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Farkas-Type Results and Duality for DC Programs with Convex Constraints

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In this paper, we are interested in new versions of Farkas lemmas for systems involving convex and DC-inequalities. These versions extend well-known Farkas-type results published recently, which were used as main tools in the study of convex optimization problems. The results are used to derive several strong duality results such as: Lagrange, Fenchel-Lagrange or Toland-Fenchel-Lagrange duality for DC and convex problems. Moreover, it is shown that for this class of problems, these versions of Farkas lemma are actually equivalent to several strong duality results of Lagrange, Fenchel-Lagrange or Toland-Fenchel-Lagrange types.

Keywords: Generalized Farkas lemmas, DC-programs, Toland, Fenchel, Lagrange duality, approximate normal cones

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1. Introduction

Farkas-type results have played very important roles in non-linear programming (see [16], [29], [31] and references therein). Recently, several versions of Farkas-type results were proposed for general cone-convex systems (see e.g., [4], [12], [13]) or for systems defined by an arbitrary family of convex (homogeneous) inequalities (see, e.g. [10], [11], [29]). An overview on the development and applications of this type of results can be found in [21]. In parallel, many applications to optimization problems are given: optimality conditions for convex programs in [4], [16], [29]; general programs with cone-convex constraints in [7], [9], [12], [13], [24]; optimal control problems in [31], for instance.

In this paper, we establish various Farkas-type results for a system involving a general

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cone-convex constraint, a geometrical constraint, and a constraint defined by a DC-function (difference of two convex functions). The results are represented in asymptotic forms, non-asymptotic forms and in dual forms (for the last two forms, a closedness condition is needed). Then, these Farkas-type results are used to study the duality of a problem of minimizing a DC function under a convex-cone constraint and a set constraint (Problem (P) in Section 5). Several duality results of Toland-Fenchel-Lagrange type are obtained. This type of duality for DC programs was studied in [13] and [28]. In [13], a Toland-Fenchel-Lagrange duality theorem was established and Farkas lemmas of dual forms were derived. In this paper, all of these results and some more of asymptotic and non-asymptotic forms, are developed in the converse way. This approach allows us to point out that in fact, all these forms and duality results are actually equivalent together (Theorem 5.8).

The results yield corresponding results in convex setting which recapture several known version of generalized Farkas lemmas and duality results (Fenchel-Lagrange duality, Lagrange duality) in the literature (for instance, [4], [5], [10], [11], [12], [16], [21]). Moreover, necessary and sufficient conditions for stable Fenchel-Lagrange duality are also obtained for a class of convex programs.

The paper is organized as follows. In Section 2, we fix some notations and recall some necessary results for the paper. In Section 3, we revisit the Farkas-type constraint qualifications called (FM), (CC) and (CC1). We recall some characterizations for (CC) and (FM) which were introduced in [13]. Several other characterizations for (CC) are proposed and characterizations for normal cones to convex-constrained sets are given. In particular, standard representations of normal cone to a set defined by finite number of convex functions are reformulated under a constraint qualification, which is weaker than Slater-type ones, often used in the literature. In Section 4, we will introduce several versions of Farkas lemmas (asymptotic forms, non-asymptotic forms and in dual forms) for systems involving convex and DC functions. A DC program with convex constraints (P) is examined in Section 5. For this problem, we first give a formula and then, we propose an upper bound for its optimal value. Next, we consider a dual problem to (P) called "Toland-Fenchel-Lagrange" dual problem (see [13]). Already considered in [28], it is in some sense, a "combination" of Toland dual problem for DC problem (see [33]), Fenchel-Lagrange and Lagrange dual problems (see [4, 5, 23]). Several duality results of this type for (P) are obtained, which extend the ones given in [28]. The results were formulated in [13] and then were used to construct dual forms of Farkas lemmas. However, here these results are developed in the converse way. This approach allows us to conclude that in fact all the form of Farkas lemmas and Toland-Fenchel-Lagrange duality results are equivalent together. Section 6 is left for an application of the results obtained in the previous sections to the convex setting: convex systems and convex programs. Various versions of Farkas lemma for convex systems and duality results for convex programs established recently in the literature such as, [4], [5], [10], [11], [12], [13], [16], are recaptured and extended. Moreover, it is shown that in this setting, several Farkas-type results, Lagrange duality, Fenchel-Lagrange duality results are actually equivalent together. The last part of the section is left for some necessary and sufficient conditions for the stable Fenchel-Lagrange duality.

2. Preliminaries

Let X, Z be locally convex Hausdorff topological spaces (except in Theorem 4.1 and Corollary 6.1). X^* and Z^* denote their topological dual (respectively), endowed with the weak*-topology.

We now fix some definitions and preliminaries that will be used later on. For a set $D \subset X$, the **closure** of D (the cone generated by D, resp.) will be denoted by cl D (cone D, resp.). The **support function** of D, σ_D, is defined by $\sigma_D(u) = \sup_{x\in D} u(x)$. δ_D denotes the **indicator function** of D, defined by: $\delta_D(x) := 0$ if $x \in D$ and $\delta_D(x) := +\infty$ if $x \notin D$. If $D \subset X^*$ then $\mathrm{cl}\, D$ stands for the closure of D with respect to the weak*-topology.

For any cone S in Z, let us set S^+ the dual cone of S, by

$$S^+ := \{\theta \in Z^* \,|\, \theta(s) \geq 0, \forall s \in S\}.$$

In the sequel of the paper, S is assumed to be a closed convex cone.

Let $f, g : X \to \mathbb{R}\cup\{+\infty\}$ be proper lower semi-continuous (l.s.c.) convex functions. Then, the conjugate function of f, $f^* : X^* \to \mathbb{R}\cup\{+\infty\}$, is defined, for any v in X^*, by

$$f^*(v) = \sup\{v(x) - f(x) \mid x \in \mathrm{dom}\, f\},$$

where the domain of f is given by $\mathrm{dom}\, f := \{x \in X \mid f(x) < +\infty\}$. It is also worth noticing that we have

$$\mathrm{epi}(f+g)^* = \mathrm{cl}\,\{\mathrm{epi}\, f^* + \mathrm{epi}\, g^*\}\,, \tag{1}$$

where the epigraph of f is given by

$$\mathrm{epi}\ f := \{(x,r) \in X \times \mathbb{R} \mid x \in \mathrm{dom}\, f,\ f(x) \leq r\}.$$

For any nonnegative ϵ, the ϵ-subdifferential of f at a given a in $\mathrm{dom}\, f$ is defined as the (possibly empty) closed convex set

$$\partial_\epsilon f(a) = \{v \in X^* \mid f(x) - f(a) \geq v(x-a) - \epsilon,\ \forall x \in \mathrm{dom}\, f\},$$

and note that if $\epsilon > 0$ then $\partial_\epsilon f(a) \neq \emptyset$. See J. B. Hiriart-Urruty [17, 18] and C. Zalinescu [36] for a detailed discussion on this set and its properties.

Moreover, if $a \in \mathrm{dom}\, f$, then

$$\mathrm{epi}\, f^* = \bigcup_{\epsilon\geq 0} \{(v, v(a) + \epsilon - f(a)) \mid v \in \partial_\epsilon f(a)\}\,. \tag{2}$$

For details, see for example V. Jeyakumar [20].

For a closed convex subset $D \subset X$ and an arbitrary nonnegative ϵ, the ϵ-normal cone to D at a point $a \in D$ is defined by (see J. B. Hiriart-Urruty [17, 18])

$$N_\epsilon(D,a) := \partial_\epsilon \delta_D(a) = \{x^* \in X^* \,|\, (x^*, x-a) \leq \epsilon, \forall x \in D\}.$$

Here, $N_0(D,a) = N_D(a)$ is the normal cone to D at $a \in D$ in the sense of convex analysis, i.e.,

$$N_D(a) := \{x^* \in X^* \mid \langle x^*, x-a\rangle \leq 0, \forall x \in D\}.$$

Let h be a S-convex mapping, that is,

$$\forall u,\, v \in X, \forall t \in [0,\, 1],\ h[tu + (1-t)v] - th(u) - (1-t)h(v) \in -S,$$

such that $\lambda \circ h$ is l.s.c. for each λ in S^+.

For convenience, the composition of $\lambda \in S^+$ and h, i.e., $\lambda \circ h$, will be denoted by λh.

Throughout this paper, C would be a closed convex subset of X and one considers the system

$$\sigma := \{x \in C,\ h(x) \in -S\}$$

and the set of its solution A, i.e.,

$$A := \{x \in X \mid x \in C,\ h(x) \in -S\} = C \cap h^{-1}(-S),$$

with assumption that $A \cap \operatorname{dom} f$ is non-empty.

For convenience, we shall use the convention $+\infty - (+\infty) = +\infty$.

The following lemma, proved in [8], will be useful in the sequel.

Lemma 2.1 ([8]). *Suppose that X is a locally convex Hausdorff topological vector space and $f, g : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ are proper, l.s.c. and convex functions such that $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. The following statements are equivalent:*

(*i*) $\operatorname{epi} f^* + \operatorname{epi} g^*$ *is weak***-closed,*
(*ii*) $\operatorname{epi}(f+g)^* = \operatorname{epi} f^* + \operatorname{epi} g^*$,
(*iii*) *for each* $\epsilon \geq 0$ *and each* $x \in \operatorname{dom} f \cap \operatorname{dom} g$,

$$\partial_\epsilon (f+g)(x) = \bigcup_{\substack{\epsilon_1+\epsilon_2=\epsilon \\ \epsilon_1,\epsilon_2 \geq 0}} \partial_{\epsilon_1} f(x) + \partial_{\epsilon_2} g(x).$$

3. Farkas-type constraint qualifications

Let us denote by K, the convex cone

$$K := \bigcup_{\lambda \in S^+} \operatorname{epi}(\lambda h)^* + \operatorname{epi} \delta_C^*, \tag{3}$$

and note that $\cup_{\lambda \in S^+} \operatorname{epi}(\lambda h)^*$ is a cone (see V. Jeyakumar et al. [25]).
Moreover, A is a convex subset of X such that (see V. Jeyakumar et al. [26]),

$$\operatorname{cl} K = \operatorname{epi} \delta_A^*. \tag{4}$$

We say that the system σ is **Farkas-Minkovski** (FM, in brief) if it satisfies:

(FM) The cone K is weak*-closed.

The following **closedness conditions** (CC) and (CC1) involving the function $f : X \to \mathbb{R} \cup \{+\infty\}$ and the system σ, which will be the main assumptions of most of the results in this paper, are:

(CC) $\operatorname{epi} f^* + K$ is weak*-closed.

(CC1) $\operatorname{epi} f^* + \operatorname{cl}(K)$ is weak*-closed.

If the set constraint "$x \in C$" is absent, i.e. $\sigma = \{h(x) \in -S\}$, the condition (CC) (resp. (CC1)) becomes $\operatorname{epi} f^* + \bigcup_{\lambda \in S^+} \operatorname{epi}(\lambda h)^*$ (resp. $\operatorname{epi} f^* + \operatorname{cl}\left\{\bigcup_{\lambda \in S^+} \operatorname{epi}(\lambda h)^*\right\}$) is weak*-closed.

3.1. Characterizations of (CC)

The closedness condition (CC) was proposed for the first time by R. S. Burachik et al. in [7] and it has been used after in several papers (see, for instance, [10], [11], [13]) to establish optimality conditions of Karush-Kuhn-Tucker form, duality and stability results for convex cone-constrained programs or convex infinite programs. Then, a relaxed version of this condition, (CC1), has been introduced in [11] for convex infinite programming problems.

It is clear that (CC) implies (CC1). Indeed, let $E := \operatorname{epi} f^* + K$ and $F := \operatorname{epi} f^* + \operatorname{cl}(K)$. Then $E \subset F$ and $\operatorname{cl}(E) = \operatorname{cl}(F)$. If E is closed then $\operatorname{cl}(E) = E \subset F \subset \operatorname{cl}(F) = \operatorname{cl}(E)$ which implies that $\operatorname{cl}(F) = F$. The converse is not true as the following simple example shows.

Example 3.1. Let $X = Z = C = \mathbb{R}$, $S = \mathbb{R}_+$ and $h : \mathbb{R} \to \mathbb{R}$ be the function defined, for any real x, by $h(x) = [\max(x, 0)]^2$.
It is easy to check that for any positive λ, $\operatorname{epi}(\lambda h)^* = \{(x, \alpha) \mid x \geq 0, \alpha \geq \frac{x^2}{4\lambda}\}$. Thus, it yields $K := \cup_{\lambda \in S^+} \operatorname{epi}(\lambda h)^* = \{(0,0)\} \cup [\mathbb{R}_+ \times \mathbb{R}_+^*]$.
Consider the function f, defined for any real x by $f(x) = 0$ if $x \in [0,1]$ and $+\infty$ else.

Obviously, $\operatorname{epi} f^* = \{(x, y) \mid y \geq \max(0, x)\}$. Note that if $(x, \gamma) \in \operatorname{epi} f^* + K$, then $\gamma \geq 0$ and $\gamma = 0$ implies that $x \leq 0$. Thus, it leads to $\operatorname{epi} f^* + K = \mathbb{R} \times \mathbb{R}_+ \setminus \{(x, 0) \,|\, x > 0\}$ which is not a closed subset in $\mathbb{R}^2$. However, it is clear that $\operatorname{epi} f^* + \operatorname{cl} K = \mathbb{R} \times \mathbb{R}_+$ is a closed subset in $\mathbb{R}^2$.

Let us mention in the following lemma, simple sufficient conditions for (CC) and (CC1).

Lemma 3.2. *If either*

(i) *f is continuous at one point in A, or*

(ii) $\operatorname{cone}(\operatorname{dom} f - A)$ *is a closed subspace of X,*

then (CC1) *holds. If, in addition, σ is* (FM) *then* (CC) *holds.*

Proof. If either (i) or (ii) holds then it follows from [7, Proposition 3.1], that $\operatorname{epi} f^* + \operatorname{epi} \delta_A^*$ is weak*-closed and that $\operatorname{epi}(f + \delta_A)^* = \operatorname{epi} f^* + \operatorname{epi} \delta_A^*$. Since $\operatorname{epi} \delta_A^* = \operatorname{cl}(K)$ (see (4)), $\operatorname{epi} f^* + \operatorname{cl}(K)$ is weak*-closed, and (CC1) holds. Then, the last assertion is obvious. $\square$

Let us now recall some characterizations for (CC) established recently in [13].

Proposition 3.3 ([13]). *The following statements are equivalent:*

(i) *condition* (CC) *holds,*

(ii) *for all $x^* \in X^*$,*

$$(f + \delta_A)^*(x^*) = \min_{\lambda \in S^+} \min_{u, v \in X^*} \left[f^*(u) + (\lambda h)^*(v) + \delta_C^*(x^* - u - v)\right],$$

(the infimum in the right-hand side is attained at some $\lambda \in S^+$ and $u, v \in X^*$),

(*iii*) *for any* $\bar{x} \in A \cap \operatorname{dom} f$ *and each* $\epsilon \geq 0$,

$$\partial_\epsilon (f + \delta_A)(\bar{x}) = \bigcup_{\lambda \in S^+} \bigcup_{\substack{\epsilon_1, \epsilon_2, \epsilon_3 \geq 0 \\ \epsilon_1 + \epsilon_2 + \epsilon_3 = \epsilon + \lambda h(\bar{x})}} \left\{ \partial_{\epsilon_1} f(\bar{x}) + \partial_{\epsilon_2} \lambda h(\bar{x}) + N_{\epsilon_3}(C, \bar{x}) \right\}. \quad (5)$$

The following proposition gives other alternatives to the above characterizations.

Proposition 3.4. *The following statements are equivalent:*

(*i*) *condition* (CC) *holds*,

(*ii′*) $\forall x^* \in X^*, \quad (f + \delta_A)^*(x^*) = \min_{\lambda \in S^+} \min_{u \in X^*} \left[(f + \lambda h)^*(u) + \delta_C^*(x^* - u) \right],$ (6)

(*ii″*) $\forall x^* \in X^*, \quad (f + \delta_A)^*(x^*) = \min_{\lambda \in S^+} \min_{u \in X^*} \left[(f^*(u) + (\lambda h + \delta_C)^*(x^* - u) \right],$

(*iii′*) $\forall \bar{x} \in A \cap \operatorname{dom} f, \ \forall \epsilon \geq 0,$

$$\partial_\epsilon (f + \delta_A)(\bar{x}) = \bigcup_{\lambda \in S^+} \bigcup_{\substack{\epsilon_1, \epsilon_2 \geq 0 \\ \epsilon_1 + \epsilon_2 = \epsilon + \lambda h(\bar{x})}} \left\{ \partial_{\epsilon_1} (f + \lambda h)(\bar{x}) + N_{\epsilon_2}(C, \bar{x}) \right\}.$$

Proof. (*i*) $\Longrightarrow$ (*ii′*) Assume that (*i*) holds and let us consider any $x^* \in X^*$.

For all $u \in X^*$, $\lambda \in S^+$ and $x \in A$, we have (note that $-\infty < \lambda h(x) \leq 0$),

$$(f + \lambda h)^*(u) \geq (u, x) - f(x) - \lambda h(x) \quad \text{and} \quad \delta_C^*(x^* - u) \geq (x^* - u, x), \quad (7)$$

which implies that $(f + \lambda h)^*(u) + \delta_C^*(x^* - u) \geq (x^*, x) - f(x) - \lambda h(x) \geq (x^*, x) - f(x)$. Thus, for all $u \in X^*$, $\lambda \in S^+$ and $x \in A \cap \operatorname{dom} f$,

$$(f + \lambda h)^*(u) + \delta_C^*(x^* - u) \geq (x^*, x) - f(x) - \delta_A(x).$$

This implies that $(f + \lambda h)^*(u) + \delta_C^*(x^* - u) \geq (f + \delta_A)^*(x^*)$, and hence,

$$\inf_{\lambda \in S^+} \inf_{u \in X^*} [(f + \lambda h)^*(u) + \delta_C^*(x^* - u)] \geq (f + \delta_A)^*(x^*). \quad (8)$$

If $x^* \notin \operatorname{dom}(f + \delta_A)^*$ then $(f + \delta_A)^*(x^*) = +\infty$ and by (8), (6) holds.

We now suppose that $x^* \in \operatorname{dom}(f + \delta_A)^*$. Combining (1), (4) and (CC), we get

$$\begin{aligned} \operatorname{epi}(f + \delta_A)^* &= \operatorname{cl}(\operatorname{epi} f^* + \operatorname{epi} \delta_A^*) = \operatorname{cl}(\operatorname{epi} f^* + \operatorname{cl} K) = \operatorname{cl}(\operatorname{epi} f^* + K) \\ &= \operatorname{epi} f^* + \cup_{\lambda \in S^+} \operatorname{epi}(\lambda h)^* + \operatorname{epi} \delta_C^*. \end{aligned} \quad (9)$$

So, one has that

$$(x^*, (f + \delta_A)^*(x^*)) \in \operatorname{epi}(f + \delta_A)^* = \operatorname{epi} f^* + \cup_{\lambda \in S^+} \operatorname{epi}(\lambda h)^* + \operatorname{epi} \delta_C^*. \quad (10)$$

Thus, there exist $\lambda \in S^+$,$(u, r) \in \operatorname{epi} f^*$, $(v, s) \in \operatorname{epi}(\lambda h)^*$ and $(w, t) \in \operatorname{epi} \delta_C^*$, such that

$$(x^*, (f + \delta_A)^*(x^*)) = (u, r) + (v, s) + (w, t).$$

This implies that $u + v + w = x^*$ and it comes that

$$(f + \delta_A)^*(x^*) \geq f^*(u) + (\lambda h)^*(v) + \delta_C^*(w).$$

Thus, for all $x \in \operatorname{dom} f = \operatorname{dom}(f + \lambda h)$, one gets that

$$\begin{aligned}(f+\delta_A)^*(x^*) &\geq (u,x) - f(x) + (v,x) - (\lambda h)(x) + \delta_C^*(x^* - u - v)\\ &\geq (u',x) - (f+\lambda h)(x) + \delta_C^*(x^* - u'),\end{aligned}$$

where $u' = u + v$. The last inequality implies that

$$(f+\delta_A)^*(x^*) \geq (f+\lambda h)^*(u') + \delta_C^*(x^* - u'),$$

which, together with (8), shows that (6) holds and that the infimum, in the right-hand side of (6), is attained at some $\lambda \in S^+$ and some $u' \in X^*$. Therefore, (ii') has been proved.

$(iii') \Longrightarrow (i)$ Let $(x^*, r) \in \operatorname{cl}\{\operatorname{epi} f^* + K\}$. Then, thanks to (9), it comes that $(x^*, r) \in \operatorname{epi}(f+\delta_A)^*$. Let $\bar{x} \in \operatorname{dom}(f+\delta_A)$ (this set is nonempty by assumption). It now follows from (2) that $\epsilon \geq 0$ exists such that $x^* \in \partial_\epsilon(f+\delta_A)(\bar{x})$ with $r = (x^*, \bar{x}) - f(\bar{x}) + \epsilon$.

Then, (iii') implies that there exist $\lambda \in S^+$, $u, v \in X^*$ and $\epsilon_1, \epsilon_2 \geq 0$ satisfying: $x^* = u+v$ and $\epsilon_1 + \epsilon_2 = \epsilon + \lambda h(\bar{x})$ with $u \in \partial_{\epsilon_1}(f+\lambda h)(\bar{x})$ and $v \in N_{\epsilon_2}(C, \bar{x})$.

Set $s = (u, \bar{x}) - (f+\lambda h)(\bar{x}) + \epsilon_1$ and $t = (v, \bar{x}) + \epsilon_2$ and note that in a similar way, it comes that $(u, s) \in \operatorname{epi}(f+\lambda h)^*$ and that $(v, t) \in \operatorname{epi} \delta_C^*$. Then, since $x^* = u + v$ and as

$$s + t = (u, \bar{x}) - (f+\lambda h)(\bar{x}) + \epsilon_1 + (v, \bar{x}) + \epsilon_2 = (x^*, \bar{x}) - f(\bar{x}) + \epsilon = r,$$

we get that

$$(x^*, r) = (u, s) + (v, t) \in \operatorname{epi}(f+\lambda h)^* + \operatorname{epi} \delta_C^*. \tag{11}$$

Since $\operatorname{dom}(\lambda h) = X$, $\operatorname{cone}(\operatorname{dom}(\lambda h) - \operatorname{dom} f)$ is a closed subspace and hence, $\operatorname{epi}(f+\lambda h)^* = \operatorname{epi} f^* + \operatorname{epi}(\lambda h)^*$ (see [7, 8]). Then, it follows from (11) that

$$(x^*, r) \in \operatorname{epi} f^* + \operatorname{epi}(\lambda h)^* + \operatorname{epi} \delta_C^* \subset \operatorname{epi} f^* + K,$$

which proves that $\operatorname{epi} f^* + K$ is weak*-closed. In other words, (CC) holds.

The proofs of the other implications are quite similar to the corresponding ones in [13] (Theorem 3.1) and so, they are omitted. □

Proposition 3.5. *The following statements are equivalent:*

(i') $\operatorname{epi}(f+\delta_C)^* + \bigcup_{\lambda \in S^+} \operatorname{epi}(\lambda h)^*$ *is weak*-closed,*

(ii''') *for all* $x^* \in X^*$,

$$(f+\delta_A)^*(x^*) = \min_{\lambda \in S^+} \min_{u \in X^*} \left[(f+\delta_C)^*(u) + (\lambda h)^*(x^* - u)\right],$$

(the infimum in the right-hand side is attained at some $\lambda \in S^+$ *and* $u \in X^*$*),*

(iii') *for any* $\bar{x} \in A \cap \operatorname{dom} f$ *and each* $\epsilon \geq 0$,

$$\partial_\epsilon(f+\delta_A)(\bar{x}) = \bigcup_{\lambda \in S^+} \bigcup_{\substack{\epsilon_1, \epsilon_2 \geq 0 \\ \epsilon_1 + \epsilon_2 = \epsilon + \lambda h(\bar{x})}} \{\partial_{\epsilon_1}(f+\delta_C)(\bar{x}) + \partial_{\epsilon_2} \lambda h(\bar{x})\}.$$

Proof. The conclusion follows from Proposition 3.3 where $\bar{f} = f + \delta_C$ and $\bar{C} = X$ play the roles of f and C, respectively. □

3.2. Characterizations of (FM) and approximate normal cones to convex-constrained sets

Known as the "closed cone constraint qualification" (CCCQ), the condition (FM) was first proposed in [23]. Then, an alternative formulation was also given recently in [5]. It has been used as a constraint qualification in [10, 11, 24] for optimality conditions, duality and stability for convex (infinite) programming problems. It is shown in [5] and [23] that the condition (FM) is strictly weaker than several generalized Slater type constraint qualifications, and is weaker than Robinson type constraint qualification stating that $\mathbb{R}_+[S + h(C)]$ is a closed subspace (see [5, 23] for more details).

In this subsection, we first recall some characterizations of (FM) proposed recently in [13] and derive some corollaries. Among them, the last one comes back to the standard representation of normal cone to a convex subset defined by a finite number of convex functions and a convex subset. In this case, the classical formula is re-established under a weaker constraint qualification.

Proposition 3.6 ([13]). *The following statements are equivalent:*

(*i*) *the system σ is* (FM),

(*ii*) *for each $x^* \in X^*$,*

$$\sigma_A(x^*) = \min_{\lambda\in S^+} \min_{u\in X^*} [(\lambda h)^*(u) + \delta_C^*(x^* - u)] = \min_{\lambda\in S^+} [(\lambda h)^* \oplus \delta_C^*](x^*),$$

(*iii*) *for any $a \in A$ and each $\epsilon \geq 0$,*

$$N_\epsilon(A, a) = \bigcup_{\lambda\in S^+} \bigcup_{\substack{\epsilon_1,\epsilon_2\geq 0\\ \epsilon_1+\epsilon_2=\epsilon+\lambda h(a)}} \{\partial_{\epsilon_1}\lambda h(a) + N_{\epsilon_2}(C, a)\}.$$

Here $(\lambda h)^* \oplus \delta_C^*$ denotes the infimum convolution defined by:

$$[(\lambda h)^* \oplus \delta_C^*](x^*) := \min_{u\in X^*} [(\lambda h)^*(u) + \delta_C^*(x^* - u)], \ \forall x^* \in X^*.$$

Let us now propose some corollaries that are direct consequences of the previous proposition when $\epsilon = 0$ and $C = X$ respectively.

Corollary 3.7. *Let $a \in A$. Suppose that* (FM) *holds. Then*

$$N_A(a) = \bigcup_{\substack{\lambda\in S^+\\ \lambda h(a)=0}} \{\partial(\lambda h)(a) + N(C, a)\}.$$

Corollary 3.8. *If $C = X$, the following statements are equivalent*

(*i*) $\bigcup_{\lambda\in S^+} \mathrm{epi}(\lambda h)^*$ *is weak*-closed,*

(*ii*) *for any $a \in h^{-1}(-S)$ and each $\epsilon \geq 0$,*

$$N_\epsilon(h^{-1}(-S), a) = \bigcup_{\substack{\lambda\in S^+\\ \epsilon+\lambda h(a)\geq 0}} \partial_{\epsilon+\lambda h(a)}(\lambda h)(a).$$

In particular, if (i) holds then $N_{h^{-1}(-S)}(a) = \bigcup_{\substack{\lambda \in S^+ \\ \lambda h(a)=0}} \partial(\lambda h)(a)$.

We now consider the case where $h = (h_1, h_2, \dots, h_n) : X \to \mathbb{R}^n$ is constituted as n continuous convex functions. Let $Z = \mathbb{R}^n$, $S = \mathbb{R}^n_+$, and C be a closed convex subset of X. Then $Z^* = \mathbb{R}^n$ and $S^+ = \mathbb{R}^n_+$. Note that, in this case, $h^{-1}(-S) = \{x \in X \mid h_i(x) \leq 0,\ \forall i \in I\}$ where $I := \{1, 2, \dots, n\}$. Let us denote by: $\sigma := \{h(x) \in -S, x \in C\}$ and $A := h^{-1}(-S) \cap C$ and suppose that A is non-empty. Note also that for each $\lambda = (\lambda_i)_{i \in I} \in S^+$, $\lambda h(x) = \sum_{i \in I} \lambda_i h_i(x)$. The following result is then a consequence of Proposition 3.6.

Corollary 3.9. *The following statements are equivalent*

(i') $\operatorname{cone} \bigcup_{i \in I} \operatorname{epi} h_i^* + \operatorname{epi} \delta_C^*$ *is weak***-closed.*

(ii') *for any* $a \in A$ *and for each* $\epsilon \geq 0$,

$$N_\epsilon(A, a) = \bigcup_{(\lambda_i) \in \mathbb{R}^n_+} \bigcup_{\substack{\sum_{i \in I} \epsilon_i + \beta = \epsilon + \sum_{i \in I} \lambda_i h_i(a) \\ (\epsilon_i) \in \mathbb{R}^n_+,\ \beta \geq 0}} \sum_{i \in I} \left\{\partial_{\epsilon_i}(\lambda_i h_i)(a) + N_\beta(C, a)\right\}.$$

In particular, if (i') holds then, for any $a \in A$, if $I(a) = \{i \in I \mid h_i(a) = 0\}$,

$$N_A(a) = \sum_{i \in I(a)} \operatorname{cone} \partial h_i(a) + N_C(a). \tag{12}$$

Proof. Since $\lambda_i h_i$ is continuous for each i, Lemma 2.1 gives

$$\begin{aligned}
\bigcup_{\lambda \in S^+} \operatorname{epi}(\lambda h)^* + \operatorname{epi} \delta_C^* &= \bigcup_{(\lambda_i) \in \mathbb{R}^n_+} \operatorname{epi} \left(\sum_{i \in I} \lambda_i h_i\right)^* + \operatorname{epi} \delta_C^* \\
&= \bigcup_{(\lambda_i) \in \mathbb{R}^n_+} \sum_{i \in I} \operatorname{epi}(\lambda_i h_i)^* + \operatorname{epi} \delta_C^* = \bigcup_{(\lambda_i) \in \mathbb{R}^n_+} \sum_{i \in I} \lambda_i \operatorname{epi} h_i^* + \operatorname{epi} \delta_C^* \\
&= \sum_{i \in I} \operatorname{cone}\ \operatorname{epi} h_i^* + \operatorname{epi} \delta_C^* = \operatorname{cone} \bigcup_{i \in I} \operatorname{epi} h_i^* + \operatorname{epi} \delta_C^*,
\end{aligned}$$

which means that σ is (FM).

By Proposition 3.6, (i') is equivalent to the fact that, for any $a \in A$, the following equalities hold for any $\epsilon \geq 0$:

$$\begin{aligned}
N_\epsilon(A, a) &= \bigcup_{(\lambda_i) \in \mathbb{R}^n_+} \bigcup_{\substack{\alpha + \beta = \epsilon + \lambda h(a) \\ \alpha, \beta \geq 0}} \{\partial_\alpha(\lambda h)(a) + N_\beta(C, a)\} \\
&= \bigcup_{(\lambda_i) \in \mathbb{R}^n_+} \bigcup_{\substack{\alpha + \beta = \epsilon + \sum_{i \in I} \lambda_i h_i(a) \\ \alpha \geq 0,\ \beta \geq 0}} \left\{\partial_\alpha \left(\sum_{i \in I} \lambda_i h_i\right)(a) + N_\beta(C, a)\right\}.
\end{aligned}$$

Since for each $i \in I$, $\lambda_i h_i$ is continuous, Lemma 2.1 leads to

$$\begin{aligned} N_\epsilon(A,a) &= \bigcup_{(\lambda_i)\in\mathbb{R}^n_+} \ \bigcup_{\substack{\alpha+\beta=\epsilon+\sum_{i\in I}\lambda_i h_i(a)\\ \alpha,\beta\geq 0}} \ \bigcup_{\substack{\sum_{i\in I}\epsilon_i=\alpha\\ (\epsilon_i)\in\mathbb{R}^n_+}} \left\{\sum_{i\in I}\partial_{\epsilon_i}(\lambda_i h_i)(a) + N_\beta(C,a)\right\} \\ &= \bigcup_{(\lambda_i)\in\mathbb{R}^n_+} \ \bigcup_{\substack{\sum_{i\in I}\epsilon_i+\beta=\epsilon+\sum_{i\in I}\lambda_i h_i(a)\\ (\epsilon_i)\in\mathbb{R}^n_+,\ \beta\geq 0}} \left\{\sum_{i\in I}\partial_{\epsilon_i}(\lambda_i h_i)(a) + N_\beta(C,a)\right\}, \end{aligned}$$

which shows the equivalence between (i') and (ii'). For the proof of (12), let us take $\epsilon = 0$ in the previous equalities. Then $\alpha = \beta = \epsilon_i = 0$ for all $i \in I$ and the last inequality collapses to

$$\begin{aligned} N_A(a) &= \bigcup_{\substack{\sum_{i\in I}\lambda_i h_i(a)=0\\ (\lambda_i)\in\mathbb{R}^n_+}} \left\{\sum_{i\in I}\partial(\lambda_i h_i)(a) + N_C(a)\right\} \\ &= \bigcup_{\substack{\sum_{i\in I}\lambda_i h_i(a)=0\\ (\lambda_i)\in\mathbb{R}^n_+}} \left\{\sum_{i\in I}\lambda_i\partial h_i(a) + N_C(a)\right\} = \sum_{i\in I(a)} \text{cone } \partial h_i(a) + N_C(a), \end{aligned}$$

and the proof is complete. □

It is worth mentioning that (12) is the standard formula in convex analysis that is often established under the Slater constraint qualification (see, e.g., [2]). Here, this formula is proved under the assumption that the cone $\text{cone}\bigcup_{i\in I}\text{epi}\, h_i^* + \text{epi}\,\delta_C^*$ is weak*-closed. It is shown in [26, 23] that this condition is strictly weaker than various generalized Slater type constraint qualifications. Thus, Corollary 3.9 improves known results, even for finite convex systems.

4. Generalized Farkas lemmas for systems involving convex and DC functions

As mentioned in the introduction, Farkas-type results have played a very important role in non-linear programming (see [16], [21], [29], [31] and references therein). In the recent years, several versions of Frakas-type results were proposed for general cone-convex systems (see e.g., [4], [12], [13]) or for systems defined by an arbitrary family of convex inequalities (see, e.g. [10], [11]) with applications to optimization problems.

In this section, we will introduce several versions of Farkas lemmas (in asymptotic forms, non-asymptotic forms, and in dual forms) for systems involving convex and DC functions. It will be shown in Section 6 that for convex systems, these results yield the corresponding ones that recapture and extend several known ones in the literature.

4.1. Generalized Farkas lemmas

Theorem 4.1 (Asymptotic Farkas lemma). *Assume that X is a reflexive Banach space*[1] *and that $\alpha \in \mathbb{R}$. If* (CC1) *is satisfied, then the following statements are equivalent:*

[1]In order to avoid the use of nets, we will restrict ourselves to reflexive Banach space.

(AF1) $h(x) \in -S,\ x \in C$ (*i.e.* $x \in A$) $\quad\Longrightarrow\quad f(x) - g(x) \geq \alpha,$
(AF2) $(0, -\alpha) + \operatorname{epi} g^* \subset \operatorname{epi} f^* + \operatorname{cl} K,$
(AF3) $(\forall x^* \in X^*)(\exists (\lambda_n)_n \subset S^+)(\forall x \in C)$

$$f(x) + \liminf_{n\to\infty} \lambda_n h(x) \geq (x^*, x) - g^*(x^*) + \alpha,$$

(AF4) $(\forall \epsilon > 0)$ $(\forall x \in C)$ $(\exists (\lambda_n)_n \subset S^+)$

$$f(x) + \liminf_{n\to\infty} \lambda_n h(x) - g(x) \geq \alpha - \epsilon.$$

Proof. (AF1) $\Rightarrow$ (AF2) Suppose that (AF1) holds. Then, for each $x \in A$, $f(x) \geq g(x)+\alpha$. It follows that $f + \delta_A \geq g + \alpha$, which implies that $(g+\alpha)^* \geq (f+\delta_A)^*$. In turn, this gives that

$$\operatorname{epi}(f + \delta_A)^* \supset \operatorname{epi}(g+\alpha)^* = \operatorname{epi} g^* + (0, -\alpha).$$

Since (CC1) holds, it follows from (1) and (4), that

$$\operatorname{epi}(f + \delta_A)^* = \operatorname{cl}(\operatorname{epi} f^* + \operatorname{epi} \delta_A^*) = \operatorname{cl}(\operatorname{epi} f^* + \operatorname{cl} K) = \operatorname{epi} f^* + \operatorname{cl} K.$$

Consequently, it comes that

$$\operatorname{epi} g^* + (0, -\alpha) \subset \operatorname{epi} f^* + \operatorname{cl} K,$$

which is (AF2).

(AF2) $\Rightarrow$ (AF3) Suppose (AF2) holds.
Then, on the one hand, for each $x^* \in \operatorname{dom} g^*$, there exist $(u, \beta) \in \operatorname{epi} f^*$, $(u_n), (v_n) \subset X^*$, $(\lambda_n) \subset S^+$ and $(\alpha_n), (\beta_n) \subset \mathbb{R}$, such that

$$\begin{aligned} &x^* = u + \lim_{n\to\infty}(u_n + v_n), \quad g^*(x^*) - \alpha = \beta + \lim_{n\to\infty}(\alpha_n + \beta_n), \quad f^*(u) \leq \beta, \\ &(\lambda_n h)^*(u_n) \leq \alpha_n, \quad \text{and} \quad \delta_C^*(v_n) \leq \beta_n. \end{aligned} \tag{13}$$

It follows that for each $n \in \mathbf{N}$ and any $x \in C$,

$$\alpha_n + \beta_n \geq (u_n, x) - \lambda_n h(x) + (v_n, x).$$

Thus, for each $x \in C$, it comes that

$$\begin{aligned} \lim_{n\to\infty}(\alpha_n + \beta_n) = \limsup_{n\to\infty}(\alpha_n + \beta_n) &\geq \limsup_{n\to\infty}[(u_n, x) - \lambda_n h(x) + (v_n, x)] \\ &\geq \lim_{n\to\infty}(u_n + v_n, x) + \limsup_{n\to\infty}[-\lambda_n h(x)] \\ &\geq \lim_{n\to\infty}(u_n + v_n, x) - \liminf_{n\to\infty}[\lambda_n h(x)]. \end{aligned} \tag{14}$$

Combining (13) and (14), we get that

$$\begin{aligned} g^*(x^*) - \alpha &= \beta + \limsup_{n\to\infty}(\alpha_n + \beta_n) \\ &\geq (u, x) - f(x) + \lim_{n\to\infty}(u_n + v_n, x) - \liminf_{n\to\infty}[\lambda_n h(x)] \\ &\geq (u, x) - f(x) + (x^* - u, x) - \liminf_{n\to\infty}[\lambda_n h(x)] \\ &\geq -f(x) + (x^*, x) - \liminf_{n\to\infty}[\lambda_n h(x)], \end{aligned}$$

which implies that $f(x) + \liminf_{n\to\infty} \lambda_n h(x) \geq (x^*, x) - g^*(x^*) + \alpha,\ \forall x \in C$.

If, on the other hand $x^* \notin \operatorname{dom} g^*$, then, taking $\lambda_n = 0$ for each n leads to (AF3).

(AF3) $\Rightarrow$ (AF1) For any $x^* \in \operatorname{dom} g^*$, (AF3) ensures that $(\lambda_n) \subset S^+$ exists such that

$$f(x) + \liminf_{n\to\infty} \lambda_n h(x) \geq (x^*, x) - g^*(x^*) + \alpha,\ \forall x \in C.$$

Thus, for any $x \in A$, it comes that $f(x) \geq f(x) + \liminf_{n\to\infty} \lambda_n h(x) \geq (x^*, x) - g^*(x^*) + \alpha$. Since this inequality holds for any $x^* \in \operatorname{dom} g^*$ and for all $x \in A$, one gets that

$$f(x) \geq f(x) + \liminf_{n\to\infty} \lambda_n h(x) \geq g^{**}(x) + \alpha = g(x) + \alpha. \tag{15}$$

Then, if on the one hand $x \in \operatorname{dom} g$, then $f(x) - g(x) \geq \alpha$ and (AF1) holds.
But, if on the other hand $x \notin \operatorname{dom} g$, then by (15), it comes that $x \notin \operatorname{dom} f$.
Therefore, $f(x) - g(x) = \infty - \infty = +\infty \geq \alpha$ and (AF1) follows.

(AF4) $\Rightarrow$ (AF1) Let $\epsilon > 0$ be arbitrary and let $x \in A = h^{-1}(-S) \cap C$. Thanks to (AF4), $(\lambda_n) \subset S^+$ exists such that

$$f(x) - g(x) \geq f(x) + \liminf_{n\to\infty} \lambda_n h(x) - g(x) \geq \alpha - \epsilon. \tag{16}$$

Since $h(x) \in -S$ and $\lambda_n \in S^+$ for each $n \in \mathbf{N}$, $\liminf_{n\to\infty} \lambda_n h(x) \leq 0$. It follows from (16) that

$$f(x) - g(x) \geq \alpha - \epsilon,\ \forall x \in A, \tag{17}$$

which shows that

$$\inf\{f(x) - g(x) \mid x \in A\} \geq \alpha,$$

since $\epsilon > 0$ is arbitrary, and (AF1) holds.

(AF1) $\Rightarrow$ (AF4) Assume (AF1). Then, by the previous proof, (AF3) holds.
Let us fix $x \in C$, $\epsilon > 0$ and note that if on the one hand $x \in \operatorname{dom} g$, then $\partial_\epsilon g(x) \neq \emptyset$. Thus, for any $x^* \in \partial_\epsilon g(x)$, $g(y) - g(x) \geq (x^*, y - x) - \epsilon$ for all $y \in \operatorname{dom} g$, which implies that

$$\epsilon + (x^*, x) - g(x) \geq (x^*, y) - g(y),\ \forall y \in \operatorname{dom} g.$$

In turn, this gives that

$$\epsilon + (x^*, x) - g(x) \geq g^*(x^*),$$

which shows that $x^* \in \operatorname{dom} g^*$ and that $(x^*, x) - g^*(x^*) \geq g(x) - \epsilon$.
Then, thanks to (AF3), $(\lambda_n) \subset S^+$ exists such that

$$f(x) + \liminf_{n\to\infty} \lambda_n h(x) \geq (x^*, x) - g^*(x^*) + \alpha \geq g(x) + \alpha - \epsilon.$$

Thus, for all $x \in C \cap \operatorname{dom} g$,

$$f(x) + \liminf_{n\to\infty} \lambda_n h(x) - g(x) \geq \alpha - \epsilon. \tag{18}$$

Suppose, on the other hand, that $x \notin \operatorname{dom} g$.
If on the one hand $x \in \operatorname{dom} f$, then by (AF1), $x \notin A$, which means that $h(x) \notin -S$ (as

$x \in C$). Thus, $\bar{\lambda} \in S^+$ exists such that $\bar{\lambda}h(x) > 0$. By taking $\lambda_n = n\bar{\lambda}$ for each n, it comes that $\liminf \bar{\lambda}_n h(x) = +\infty$. Thus, in this case, by our convention,

$$f(x) + \liminf_{n\to\infty} \lambda_n h(x) - g(x) = \infty - \infty = +\infty \geq \alpha - \epsilon.$$

If on the other hand $x \notin \operatorname{dom} f$, then $f(x) - g(x) = \infty - \infty = +\infty \geq \alpha - \epsilon$ too. Then, inequality (18) holds in this case for $\lambda_n = 0$ for all n and (AF4) has been proved. □

Theorem 4.2 (Non-asymptotic Farkas lemma). *Let $\alpha \in \mathbb{R}$. If (CC) is satisfied, then the following statements are equivalent:*

(F1) $h(x) \in -S,\ x \in C \implies f(x) - g(x) \geq \alpha,$
(F2) $(0, -\alpha) + \operatorname{epi} g^* \subset \operatorname{epi} f^* + K,$
(F3) $(\forall x^* \in X^*)(\exists \lambda \in S^+)(\forall x \in C) \quad f(x) + \lambda h(x) \geq (x^*, x) - g^*(x^*) + \alpha,$
(F4) $(\forall \epsilon > 0)\ (\forall x \in B)\ (\exists \lambda \in S^+) \quad f(x) + \lambda h(x) - g(x) \geq \alpha - \epsilon,$
where $B := (C \cap \operatorname{dom} g) \cup A.$

Proof. The proofs of the implications (F1) ⇒ (F2), (F2) ⇒ (F3), (F3) ⇒ (F1) and (F4) ⇒ (F1) are similar to the corresponding implications in Theorem 4.1, by using (CC) instead of (CC1). Let us prove now that (F1) ⇒ (F4).

Since (F1) is assumed, (F3) holds and, for any $x \in C \cap \operatorname{dom} g$, similarly to the proof of (AF1) ⇒ (AF4), (F3) ensures that $\lambda \in S^+$ exists such that

$$f(x) + \lambda h(x) - g(x) \geq \alpha - \epsilon. \tag{19}$$

Now, if $x \in (A \setminus \operatorname{dom} g)$, then $g(x) = +\infty$ and, by (F1), $f(x) - g(x) \geq \alpha$. This implies that $x \notin \operatorname{dom} f$, i.e., $f(x) = +\infty$. In this case, by taking $\lambda = 0$, (19) holds thanks to our convention $\infty - \infty = \infty$. Thus, (19) holds for all $x \in B$ and (F4) has been proved. □

Corollary 4.3. *Let $\alpha \in \mathbb{R}$. If (CC) is satisfied and if $\operatorname{dom} f \cap C \subset \operatorname{dom} g$ holds, then (F1), (F2), (F3), and (F4′) are equivalent, where*

(F4′) $(\forall \epsilon > 0)\ (\forall x \in C)\ (\exists \lambda \in S^+) \quad f(x) + \lambda h(x) - g(x) \geq \alpha - \epsilon.$

Proof. It is obvious that (F4′) ⇒ (F4). Only the implication (F1) ⇒ (F4′) needs to be proved. Following the proof of (F1) ⇒ (F4) in Theorem 4.2, if $x \in C \cap \operatorname{dom} g$ then (19) holds. If $x \in C$ and $x \notin \operatorname{dom} g$ then by assumption, $x \notin \operatorname{dom} f$. In this case, take $\lambda = 0$ and (19) holds. Thus (19) holds for all $x \in C$. □

Corollary 4.4. *Let $\alpha \in \mathbb{R}$.*

(α) *If either (i) or (ii) in Lemma 3.2 holds, then the conclusion of Theorem 4.1 holds.*
(β) *If σ is (FM) and if either (i) or (ii) in Lemma 3.2 holds then, the conclusion of Theorem 4.2 holds.*

Proof. If either (*i*) or (*ii*) in Lemma 3.2 holds then (CC1) holds. If moreover σ is (FM), then (CC) holds too, and the conclusion follows from Theorems 4.1 and 4.2. □

4.2. Generalized Farkas lemmas in dual forms

We are now in the position to present Farkas results in dual form.

Theorem 4.5. *Let $\alpha \in \mathbb{R}$. If* (CC) *is satisfied, then the following statements are equivalent:*

(F3) *for each $x^* \in X^*$, there exist $\lambda \in S^+$, such that*

$$f(x) + \lambda h(x) \geq (x^*, x) - g^*(x^*) + \alpha, \quad \forall x \in C,$$

(F5) *for each $x^* \in X^*$, there exist $\lambda \in S^+$, $u \in X^*$ such that*

$$g^*(x^*) - f^*(u) - (\lambda h + \delta_C)^*(x^* - u) \geq \alpha,$$

(F6) *for each $x^* \in X^*$, there exist $\lambda \in S^+$, $u \in X^*$ such that*

$$g^*(x^*) - (f + \delta_C)^*(u) - (\lambda h)^*(x^* - u) \geq \alpha,$$

(F7) *for each $x^* \in X^*$, there exist $\lambda \in S^+$, $u, v \in X^*$ such that*

$$g^*(x^*) - f^*(u) - (\lambda h)^*(v) - \delta_C^*(x^* - u - v) \geq \alpha.$$

Moreover, in (F5) [(F6) *and* (F7) *respectively*], *if $x^* \in \operatorname{dom} g^*$, it comes that $u \in \operatorname{dom} f^*$ and $x^* - u \in \operatorname{dom}(\lambda h + \delta_C)^*$ [$u \in \operatorname{dom}(f + \delta_C)^*$ and $x^* - u \in \operatorname{dom}(\lambda h)^*$ in* (F6)*; $u \in \operatorname{dom} f^*$, $v \in \operatorname{dom}(\lambda h)^*$ and $x^* - u - v \in \operatorname{dom} \delta_C^*$ in* (F7) *respectively*].

Proof. (F3) $\Longrightarrow$ (F7) By assuming that (F3) holds, it follows from Theorem 4.2 that (F2) holds. Then, for any $x^* \in X^*$:
If on the one hand $x^* \in \operatorname{dom} g^*$, then it follows from (F2) that

$$(0, -\alpha) + (x^*, g^*(x^*)) \in \operatorname{epi} f^* + K.$$

Thus, there exist $\lambda \in S^+$, $(u, r) \in \operatorname{epi} f^*$, $(v, s) \in \operatorname{epi}(\lambda h)^*$ and $(w, t) \in \operatorname{epi} \delta_C^*$, such that

$$(0, -\alpha) + (x^*, g^*(x^*)) = (u, r) + (v, s) + (w, t),$$

and the last equality implies that $x^* = u + v + w$ and $-\alpha + g^*(x^*) = r + s + t$.
Thus, it comes that $f^*(u) + (\lambda h)^*(v) + \delta_C^*(x^* - u - v) \leq -\alpha + g^*(x^*)$.

Since $u \in \operatorname{dom} f^*$, $v \in \operatorname{dom}(\lambda h)^*$ and $w = x^* - u - v \in \operatorname{dom} \delta_C^*$, it follows from the last inequality that

$$g^*(x^*) - f^*(u) - (\lambda h)^*(v) - \delta_C^*(x^* - u - v) \geq \alpha.$$

If on the other hand $x^* \notin \operatorname{dom} g^*$, one just has to take $\lambda = 0$, $v = 0$ and $u \in \operatorname{dom} f^*$.
Then, the last inequality holds and (F7) has been proved.

(F7) $\Longrightarrow$ (F6) Assume that (F7) holds and consider any $x^* \in X^*$. Thanks to (F7), $\lambda \in S^+$ and $u, v \in X^*$ exist such that

$$g^*(x^*) - f^*(u) - (\lambda h)^*(v) - \delta_C^*(x^* - u - v) \geq \alpha.$$

If on the one hand $x^* \in \operatorname{dom} g^*$, then $u \in \operatorname{dom} f^*$, $v \in \operatorname{dom}(\lambda h)^*$ and $w = x^* - u - v \in \operatorname{dom} \delta_C^*$. So, for each $x \in X$, one has that

$$g^*(x^*) - (u, x) + f(x) - (\lambda h)^*(v) - (x^* - u - v, x) + \delta_C(x) \geq \alpha.$$

Therefore, for each $x \in X$, it comes that

$$g^*(x^*) - [(x^* - v, x) - (f + \delta_C)(x)] - (\lambda h)^*(v) \geq \alpha,$$

which implies that

$$g^*(x^*) - (f + \delta_C)^*(x^* - v) - (\lambda h)^*(v) \geq \alpha.$$

If on the other hand $x^* \notin \operatorname{dom} g^*$, by taking $\lambda = 0 \in S^+$ and $u = x^*$, (F6) holds.

(F7) $\Longrightarrow$ (F5) is proved by the same argument.

(F7) $\Longrightarrow$ (F3) Assume that (F7) holds. For any $x^* \in X^*$, there exist $\lambda \in S^+$ and $u, v \in X^*$ such that

$$g^*(x^*) - (u, x) + f(x) - (v, x) + \lambda h(x) - (x^* - u - v, x) \geq \alpha, \ \forall x \in C.$$

If on the one hand $x^* \in \operatorname{dom} g^*$, then it follows from the last inequality that

$$f(x) + \lambda h(x) \geq (x^*, x) - g^*(x^*) + \alpha, \ \forall x \in C.$$

Since this inequality holds if on the other hand $x^* \notin \operatorname{dom} g^*$, (F3) has been proved.

By the same argument, (F5) $\Longrightarrow$ (F3) and (F6) $\Longrightarrow$ (F3) can be proved. □

Let us summarize these results in the following theorem.

Theorem 4.6 (Non-asymptotic Farkas lemma). *Let $\alpha \in \mathbb{R}$. If* (CC) *is satisfied, then all the statements* (F1) *to* (F7) *are equivalent together.*

In order to conclude this section, it is worth observing that if $B := \{x \in X \mid f(x) - g(x) \geq \alpha\}$ then it is a DC set (see [34]), and when $g \equiv 0$, B becomes a reverse convex set. Several characterizations of the set containment of a convex set A in reverse convex set B were given in [11], [15] , [22]. The statement (AF1) (and also (F1)) means that the convex set A is included in the DC set B. The Farkas-type results (asymptotic and non-asymptotic) given in this section, may result various characterizations of the mentioned set containments, which possibly extend or go back to some results already appeared in the literature. This will be considered elsewhere.

5. Duality for DC programs with convex constraints

Let us consider the DC optimization problem with convex constraints:

$$\begin{array}{lrl} \text{(P)} & \inf & [f(x) - g(x)] \\ & \text{subject to} & x \in C, \ h(x) \in -S. \end{array}$$

Here, all the spaces X, Z, the set C, the cone S and the mappings f, g, h are as in Section 2. For Problem (P), its feasible set $A := \{x \in X \mid x \in C, h(x) \in -S\}$ is assumed non-empty.

DC problems (P) are of great interest and have been studied by many authors since the last decades (see [14], [18], [19], [25], [28], [33], [34] and references therein). Many real world problems lead to such mathematical models and various numerical methods have been developed for this class of problems as well (see [1], [19] and [34] for an overview).

At the same time, duality for DC program (P) (also for vector DC problems, convex problems with DC constraints) has also been studied since 1978, when the first result of this type appeared in [33] (see [6], [30], [35]). Duality results for DC problems have been used successfully to build numerical methods for this class of problems (see, for instance [1], [32] and references therein).

In this section, we consider a dual problem for the problem (P) called "Toland-Fenchel-Lagrange" dual problem (see [13]). This type of dual problem was considered in [28]. It is, in some sense, a "combination" of the Toland duality for DC problem (see [33]), the Fenchel-Lagrange and the Lagrange duality for convex problems (see [4, 5, 23]). Several duality results of this type are obtained for Problem (P) that cover and extend some known ones in [28]. They yield corresponding results for convex problems that go back or extend Fenchel-Lagrange duality or Lagrange duality in [4, 5, 12, 23].

In the rest of the paper, we will also be interested in the following convex program, which is a special case of (P) when $g \equiv 0$:

$$\begin{array}{lrl}(\mathrm{Q}) & \inf & f(x) \\ & \text{subject to} & x \in C,\ h(x) \in -S.\end{array}$$

5.1. On the value of the Problem (P)

In this subsection, we will be concerned by the value of the DC program (P). In order to, let us first give a technical lemma.

Lemma 5.1. *Assume that* (CC) *holds and consider the following assertions:*

(F1) $h(x) \in -S,\ x \in C \implies f(x) - g(x) \geq \alpha,$

(F8) *for any* $x^* \in \operatorname{dom} g^*$*, there exists* $\lambda \in S^+$ *such that for any* $\epsilon > 0$*, there exists* $y \in X$*,*

$$f(x) + \lambda h(x) - g(y) + (x^*, y - x) \geq \alpha - \epsilon, \quad \forall x \in C,$$

(F9) *there exists* $\lambda \in S^+$ *such that for any* $\epsilon > 0$*, there exists* $y \in X$ *satisfying*

$$f(x) + \lambda h(x) - g(y) \geq \alpha - \epsilon, \quad \forall x \in C.$$

If $\alpha := \inf(\mathrm{P}) \in \mathbb{R}$*, then* (F1) $\implies$ (F8).
If moreover g *is bounded from below,* (F8) $\implies$ (F9).

Proof. (F1) $\implies$ (F8) Since (CC) holds, it follows from Theorem 4.2-(F3), that for any $x^* \in \operatorname{dom} g^*$, $\lambda \in S^+$ exists such that

$$f(x) + \lambda h(x) \geq (x^*, x) - g^*(x^*) + \alpha,\ \forall x \in C. \tag{20}$$

By definition of g^*, for any $\epsilon > 0$, there exists $y \in \operatorname{dom} g$ satisfying $(x^*, y) - g(y) > g^*(x^*) - \epsilon$. This and (20) imply that for any $\epsilon > 0$, there exists $y \in X$ such that

$$f(x) + \lambda h(x) \geq (x^*, x) - (x^*, y) + g(y) + \alpha - \epsilon, \tag{21}$$

which implies that

$$f(x) + \lambda h(x) - g(y) + (x^*, y - x) \geq \alpha - \epsilon,$$

which is (F8).

(F8) $\implies$ (F9) Suppose that (F8) holds. Since g is bounded from below, $0 \in \operatorname{dom} g^*$ and $x^* = 0$ in (F8) leads to (F9). $\square$

Proposition 5.2. *Assume that* (CC) *holds and that* $\inf(\mathrm{P}) \in \mathbb{R}$*, then*

$$\inf(\mathrm{P}) = \inf_{x\in A} \sup_{\lambda\in S^+} [f(x) + \lambda h(x) - g(x)].$$

If moreover $\operatorname{dom} f \cap C \subset \operatorname{dom} g$*, then* $\inf(\mathrm{P}) = \inf_{x\in C} \sup_{\lambda\in S^+}[f(x) + \lambda h(x) - g(x)]$.

Proof. Let $\alpha = \inf(\mathrm{P}) \in \mathbb{R}$. Then, for any $x \in A$, one gets that $f(x) - g(x) \geq \alpha$. Since (CC) holds, it follows from Theorem 4.2-(F4) that for each $\epsilon > 0$ and each $x \in A$, $\lambda \in S^+$ exists satisfying

$$f(x) + \lambda h(x) - g(x) \geq \alpha - \epsilon. \tag{22}$$

This implies that for any $\epsilon > 0$, $\inf_{x\in A} \sup_{\lambda\in S^+}[f(x) + \lambda h(x) - g(x)] \geq \alpha - \epsilon$, which gives rise to (as $\epsilon > 0$ is arbitrary)

$$\inf_{x\in A} \sup_{\lambda\in S^+} [f(x) + \lambda h(x) - g(x)] \geq \alpha. \tag{23}$$

Note that if $\operatorname{dom} f \cap C \subset \operatorname{dom} g$ is assumed, then Corollary 4.3-(F4′) leads to:

$$\inf_{x\in C} \sup_{\lambda\in S^+} [f(x) + \lambda h(x) - g(x)] \geq \alpha. \tag{24}$$

On the other hand, for a fixed $x \in A$ and for any $\lambda \in S^+$, one has

$$f(x) - g(x) \geq f(x) + \lambda h(x) - g(x),$$

which gives $f(x) - g(x) \geq \sup_{\lambda\in S^+}[f(x) + \lambda h(x) - g(x)]$.
In turn, this implies that

$$\begin{aligned} \inf(\mathrm{P}) &= \inf_{x\in A}[f(x) - g(x)] \geq \inf_{x\in A} \sup_{\lambda\in S^+} [f(x) + \lambda h(x) - g(x)] \\ &\geq \inf_{x\in C} \sup_{\lambda\in S^+} [f(x) + \lambda h(x) - g(x)]. \end{aligned} \tag{25}$$

Then, the conclusion follows from (23), (24) and (25). □

Proposition 5.3. *Assume that* (CC) *holds, that* $\inf(\mathrm{P}) \in \mathbb{R}$ *and that* g *is bounded from below, then*

$$\inf_{x\in C} \sup_{\lambda\in S^+} [f(x) + \lambda h(x) - g(x)] = \inf(\mathrm{P}) \leq \sup_{\lambda\in S^+} \sup_{y\in X} \inf_{x\in C}[f(x) + \lambda h(x) - g(y)]. \tag{26}$$

Proof. The equality holds by Proposition 5.2. For the inequality, observe that if $\alpha := \inf(\mathrm{P}) \in \mathbb{R}$ then $x \in X, h(x) \in -S \implies f(x) - g(x) \geq \alpha$.

Since (CC) holds, by Lemma 5.1, $\lambda \in S^+$ exists such that for any $\epsilon > 0$, there exists $y \in X$ satisfying

$$f(x) + \lambda h(x) - g(y) \geq \alpha - \epsilon, \ \forall x \in C,$$

which implies that

$$\inf_{x\in C}[f(x) + \lambda h(x) - g(y)] \geq \alpha - \epsilon.$$

In turn, this gives rise to

$$\sup_{\lambda \in S^+} \sup_{y \in X} \inf_{x \in C} [f(x) + \lambda h(x) - g(y)] \geq \alpha - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have that

$$\alpha = \inf(\mathrm{P}) \leq \sup_{\lambda \in S^+} \sup_{y \in X} \inf_{x \in C} [f(x) + \lambda h(x) - g(y)],$$

which is desired. □

By the same argument, we also get that

$$\inf_{x \in C} \sup_{\lambda \in S^+} [f(x) + \lambda h(x) - g(x)] = \inf(\mathrm{P}) \leq \sup_{\lambda \in S^+} \inf_{x \in C} \sup_{y \in X} [f(x) + \lambda h(x) - g(y)]. \tag{27}$$

It is worth observing that, in general, the inequalities in (26) and (27) cannot be replaced by equalities. Let us consider this simple case where $f(x) = 0$, $g(x) = |x|$, $h(x) = x$, $x \in \mathbb{R}$, $S = \mathbb{R}_+$, and $C = [-1, 2]$. Then, (CC) holds obviously, but

$$\sup_{\lambda \in S^+} \sup_{y \in X} \inf_{x \in C} [f(x) + \lambda h(x) - g(y)] = \sup_{\lambda \in S^+} \inf_{x \in C} \sup_{y \in X} [f(x) + \lambda h(x) - g(y)] = 0 > -1 = \inf(\mathrm{P}).$$

5.2. Toland-Fenchel-Lagrange duality theorems for (P) and their relations to Farkas lemmas

Theorem 5.4. *If* $\inf(\mathrm{P}) \in \mathbb{R}$ *and* (F7) *holds for* $\alpha = \inf(\mathrm{P})$*, then*

$$\inf(\mathrm{P}) = \inf_{x^* \in X^*} \left\{ \sup_{\lambda \in S^+} \sup_{u,v \in X^*} \{ g^*(x^*) - f^*(u) - (\lambda h)^*(v) - \delta_C^*(x^* - u - v) \} \right\}. \tag{28}$$

Proof. Assume that (F7) holds for $\alpha = \inf(\mathrm{P})$. Then, for each $x^* \in X^*$, there exist $\lambda \in S^+$ and $u, v \in X^*$ such that

$$\alpha \leq g^*(x^*) - f^*(u) - (\lambda h)^*(v) - \delta_C^*(x^* - u - v),$$

which gives rise to

$$\inf(\mathrm{P}) = \alpha \leq \inf_{x^* \in X^*} \left\{ \sup_{\lambda \in S^+} \sup_{u,v \in X^*} \{ g^*(x^*) - f^*(u) - (\lambda h)^*(v) - \delta_C^*(x^* - u - v) \} \right\}. \tag{29}$$

On the other hand, if $x \in A \cap \operatorname{dom} f$, $\lambda \in S^+$ and $u, v \in X^*$, then

$$\begin{aligned} & g^*(x^*) - f^*(u) - (\lambda h)^*(v) - \delta_C^*(x^* - u - v) \\ \leq\ & g^*(x^*) - (u, x) + f(x) - (v, x) + \lambda h(x) - (x^* - u - v, x) \\ \leq\ & g^*(x^*) - (x^*, x) + f(x). \end{aligned}$$

Thus, for each $x \in A \cap \operatorname{dom} f$, it comes that

$$\begin{aligned} & \inf_{x^* \in X^*} \left\{ \sup_{\lambda \in S^+} \sup_{u,v \in X^*} \{ g^*(x^*) - f^*(u) - (\lambda h)^*(v) - \delta_C^*(x^* - u - v) \} \right\} \\ \leq\ & \inf_{x^* \in X^*} \{ g^*(x^*) - (x^*, x) + f(x) \} \leq f(x) - g^{**}(x) = f(x) - g(x). \end{aligned} \tag{30}$$

If $x \in A$ but $x \notin \operatorname{dom} f$, then (30) holds even if $x \in \operatorname{dom} g$ or not (by our convention). Consequently, (30) holds for all $x \in A$, which yields

$$\inf_{x^* \in X^*} \left\{ \sup_{\lambda \in S^+} \sup_{u,v \in X^*} \{g^*(x^*) - f^*(u) - (\lambda h)^*(v) - \delta_C^*(x^* - u - v)\} \right\} \leq \inf(\mathrm{P}) = \alpha,$$

which, together with (29), gives the desired equality. □

Theorem 5.5. *Assume that the condition* (CC) *holds and that* $\inf(\mathrm{P}) \in \mathbb{R}$*, then* (F7) *holds for* $\alpha = \inf(\mathrm{P})$ *if and only if*

$$\inf(\mathrm{P}) = \inf_{x^* \in X^*} \left\{ \max_{\lambda \in S^+} \max_{u,v \in X^*} \{g^*(x^*) - f^*(u) - (\lambda h)^*(v) - \delta_C^*(x^* - u - v)\} \right\}. \tag{31}$$

Proof. If the condition (CC) holds, then by Proposition 3.3,

$$\forall x^* \in X^*, \quad \inf_{\lambda \in S^+} \inf_{u,v \in X^*} \{f^*(u) + (\lambda h)^*(v) + \delta_C^*(x^* - u - v)\}$$

is attained at some $\lambda \in S^+$ and some $u, v \in X^*$. Thus, (29) can be rewritten as

$$\inf(\mathrm{P}) = \inf_{x^* \in X^*} \left\{ \max_{\lambda \in S^+} \max_{u,v \in X^*} \{g^*(x^*) - f^*(u) - (\lambda h)^*(v) - \delta_C^*(x^* - u - v)\} \right\},$$

which shows that (F7) holds for $\alpha = \inf(\mathrm{P})$. □

As a consequence of Theorem 5.5, we get

Corollary 5.6 ([13], Toland-Fenchel-Lagrange duality). *Assume that* (CC) *holds. Then*

$$\text{(TFL1)} \quad \inf(\mathrm{P}) = \inf_{x^* \in X^*} \left\{ \max_{\lambda \in S^+} \max_{u,v \in X^*} \{g^*(x^*) - f^*(u) - (\lambda h)^*(v) - \delta_C^*(x^* - u - v)\} \right\}.$$

Proof. If $\alpha = \inf(\mathrm{P})$ is finite, then for any $x \in A$, $f(x) - g(x) \geq \alpha$. Since (CC) holds, by Theorem 4.6, this is equivalent to (F7) and the conclusion follows from Theorem 5.5.

If $\inf(\mathrm{P}) = -\infty$, then note that for each $\lambda \in S^+$, $x^*, u, v \in X^*$ and $x \in A$, we get

$$f^*(u) + (\lambda h)^*(v) + \delta_C^*(x^* - u - v) \geq (x^*, x) - f(x) - \lambda h(x) - \delta_C(x) \geq (x^*, x) - f(x). \tag{32}$$

Since (CC) holds, by Proposition 3.3, for each $x^* \in X^*$, the expression

$$\max_{\lambda \in S^+} \max_{u,v \in X^*} \{g^*(x^*) - f^*(u) - (\lambda h)^*(v) - \delta_C^*(x^* - u - v)\} \tag{33}$$

is attained at some $\lambda \in S^+$ and $u, v \in X^*$. It is also worth noticing that if $x^* \in \operatorname{dom}(f + \delta_A)^*$ then u, v can be chosen such that $u \in \operatorname{dom} f^*$, $v \in \operatorname{dom}(\lambda h)^*$ and $x^* - u - v \in \operatorname{dom} \delta_C^*$ (see the proof of Theorem 3.1 in [13] and that of Proposition 3.4).

If $x^* \notin \operatorname{dom}(f + \delta_A)^*$, we consider two possibilities:

(i) If $F^* := [X^* \setminus \mathrm{dom}(f+\delta_A)^*] \cap \mathrm{dom} g^* \neq \emptyset$, then for each $x^* \in F^*$, the expression in (33) is equal to $-\infty$. Therefore, it comes that

$$\inf_{x^* \in X^*} \left\{ \max_{\lambda \in S^+} \max_{u,v \in X^*} \{g^*(x^*) - f^*(u) - (\lambda h)^*(v) - \delta_C^*(x^* - u - v)\} \right\} = -\infty = \inf(\mathrm{P}).$$

(ii) If $F^* = \emptyset$ (i.e., $\mathrm{dom}\, g^* \subset \mathrm{dom}(f+\delta_A)^*$) then for each $x^* \notin \mathrm{dom}(f+\delta_A)^*$, the expression in (33) is equal to $+\infty$ by our convention. Therefore, taking (32) into account,

$$\begin{aligned}
&\inf_{x^* \in X^*} \left\{ \max_{\lambda \in S^+} \max_{u,v \in X^*} \{g^*(x^*) - f^*(u) - (\lambda h)^*(v) - \delta_C^*(x^* - u - v)\} \right\} \\
= &\inf_{x^* \in \mathrm{dom}(f+\delta_A)^*} \left\{ \max_{\lambda \in S^+} \max_{u,v \in X^*} \{g^*(x^*) - f^*(u) - (\lambda h)^*(v) - \delta_C^*(x^* - u - v)\} \right\} \\
\leq &\inf_{x^* \in \mathrm{dom}(f+\delta_A)^*} \{g^*(x^*) - (x^*, x) + f(x)\} \leq \inf_{x^* \in \mathrm{dom}\, g^*} \{g^*(x^*) - (x^*, x) + f(x)\} \\
= & f(x) - g^{**}(x) = f(x) - g(x), \ \forall x \in A.
\end{aligned}$$

The last equality follows from the semi-continuity of g. This implies that

$$\inf_{x^* \in X^*} \left\{ \max_{\lambda \in S^+} \max_{u,v \in X^*} \{g^*(x^*) - f^*(u) - (\lambda h)^*(v) - \delta_C^*(x^* - u - v)\} \right\} = \inf(\mathrm{P}) = -\infty.$$

The proof is complete. □

The Toland-Fenchel-Lagrange duality in Corollary 5.6 was established in [13]. It was also proved in [28] under either one of the following conditions:

(α) f is continuous at one point in C and there exists $\bar{x} \in C$ such that $h(\bar{x}) \in -\mathrm{int} S$ (Slater constraint qualification), or

(β) f is continuous at one point in C and $\mathbb{R}_+[S + h(\mathrm{dom}\, f \cap C)]$ is a closed subspace.

It was proved in [13] that condition (CC) is strictly weaker than either (α) or (β).

The same line of arguments as in Corollary 5.6 leads to various versions of the Toland-Fenchel-Lagrange duality and their proofs are similar to that of Theorem 5.5 (see also Theorem 4.6) and will be omitted.

Corollary 5.7 ([13], Toland-Fenchel-Lagrange duality). *Assume that* (CC) *holds. Then*

$$\text{(TFL2)} \quad \inf(\mathrm{P}) = \inf_{x^* \in X^*} \max_{\lambda \in S^+} \{g^*(x^*) - (f + \lambda h + \delta_C)^*(x^*)\}, \tag{34}$$

$$\text{(TFL3)} \quad \inf(\mathrm{P}) = \inf_{x^* \in X^*} \left\{ \max_{\lambda \in S^+} \max_{u \in X^*} \{g^*(x^*) - f^*(u) - (\lambda h + \delta_C)^*(x^* - u)\} \right\}, \tag{35}$$

$$\text{(TFL4)} \quad \inf(\mathrm{P}) = \inf_{x^* \in X^*} \left\{ \max_{\lambda \in S^+} \max_{u \in X^*} \{g^*(x^*) - (f + \delta_C)^*(u) - (\lambda h)^*(x^* - u)\} \right\}. \tag{36}$$

It is also worth noticing that the Toland-Fenchel-Lagrange duality result (TFL1) can be proved by using the Toland duality theorem (see [33]) and Propositions 3.3, as it was done in [13], where Farkas lemmas of dual forms were then derived. Here, all of these results are developed in the converse way. This allows us to point out an important fact: these results are actually equivalent together as it is shown in the following theorem.

Theorem 5.8. *Assume that* (CC) *holds and that* $\alpha := \inf(\mathrm{P}) \in \mathbb{R}$, *then, all the statements* (F1) *to* (F7), (TFL1) *to* (TFL4) *are all equivalent to each other.*

Proof. An argument similar to the proof of Corollary 5.6 shows that each statements from (F1) to (F7) are equivalent together when $\alpha = \inf(\mathrm{P})$. The rest follows from Corollary 5.7 and Theorem 5.5. □

Example 5.1 Consider the following problem (E):

$$\begin{array}{lrl} (\mathrm{E}) & \inf & (x_1^2 - 2x_2^2 - 4x_1x_2) \\ & \text{subject to} & x = (x_1, x_2) \in \mathbb{R}^2, x_1^2 + x_2^2 \leq 1. \end{array}$$

Let $f(x) = 2x_1^2 + 2x_2^2$, $g(x) = (x_1 + 2x_2)^2$, $X = \mathbb{R}^2$, $C = X$, $Z = \mathbb{R}$, $S = S^+ = \mathbb{R}_+$ and $h(x) = x_1^2 + x_2^2 - 1$. It is clear that f, g are convex, continuous functions on $\mathbb{R}^2$ while h is S-convex and continuous. Then, (E) has the form of (P).

For each $a = (a_1, a_2) \in \mathbb{R}^2$, one has that

$$g^*(a) = \sup_{x \in X} \left\{a_1x_1 + a_2x_2 - (x_1 + 2x_2)^2\right\} = \begin{cases} \frac{a_1^2}{4} & \text{if } 2a_1 = a_2, \\ +\infty & \text{otherwise.} \end{cases}$$

For each $\lambda \geq 0$ and $a = (a_1, a_2) \in \mathbb{R}^2$, it comes that

$$\begin{aligned} (f + \lambda h + \delta_C)^*(a) &= \sup_{x \in X} \left\{a_1x_1 + a_2x_2 - (2 + \lambda)(x_1^2 + x_2^2) + \lambda\right\} = \frac{a_1^2 + a_2^2}{4(\lambda + 2)} + \lambda + 2 - 2 \\ &\geq \sqrt{a_1^2 + a_2^2} - 2 \end{aligned}$$

(the last inequality follows from the inequality of Cauchy).

For each $\lambda \geq 0$ and $(u_1, u_2) \in \mathbb{R}^2$,

$$\begin{aligned} (\lambda h)^*(u) &= \sup_{x \in X} \left\{u_1x_1 + u_2x_2 - \lambda(x_1^2 + x_2^2) + \lambda\right\} \\ &= \begin{cases} \frac{u_1^2 + u_2^2}{4\lambda} + \lambda & \text{if } \lambda > 0, \\ 0 & \text{if } \lambda = 0 \text{ and } u = (0, 0), \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

Note that $\frac{u_1^2 + u_2^2}{4\lambda} + \lambda \geq \sqrt{u_1^2 + u_2^2}$ for all $\lambda > 0$, with equality if $\lambda = \frac{\sqrt{u_1^2 + u_2^2}}{2} > 0$, then

$$\begin{aligned} K &= \cup_{\lambda \in S^+} \operatorname{epi}(\lambda h)^* = \{(u_1, u_2, \frac{u_1^2 + u_2^2}{4\lambda} + \lambda + r) | \lambda > 0, r \geq 0\} \cup \{(0.0)\} \times \mathbb{R}_+ \\ &= \{(u_1, u_2, \sqrt{u_1^2 + u_2^2} + r) | r \geq 0\},. \end{aligned}$$

which is a closed subset of $\mathbb{R}^3$. Since f is continuous on $\mathbb{R}^2$, (CC) holds. Moreover $A := h^{-1}(-S) \cap C$ is compact and $f - g$ is continuous on $\mathbb{R}^2$, then $\inf(\mathrm{E})$ is finite. By the Corollary 5.7-(TFL2),

$$\begin{aligned}
\inf(\mathrm{E}) &= \inf_{a\in\mathbb{R}} \max_{\lambda\in S^+} \{g^*(a,2a) - (f+\lambda+\delta_C)^*(a,2a)\} \\
&= \inf_{a\in\mathbb{R}} \max_{\lambda\in S^+} \left(\frac{a^2}{4} - \left\{ \frac{a^2+4a^2}{4(\lambda+2)} + \lambda + 2 - 2 \right\} \right) \\
&= \inf_{a\in\mathbb{R}} \left\{ \frac{a^2}{4} - \sqrt{5}|a| + 2 \right\} = -3.
\end{aligned}$$

6. Generalized Farkas Lemmas and duality in convex setting

In this section, we will apply the results obtained in Sections 3 and 4, to derive various versions of Farkas lemma for convex systems and duality results for convex programs. These results recapture several ones established recently in [4], [5], [10], [11], [12], [13]. It is also shown that in fact, these versions of Farkas lemma and duality results are equivalent together. Moreover, stable duality results are also established for convex programs.

6.1. Generalized Farkas lemma for convex systems

Let us give some asymptotic generalized Farkas lemmas for convex systems.

Corollary 6.1 (Asymptotic Farkas lemma). *Let $\alpha \in \mathbb{R}$ and let X be a reflexive Banach space*[2]*. If* (CC1) *holds, then the following statements are equivalent:*

(ACF1) $h(x) \in -S,\ x \in C \implies f(x) \geq \alpha,$
(ACF2) $(0, -\alpha) \in \operatorname{epi} f^* + \operatorname{cl}(\cup_{\lambda\in S^+} \operatorname{epi}(\lambda h)^* + \operatorname{epi} \delta_C^*),$
(ACF3) $(\exists(\lambda_n)_n \subset S^+)(\forall x \in C)\ f(x) + \liminf_{n\to\infty} \lambda_n h(x) \geq \alpha.$

Proof. Note that if $g \equiv 0$ then $\operatorname{dom} g^* = \{0\}$, $g^*(0) = 0$ and so $\operatorname{epi} g^* = \{0\} \times \mathbb{R}_+$ with $(0, -\alpha) + \operatorname{epi} g^* = \{0\} \times [-\alpha, +\infty)$. The conclusions of the corollary now follow from Theorem 4.1. □

The following non-asymptotic version of Farkas lemma follows from Theorem 4.6.

Corollary 6.2 (Non-asymptotic Farkas lemma). *Let $\alpha \in \mathbb{R}$. If* (CC) *is satisfied, then the following statements are equivalent:*

(CF1) $h(x) \in -S,\ x \in C \implies f(x) \geq \alpha,$
(CF2) $(0, -\alpha) \in \operatorname{epi} f^* + \cup_{\lambda\in S^+} \operatorname{epi}(\lambda h)^* + \operatorname{epi} \delta_C^*,$
(CF3) *there exists $\lambda \in S^+$ such that $f(x) + \lambda h(x) \geq \alpha$, $\forall x \in C$,*
(CF4) *there exist $\lambda \in S^+$, $u \in X^*$ such that*

$$-f^*(u) - (\lambda h + \delta_C)^*(-u) \geq \alpha,$$

(CF5) *there exist $\lambda \in S^+$, $u \in X^*$ such that*

$$-(f + \delta_C)^*(u) - (\lambda h)^*(-u) \geq \alpha,$$

[2]In order to avoid the use of nets, we will restrict ourselves to reflexive Banach space.

(CF6) *there exist* $\lambda \in S^+,\ u, v \in X^*$ *such that*

$$-f^*(u) - (\lambda h)^*(v) - \delta_C^*(-u-v) \geq \alpha.$$

Proof. The equivalence between (CF1), (CF2) and (CF3) follows from Theorem 4.2 by taking $g \equiv 0$ (see also the proof of Corollary 6.1).

Note that when $g \equiv 0$, (F3) collapses to (CF3). The equivalence between (CF3), (CF4), (CF5) and (CF6) follows easily from Theorem 4.5. □

The asymptotic Farkas lemma given in Corollary 6.1 and the equivalence between (CF1), (CF2) and (CF3) recapture the ones established in [12] where $C = X$, Z is a Banach space and f is a continuous function. In [12], the asymptotic result was used as a main tool to establish the (sequentially) Lagrangian minimax Theorems and the perfect duality for the convex Problem (Q). Note also that the equivalence between (ACF1) and (ACF3) was established in [16] with $\alpha = 0$. However, this yields a non-asymptotic equivalence [(CF1) and (CF3)] under a closedness condition, which is stronger than (CC) (see [12]). The inequality (CF4) is the Farkas lemma established in [4] where $X = \mathbb{R}^n$, $h = (g_1, g_2, \cdots, g_m)$ with $g_i : \mathbb{R}^n \to \mathbb{R}$ are convex functions, and under Slater constraint qualification.

6.2. Duality for convex programs

We shall now apply the results of Section 5 to the convex program (Q) to obtain various corresponding duality results that bring back the Fenchel-Lagrange duality or the Lagrange duality results proposed in [4], [5], [12], and [23]. Stable duality results for convex programs are also established.

Fenchel-Lagrange duality.

Corollary 6.3. *Assume that* $\inf(\mathrm{Q}) \in \mathbb{R}$.

(*i*) *If* (CF6) *holds for* $\alpha = \inf(\mathrm{Q})$, *then*

$$\inf(\mathrm{Q}) = \sup_{\lambda \in S^+} \sup_{u,v \in X^*} \{-f^*(u) - (\lambda h)^*(v) - \delta_C^*(-u-v)\}.$$

(*ii*) *If* (CC) *holds, then* (CF6) *holds for* $\alpha = \inf(\mathrm{Q})$ *if and only if*

$$\inf(\mathrm{Q}) = \max_{\lambda \in S^+} \max_{u,v \in X^*} \{-f^*(u) - (\lambda h)^*(v) - \delta_C^*(-u-v)\}.$$

Proof. We first prove (*i*). Take $g \equiv 0$ and $x^* = 0$. Then (F7) collapses to (CF6) and $\inf(\mathrm{P}) = \inf(\mathrm{Q}) = \alpha$. Since $\operatorname{dom} g^* = \{0\}$, the infimum with respect to $x^* \in X^*$ in the right-hand side of (28) attains at $x^* = 0$ and the assertion (*i*) follows from Theorem 5.4. By the same argument, (*ii*) follows from Theorem 5.5. □

Corollary 6.4 ([13], Fenchel-Lagrange duality). *Assume that* (CC) *holds. Then*

(FL1) $$\inf(\mathrm{Q}) = \max_{\lambda \in S^+} \max_{u,v \in X^*} \{-f^*(u) - (\lambda h)^*(v) - \delta_C^*(-u-v)\}, \quad (37)$$

(FL3) $$\inf(\mathrm{Q}) = \max_{\lambda \in S^+} \max_{u \in X^*} \{-f^*(u) - (\lambda h + \delta_C)^*(-u)\}, \quad (38)$$

(FL4) $$\inf(\mathrm{Q}) = \max_{\lambda \in S^+} \max_{u \in X^*} \{-(f + \delta_C)^*(u) - (\lambda h)^*(-u)\}. \quad (39)$$

Proof. It is easy to see that (FL1) follows from Corollary 5.6, while (FL3) and (FL4) follow from Corollary 5.7, by using arguments similar to the proof of Corollary 6.3. $\square$

The duality result (FL1) was introduced for the first time in [4] where $X = \mathbb{R}^n$, $h = (h_1, h_2, \cdots, h_m)$, $h_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ are convex ($S = \mathbb{R}^m_+$) and under the Slater constraint qualification condition stating that $\mathrm{ri}C \cap \mathrm{ri}(\mathrm{dom}\, f) \neq \emptyset$ and that $x' \in \mathrm{ri}C \cap \mathrm{ri}(f)$ such that $h_i(x') < 0$, $i = 1, 2, ..., m$ exists. So, (FL1) relaxes assumptions in [4]. Such a result was also established in [3] for the so-called nearly-convex problems under a Slater-type constraint qualification.

Lagrange duality.

The Lagrange dual problem of (P) is defined by

$$(\mathrm{DQ}) \qquad \sup_{\lambda \in S^+} \left\{ \inf_{x \in C} (f + \lambda h)(x) \right\}.$$

The strong duality for the Lagrange dual problem is then given below.

Corollary 6.5 (Lagrange duality). *Suppose that* (CC) *holds, then the Lagrange strong duality holds for* (Q)*, that is,* (DQ) *is solvable and*

$$(\mathrm{LD}) \qquad \inf(\mathrm{Q}) = \max_{\lambda \in S^+} \left\{ \inf_{x \in C} (f + \lambda h)(x) \right\}.$$

Proof. The result comes from Corollary 5.6-(TFL2) with $g \equiv 0$. In fact, (TFL2) becomes

$$\begin{aligned} \inf(\mathrm{Q}) &= \max_{\lambda \in S^+} \{g^*(0) - (f + \lambda h + \delta_C)^*(0)\} = \max_{\lambda \in S^+} \{-(f + \lambda h + \delta_C)^*(0)\} \\ &= \max_{\lambda \in S^+} \left\{ \inf_{x \in C} (f + \lambda h)(x) \right\}. \end{aligned}$$

$\square$

Corollary 6.5 was established in [12] where X, Z were Banach spaces, f and h were continuous, $C = X$ and under the (FM) condition. A similar result for convex semi-infinite programs was also proposed in [11] assuming (CC1) and (FM).

Taking into account all the results just obtained, we are now in the position to conclude that all the duality formulas and all forms of Farkas lemmas that we have proposed are equivalent together.

Theorem 6.6. *For Problem* (Q)*, assume that* (CC) *holds and that* $\inf(\mathrm{Q}) \in \mathbb{R}$*, then the statements* (CF1) *to* (CF6)*,* (FL1)*,* (FL3)*,* (FL4) *and* (LD) *are all equivalent to each other.*

Proof. This is a direct consequence of Corollaries 6.3, 6.4 and 6.5. $\square$

To conclude this section, we establish some results concerning the stable Fenchel-Lagrange duality for the convex problems (Q).

6.3. Necessary and sufficient conditions for stable Fenchel-Lagrange duality

Adopting the notion of stable duality developed in [9], we say that the strong Fenchel-Lagrange duality

$$\inf_{\substack{h(x)\in -S\\ x\in C}} f(x) = \max_{\lambda\in S^+}\max_{u,v\in X^*}\Big\{-f^*(u)-(\lambda h)^*(v)-\delta_C^*(-u-v)\Big\}$$

is stable if, for any $x^*\in X^*$,

$$\inf_{\substack{h(x)\in -S\\ x\in C}}[f(x)+(x^*,x)] = \max_{\lambda\in S^+}\max_{u,v\in X^*}\Big\{-f^*(u)-(\lambda h)^*(v)-\delta_C^*(-x^*-u-v)\Big\}.$$

Stable duality of other forms of Fenchel-Lagrange duality (see Subsection 6.1) will be understood by the same way.

The following theorems give necessary and sufficient conditions for the stable duality of the Fenchel-Lagrange duality of the convex Problem (Q).

Theorem 6.7. *The following statements are equivalent:*

(i) *condition* (CC) *holds,*

(ii) *for all* $x^*\in X^*$,

$$\inf_{\substack{h(x)\in -S\\ x\in C}}[f(x)+(x^*,x)] = \max_{\lambda\in S^+}\max_{u,v\in X^*}\{-f^*(u)-(\lambda h)^*(v)-\delta_C^*(-x^*-u-v)\},$$

(iii) *for all* $x^*\in X^*$,

$$\inf_{\substack{h(x)\in -S\\ x\in C}}[f(x)+(x^*,x)] = \max_{\lambda\in S^+}\max_{u\in X^*}\{-f^*(u)-(\lambda h+\delta_C)^*(-x^*-u)\}.$$

Proof. Note that (ii) is equivalent to

$$(f+\delta_A)^*(-x^*) = \min_{\lambda\in S^+}\min_{u,v\in X^*}\big[f^*(u)+(\lambda h)^*(v)+\delta_C^*(-x^*-u-v)\big],\ \forall x^*\in X^*.$$

By Proposition 3.3, the last equality is equivalent to (i). This shows that (i) is equivalent to (ii). The other assertion follows by the same argument, by using Proposition 3.4. □

Theorem 6.8. *The following statements are equivalent:*

(i') $\mathrm{epi}(f+\delta_C)^*+\bigcup_{\lambda\in S^+}\mathrm{epi}(\lambda h)^*$ *is weak*-closed,*

(ii') *for all* $x^*\in X^*$,

$$\inf_{\substack{h(x)\in -S\\ x\in C}}[f(x)+(x^*,x)] = \max_{\lambda\in S^+}\max_{u\in X^*}\{-(f+\delta_C)^*(u)-(\lambda h)^*(-x^*-u)\}.$$

Proof. The proof is similar to that of Theorem 6.7, using Proposition 3.5. □

As an example, we consider the problem:

$$\text{(Q1)} \qquad \begin{array}{rl} \inf & f(x) \\ \text{subject to} & x \in C,\ Bx - b \in S, \end{array}$$

where $B : X \to Z$ is a continuous linear mapping and b is a given element of Z.

The Problem (Q1) is a special case of Problem (Q) with $h(x) = -Bx + b$. Recalling that $\sigma := \{-Bx+b \in -S,\ x \in C\}$ and that the solution set A of σ is $A := \{x \in X \mid -Bx+b \in -S,\ x \in C\}$. It is easy to see that $A \neq \emptyset$ if and only if $B(C) \cap (b+S) \neq \emptyset$.

The following corollary establishes a necessary and sufficient condition for the stable duality for Problem (Q1).

Corollary 6.9 ([9]). *Assume that* $B(C) \cap (S+b) \neq \emptyset$ *and* $\inf(\mathrm{Q}) \in \mathbb{R}$. *The following statements are equivalent:*

(i) $\operatorname{epi}(f+\delta_C)^* + \bigcup_{\lambda \in S^+} \{-B^*\lambda\} \times [-\lambda b, +\infty)$ *is weak***-closed,*

(ii) *for all* $x^* \in X^*$,

$$\inf_{\substack{Bx-b\in S\\ x\in C}} [f(x) + (x^*, x)] = \max_{\lambda \in S^+} \big[-(f+\delta_C)^*(B^*\lambda - x^*) + \lambda b \big],$$

where B^* *denotes the adjoint mapping of* B.

Proof. For each $u \in X^*$, we have

$$(\lambda h)^*(u) = \sup_{x \in X} \big[(u,x) + (B^*\lambda, x) - \lambda b\big] = \begin{cases} -\lambda b & \text{if } u = -B^*\lambda, \\ +\infty & \text{otherwise.} \end{cases}$$

So, the condition (i') in Theorem 6.8 means that the set

$$\operatorname{epi}(f+\delta_C)^* + \bigcup_{\lambda \in S^+} \{-B^*\lambda\} \times [-\lambda b, +\infty)$$

is weak*-closed. The conclusion follows from Theorem 6.8. □

The Corollary 6.9 was established recently in [9]. In fact, if we set $(\operatorname{epi}\delta^*_{b+S})_D := \{(D^*x^*, r) \mid (x^*, r) \in \operatorname{epi}\delta^*_{b+S}\}$ then the set involved in Corollary 6.9-(i) is nothing else than $\operatorname{epi}(f+\delta_C)^* + (\operatorname{epi}\delta^*_{b+S})_D$ that appeared in [9, Corollary 3.2][3]

In the Corollary 6.9, if we take $x^* = 0$ then we get the standard duality result for Problem (Q) under the condition (CC).

Corollary 6.10. *Assume that* $B(C) \cap (S+b) \neq \emptyset$ *and* $\inf(\mathrm{Q}) \in \mathbb{R}$. *Assume further that the condition* (i) *in Corollary* 6.9 *holds. Then*

$$\inf(\mathrm{Q}) = \max_{\lambda \in S^+} \big[-(f+\delta_C)^*(B^*\lambda) + \lambda b \big].$$

[3]It seems that in the statement (i) of this corollary, the maximum should be taken over the set $-K^0$ instead of K^0.

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