BIC-cover condition is trivially true.

Our recipe concerning the construction of a bipotential is based on the following result.

Theorem 6.7. *Let $\lambda \mapsto \phi_\lambda$ be a BIC-cover of the BB-graph M and $b : X \times Y \to R$ defined by*

$$b(x, y) = \inf \{\phi_\lambda(x) + \phi^*_\lambda(y) \ \mid \ \lambda \in \Lambda\}. \tag{4}$$

Then b is a bipotential and $M = M(b)$.

Proof. *Step 1.* We prove first that for any $x \in X$ and for any $y \in Y$, the functions $b(x, \cdot)$ and $b(\cdot, y)$ are convex.

For fixed $y \in Y$, for any $x_1, x_2 \in X$ and for any $\varepsilon > 0$, there are $\lambda_1, \lambda_2 \in \Lambda$ such that $(i = 1, 2)$

$$b(x_i, y) + \varepsilon \geq f(\lambda_i, x_i, y).$$

For the pairs $(\lambda_1, x_1), (\lambda_2, x_2)$ we use the implicit convexity of $f(\cdot, \cdot, y)$ to find that there is $\lambda \in \Lambda$ such that

$$f(\lambda, \alpha x_1 + \beta x_2, y) \leq \alpha f(\lambda_1, x_1, y) + \beta f(\lambda_2, x_2, y).$$

All in all we have:

$$\begin{aligned} b(\alpha x_1 + \beta x_2, y) &\leq f(\lambda, \alpha x_1 + \beta x_2, y) \\ &\leq \alpha f(\lambda_1, x_1, y) + \beta f(\lambda_2, x_2, y) \leq \alpha b(x_1, y) + \beta b(x_2, y) + \varepsilon. \end{aligned}$$

As $\varepsilon > 0$ is an arbitrary chosen positive number, the convexity of the function $b(\cdot, y)$ is proven. The proof for the convexity of $b(x, \cdot)$ is similar.

Step 2. We shall prove now that for any $x \in X$ and for any $y \in Y$, the functions $b(\cdot, x)$ and $b(\cdot, y)$ are lower semicontinuous. Consider a net $(x_n)_n \in X$ which converges to x. We use the same reasoning as in the proof of Proposition 6.4(a) to deduce that for each $n \in \mathbb{N}$ there exists a $\lambda_n \in \Lambda$ such that

$$b(x_n, y) = \phi_{\lambda_n}(x_n) + \phi_{\lambda_n}(y) = f(\lambda_n, x_n, y).$$

Λ is compact, therefore up to the choice of a subnet, there exists a $\lambda \in \Lambda$ such that $(\lambda_n)_n$ converges to λ. We use now the lower semicontinuity of $f(\cdot, \cdot, y)$ in order to get that

$$b(x, y) \leq f(\lambda, x, y) \leq \liminf_{n \to \infty} f(\lambda_n, x_n, y),$$

therefore the lower semicontinuity of $b(x, \cdot)$ is proven. For the function $b(\cdot, y)$ the proof is similar.

Step 3. $M = M(b)$. Indeed, this is true, by Propositions 6.3(a) and 6.4(b).

Step 4. By Proposition 6.3(b) we have that for any $(x, y) \in X \times Y$ the inequality $b(x, y) \geq \langle x, y \rangle$ is true. Therefore the conditions (a), (b), from the Definition 2.2 of a bipotential, are verified.

Step 5. The only thing left to prove is the string of equivalences from Definition 2.2(c). Using the knowledge that b is separately convex and lower semicontinuous, we remark that in fact we only have to prove two implications.

The first is: for any $x \in X$ suppose that $y \in Y$ has the minimum property

$$b(x, y) - \langle x, y \rangle \leq b(x, z) - \langle x, z \rangle$$

for any $z \in Y$. Then $b(x, y) = \langle x, y \rangle$.

The second implication is similar, only that we start with an arbitrary $y \in Y$ and with $x \in X$ satisfying the minimum property

$$b(x, y) - \langle x, y \rangle \leq b(z, y) - \langle z, y \rangle$$

for any $z \in X$. Then $b(x, y) = \langle x, y \rangle$.

The first implication is just Proposition 6.4(a). The second implication has a similar proof. □

The next proposition makes easier to check if a convex lagrangian cover satisfies the BIC condition.

Proposition 6.8. *Let $\lambda \mapsto \phi_\lambda$ be a BIC-cover of the BB-graph M. Consider any $\alpha, \beta \in [0, 1]$, $\alpha + \beta = 1$, any $y \in im(M)$, any $\lambda_1, \lambda_2 \in \Lambda$ and any $x_1 \in \partial\phi^*_{\lambda_1}(y)$, $x_2 \in \partial\phi^*_{\lambda_2}(y)$. According to the BIC condition there exists $\lambda \in \Lambda$ such that*

$$f(\lambda, \alpha x_1 + \beta x_2, y) \leq \alpha f(\lambda_1, x_1, y) + \beta f(\lambda_2, x_2, y). \tag{5}$$

Then λ has the property:

$$\alpha x_1 + \beta x_2 \ \in \ \partial\phi^*_\lambda(y).$$

Proof. Inequality (5) expresses as:

$$\phi_\lambda(\alpha x_1 + \beta x_2) + \phi^*_\lambda(y) \leq \alpha\phi_{\lambda_1}(x_1) + \beta\phi_{\lambda_2}(x_2) + \alpha\phi^*_{\lambda_1}(y) + \beta\phi^*_{\lambda_2}(y). \tag{6}$$

We have also $(i = 1, 2)$

$$\phi_{\lambda_i}(x_i) + \phi^*_{\lambda_i}(y) = \langle x_i, y\rangle.$$

We use this in the inequality (6) to get

$$\phi_\lambda(\alpha x_1 + \beta x_2) + \phi^*_\lambda(y) \leq \langle \alpha x_1 + \beta x_2, y\rangle,$$

which shows that $\alpha x_1 + \beta x_2 \in \partial\phi^*_\lambda(y)$. Therefore the $\lambda \in \Lambda$ given by the implicit convexity inequality satisfies the conclusion of the proposition. □

Remark 6.9. Enforcing the satisfaction of the implicit convexity inequality for all values of λ which satisfy the conclusion of Proposition 6.8 would be too strong. This remark is supported by the second example in section 7, involving a family of non differentiable potentials for which there is no uniqueness for λ.

7. Reconstruction of the Cauchy bipotential

In this section we shall reconstruct the Cauchy bipotential from two different convex lagrangian covers. As explained in Remark 2.3, it is important for applications that we are able to reconstruct the expression of the Cauchy bipotential from the graph of its associated law.

We shall take $X = Y = \mathbb{R}^n$ and the duality product is the usual scalar product in $\mathbb{R}$. The Cauchy bipotential is $\bar{b}(x, y) = \|x\|\|y\|$. By Cauchy-Schwarz-Bunyakovsky inequality the set $M = M(\bar{b})$ is

$$M = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \exists\lambda > 0,\ x = \lambda y\} \cup (\{0\} \times \mathbb{R}^n) \times (\mathbb{R}^n \times \{0\}).$$

Let us consider the topological compact set $\Lambda = [0, \infty]$ (with usual topology) and the function $\lambda \in \Lambda \mapsto \phi_\lambda$ defined as:

- if $\lambda \in [0, \infty)$ then $\phi_\lambda(x) = \dfrac{\lambda}{2}\|x\|^2$,
- if $\lambda = \infty$ then

$$\phi_\infty(x) = \chi_0(x) = \begin{cases} 0 & \text{if } x = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

A straightforward computation shows that the associated function f has the expression:

$$f(\lambda, x, y) = \begin{cases} \frac{\lambda}{2}\|x\|^2 + \frac{1}{2\lambda}\|y\|^2 & \text{if } \lambda \in (0, \infty) \\ \chi_0(y) & \text{if } \lambda = 0 \\ \chi_0(x) & \text{if } \lambda = \infty. \end{cases} \tag{7}$$

It is easy to check that we have here a convex lagrangian cover of the set M. We shall prove now that we have a BIC-cover, according to Definition 6.6.

The cases $\lambda = 0$ and $\lambda = \infty$ will be treated separately.

Consider $y \in im(M) = \mathbb{R}^n$, $x_1, x_2 \in X$, $\alpha, \beta \in [0,1]$, $\alpha + \beta = 1$, and $\lambda_1, \lambda_2 \in (0, \infty)$. We have to find $\lambda \in \Lambda$ such that

$$f(\lambda, \alpha x_1 + \beta x_2, y) \leq \alpha f(\lambda_1, x_1, y) + \beta f(\lambda_2, x_2, y). \tag{8}$$

We use Proposition 6.8, for $i = 1, 2$ and $x_i \in \partial \phi^*_{\lambda_i}(y)$ in order to find the value of λ. Computation shows that there is only one such $\lambda \in \Lambda$, given by

$$\frac{1}{\lambda} = \frac{\alpha}{\lambda_1} + \frac{\beta}{\lambda_2}. \tag{9}$$

As this value depends only on λ_1, λ_2, we shall try to see if this λ is good for any choice of x_1, x_2.

This is indeed the case: with λ given by (9) the relation (8) (multiplied by 2) becomes:

$$\lambda \|\alpha x_1 + \beta x_2\|^2 \leq \alpha \lambda_1 \|x_1\|^2 + \beta \lambda_2 \|x_2\|^2. \tag{10}$$

Remark that (9) can be written as:

$$\frac{\alpha\lambda}{\lambda_1} + \frac{\beta\lambda}{\lambda_2} = 1.$$

Write then the fact that the square of the norm is convex, for the convex combination of $\lambda_1 x_1, \lambda_2 x_2$, with the coefficients $\frac{\alpha\lambda}{\lambda_1}, \frac{\beta\lambda}{\lambda_2}$. We get, after easy simplifications, the inequality (10).

If $\lambda_1 = 0$, $\lambda_2 \in (0, \infty)$ then y has to be equal to 0 and x_1 is arbitrary, $x_2 = 0$ and $\lambda = 0$. The inequality (8) is then trivial.

All other exceptional cases lead to trivial inequalities.

Remark that for any $\lambda \in \Lambda$ and any $x, y \in \mathbb{R}^n$ we have

$$f(\lambda, x, y) = f(\frac{1}{\lambda}, y, x)$$

with the conventions $1/0 = \infty$, $1/\infty = 0$. This symmetry and previous proof imply that we have a BIC-cover.

We compute now the function b from Definition 6.2. We know from Theorem 6.7 that b is a bipotential for the set M.

We have:

$$b(x, y) = \inf \{ f(\lambda, x, y) \, : \, \lambda \in [0, \infty] \} .$$

From the relation (7) we see that actually

$$b(x, y) = \inf \left\{ \frac{\lambda}{2} \|x\|^2 + \frac{1}{2\lambda} \|y\|^2 \, : \, \lambda \in (0, \infty) \right\} .$$

By the arithmetic-geometric mean inequality we obtain that $b(x, y) = \|x\| \|y\|$, that is the Cauchy bipotential.

Here is a second example, supporting the Remark 6.9. We shall reconstruct the Cauchy bipotential starting from a family of non differentiable convex potentials.

Let $\lambda \geq 0$ be non negative and the closed ball of center 0 and radius λ be defined by

$$B(\lambda) = \{y \in Y \, : \, \|y\| \leq \lambda\}$$

Defining $B(+\infty)$ as the whole space Y, one can suppose that λ belongs to the compact set $\Lambda = [0, +\infty]$.

For $\lambda \in [0, +\infty)$ we define the set:

$$M_\lambda = \{(0, y) \in X \times Y \, : \, \|y\| < \lambda\} \cup \{(x, y) \in X \times Y \, : \, \|y\| = \lambda \text{ and } \exists \eta \geq 0 \;\; x = \eta y\} \, .$$

One can recognize M_λ as the graph of the yielding law of a plastic material with a yielding threshold equal to λ. For $\lambda = +\infty$ we set $M_{+\infty} = \{0\} \times Y$.

It can be easily verified that the family $(M_\lambda)_{\lambda \in \Lambda}$ of maximal cyclically monotone graphs provides us a convex lagrangian cover of the set:

$$M = \{(x, y) \in X \times Y \, : \, \exists \alpha, \beta \geq 0 \;\; \alpha x = \beta y\} \, .$$

The corresponding convex lagrangian cover is given by:

- for $\lambda \in [0, +\infty)$, $\phi_\lambda(x) = \lambda \|x\|$, $\phi^*_\lambda(y) = \chi_{B(\lambda)}(y)$,
- $\phi_{+\infty}(x) = \chi_0(x)$, $\phi^*_{+\infty}(y) \; = 0$.

The associated function f has the expression:

$$f(\lambda, x, y) = \begin{cases} \lambda \|x\| + \chi_{B(\lambda)}(y) & \text{if } \lambda \in (0, \infty) \\ \chi_0(y) & \text{if } \lambda = 0 \\ \chi_0(x) & \text{if } \lambda = +\infty. \end{cases} \tag{11}$$

All hypothesis excepting the BIC-cover condition are obviously satisfied. We check this condition further. Let $\lambda_1 < \lambda_2$, both in $[0, +\infty)$. We want first to determine the values of λ fulfilling the conclusion of Proposition 6.8. Let us recall that:

- if $\|y\| < \lambda$ then $\partial \phi^*_\lambda(y) = \{0\}$,
- if $\|y\| = \lambda$ then $x \in \partial \phi^*_\lambda(y)$ is equivalent to: $\exists \eta \geq 0$ such that $x = \eta y$,
- if $\|y\| > \lambda$ then $\partial \phi^*_\lambda(y) = \emptyset$.

Then the following events have to be considered:

(1) if $\|y\| < \lambda_1 < \lambda_2$ then $x_1 \in \partial \phi_{\lambda_1}(y)$ and $x_2 \in \partial \phi_{\lambda_2}(y)$ imply $x_1 = x_2 = 0$,

(2) if $\|y\| = \lambda_1 < \lambda_2$ then $x_1 \in \partial \phi_{\lambda_1}(y)$ and $x_2 \in \partial \phi_{\lambda_2}(y)$ imply: $\exists \eta \geq 0$ such that $x_1 = \eta y$ and $x_2 = 0$. Thus

$$\alpha x_1 + \beta x_2 = \alpha \eta x_1 \in \partial \phi^*_\lambda(y)$$

occurs for any $\lambda \geq \|y\|$ when $x_1 = 0$ and $\lambda = \|y\|$ otherwise.

(3) If $\lambda_1 < \|y\|$ then there is no x_1 such that $x_1 \in \partial \phi^*_{\lambda_1}(y)$. Likewise, if $\lambda_2 < \|y\|$ then there is no x_2 such that $x_2 \in \partial \phi^*_{\lambda_2}(y)$.

Consider $y \in im(M) = \mathbb{R}^n$, $x_1, x_2 \in X$, $\alpha, \beta \in [0,1]$, $\alpha + \beta = 1$, and $\lambda_1, \lambda_2 \in [0, \infty)$. For the verification of the implicit convexity inequality (8), we need only to consider the case $\|y\| \leq \min\{\lambda_1, \lambda_2\}$ and we shall choose $\lambda = \min\{\lambda_1, \lambda_2\} \geq \|y\|$. The relation (8) becomes

$$\min\{\lambda_1, \lambda_2\} \|\alpha x_1 + \beta x_2\| \leq \alpha \lambda_1 \|x_1\| + \beta \lambda_2 \|x_2\|,$$

which is true by the convexity of the norm. All the other cases turn out to be trivial.

The other half of the BIC-cover condition has a similar proof (remark though that the associated function f is not symmetric, as in the previous case).

By virtue of Theorem 6.7, the function given by Definition 6.2, namely

$$b(x, y) = \inf\{\phi_\lambda(x) + \phi^*_\lambda(y) \, : \, \lambda \in [0, +\infty]\},$$

is a bipotential. Computation shows that b is the Cauchy bipotential. Indeed:

$$\begin{aligned} b(x, y) &= \inf\left\{\lambda\|x\| + \chi_{B(\lambda)}(y) \, : \, \lambda \in [0, \infty)\right\} \\ &= \inf\{\lambda\|x\| \, : \, \lambda \geq \|y\|\} = \|y\|\|x\|. \end{aligned}$$

8. Conclusion and perspectives

Given (the graph of) a multivalued constitutive law M, there is a bipotential b such that $M = M(b)$ if and only if M is a BB-graph (Definition 3.1 and Theorem 3.2). If the BB-graph M admits a convex lagrangian cover (Definition 4.1) which is bi-implicitly convex (Definition 6.6) then we are able to construct an associated bipotential (Theorem 6.7).

Remarks 5.1 and 5.2 show that not any BB-graph admits a convex lagrangian cover. We would like to elaborate on the obstructions to the existence of such covers. We start with the example from the Remark 5.2, due to E. Ernst.

From a mechanical point of view, multivalued laws M with the property that for any two different pairs $(x_1, y_1), (x_2, y_2) \in M$ we have

$$\langle x_1 - x_2, y_1 - y_2 \rangle < 0$$

are not very interesting. Indeed, suppose that the evolution of a mechanical system is described by a sequence of states $(x_n, y_n) \in M$. Then, as the system passes from one state to another, the work done is always negative. Much more interesting seem to be multivalued laws with the property that for any $(x, y) \in M$ there is at least a different pair $(x', y') \in M$ such that

$$\langle x - x', y - y' \rangle \geq 0.$$

The BB-graphs admitting a convex lagrangian cover have this property.

There is another aspect, concerning the linear transformation A from the Remark 5.2. In the example given the transformation $A(x, y) = (x, -y)$ is not symplectic, but still it transforms lagrangian sets into lagrangian sets. In general, if the dimension of X is strictly greater than one then we can find linear endomorphisms of $X \times Y$ transforming lagrangian subsets of $X \times Y$ into sets which are not lagrangian, thus destroying lagrangian covers. Moreover, we can find linear symplectic transformations which transforms a convex lagrangian cover into a lagrangian cover which is no longer convex. For example, take $X = Y = \mathbb{R}$, $A(x, y) = (x, y - x)$ and the BB-graph $M = \mathbb{R} \times \{0\}$. Then $det(A) = 1$, therefore A is symplectic, and $A(M) = \{(x, -x) : x \in \mathbb{R}\}$. The set $A(M)$ is a BB-graph and a lagrangian set, but it does not admit a convex lagrangian cover. The reason for this phenomenon is that convexity is not a symplectic invariant. Nevertheless, there are famous theorems in Hamiltonian Dynamics which have a convexity assumption in the hypothesis, like the theorem of Rabinowitz stating that the Reeb vector field on the boundary of a convex domain which is bounded has at least a closed orbit (equivalently, a convex and coercive hamiltonian on $\mathbb{R}^{2n}$ admits a closed orbit on every level set). We can easily destroy the convexity assumption of this theorem but not the conclusion, by applying a nonlinear symplectomorphism.

For the notion of convex lagrangian cover we had the following source of inspiration. If M is a symplectic manifold and with convexity assumptions left aside, lagrangian covers as described in this paper resemble to (real) symplectic polarizations, which are a basic tool in some problems of symplectic geometry.

Much more interesting are cases relating with Remark 5.1. We may consider BB-graphs M not admitting convex lagrangian covers, but with the property that there is a family of convex, lower semicontinuous functions ϕ_λ, $\lambda \in \Lambda$ such that

$$M \subset \bigcup_{\lambda \in \Lambda} M(\phi_\lambda)$$

with strict inclusion. This is the case, for example, of the bipotential which appears in [21], related to contact with friction. In a future paper we shall extend this method of convex lagrangian cover to lagrangian covers by graphs which are cyclically monotone but not necessarily maximal cyclically monotone.

Acknowledgements. The first author acknowledges partial support from the Romanian Ministry of Education and Research, through the grant CEEX06-11-12/2006. Part of this work has been done in 2005, when the first author has been invited at the Laboratoire de Mécanique de Lille, UMR CNRS 8107, Université des Sciences et Technologies de Lille.

The authors thanks E. Ernst for pointing out an error in a first version of the paper, and for some examples related to Remarks 5.1 and 5.2. We thank also the anonymous referee for comments and suggestions leading hopefully to a better paper.

References

[1] A. Berga, G. de Saxcé: Elastoplastic finite element analysis of soil problems with implicit standard material constitutive laws, Rev. Eur. Élém. Finis 3(3) (1994) 411–456.

[2] G. Bodovillé: On damage and implicit standard materials, C. R. Acad. Sci., Paris, Sér. II, Fasc. b, Méc. Phys. Astron. 327(8) (1999) 715–720.

[3] G. Bodovillé, G. de Saxcé: Plasticity with non linear kinematic hardening: modelling and shakedown analysis by the bipotential approach, Eur. J. Mech., A/Solids 20 (2001) 99–112.

[4] C. Bouby, G. de Saxcé, J.-B. Tritsch: A comparison between analytical calculations of the shakedown load by the bipotential approach and step-by-step computations for elastoplastic materials with nonlinear kinematic hardening, Int. J. Solids Struct. 43(9) (2006) 2670–2692.

[5] L. Bousshine, A. Chaaba, G. de Saxcé: Softening in stress-strain curve for Drücker-Prager non-associated plasticity, Int. J. Plast. 17(1) (2001) 21–46.

[6] L. Bousshine, A. Chaaba, G. de Saxcé: Plastic limit load of plane frames with frictional contact supports, Int. J. Mech. Sci. 44(11) (2002) 2189–2216.

[7] L. Bousshine, A. Chaaba, G. de Saxcé: A new approach to shakedown analysis for non-standard elastoplastic material by the bipotential, Int. J. Plast. 19(5) (2003) 583–598.

[8] K. Dang Van, G. de Saxcé, G. Maier, C. Polizzotto, A. Ponter, A. Siemaszko, D. Weichert: Inelastic Behaviour of Structures under Variable Repeated Loads, D. Weichert, G. Maier (eds.), CISM Courses and Lectures 432, Springer, Wien (2002).

[9] Z.-Q. Feng, M. Hjiaj, G. de Saxcé, Z. Mróz: Effect of frictional anisotropy on the quasistatic motion of a deformable solid sliding on a planar surface, Comput. Mech. 37 (2006) 349–361.

[10] J. Fortin, G. de Saxcé: Modélisation numérique des milieux granulaires par l'approche du bipotentiel, C. R. Acad. Sci., Paris, Sér. II, Fasc. b, Méc. Phys. Astron. 327 (1999) 721–724.

[11] J. Fortin, M. Hjiaj, G. de Saxcé: An improved discrete element method based on a variational formulation of the frictional contact law, Comput. Geotech. 29(8) (2002) 609–640.

[12] B. Halphen, Q. S. Nguyen: Sur les matériaux standard généralisés, J. Mec., Paris 14 (1975) 39–63.

[13] J. Fortin, O. Millet, G. de Saxcé: Numerical simulation of granular materials by an improved discrete element method, Int. J. Numer. Methods Eng. 62 (2005) 639–663.

[14] M. Hjiaj, G. de Saxcé, N.-E. Abriak: The bipotential approach: some applications, in: Advances in Finite Element Technology, B. H. V. Topping (ed.), Civil-Comp Press, Edinburgh (1996) 341–348.

[15] M. Hjiaj, N.-E. Abriak, G. de Saxcé: The bipotential approach for soil plasticity, in: Numerical Methods in Engineering '96, J.-A. Désidéri et al. (ed.), John Wiley & Sons, Chichester (1996) 918–923.

[16] M. Hjiaj, G. Bodovillé, G. de Saxcé: Matériaux viscoplastiques et lois de normalité implicites, C. R. Acad. Sci., Paris, Sér. II, Fasc. b, Méc. Phys. Astron. 328 (2000) 519–524.

[17] M. Hjiaj, J. Fortin, G. de Saxcé: A complete stress update algorithm for the non-associated Drücker-Prager model including treatment of the apex, Int. J. Eng. Sci. 41(10) (2003) 1109–1143.

[18] M. Hjiaj, Z.-Q. Feng, G. de Saxcé, Z. Mróz: Three dimensional finite element computations for frictional contact problems with on-associated sliding rule, Int. J. Numer. Methods Eng. 60(12) (2004) 2045–2076.

[19] J.-J. Moreau: Fonctionnelles Convexes, Istituto Poligrafico e Zecca dello Stato, Rome (2003) pp. 71, 97.

[20] R. T. Rockafellar: Convex Analysis, Princeton University Press, Princeton (1970) p. 238.

[21] G. de Saxcé, Z.Q. Feng: New inequation and functional for contact with friction: the implicit standard material approach, Mech. Struct. Mach. 19(3) (1991) 301–325.

[22] G. de Saxcé: Une généralisation de l'inégalité de Fenchel et ses applications aux lois constitutives, C. R. Acad. Sci., Paris, Sér. II 314 (1992) 125–129.

[23] G. de Saxcé, L. Bousshine: On the extension of limit analysis theorems to the non associated flow rules in soils and to the contact with Coulomb's friction, in: Proc. XI Polish Conference on Computer Methods in Mechanics (Kielce, 1993), Vol. 2 (1993) 815–822.

[24] G. de Saxcé: The bipotential method, a new variational and numerical treatment of the dissipative laws of materials, in: Proc. 10th Int. Conf. on Mathematical and Computer Modelling and Scientific Computing (Boston, 1995).

[25] G. de Saxcé, M. Hjiaj: Sur l'intégration numérique de la loi d'écrouissage cinématique non-linéaire, in: Actes du 3ème Colloque National en Calcul des Structures (Giens, 1997), Presses Académiques de l'Ouest (1997) 773–778.

[26] G. de Saxcé, L. Bousshine: Limit analysis theorems for the implicit standard materials: application to the unilateral contact with dry friction and the non associated flow rules in soils and rocks, Int. J. Mech. Sci. 40(4) (1998) 387–398.

[27] G. de Saxcé, Z.-Q. Feng: The bipotential method: a constructive approach to design the complete contact law with friction and improved numerical algorithms, Math. Comput. Modelling 28(4–8) (1998) 225–245.

[28] G. de Saxcé, J.-B. Tritsch, M. Hjiaj: Shakedown of elastoplastic materials with non linear kinematical hardening rule by the bipotential approach, in: Solid Mechanics and its Applications 83, D. Weichert, G. Maier (eds.), Kluwer, Dordrecht (2000) 167–182.

[29] G. de Saxcé, J. Fortin, O. Millet: About the numerical simulation of the dynamics of granular media and the definition of the mean stress tensor, Mech. Mater. 36(12) (2004) 1175–1184.

[30] C. Vallée, C. Lerintiu, D. Fortuné, M. Ban, G. de Saxcé: Hill's bipotential, in: New Trends in Continuum Mechanics, M. Mihailescu-Suliciu (ed.), Theta Series in Advanced Mathematics, Theta Foundation, Bucarest (2005) 339–351.