Journal of Convex Analysis
Volume 15 (2008), No. 2, 285–297

Comparing Fenchel-Moreau Conjugates with Level Set Conjugates

Juan-Enrique Martínez-Legaz*
Departament d'Economia i d'Història Econòmica, Universitat Autònoma de Barcelona, 08193 Bellaterra, Spain

Ivan Singer†
Institute of Mathematics, P.O. Box 1-764, 70700 Bucharest, Romania

Received: November 3, 2006
Revised manuscript received: June 18, 2007

We compare the Fenchel-Moreau second conjugates associated to an arbitrary coupling function $\varphi : X \times W \to \overline{R} = [-\infty, +\infty]$ between two sets X and W with the second level set conjugates associated to the same coupling. For a coupling $\varphi : R^n \times R^n \to R = (-\infty, +\infty)$ that is additively homogeneous in one (or both) of the variables we also compare the first conjugates associated to the same coupling. We give an application to the "min-type" coupling function arising in the study of topical functions.

Keywords: Generalized conjugation; Topical functions; Hull operators

2000 Mathematics Subject Classification: 49N15, 26B25, 52A01

1. Introduction

Let us first recall some concepts from abstract convex analysis (see e.g. [8]).

If E and F are two complete lattices (assumed nonempty, throughout the sequel), a mapping $\Delta : E \to F$ is called a polarity (or, as in [8], a "duality"), if for any index set I (including $I = \emptyset$) we have

$$\Delta(\inf_{i\in I} e_i) = \sup_{i\in I} \Delta(e_i), \tag{1}$$

with the usual conventions $\inf_{i\in I} \emptyset = +\infty$, the least element of E, and $\sup_{i\in I} \emptyset = -\infty$, the greatest element of F. The dual of any mapping $\Delta : E \to F$ is the mapping $\Delta' : F \to E$ defined by

$$\Delta'(z) := \inf\{e \in E \,|\, \Delta(e) \leq z\} \qquad (z \in F). \tag{2}$$

If $\Delta : E \to F$ is a polarity, then e.g. by [8], Corollary 5.5, the composition $\Delta'\Delta : E \to E$ is a "hull operator". We recall that a mapping $u : E \to E$ is a hull operator (see e.g. [8], Definition 1.4) if for any $x, \widetilde{x} \in E$ we have

(a) $x \leq \widetilde{x} \Rightarrow u(x) \leq u(\widetilde{x})$;
(b) $u(x) \leq x$;
(c) $u(u(x)) = u(x)$.

*The research of this author has been partially supported by the Ministerio de Ciencia y Tecnología, Project MTM2005-08572-C03-03, and by the Barcelona Economics Program of XREA.
†The research of this author has been partially supported by grant CEx05-D11-23/2005.

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

Definition 1.1. Let E and F be two complete lattices and $\Delta : E \to F$ a polarity. An element $x \in E$ is said to be Δ-*convex* if

$$x = \Delta'\Delta(x). \tag{3}$$

Also, for each $x \in E$, the element $\Delta'\Delta(x)$ will be called *the* Δ-*convex hull of* x, and the mapping $x \longmapsto \Delta'\Delta(x)$ will be called *the* Δ-*convex hull operator.*

Remark 1.2. In [8], p. 180, the elements $x \in E$ satisfying (3) have been called "$\Delta'\Delta$-convex", and for each $x \in E$ the element $\Delta'\Delta(x)$ has been called the "$\Delta'\Delta$-convex" hull of x, but the simpler terms "Δ-convex" and "Δ-convex hull of x" introduced here, will lead to no confusion. For the motivation of the terms "hull" and "hull operator", see e.g. [8], Definitions 1.4 and 5.4, and Corollary 5.5.

By [8], Definition 5.6 and Proposition 5.7, if E and F are two complete lattices and $\Delta_1 : E \to F_1$ and $\Delta_2 : E \to F_2$ are two polarities, then Δ_1 is said to be *equivalent to* Δ_2, in symbols, $\Delta_1 \sim \Delta_2$, if

$$\Delta_1'\Delta_1 = \Delta_2'\Delta_2, \tag{4}$$

that is, if $\Delta_1'\Delta_1(x) = \Delta_2'\Delta_2(x)$ for all $x \in E$.

In the present paper we shall consider the particular case of the complete lattice $E = \overline{R}^X$, the set of all functions $f : X \to \overline{R} = [-\infty, +\infty]$, where X is any set (assumed nonempty throughout the sequel, without any special mention) endowed with the partial order $\leq$ and the lattice operations sup and inf defined pointwise on X, that is, $f \leq h$ if and only if $f(x) \leq h(x)$ $(x \in X)$, and $(\sup_{i\in I} f_i)(x) :=\sup_{i\in I} f_i(x)$, $(\inf_{i\in I} f_i)(x) :=\inf_{i\in I} f_i(x)$ $(x \in X)$. Thus, the elements of E are now functions $f : X \to \overline{R}$. We recall that if X and W are two (nonempty) sets, any function $\varphi : X \times W \to \overline{R}$ is called a *coupling function.* For a mapping $\Delta : \overline{R}^X \to \overline{R}^W$ it is usual to denote $\Delta(f)$ by f^Δ. Then, for example, for any mapping $\Delta : \overline{R}^X \to \overline{R}^W$, formula (2) becomes

$$g^{\Delta'} = \inf\{h \in \overline{R}^X \,|\, h^\Delta \leq g\} \quad (g \in \overline{R}^W). \tag{5}$$

We shall be concerned with the polarities $\Delta = c(\varphi) : \overline{R}^X \to \overline{R}^W$, the "Fenchel-Moreau conjugation", and $\Delta = L(\varphi) : \overline{R}^X \to \overline{R}^W$, the "level set conjugation", with respect to a coupling function $\varphi : X \times W \to \overline{R}$. For the first one, let us recall that the usual addition $+$ on $R = (-\infty, +\infty)$ admits two natural extensions to $\overline{R}$, $\dot{+}$ and $\underset{\cdot}{+}$, called *upper* and *lower addition*, respectively, defined by

$$a\dot{+}b = a\underset{\cdot}{+}b = a + b \quad \text{if either } R \cap \{a, b\} \neq \emptyset \text{ or } a = b = \pm\infty, \tag{6}$$

$$a\dot{+}b = +\infty, \quad a\underset{\cdot}{+}b = -\infty \quad \text{if } a = -b = \pm\infty; \tag{7}$$

as usual, we shall keep the notation $a + b$ also for $a, b \in \overline{R}$ with $R \cap \{a, b\} \neq \emptyset$. If X, W are two sets and $\varphi : X \times W \to \overline{R}$ is a coupling function, then for a function $f : X \to \overline{R}$ the *Fenchel-Moreau conjugate function of* f *with respect to* φ is the function $f^{c(\varphi)} : W \to \overline{R}$ defined by

$$f^{c(\varphi)}(w) := \sup_{x\in X} \{\varphi(x, w) \underset{\cdot}{+} -f(x)\} \quad (w \in W). \tag{8}$$

The main example corresponds to the case where X is a Banach space (or more generally, a locally convex space), W is X^* (the dual of X), and φ is the usual pairing function ([5], [3]) between X and X^* :

$$\varphi(x, x^*) = x^*(x) \quad (x \in X, x^* \in X^*).$$

It is well-known and immediate that the mapping $\Delta = c(\varphi) : f \longmapsto f^{c(\varphi)}$ is a polarity from $\overline{R}^X$ into $\overline{R}^W$, and that φ is uniquely determined by $\Delta = c(\varphi)$. Indeed (see e.g. [8], Theorem 8.2), if $\varphi : X \times W \to \overline{R}$ and $\psi : X \times W \to \overline{R}$ are two coupling functions such that $f^{c(\varphi)} = f^{c(\psi)}$ for all $f \in \overline{R}^X$, then by (8), for $f = \chi_{\{x\}}$, where χ_A denotes the indicator function of A, for any set A (that is $= 0$ on A and $= +\infty$ outside A), we have

$$\begin{aligned} \varphi(x, w) &= \sup_{x' \in X} \{\varphi(x', w) \dot{+} -\chi_{\{x\}}(x')\} = (\chi_{\{x\}})^{c(\varphi)}(w) = (\chi_{\{x\}})^{c(\psi)}(w) \\ &= \sup_{x' \in X} \{\psi(x', w) \dot{+} -\chi_{\{x\}}(x')\} = \psi(x, w) \quad (x \in X, w \in W). \end{aligned}$$

The Fenchel-Moreau biconjugate of $f : X \to \overline{R}$ with respect to φ is the function $f^{c(\varphi)c(\varphi)'} := (f^{c(\varphi)})^{c(\varphi)'} : X \to \overline{R}$, and the mapping $f \longmapsto f^{c(\varphi)c(\varphi)'}$ is the corresponding *Fenchel-Moreau hull operator*. By (5) for $\Delta = c(\varphi)$ and (8), we have

$$f^{c(\varphi)c(\varphi)'}(x) = \sup_{w \in W} \{\varphi(x, w) \dot{+} - \sup_{x' \in X} \{\varphi(x', w) \dot{+} - f(x')\}\} \quad (f \in \overline{R}^X, x \in X). \tag{9}$$

If X, W are two sets and $\varphi : X \times W \to \overline{R}$ is a coupling function, then for each function $f : X \to \overline{R}$ one defines $f^{L(\varphi)} : W \to \overline{R}$, the *level set conjugate of f with respect to φ*, by

$$f^{L(\varphi)}(w) := \sup_{\substack{x \in X \\ \varphi(x,w) > 0}} (-f)(x) \quad (w \in W). \tag{10}$$

The level set biconjugate of $f : X \to \overline{R}$ with respect to φ is the function $f^{L(\varphi)L(\varphi)'} := (f^{L(\varphi)})^{L(\varphi)'} : X \to \overline{R}$, and the mapping $f \longmapsto f^{L(\varphi)L(\varphi)'}$ is the corresponding *level set hull operator.*

Remark 1.3. In [8], [9] and other papers, the level set conjugate of f with respect to φ was called "conjugate of type Lau" (this term, suggested by the work of Lau [2] and Crouzeix [1], was motivated in [8], p. 435). This notion was defined (see e.g. [8], Definition 8.5) by the slightly different formula

$$f^{\widetilde{L}(\varphi)}(w) := \sup_{\substack{x \in X \\ \varphi(x,w) > -1}} (-f)(x) \quad (w \in W); \tag{11}$$

however the results on $\widetilde{L}(\varphi)$, given in [8], remain valid also for $L(\varphi)$, mutatis mutandis. Our present term "level set conjugate", suggested by some particular cases given in [7], was also used in the literature (see e.g. [10], [4]).

We recall (see e.g. [8], Theorem 8.15) that for the level set biconjugates we have

$$f^{L(\varphi)L(\varphi)'}(x) = \sup_{\substack{w' \in W \\ \varphi(x,w') > 0}} \inf_{\substack{x' \in X \\ \varphi(x',w') > 0}} f(x') \quad (f \in \overline{R}^X, x \in X). \tag{12}$$

In the present paper our main focus will be on the comparison of the Fenchel-Moreau conjugate $f^{c(\varphi)}$ and biconjugate $f^{c(\varphi)c(\varphi)'}$ with the level set conjugate $f^{L(\varphi)}$ and biconjugate $f^{L(\varphi)L(\varphi)'}$ respectively, with respect to the same coupling function $\varphi : X \times W \to \overline{R}$. Some results in this direction have been obtained in [9]. For example, by [9], Remark 4.1, if X, W are two sets and $\varphi : X \times W \to \overline{R}$ is a coupling function, then for any function $f : X \to \overline{R}$ we have

$$f^{c(\varphi)} \geq f^{L(\varphi)}; \tag{13}$$

indeed,

$$\begin{aligned} f^{c(\varphi)}(w) &= \sup_{x\in X}\{\varphi(x,w) \dot{+} -f(x)\} \geq \sup_{\substack{x\in X\\ \varphi(x,w)>0}}\{\varphi(x,w) \dot{+} - f(x)\} \\ &\geq \sup_{\substack{x\in X\\ \varphi(x,w)>0}}(-f(x)) = f^{L(\varphi)}(w) \quad (w \in W). \end{aligned}$$

Note that, similarly, for any function $g : W \to \overline{R}$ we have

$$g^{c(\varphi)'} \geq g^{L(\varphi)'}; \tag{14}$$

indeed, by [8], formulas (8.54) and (8.178),

$$\begin{aligned} g^{c(\varphi)'}(x) &= \sup_{w\in W}\{\varphi(x,w) \dot{+} -g(w)\} \geq \sup_{\substack{w\in W\\ \varphi(x,w)>0}}\{\varphi(x,w) \dot{+} - g(w)\} \\ &\geq \sup_{\substack{w\in W\\ \varphi(x,w)>0}}(-g(w)) = g^{L(\varphi)'}(x) \quad (x \in X). \end{aligned}$$

Every level set conjugation $L(\varphi)$ is also a Fenchel-Moreau conjugation with respect to another coupling function, and hence a polarity. Indeed, we have

Lemma 1.4 (see e.g. [8], Corollary 8.12). *Let X, W be two sets and $\varphi : X \times W \to \overline{R}$ a coupling function. Then for the level set conjugation $L(\varphi) : \overline{R}^X \to \overline{R}^W$ there exists a unique coupling function $\varphi_1 : X \times W \to \{0, -\infty\}$ such that*

$$L(\varphi) = c(\varphi_1), \tag{15}$$

namely

$$\varphi_1 := -\chi_P, \tag{16}$$

where

$$P := \{(x, w) \in X \times W \mid \varphi(x, w) > 0\}. \tag{17}$$

Therefore, we shall use the following

Basic observation: *The problem of comparing $f^{c(\varphi)}$ and $f^{L(\varphi)}$) respectively, $f^{c(\varphi)c(\varphi)'}$ and $f^{L(\varphi)L(\varphi)'}$) amounts to the problem of comparing the two Fenchel-Moreau conjugates $f^{c(\varphi)}$ and $f^{c(\varphi_1)}$ (respectively, biconjugates $f^{c(\varphi)c(\varphi)'}$ and $f^{c(\varphi_1)c(\varphi_1)'}$), with φ_1 of* (16), (17).

According to Definition 1.1 above, applied to $\Delta = c(\varphi)$, we shall say that a function $f : X \to \overline{R}$ is $c(\varphi)$-convex, if

$$f = f^{c(\varphi)c(\varphi)'}. \tag{18}$$

We recall the following well-known result, which we shall use in the sequel:

Lemma 1.5 (see e.g. [8], Corollary 8.5 and Remark 1.1(c)). *Let X, W be two sets and $\varphi : X \times W \to \overline{R}$ a coupling function. For a function $f : X \to \overline{R}$ the following statements are equivalent:*

1°. *f is $c(\varphi)$-convex.*

2°. *There exists a subset M of $W \times R$ such that*

$$f = \sup_{(w,d)\in M} \{\varphi(., w) + d\}. \tag{19}$$

3°. *We have*

$$f = \sup_{\substack{w\in W, d\in R\\ \varphi(.,w)+d\leq f}} \{\varphi(., w) + d\}. \tag{20}$$

We have the following useful consequence of Lemma 1.5:

Corollary 1.6. *Let X, W be two sets and $\varphi : X \times W \to \overline{R}$ a coupling function. Then for each $w \in W$, the function $\varphi(., w) : X \to \overline{R}$ is $c(\varphi)$-convex.*

Proof. For any $w \in W$, we have

$$\varphi(., w) = \sup_{(w',d)\in\{(w,0)\}} \{\varphi(., w') + d\},$$

that is, (19) for $f = \varphi(., w)$, with $M \subset W \times R$ defined by $M = \{(w, 0)\}$. Hence, by Lemma 1.5, the conclusion follows. □

2. Comparison of Fenchel-Moreau hull operators with level set hull operators

We have the following result on comparison of Fenchel-Moreau hull operators with respect to two different coupling functions:

Proposition 2.1. *Let X, W_1, W_2 be three sets and $\varphi : X \times W_1 \to \overline{R}, \psi : X \times W_2 \to \overline{R}$ two coupling functions. The following statements are equivalent:*

1°. *We have $c(\varphi)'c(\varphi) \leq c(\psi)'c(\psi)$, that is,*

$$f^{c(\varphi)c(\varphi)'} \leq f^{c(\psi)c(\psi)'} \quad (f \in \overline{R}^X). \tag{21}$$

2°. *For each $w \in W_1$, the function $\varphi(., w)$ is $c(\psi)$-convex.*

Proof. *1°* ⇒ *2°*. Assume *1°* and let $w \in W_1$. Then, by Corollary 1.6 and *1°* (for $f = \varphi(., w)$), we have

$$\varphi(., w) = \varphi(., w)^{c(\varphi)c(\varphi)'} \leq \varphi(., w)^{c(\psi)c(\psi)'} \leq \varphi(., w),$$

whence $\varphi(., w)^{c(\psi)c(\psi)'} = \varphi(., w)$.

2° ⇒ *1°*. Assume *2°* and let $f \in \overline{R}^X$. Then, since $f^{c(\varphi)c(\varphi)'}$ is $c(\varphi)$-convex (indeed, $f^{c(\varphi)c(\varphi)'c(\varphi)c(\varphi)'} = f^{c(\varphi)c(\varphi)'}$), by Lemma 1.5 there exists a subset P of $W_1 \times R$ such that

$$f^{c(\varphi)c(\varphi)'} = \sup_{(w,d)\in P} \{\varphi(., w) + d\}. \tag{22}$$

But, by *2°* and Lemma 1.5, for each $w \in W_1$ there exists a subset P_w of $W_2 \times R$ such that

$$\varphi(.,w) = \sup_{(w',d')\in P_w} \{\psi(.,w') + d'\}. \tag{23}$$

Consequently, by (22) and (23),

$$f^{c(\varphi)c(\varphi)'} = \sup_{\substack{(w,d)\in P\\ (w',d')\in P_w}} \{\psi(.,w') + d' + d\}. \tag{24}$$

Therefore, again by Lemma 1.5, $f^{c(\varphi)c(\varphi)'}$ is $c(\psi)$-convex. Hence, since $f^{c(\varphi)c(\varphi)'} \leq f$, it follows that $f^{c(\varphi)c(\varphi)'}$ is less than or equal to the greatest $c(\psi)$-convex function majorized by f, i.e., $f^{c(\varphi)c(\varphi)'} \leq f^{c(\psi)c(\psi)'}$. □

Remark 2.2. *The above conditions are equivalent to the following one:*

3°. *Every $c(\varphi)$-convex function is $c(\psi)$-convex.*

Indeed, more generally, the equivalence *1°* ⇔ 3° holds for any two hull operators on a complete lattice. For, let us recall that if u is a hull operator on a complete lattice E, an element $x \in E$ is said to be *u-convex* (see e.g. [8], Definition 1.9) if $u(x) = x$. *For any two hull operators $u_1, u_2 : E \to E$ the following statements are equivalent:*

1′. *We have $u_1 \leq u_2$ (that is, $u_1(x) \leq u_2(x)$ for all $x \in E$).*
2′. *Every u_1-convex element $x \in E$ is u_2-convex (that is, $u_1(x) = x \Rightarrow u_2(x) = x$).*

Indeed, if 1′ holds and $x \in E, u_1(x) = x$, then $x = u_1(x) \leq u_2(x) \leq x$, whence $u_2(x) = x$. Conversely, if 2′ holds and $x \in E$, then since $u_1(u_1(x)) = u_1(x)$, by 2′ applied to $u_1(x)$ we have $u_2(u_1(x)) = u_1(x)$. But, since $u_1(x) \leq x$, we have $u_2(u_1(x)) \leq u_2(x)$. Hence, finally, $u_1(x) \leq u_2(x)$. Applying this fact to $E = \overline{R}^X$ (with the natural order $\leq$) and the hull operators $u_1 = c(\varphi)'c(\varphi), u_2 = c(\psi)'c(\psi)$, we obtain the equivalence *1°* ⇔ 3° mentioned above.

Corollary 2.3. *Let X, W_1, W_2 be three sets and $\varphi : X \times W_1 \to \overline{R}, \psi : X \times W_2 \to \overline{R}$ two coupling functions. The following statements are equivalent:*

1°. *We have $c(\varphi)'c(\varphi) = c(\psi)'c(\psi)$, that is,*

$$f^{c(\varphi)c(\varphi)'} = f^{c(\psi)c(\psi)'} \quad (f \in \overline{R}^X). \tag{25}$$

2°. *For each $w \in W_1$, the function $\varphi(.,w)$ is $c(\psi)$-convex, and for each $w \in W_2$, the function $\psi(.,w)$ is $c(\varphi)$-convex.*

Remark 2.4. By Remark 2.2 the above conditions are equivalent to the following one:

3°. *The classes of $c(\varphi)$-convex functions and of $c(\psi)$-convex functions coincide.*

In the next two propositions we shall apply Proposition 2.1 to the particular case where $W_1 = W_2 = W$, the first one of the conjugates is a Fenchel-Moreau conjugate and the second one is a level set conjugate (which, as was observed above, is also a Fenchel-Moreau conjugate).

Proposition 2.5. *Let X, W be two sets and $\varphi : X \times W \to \overline{R}$ a coupling function. For a function $f : X \to \overline{R}$ the following statements are equivalent:*

1°. *We have $c(\varphi)'c(\varphi) \leq L(\varphi)'L(\varphi)$, that is,*

$$f^{c(\varphi)c(\varphi)'} \leq f^{L(\varphi)L(\varphi)'} \quad (f \in \overline{R}^X). \tag{26}$$

2°. *We have*

$$\varphi(x, w) = \sup_{\substack{w' \in W \\ \varphi(x,w')>0}} \inf_{\substack{x' \in X \\ \varphi(x',w')>0}} \varphi(x', w) \quad ((x, w) \in X \times W). \tag{27}$$

3°. *For each $w \in W$ the function $\varphi(., w) : X \to R$ is $L(\varphi)$-convex, that is, we have*

$$\varphi(., w) = \varphi(., w)^{L(\varphi)L(\varphi)'}. \tag{28}$$

Proof. *1° ⇔ 2°*. By Lemma 1.4, we have (15), with $\varphi_1 : X \times W \to \{0, -\infty\}$ of (16), so (26) can be written as

$$f^{c(\varphi)c(\varphi)'} \leq f^{c(\varphi_1)c(\varphi_1)'} \quad (f \in \overline{R}^X). \tag{29}$$

Hence, by Proposition 2.1 (with $\psi = \varphi_1$), we have *1°* if and only if for each $w \in W$ the function $\varphi(., w)$ is $c(\varphi_1)$-convex, that is (by (18), mutatis mutandis),

$$\varphi(., w) = \varphi(., w)^{c(\varphi_1)c(\varphi_1)'}. \tag{30}$$

But, by (15) and (12) (applied to $f = \varphi(., w)$), formula (30) is equivalent to

$$\varphi(x, w) = \sup_{\substack{w' \in W \\ \varphi(x,w')>0}} \inf_{\substack{x' \in X \\ \varphi(x',w')>0}} \varphi(x', w) \quad (x \in X). \tag{31}$$

2° ⇔ 3°. By (12) applied to $f = \varphi(., w)$ we have

$$\varphi(x, w)^{L(\varphi)L(\varphi)'} = \sup_{\substack{w' \in W \\ \varphi(x,w')>0}} \inf_{\substack{x' \in X \\ \varphi(x',w')>0}} \varphi(x', w) \qquad ((x, w) \in X \times W), \tag{32}$$

and hence *2° ⇔ 3°*. □

In order to consider the opposite inequality to (26), it will be convenient to introduce the following definition:

Definition 2.6. Let X, W be two sets and $\varphi : X \times W \to \overline{R}$. We shall say that an element $w \in W$ is *strictly positive* if $\varphi(x, w) > 0$ (that is, $\chi_P(x, w) = 0$, with P of (17)), for all $x \in X$.

Remark 2.7. An element $w \in W$ is not strictly positive if and only if one of the following equivalent conditions holds:

$$\exists x' \in X, (x', w) \notin P \Leftrightarrow \exists x' \in X, \chi_P(x', w) = +\infty \Leftrightarrow \sup_{x \in X} \chi_P(x, w) = +\infty. \tag{33}$$

Proposition 2.8. *Let X and W be two sets and $\varphi : X \times W \to R$ a finite coupling function. The following statements are equivalent:*

1°. *We have $L(\varphi)' L(\varphi) \leq c(\varphi)' c(\varphi)$, that is,*

$$f^{L(\varphi)L(\varphi)'} \leq f^{c(\varphi)c(\varphi)'} \quad \left(f \in \overline{R}^X\right). \tag{34}$$

2°. *The following two conditions hold:*

(a) *For any* $w \in W$*, either* w *is strictly positive or we have* $\varphi(x, w) \leq 0$ *for every* $x \in X$.

(b) *If there exists some strictly positive element in* W *then*

$$\sup_{w \in W} \left\{ \varphi(x, w) - \sup_{x' \in X} \varphi(x', w) \right\} = 0 \quad (x \in X). \tag{35}$$

Proof. By Lemma 1.4, we have (15), with $\varphi_1 : X \times W \to \{0, -\infty\}$ of (16), so (34) can be written as

$$f^{c(\varphi_1)c(\varphi_1)'} \leq f^{c(\varphi)c(\varphi)'} \quad (f \in \overline{R}^X). \tag{36}$$

Hence, by Proposition 2.1 (with φ and ψ replaced by φ_1 and φ respectively), we have *1°* if and only if for each $w \in W$ the function $\varphi_1(., w)$ is $c(\varphi)$-convex, that is (by (16), with P of (17)),

$$-\chi_P(\cdot, w) = [-\chi_P(\cdot, w)]^{c(\varphi)c(\varphi)'} \quad (w \in W). \tag{37}$$

1° $\Rightarrow$ *2°*. Assume that (37) holds and that $w \in W$ is not strictly positive, and let $x \in X$. Then, by (37), (9) for $f = -\chi_P$, and (33), we have

$$\begin{aligned} -\chi_P(x, w) = &\ [-\chi_P(\cdot, w)](x) = [-\chi_P(\cdot, w)]^{c(\varphi)c(\varphi)'}(x) \\ = &\ \sup_{w' \in W} \left\{ \varphi(x, w') - \sup_{x' \in X} \{\varphi(x', w') + \chi_P(x', w)\} \right\} \\ = &\ \sup_{w' \in W} \{\varphi(x, w') - (+\infty)\} = -\infty. \end{aligned}$$

Hence $\chi_P(x, w) = +\infty$, that is, $(x, w) \notin P$, which means that $\varphi(x, w) \leq 0$. This proves (a).

To prove (b), assume that there is some strictly positive $w_0 \in W$ and let $x \in X$. Then, by $\chi_P(x', w_0) = 0$ for all $x' \in X$, (37), and (9) for $f = -\chi_P$, we have

$$\begin{aligned} 0 = &\ -\chi_P(x, w_0) = \sup_{w \in W} \left\{ \varphi(x, w) - \sup_{x' \in X} \{\varphi(x', w) + \chi_P(x', w_0)\} \right\} \\ = &\ \sup_{w \in W} \left\{ \varphi(x, w) - \sup_{x' \in X} \varphi(x', w) \right\}, \end{aligned}$$

which proves (35).

2° $\Rightarrow$ *1°*. Let $w \in W$ and $x \in X$. Assume first that w is not strictly positive. In this case, by (a) of *2°*, we have $\varphi(x', w) \leq 0$, so $\chi_P(x', w) = +\infty$, for all $x' \in X$. Hence, by (9) for $f = -\chi_P$, we obtain

$$\begin{aligned} [-\chi_P(\cdot, w)]^{c(\varphi)c(\varphi)'}(x) = &\ \sup_{w' \in W} \left\{ \varphi(x, w') - \sup_{x' \in X} \{\varphi(x', w') + \chi_P(x', w)\} \right\} \\ = &\ \sup_{w' \in W} \{\varphi(x, w') + (-\infty)\} = -\infty = -\chi_P(x, w) \\ = &\ -\chi_P(\cdot, w)(x). \end{aligned}$$

Assume now that w is strictly positive, and let $x \in X$. Then by (9) for $f = -\chi_P$, (b) of $2°$, and since $f \geq f^{c(\varphi)c(\varphi)'}$ for all $f \in \overline{R}^X$, we obtain

$$\begin{aligned}[-\chi_P(\cdot, w)]^{c(\varphi)c(\varphi)'}(x) &= \sup_{w' \in W} \left\{ \varphi(x, w') - \sup_{x' \in X} \{\varphi(x', w') + \chi_P(x', w)\} \right\} \\ &= \sup_{w' \in W} \left\{ \varphi(x, w') - \sup_{x' \in X} \varphi(x', w') \right\} = 0 \\ &\geq -\chi_P(x, w) = -\chi_P(\cdot, w)(x) \\ &\geq [-\chi_P(\cdot, w)]^{c(\varphi)c(\varphi)'}(x);\end{aligned}$$

therefore, in both cases, $-\chi_P(\cdot, w)(x) = [-\chi_P(\cdot, w)]^{c(\varphi)c(\varphi)'}(x)$. We have thus proved (37). □

Here are some examples of the situation of Proposition 2.8, which have applications in optimization:

Example 2.9 (see [4], Example 6.3). Let X be a Hilbert space, $W = (0, +\infty) \times X$, and

$$\varphi(x, (\rho, y)) := -\rho \|x - y\|^2 \quad ((x, (\rho, y)) \in X \times W). \tag{38}$$

Example 2.10 (see [4], Example 6.4). Let $X = W = R^n, 0 < \alpha \leq 1, N > 0$, and

$$\varphi(x, w) = -N \|x - w\|^\alpha \quad ((x, w) \in X \times W). \tag{39}$$

Example 2.11 (see [4], Example 6.9). Let $X = W = [0, +\infty)$, and

$$\varphi(x, w) := -\max_{1 \leq i \leq n} x_i w_i \quad (x = (x_1, ..., x_n), w = (w_1, ..., w_n) \in X). \tag{40}$$

In these examples $\varphi : X \times W \to R$ satisfies conditions $2°(a)$ and (vacuously) $2°(b)$ of Proposition 2.8, since it takes nonpositive values. In contrast with the above examples, the situation of Proposition 2.8 does not hold in the convex analytic case where X is a locally convex space, W is its dual and φ is the usual pairing function between X and X^*.

3. Conjugates and biconjugates of a function $f : R^n \to \overline{R}$

In this section we shall consider the particular case where

$$X = W = R^n, \tag{41}$$

and, for simplicity, $\varphi : R^n \times R^n \to R$ is a finite coupling function.

We recall (see e.g. [6], [8]) that a function $f : R^n \to \overline{R}$ is called

(a) *plus-homogeneous* (or, homogeneous with respect to addition), if

$$f(x + \lambda \mathbf{1}) = f(x) + \lambda \quad (x \in R^n, \lambda \in R), \tag{42}$$

where $\mathbf{1} = (1, ..., 1)$ is the vector of dimension n with all coordinates equal to one;

(b) *increasing*, if

$$x', x'' \in X, x' \leq x'' \Rightarrow f(x') \leq f(x''); \tag{43}$$

(c) *topical*, if it is both plus-homogeneous and increasing.

Proposition 3.1. *Let $\varphi : R^n \times R^n \to R$. If for each $x \in R^n$ the function $\varphi(x, .) : R^n \to R$ is plus-homogeneous, then we have* (27)*, or, equivalently,* (26).

Proof. Clearly, we have the inequality $\geq$ in (27). To prove the opposite inequality, let $(x, w) \in R^n \times R^n$ and $\varepsilon > 0$ be arbitrary, and let

$$w_\varepsilon := w - (\varphi(x, w) - \varepsilon)\mathbf{1}. \tag{44}$$

Then, by (44) and since $\varphi(x, .)$ is plus-homogeneous, we have

$$\begin{aligned} \varphi(x, w_\varepsilon) &= \varphi(x, w - (\varphi(x, w) - \varepsilon)\mathbf{1}) \\ &= \varphi(x, w) - \varphi(x, w) + \varepsilon = \varepsilon > 0, \end{aligned} \tag{45}$$

whence, by (45), (44), and since $\varphi(x, .)$ is plus-homogeneous,

$$\begin{aligned} \sup_{\substack{w' \in R^n \\ \varphi(x,w')>0}} \inf_{\substack{x' \in R^n \\ \varphi(x',w')>0}} \varphi(x', w) &\geq \inf_{\substack{x' \in R^n \\ \varphi(x',w_\varepsilon)>0}} \varphi(x', w_\varepsilon + (\varphi(x, w) - \varepsilon)\mathbf{1}) \\ &= \inf_{\substack{x' \in R^n \\ \varphi(x',w_\varepsilon)>0}} \{\varphi(x', w_\varepsilon) + \varphi(x, w) - \varepsilon\} \\ &\geq \varphi(x, w) - \varepsilon. \end{aligned}$$

Consequently, since $\varepsilon > 0$ was arbitrary, we obtain the inequality $\leq$ in (27), and hence the equality. $\square$

Proposition 3.2. *Let $\varphi : R^n \times R^n \to R$. If for each $w \in R^n$ the function $\varphi(., w) : R^n \to \overline{R}$ is plus-homogeneous, then for every plus-homogeneous function $f : R^n \to \overline{R}$ we have*

$$f^{c(\varphi)} = f^{L(\varphi)}. \tag{46}$$

Proof. Let $x, w \in R^n$, and $\varepsilon > 0$, be arbitrary, and let

$$x_\varepsilon := x - (\varphi(x, w) - \varepsilon)\mathbf{1}. \tag{47}$$

Then, since by our assumption $\varphi(., w)$ is plus-homogeneous,

$$\varphi(x_\varepsilon, w) = \varphi(x - (\varphi(x, w) - \varepsilon)\mathbf{1}, w) = \varphi(x, w) - (\varphi(x, w) - \varepsilon) > 0,$$

and hence, for any plus-homogeneous function $f : R^n \to \overline{R}$,

$$\begin{aligned} \varphi(x, w) - f(x) &= \varphi(x, w) - f(x_\varepsilon + (\varphi(x, w) - \varepsilon)\mathbf{1}) \\ &= \varphi(x, w) - [f(x_\varepsilon) + \varphi(x, w) - \varepsilon] = -f(x_\varepsilon) + \varepsilon \\ &\leq \sup_{\substack{x \in R^n \\ \varphi(x,w)>0}} (-f(x)) + \varepsilon = f^{L(\varphi)} + \varepsilon. \end{aligned}$$

Consequently, since $x \in R^n$ was arbitrary, we obtain $f^{c(\varphi)}(w) \leq f^{L(\varphi)}(w) + \varepsilon$, whence, since $w \in W$ and $\varepsilon > 0$ were arbitrary,

$$f^{c(\varphi)} \leq f^{L(\varphi)},$$

which, together with (13), yields (46). $\square$

Remark 3.3. (a) Concerning the assumption of Proposition 3.2, let us observe that *for any* $\varphi : R^n \times R^n \to R$, *the following statements are equivalent:*

1°. *For each* $w \in W$ *the function* $\varphi(., w)$ *is plus-homogeneous.*
2°. *Every* $c(\varphi)$*-convex function* $f : R^n \to R$ *is plus-homogeneous.*

Indeed, if for each $w \in W$ the function $\varphi(., w)$ is plus-homogeneous, then so is the function $\varphi(., w) + d$, for each $d \in R$. Consequently, by Lemma 1.5 and since the supremum of any family of plus-homogeneous functions is plus-homogeneous, it follows that every $c(\varphi)$-convex function $f : R^n \to R$ is plus-homogeneous. Thus, 1° ⇒ 2°. The reverse implication 2° ⇒ 1° follows from Corollary 1.6.

(b) In the particular case where $\varphi = \mu$, the so-called "min-type coupling function" $\mu : R^n \times R^n \to R$ defined by

$$\mu(x, w) := \min_{1 \leq i \leq n} (x_i + w_i) \quad (x = (x_1, ..., x_n), w = (w_1, ..., w_n) \in R^n), \tag{48}$$

and f is a topical function, Proposition 3.2 was proved, with a different method, in [9], Theorem 4.2.

From Proposition 3.2 and Remark 3.3(a) we obtain:

Corollary 3.4. *Under the assumptions of Proposition* 3.2*, for every* $c(\varphi)$*-convex function* $f : R^n \to \overline{R}$ *we have* (46)*.*

Theorem 3.5. *Let* $\varphi : R^n \times R^n \to R$*. If for each* $x, w \in R^n$ *the functions* $\varphi(x, .) : R^n \to R$ *and* $\varphi(., w) : R^n \to R$ *are plus-homogeneous, then for any function* $f : R^n \to \overline{R}$ *the following statements are equivalent:*

1°. *We have* (46)*.*
2°. *We have*
$$f^{c(\varphi)c(\varphi)'} = f^{L(\varphi)L(\varphi)'}. \tag{49}$$
3°. $f^{L(\varphi)L(\varphi)'}$ *is* $c(\varphi)$*-convex.*
4°. *We have*
$$f^{L(\varphi)L(\varphi)'c(\varphi)} = f^{L(\varphi)}. \tag{50}$$

Proof. *1°* ⇒ *2°*. Assume *1°*. Then by the plus-homogeneity of $\varphi(x, .)$ for each $x \in R^n$, Proposition 3.1, (46) and (14) applied to $g = f^{c(\varphi)}$, we have

$$f^{c(\varphi)c(\varphi)'} \leq f^{L(\varphi)L(\varphi)'} = f^{c(\varphi)L(\varphi)'} \leq f^{c(\varphi)c(\varphi)'},$$

and hence (49).

2° ⇒ *3°*. If *2°* holds, then

$$f^{L(\varphi)L(\varphi)'} = f^{c(\varphi)c(\varphi)'} = f^{c(\varphi)c(\varphi)'c(\varphi)c(\varphi)'} = f^{L(\varphi)L(\varphi)'c(\varphi)c(\varphi)'}.$$

3° ⇒ *4°*. If *3°* holds, then by the plus-homogeneity of $\varphi(., w)$ for each $w \in R^n$ and Proposition 3.2 applied to the $c(\varphi)$-convex function $f^{L(\varphi)L(\varphi)'}$, we have

$$f^{L(\varphi)L(\varphi)'c(\varphi)} = f^{L(\varphi)L(\varphi)'L(\varphi)} = f^{L(\varphi)}.$$

4° ⇒ *1°*. Assume *4°*. Then, by $f^{L(\varphi)L(\varphi)'} \leq f$, *4°*, and (13), we obtain

$$f^{c(\varphi)} \leq f^{L(\varphi)L(\varphi)'c(\varphi)} = f^{L(\varphi)} \leq f^{c(\varphi)},$$

and hence (46). □

Remark 3.6. There exist some useful coupling functions satisfying the assumptions of Theorem 3.5 (and hence those of Propositions 3.1 and 3.2), for example, the "min-type" coupling function $\varphi = \mu$ of (48), and the "max-type" coupling function $\varphi_{\max} : R^n \times R^n \to R$ defined by

$$\varphi_{\max}(x, w) := \max_{1 \leq i \leq n} (x_i + w_i) \quad (x = (x_1, ..., x_n), w = (w_1, ..., w_n) \in R^n). \tag{51}$$

Let us recall that by [6], Remark 5.4(c) and [9], Proposition 3.1(b), for the min-type coupling function $\varphi = \mu$ of (48) and any function $f : R^n \to \overline{R}$, $f^{c(\mu)c(\mu)'}$ is the topical hull of f (i.e., the greatest topical minorant of f), and

$$f^{L(\mu)L(\mu)'} = \overline{f}_{\leq}, \tag{52}$$

the increasing lower semicontinuous hull of f. Hence, by Proposition 2.5 for $\varphi = \mu$ (or, alternatively, since by [6], Corollary 5.2, every topical function is increasing and lower semicontinuous), we have

$$f^{c(\mu)c(\mu)'} \leq f^{L(\mu)L(\mu)'}, \tag{53}$$

for all functions $f : R^n \to \overline{R}$. Concerning the equality, from Theorem 3.5 for $\varphi = \mu$ we obtain:

Corollary 3.7. *For any function $f : R^n \to \overline{R}$ the following statements are equivalent:*

1°. *We have*

$$f^{c(\mu)} = f^{L(\mu)}. \tag{54}$$

2°. *We have*

$$f^{c(\mu)c(\mu)'} = f^{L(\mu)L(\mu)'}. \tag{55}$$

3°. *The increasing lower semicontinuous hull $\overline{f}_{\leq}$ of f is topical.*

4°. *$\overline{f}_{\leq}$ satisfies*

$$(\overline{f}_{\leq})^{c(\mu)} = (\overline{f}_{\leq})^{L(\mu)}. \tag{56}$$

References

[1] J.-P. Crouzeix: Contributions à l'Étude des Fonctions Quasi-Convexes, Thèse, Univ. de Clermont (1977).

[2] L. J. Lau: Duality and the structure of utility functions, J. Econ. Theory 1 (1970) 374–396.

[3] I. Ekeland, R. Temam: Convex Analysis and Variational Problems, North-Holland, Amsterdam (1976).

[4] J. E. Martínez-Legaz: Generalized convex duality and its economic applications, in: Handbook of Generalized Convexity and Generalized Monotonicity, N. Hadjisavvas et al. (ed.), Nonconvex Optim. Appl. 76, Springer, New York (2005) 237–292.

[5] R. T. Rockafellar: Convex Analysis, Princeton University Press, Princeton (1970).

[6] A. M. Rubinov, I. Singer: Topical and sub-topical functions, downward sets and abstract convexity, Optimization 50 (2001) 307–351.

[7] I. Singer: Conjugate functionals and level sets, Nonlinear Anal., Theory Methods Appl. 8 (1984) 313–320.

[8] I. Singer: Abstract Convex Analysis, Wiley, New York (1997).

[9] I. Singer: Further applications of the additive min-type coupling function, Optimization 51 (2002) 471–485.

[10] M. Volle: Conjugaison par tranches, Ann. Mat. Pura Appl., IV. Ser. 139 (1985) 279–311.