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Relaxation Results for a Class of Functionals with Linear Growth Defined on Manifold Constrained Mappings

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In this paper we study the lower semicontinuous envelope of a class of functionals with linear growth defined on mappings from the n-dimensional ball into $\mathbb{R}^N$ that are constrained to take values into a smooth submanifold $\mathcal{Y}$ of $\mathbb{R}^N$.

Let B^n be the unit ball in $\mathbb{R}^n$ and $\mathcal{Y}$ a smooth oriented Riemannian manifold of dimension $M \geq 1$, isometrically embedded in $\mathbb{R}^N$ for some $N \geq 2$. We shall assume that $\mathcal{Y}$ is compact, connected, without boundary. In addition, we assume that the integral 1-homology group $H_1(\mathcal{Y}) := H_1(\mathcal{Y}; \mathbb{Z})$ has no torsion.

In this paper we shall be concerned with manifold constrained energy relaxation problems, and we consider variational functionals $\mathcal{F} : L^1(B^n, \mathcal{Y}) \to [0, +\infty]$ of the type

$$\mathcal{F}(u) := \begin{cases} \displaystyle\int_{B^n} f(x, u, Du)\, dx & \text{if } u \in C^1(B^n, \mathcal{Y}) \\ +\infty & \text{otherwise} \end{cases} \tag{1}$$

for a suitable class of integrands $f : B^n \times \mathbb{R}^N \times M(N, n) \to [0, +\infty)$, where $M(N, n)$ is the class of real $(N \times n)$-matrices and, for $X = C^1$, L^1, BV, $W^{1,1}$, we define

$$X(B^n, \mathcal{Y}) := \{u \in X(B^n, \mathbb{R}^N) \mid u(x) \in \mathcal{Y} \text{ for } \mathcal{L}^n\text{-a.e. } x \in B^n\}\,.$$

We introduce the relaxed functional $\overline{\mathcal{F}} : L^1(B^n, \mathcal{Y}) \to [0, +\infty]$ defined for every function $u \in L^1(B^n, \mathcal{Y})$ by

$$\begin{aligned} \overline{\mathcal{F}}(u) := \inf\Big\{ &\liminf_{k\to\infty} \mathcal{F}(u_k) \mid \{u_k\} \subset C^1(B^n, \mathcal{Y}), \\ &u_k \to u \ \text{ strongly in } L^1(B^n, \mathbb{R}^N) \Big\}\,. \end{aligned} \tag{2}$$

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We restrict our analysis to the class of integrands f given by

$$f(x,u,Du) := \widetilde{f}(x,u,(|Du^1|,\dots,|Du^N|))$$

for some function $\widetilde{f} : B^n \times \mathbb{R}^N \times \mathbb{R}^N_+ \to \mathbb{R}_+$, where $\mathbb{R}_+ := [0,+\infty)$, satisfying the following properties:

(a) $z \mapsto \widetilde{f}(x,u,z)$ is convex and lower semicontinuous in $\mathbb{R}^N_+$ for every $(x,u) \in B^n \times \mathbb{R}^N$;

(b) $C_1\,|z| \le \widetilde{f}(x,u,z) \le C_2\,(1+|z|)$ for every $(x,u,z) \in B^n \times \mathbb{R}^N \times \mathbb{R}^N_+$ and some absolute constants $C_i > 0$;

(c) for every $u \in \mathbb{R}^N$ there exists a continuous function $\omega_u : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\omega_u(t) \to 0$ if $t \to 0$, and depending continuously on $u \in \mathbb{R}^N$, such that

$$|\widetilde{f}(x,u,z) - \widetilde{f}(x_0,u,z)| \le \omega_u(|x-x_0|)\cdot(1+|z|) \quad \forall\, z \in \mathbb{R}^N_+ .$$

Assuming that the *first homotopy group* $\pi_1(\mathcal{Y})$ *is commutative*, we will prove that

$$\overline{\mathcal{F}}(u) < +\infty \qquad \Longleftrightarrow \qquad u \in BV(B^n,\mathcal{Y})\,.$$

Moreover, we will show that

$$\overline{\mathcal{F}}(u) = \inf\{\mathcal{E}_f(T) \mid T \in \mathcal{T}^{1,1}_u\} \quad \forall\, u \in BV(B^n,\mathcal{Y})\,. \tag{3}$$

In this formula, $\mathcal{T}^{1,1}_u$ denotes the class of *Cartesian currents* T in $\mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ such that the underlying BV-function u_T is equal to u, according to (5), and $\mathcal{E}_f(T)$ a suitable *f-energy* on T.

To be more precise, see Sec. 3, we recall from [12], [10] that the class $\mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ agrees with the currents that arise as *weak limits* of sequences $\{G_{u_k}\}$ of currents carried by the graphs of smooth maps $u_k \in C^1(B^n,\mathcal{Y})$ with equibounded total variation energies,

$$\sup_k \int_{B^n} |Du_k|\,dx < \infty\,. \tag{4}$$

The weak convergence $G_{u_k} \rightharpoonup T$, with the energy bound (4), yields the weak convergence $u_k \rightharpoonup u_T$ in the BV-sense to some function $u_T \in BV(B^n,\mathcal{Y})$, i.e., $u_k \to u_T$ in $L^1(B^n,\mathbb{R}^N)$ and $Du_k \rightharpoonup Du_T$ weakly as vector-valued measures. This clearly yields that for every $T \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$

$$T(\phi(x,y)\,dx^1 \wedge \dots \wedge dx^n) = \int_{B^n} \phi(x,u_T(x))\,dx \quad \forall\, \phi \in C^\infty_c(B^n \times \mathcal{Y})\,. \tag{5}$$

We will define for every current T in $\mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ a suitable *f-energy* $\mathcal{E}_f(T)$ that is lower semicontinuous and satisfies a density property, see Sec. 4 and Sec. 5, that is, if T_k, $T \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ and $T_k \rightharpoonup T$, then we have

$$\mathcal{E}_f(T) \le \liminf_{k\to\infty} \mathcal{E}_f(T_k)\,,$$

and for every $T \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ we can find a sequence of smooth maps $\{u_k\} \subset C^1(B^n,\mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$, $u_k \rightharpoonup u_T$ weakly in the BV-sense, and

$$\lim_{k\to\infty} \int_{B^n} f(x,u_k,Du_k)\,dx = \mathcal{E}_f(T)\,.$$

These properties yield that *the f-energy agrees with the relaxed energy on currents*, i.e., for every $T \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$

$$\mathcal{E}_f(T) = \inf\left\{ \liminf_{k\to\infty} \int_{B^n} f(x, u_k, Du_k)\, dx \mid \{u_k\} \subset C^1(B^n, \mathcal{Y}),\ G_{u_k} \rightharpoonup T \right\},$$

the weak convergence to be precised in Sec. 1. As a consequence, in Sec. 6 we will then show that the relaxed energy $\overline{\mathcal{F}}(u)$ is finite if and only if $u \in BV(B^n, \mathcal{Y})$, and that (3) holds true.

The f-energy of a current $T \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ is defined in Sec. 3 by

$$\begin{aligned} \mathcal{E}_f(T) := &\int_{B^n} f(x, u_T(x), \nabla u_T(x))\, dx + \int_{B^n} f^\infty\left(x, \widetilde{u}_T(x), \frac{dD^C u_T}{d|D^C u_T|}(x)\right) d|D^C u_T| \\ &+ \int_{J_c(T)} f_T(x)\, d\mathcal{H}^{n-1}(x)\,, \end{aligned} \tag{6}$$

where $f^\infty : B^n \times \mathbb{R}^N \times M(N,n) \to [0, +\infty]$ denotes the *recession* function of f,

$$f^\infty(x, u, G) := \lim_{t\to+\infty} \frac{f(x, u, t\,G) - f(x, u, 0)}{t} \quad \forall\, (x, u, G) \in B^n \times \mathbb{R}^N \times M(N,n)\,.$$

In the last term of formula (6), $J_c(T)$ is the countably $\mathcal{H}^{n-1}$-rectifiable set given by the points of *jump-concentration* of T. Roughly speaking, it is the union of the *Jump set* J_{u_T} of u_T and of the $(n-1)$-rectifiable set where the "homological vertical part" of T lives. Moreover, $f_T(x)$ denotes for any $x \in J_c(T)$ the *minimal "length"* of Lipschitz curves $\gamma : [0,1] \to \mathcal{Y}$ with end points given by the one sided approximate limits $u_T^\pm(x)$ and with image current $\gamma_\# [\![(0,1)]\!]$ equal to the "vertical part" of T over x, the "length" being given for any such $\gamma = (\gamma^1, \dots, \gamma^N)$ by

$$\mathcal{L}_{f,x}(\gamma) := \int_0^1 \widetilde{f}^\infty\big(x, \gamma(t), (|\dot\gamma^1(t)|, \dots, |\dot\gamma^N(t)|)\big)\, dt\,.$$

Note that in the model case $f(x, u, G) = |G|$, or $f(x, u, G) = \sqrt{1 + |G|^2}$, we have $f^\infty(x, u, G) = |G|$ and hence $\mathcal{L}_{f,x}(\gamma)$ agrees with the standard length of γ, compare [12] and [10].

As a consequence, by (3) we will obtain that for every map $u \in BV(B^n, \mathcal{Y})$

$$\begin{aligned} \overline{\mathcal{F}}(u) = &\int_{B^n} f(x, u(x), \nabla u(x))\, dx + \int_{B^n} f^\infty\left(x, \widetilde{u}(x), \frac{dD^C u}{d|D^C u|}(x)\right) d|D^C u| \\ &+ \inf\left\{ \int_{J_c(T)} f_T(x)\, d\mathcal{H}^{n-1}(x) \mid T \in \mathcal{T}_u^{1,1} \right\}. \end{aligned}$$

We also remark that if the target manifold $\mathcal{Y}$ is simply-connected, i.e., if $\pi_1(\mathcal{Y}) = 0$, for every $u \in BV(B^n, \mathcal{Y})$ we have

$$\begin{aligned} \overline{\mathcal{F}}(u) = &\int_{B^n} f(x, u(x), \nabla u(x))\, dx + \int_{B^n} f^\infty\left(x, \widetilde{u}(x), \frac{dD^C u}{d|D^C u|}(x)\right) d|D^C u| \\ &+ \int_{J_u} \Phi_{f,u}(x)\, d\mathcal{H}^{n-1}(x)\,, \end{aligned}$$

where

$$\Phi_{f,u}(x) := \inf\{\mathcal{L}_{f,x}(\gamma) \mid \gamma \in \mathrm{Lip}([0,1],\mathcal{Y}),\ \gamma(0) = u^-(x),\ \gamma(1) = u^+(x)\}.$$

Therefore, in the model case $f(x,u,G) = |G|$, or $f(x,u,G) = \sqrt{1+|G|^2}$, $\Phi_{f,u}(x)$ agrees with the *geodesic distance* between $u^-(x)$ and $u^+(x)$, compare [2].

Moreover, we will show that the commutativity hypothesis on $\pi_1(\mathcal{Y})$ cannot be dropped. Namely, if $\pi_1(\mathcal{Y})$ is not an Abelian group we can find BV-functions $u \in BV(B^2,\mathcal{Y})$, smooth outside the origin, for which property (3) fails to hold; more precisely:

$$\overline{\mathcal{F}}(u) > \int_{B^2} f(x,u,Du)\,dx = \inf\{\mathcal{E}_f(T) \mid T \in \mathcal{T}^{1,1}_u\}.$$

We finally mention that in the case $\mathcal{Y} = \mathbb{S}^1$, the unit sphere of $\mathbb{R}^2$, these results have been obtained in [9], and that the main references for this paper are [12] and the books [8], [10].

1. Preliminary results

In this section we collect a few known facts that are relevant for the sequel.

Vector valued BV-functions. Let $\Omega \subset \mathbb{R}^n$ be an open set and $u : \Omega \to \mathbb{R}^N$ be a function in $BV(\Omega,\mathbb{R}^N)$, i.e., $u = (u^1,\dots u^N)$ with all components $u^j \in BV(\Omega)$. The *Jump set* of u is the countably $\mathcal{H}^{n-1}$-rectifiable set J_u in Ω given by the union of the complements of the Lebesgue sets of the u^j's. Let $\nu = \nu_u(x)$ be a unit vector in $\mathbb{R}^n$ orthogonal to J_u at $\mathcal{H}^{n-1}$-a.e. point $x \in J_u$. Let $u^\pm(x)$ denote the one-sided approximate limits of u on J_u, so that for $\mathcal{H}^{n-1}$-a.e. point $x \in J_u$

$$\lim_{\rho\to 0^+} \rho^{-n} \int_{B^\pm_\rho(x)} |u(x) - u^\pm(x)|\,dx = 0,$$

where $B^\pm_\rho(x) := \{y \in B_\rho(x) : \pm\langle y-x,\nu(x)\rangle \geq 0\}$. Note that a change of sign of ν induces a permutation of u^+ and u^- and that only for scalar functions there is a canonical choice of the sign of ν which ensures that $u^+(x) > u^-(x)$. The distributional derivative of u is the sum of a "gradient" measure, which is absolutely continuous with respect to the Lebesgue measure, of a "jump" measure, concentrated on a set that is σ-finite with respect to the $\mathcal{H}^{n-1}$-measure, and of a "Cantor-type" measure. More precisely,

$$Du = D^a u + D^J u + D^C u,$$

where

$$D^a u = \nabla u \cdot dx, \qquad D^J u = (u^+(x) - u^-(x)) \otimes \nu(x)\,\mathcal{H}^{n-1} \llcorner J_u,$$

$\nabla u := (\nabla_1 u,\dots,\nabla_n u)$ being the approximate gradient of u, compare e.g. [3] or [8, Vol. I]. We also recall that $\{u_k\}$ is said to converge to u *weakly in the BV-sense*, $u_k \rightharpoonup u$, if $u_k \to u$ strongly in $L^1(B^n,\mathbb{R}^N)$ and $Du_k \rightharpoonup Du$ weakly in the sense of (vector-valued) measures.

One-dimensional restrictions of BV-functions. Following [3], given $\nu \in \mathbb{S}^{n-1}$ we denote by π_ν the hyperplane in $\mathbb{R}^n$ orthogonal to ν and by Ω_ν the orthogonal projection of Ω on π_ν. For any $y \in \Omega_\nu$ we let

$$\Omega^\nu_y := \{t \in \mathbb{R} \mid y + t\nu \in \Omega\}$$

denote the (non-empty) section of Ω corresponding to y. Accordingly, for any function $u : B \subset \Omega \to \mathbb{R}^N$ and any $y \in B_\nu$ the function $u^\nu_y : B^\nu_y \to \mathbb{R}^N$ is defined by

$$u^\nu_y(t) := u(y + t\nu) .$$

Proposition 1.1. *Let $u \in L^1(\Omega, \mathbb{R}^N)$. Then $u \in BV(\Omega, \mathbb{R}^N)$ if and only if there exist n linearly independent unit vectors ν_i such that $u^{\nu_i}_y \in BV(\Omega^{\nu_i}_y, \mathbb{R}^N)$ for $\mathcal{L}^{n-1}$-a.e. $y \in \Omega_{\nu_i}$ and*

$$\int_{\Omega_{\nu_i}} |Du^{\nu_i}_y|(\Omega^{\nu_i}_y)\, d\mathcal{L}^{n-1}(y) < \infty \quad \forall\, i = 1, \dots, n .$$

Theorem 1.2. *If $u \in BV(\Omega, \mathbb{R}^N)$ and $\nu \in \mathbb{S}^{n-1}$, then*

$$\begin{aligned} \langle Du, \nu\rangle &= \mathcal{L}^{n-1} \llcorner \Omega_\nu \otimes Du^\nu_y , & \langle D^a u, \nu\rangle &= \mathcal{L}^{n-1} \llcorner \Omega_\nu \otimes D^a u^\nu_y , \\ \langle D^J u, \nu\rangle &= \mathcal{L}^{n-1} \llcorner \Omega_\nu \otimes D^J u^\nu_y , & \langle D^C u, \nu\rangle &= \mathcal{L}^{n-1} \llcorner \Omega_\nu \otimes D^C u^\nu_y . \end{aligned}$$

In addition, for $\mathcal{L}^{n-1}$-a.e. $y \in \Omega_\nu$ the precise representative *u^* has classical directional derivatives along ν $\mathcal{L}^1$-a.e. in Ω^ν_y, the function $(u^*)^\nu_y$ is a good representative in the equivalence class of u^ν_y, its Jump set is $(J_u)^\nu_y$ and*

$$\frac{\partial u^*}{\partial \nu}(y + t\nu) = \langle \nabla u(y + t\nu), \nu\rangle \quad \textit{for } \mathcal{L}^1\textit{-a.e. } t \in \Omega^\nu_y .$$

Finally, $\sigma(t) := \langle \nu, \nu_u(y + t\nu)\rangle \neq 0$ for $\mathcal{L}^{n-1}$-a.e. $y \in \Omega_\nu$ and $\mathcal{L}^1$-a.e. $t \in \Omega^\nu_y$, and

$$\begin{cases} \lim\limits_{s\downarrow t} u^*(y + s\nu) = u^+(y + t\nu) , & \lim\limits_{s\uparrow t} u^*(y + s\nu) = u^-(y + t\nu) \quad \textit{if } \sigma(t) > 0 \\ \lim\limits_{s\downarrow t} u^*(y + s\nu) = u^-(y + t\nu) , & \lim\limits_{s\uparrow t} u^*(y + s\nu) = u^+(y + t\nu) \quad \textit{if } \sigma(t) < 0 . \end{cases}$$

$\mathcal{D}_{\mathbf{n},\mathbf{1}}$-currents. Let B^n be the unit ball in $\mathbb{R}^N$ and, we recall,

$$BV(B^n, \mathcal{Y}) := \{u \in BV(B^n, \mathbb{R}^N) \mid u(x) \in \mathcal{Y} \ \text{ for } \mathcal{L}^n\text{-a.e. } x \in B^n\} .$$

To every BV-map $u \in BV(B^n, \mathcal{Y})$ we associate a suitable family of currents in the class $\mathcal{D}_{n,1}(B^n \times \mathcal{Y})$, i.e., of linear functionals acting on $\mathcal{D}^{n,1}(B^n \times \mathcal{Y})$. Here, $\mathcal{D}^{p,1}(B^n \times \mathcal{Y})$ denotes the class of smooth compactly supported p-forms in $B^n \times \mathcal{Y}$ with *at most one differential in the vertical $\mathcal{Y}$-direction.* Therefore, every $\omega \in \mathcal{D}^{n,1}(B^n \times \mathcal{Y})$ splits as

$$\omega = \omega^{(0)} + \omega^{(1)}$$

according to the number of y-differentials, where

$$\omega^{(0)} = \phi(x, y) dx , \qquad dx := dx^1 \wedge \dots \wedge dx^n , \tag{7}$$

for some $\phi \in C^\infty_c(B^n \times \mathcal{Y})$, and

$$\omega^{(1)} = \sum_{j=1}^N \sum_{i=1}^n (-1)^{n-i} \phi^j_i(x, y)\, \widehat{dx^i} \wedge dy^j \tag{8}$$

for some $\phi^j := (\phi^j_1, \dots, \phi^j_n) \in C^\infty_0(B^n \times \mathcal{Y}, \mathbb{R}^n)$, where

$$\widehat{dx^i} := dx^1 \wedge \dots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \dots \wedge dx^n .$$

Definition 1.3. A current $G \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ is said to be in BV −graph$(B^n \times \mathcal{Y})$ if it decomposes into its absolutely continuous, Cantor, and Jump parts

$$G := G^a + G^C + G^J$$

and the following holds:

i) there exists a function $u = u(G) \in BV(B^n, \mathcal{Y})$, say $u = (u^1, \dots, u^N)$, such that if $\omega^{(0)}$ is of the type (7), we have $G^C(\omega^{(0)}) = G^J(\omega^{(0)}) = 0$ and

$$G^a(\omega^{(0)}) = G^a(\phi(x, y)\, dx) := \int_{B^n} \phi(x, u(x))\, dx\,;$$

ii) if $\omega = \omega^{(1)}$ satisfies (8), and $\widetilde{u}(x)$ is a *good representative*, we have

$$G^a(\omega^{(1)}) := \sum_{j=1}^{N} \int_{B^n} \langle \nabla u^j, \phi^j(x, u(x)) \rangle\, dx$$

$$G^C(\omega^{(1)}) := \sum_{j=1}^{N} \int_{B^n} \phi^j(x, \widetilde{u}(x))\, dD^C u^j$$

$$G^J(\omega^{(1)}) := \sum_{j=1}^{N} \sum_{i=1}^{n} \int_{J_u} \left(\int_{\gamma_x} \phi_i^j(x, y)\, dy^j \right) \nu_i\, d\mathcal{H}^{n-1}(x)\,,$$

where γ_x is a 1-dimensional integral chain in $\mathcal{Y}$ satisfying $\partial\gamma_x = \delta_{u^+(x)} - \delta_{u^-(x)}$ and δ_p denotes the unit Dirac mass at the point $p \in \mathbb{R}^N$.

The previous definition clearly depends on the choice of the γ_x's connecting the one-sided approximate limits $u^\pm(x)$ at $x \in J_u$. Moreover, if u is smooth, at least $u \in W^{1,1}(B^n, \mathcal{Y})$, it turns out that $G = G^a$ and hence G agrees with the current G_u carried by the *rectifiable graph* of u, where, we recall, $G_u \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ is defined in an approximate sense by

$$G_u := (Id \bowtie u)_\# [\![B^n]\!]\,, \tag{9}$$

i.e., by letting $G_u(\omega) = (Id \bowtie u)^\#(\omega)$ for every $\omega \in \mathcal{D}^{n,1}(B^n \times \mathcal{Y})$, where $(Id \bowtie u)(x) := (x, u(x))$.

Remark 1.4. If $n \geq 2$ in general the current G has a non-zero boundary in $B^n \times \mathcal{Y}$, even if $u \in W^{1,1}(B^n, \mathcal{Y})$. Taking for example $n = 2$, $\mathcal{Y} = \mathbb{S}^1 \subset \mathbb{R}^2$, and $u(x) = x/|x|$, we have

$$\partial G \llcorner B^2 \times \mathbb{S}^1 = -\delta_0 \times [\![\mathbb{S}^1]\!]\,,$$

where δ_0 is the unit Dirac mass at the origin, see [8, Vol. I, Sec. 3.2.2].

Weak limits of smooth graphs. A first step in the study of our relaxation results is the analysis of the weak limits of sequences $\{G_{u_k}\}$ of currents carried by the graphs of smooth maps $u_k \in C^1(B^n, \mathcal{Y})$ with equibounded $W^{1,1}$-energies, $\sup_k \|Du_k\|_{L^1} < \infty$, compare [12] [10]. Possibly passing to a subsequence, we infer that $G_{u_k} \rightharpoonup T$ *weakly in* $\mathcal{D}_{n,1}$ to some current $T \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$, i.e.,

$$\lim_{k\to\infty} G_{u_k}(\omega) = T(\omega) \quad \forall\, \omega \in \mathcal{D}^{n,1}(B^n \times \mathcal{Y})\,,$$

and $u_k \rightharpoonup u_T$ weakly in the BV-sense to some function $u_T \in BV(B^n, \mathcal{Y})$. This yields that if $\omega \in \mathcal{D}^{n,1}(B^n \times \mathcal{Y})$ is completely horizontal, see (7), we clearly obtain (5).

Since by Stokes theorem the G_{u_k}'s have no boundary in $B^n \times \mathcal{Y}$, by the weak convergence we also infer

$$\partial T = 0 \quad \text{on } \mathcal{Z}^{n-1,1}(B^n \times \mathcal{Y}), \tag{10}$$

where, we recall, $\partial T(\omega) := T(d\omega)$. Here, $\mathcal{Z}^{p,1}(B^n \times \mathcal{Y})$ denotes the class of p-forms ω in $\mathcal{D}^{p,1}(B^n \times \mathcal{Y})$ for which the "vertical" d_y-differential of the component $\omega^{(1)}$ of ω with exactly one vertical differential vanishes,

$$\mathcal{Z}^{p,1}(B^n \times \mathcal{Y}) := \{\omega \in \mathcal{D}^{p,1}(B^n \times \mathcal{Y}) \mid d_y\omega^{(1)} = 0\}.$$

We may and do associate to the weak limit current T a current $G_T \in BV-\text{graph}(B^n \times \mathcal{Y})$, see Definition 1.3, where the function $u = u(G_T) \in BV(B^n, \mathcal{Y})$ is given by u_T and the γ_x's in the definition of the jump part G_T^J are e.g. the indecomposable 1-dimensional integral chains in $\mathcal{Y}$ obtained as in Definition 3.2 below. Setting then

$$S_T := T - G_T,$$

by (5) we clearly have $S_T(\phi(x,y)\,dx) = 0$ for every $\phi \in C_c^\infty(B^n \times \mathcal{Y})$. In general $\partial G_T \llcorner B^n \times \mathcal{Y} \neq 0$, see Remark 1.4. However, on account of (10) we proved in [12]:

Proposition 1.5. *$S_T(\omega) = 0$ for every form $\omega = \omega^{(1)}$ such that $\omega = d_y\widetilde{\omega}$ for some $\widetilde{\omega} \in \mathcal{D}^{n-1,0}(B^n \times \mathcal{Y})$.*

Homological facts. Since $H_1(\mathcal{Y})$ has no torsion, there are generators $[\gamma_1], \dots, [\gamma_{\overline{s}}]$, i.e. integral 1-cycles in $\mathcal{Z}_1(\mathcal{Y})$, such that

$$H_1(\mathcal{Y}) = \left\{ \sum_{s=1}^{\overline{s}} n_s\,[\gamma_s] \mid n_s \in \mathbb{Z} \right\},$$

see e.g. [8], Vol. I, Sec. 5.4.1. By de Rham's theorem the first real homology group is in duality with the first cohomology group $H^1_{dR}(\mathcal{Y})$, the duality being given by the natural pairing

$$\langle [\gamma], [\omega] \rangle := \gamma(\omega) = \int_\gamma \omega, \quad [\gamma] \in H_1(\mathcal{Y}; \mathbb{R}), \quad [\omega] \in H^1_{dR}(\mathcal{Y}).$$

We will then denote by $[\omega^1], \dots, [\omega^{\overline{s}}]$ a dual basis in $H^1_{dR}(\mathcal{Y})$ so that $\gamma_s(\omega^r) = \delta_{sr}$, where δ_{sr} denotes the Kronecker symbols. Finally, in the sequel $\pi : \mathbb{R}^{n+N} \to \mathbb{R}^n$ and $\widehat{\pi} : \mathbb{R}^{n+N} \to \mathbb{R}^N$ shall denote the projections onto the first n and the last N coordinates, respectively.

By Proposition 1.5, similarly to [8], Vol. II, Sec. 5.4.3, we infer that the weak limit current T is given by

$$T = G_T + S_T, \quad \text{where } S_T = \sum_{s=1}^{\overline{s}} \mathbb{L}_s(T) \times \gamma_s \quad \text{on } \mathcal{Z}^{n,1}(B^n \times \mathcal{Y}),$$

$\mathbb{L}_s(T) \in \mathcal{D}_{n-1}(B^n)$ being defined by

$$\mathbb{L}_s(T) := (-1)^{n-1}\pi^\#(S_T \llcorner \widehat{\pi}^\#\omega^s), \quad s = 1, \dots, \overline{s},$$

so that

$$\mathbb{L}_s(T)(\phi) = S_T(\pi^{\#}\phi \wedge \widehat{\pi}^{\#}\omega^s) \quad \forall\, \phi \in \mathcal{D}^{n-1}(B^n)\,.$$

By the equiboundedness of the $W^{1,1}$-energies of the u_k's, and by the lower semicontinuity of the $\mathcal{E}_{1,1}$-*norm* in $\mathcal{D}_{n,1}$, we finally infer that the weak limit current T has finite $\mathcal{E}_{1,1}$-norm, $\|T\|_{\mathcal{E}_{1,1}} < +\infty$, where, for $\omega \in \mathcal{D}^{n,1}(B^n \times \mathcal{Y})$ and $T \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$,

$$\begin{aligned} \|\omega\|_{\mathcal{E}_{1,1}} &:= \max\left\{ \sup_{x,y} \frac{|\omega^{(0)}(x,y)|}{1+|y|}\,,\, \int_{B^n} \sup_y |\omega^{(1)}(x,y)|\,dx \right\}\,, \\ \|T\|_{\mathcal{E}_{1,1}} &:= \sup\Big\{ T(\omega) \mid \omega \in \mathcal{D}^{n,1}(B^n \times \mathcal{Y})\,,\ \|\omega\|_{\mathcal{E}_{1,1}} \le 1 \Big\}\,. \end{aligned} \tag{11}$$

Remark 1.6. Setting

$$S_{T,sing} := T - G_T - \sum_{s=1}^{\bar{s}} \mathbb{L}_s(T) \times \gamma_s\,,$$

it turns out that $S_{T,sing}$ is nonzero only possibly on forms ω with non-zero vertical component, $\omega^{(1)} \neq 0$, and such that $d_y\omega^{(1)} \neq 0$. Therefore, $S_{T,sing} \equiv 0$ on forms in $\mathcal{Z}^{n,1}(B^n \times \mathcal{Y})$, hence $S_{T,sing}$ does not carry homology. However, even if T is the weak limit of a sequence of graphs of smooth maps with equibounded BV-norms, in principle $S_{T,sing}$ may be any measure.

Vertical homology classes. Finally, it is convenient to consider *vertical homology equivalence classes* of currents satisfying the same structure properties as weak limits of graphs of smooth maps $u_k : B^n \to \mathcal{Y}$ with equibounded total variation, $\sup_k \|Du_k\|_{L^1} < \infty$. More precisely, we say that

$$T \sim \widetilde{T} \iff T(\omega) = \widetilde{T}(\omega) \quad \forall\, \omega \in \mathcal{Z}^{n,1}(B^n \times \mathcal{Y})\,. \tag{12}$$

Moreover, we will say that $T_k \rightharpoonup T$ *weakly in* $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ if $T_k(\omega) \to T(\omega)$ for every $\omega \in \mathcal{Z}^{n,1}(B^n \times \mathcal{Y})$.

Definition 1.7. We denote by $\mathcal{E}_{1,1}$ $-$graph$(B^n \times \mathcal{Y})$ the set of equivalence classes, in the sense of (12), of currents T in $\mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ which have no interior boundary,

$$\partial T = 0 \quad \text{on } \mathcal{Z}^{n-1,1}(B^n \times \mathcal{Y})\,,$$

finite $\mathcal{E}_{1,1}$-norm, i.e.

$$\|T\|_{\mathcal{E}_{1,1}} := \sup\Big\{ T(\omega) \mid \omega \in \mathcal{Z}^{n,1}(B^n \times \mathcal{Y})\,,\ \|\omega\|_{\mathcal{E}_{1,1}} \le 1 \Big\} < \infty\,,$$

and decompose as

$$T = G_T + S_T\,, \qquad S_T = \sum_{s=1}^{\bar{s}} \mathbb{L}_s(T) \times \gamma_s \quad \text{on } \mathcal{Z}^{n,1}(B^n \times \mathcal{Y})\,, \tag{13}$$

where $G_T \in BV$ $-$graph$(B^n \times \mathcal{Y})$, see Definition 1.3, and $\mathbb{L}_s(T)$ is an i.m. rectifiable current in $\mathcal{R}_{n-1}(B^n)$ for every s.

Remark 1.8. If $\widetilde{T} \sim T$, in general $G_{\widetilde{T}} \neq G_T$. However, the corresponding BV-functions coincide, i.e., $u(G_T) = u(G_{\widetilde{T}})$, see Definition 1.3. This yields that we may refer to the underlying functions $u_T \in BV(B^n, \mathcal{Y})$ associated to currents T in $\mathcal{E}_{1,1}$ −graph$(B^n \times \mathcal{Y})$.

2. Parametric variational integrals on currents

In this section we shall consider integrands $f : B^n \times \mathbb{R}^N \times M(N,n) \to \mathbb{R}_+$, where $M(N,n)$ is the class of real $(N \times n)$-matrices and $\mathbb{R}_+ := [0, +\infty)$, satisfying the following properties:

(a') $G \mapsto f(x,u,G)$ is convex and lower semicontinuous in $M(N,n)$ for every $(x,u) \in B^n \times \mathbb{R}^N$;

(b') $C_1 \, |G| \le f(x,u,G) \le C_2 \, (1 + |G|)$ for every $(x,u,G) \in B^n \times \mathbb{R}^N \times M(N,n)$ and for some absolute constants $C_i > 0$;

(c') for every $(x_0, u_0) \in B^n \times \mathbb{R}^N$ and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(x,u,G) \ge (1 - \varepsilon) \, f(x_0, u_0, G) \quad \forall \, G \in M(N,n)$$

for every $(x,u) \in B^n \times \mathbb{R}^N$ such that $|x - x_0| < \delta$, and $|u - u_0| < \delta$.

We shall discuss the *parametric polyconvex l.s.c. extension* of f for mappings from B^n into $\mathbb{R}^N$ that are constrained to take values into a smooth submanifold $\mathcal{Y} \subset \mathbb{R}^N$, and the related *parametric variational integral* for currents in $\mathcal{E}_{1,1}$ −graph$(B^n \times \mathcal{Y})$.

The recession function. Property (a') allows us to give the

Definition 2.1. The *recession function* $f^\infty : B^n \times \mathbb{R}^N \times M(N,n) \to \overline{\mathbb{R}}_+ := [0, +\infty]$ of f is defined by

$$f^\infty(x,u,G) := \lim_{t \to +\infty} \frac{f_{x,u}(t\,G) - f_{x,u}(0)}{t} \quad \forall \, (x,u,G) \in B^n \times \mathbb{R}^N \times M(N,n) \, .$$

It turns out that for every (x,u) the function $G \mapsto f^\infty_{x,u}(G) := f^\infty(x,u,G)$ is positively homogeneous of degree one, convex and lower semicontinuous. Moreover, property (b') yields that $G \mapsto f^\infty_{x,u}(G) := f^\infty(x,u,G)$ is actually real valued and hence continuous for every (x,u).

Notation on multivectors. Let $\Lambda_n \mathbb{R}^{n+N}$ denote the space of n-vectors in $\mathbb{R}^{n+N}$. For $0 \le k \le \min(n,N)$, we let

$$V_{n,k} := \Lambda_{n-k} \mathbb{R}^n \otimes \Lambda_k \mathbb{R}^N$$

and denote by $\xi_{(k)} \in V_{n,k}$ the "component" of an n-vector $\xi \in \Lambda_n \mathbb{R}^{n+N}$ with k "vertical" components. For example, denoting by $(e_1, \dots, e_n)$ and $(\varepsilon_1, \dots, \varepsilon_N)$ the canonical basis in $\mathbb{R}^n$ and $\mathbb{R}^N$, respectively,

$$\xi_{(0)} = \xi^{\bar{0}0} e_1 \wedge \cdots \wedge e_n \, ,$$

$\xi^{\bar{0}0} \in \mathbb{R}$ being the *first component* of $\xi \in \Lambda_n \mathbb{R}^{n+N}$. Moreover, we have

$$\xi_{(1)} = \sum_{j=1}^N \sum_{i=1}^n \xi_i^j \, \widehat{e_i} \wedge \varepsilon_j \, , \qquad \widehat{e_i} := e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_n \, ,$$

for some $\xi_i^j \in \mathbb{R}$. We set

$$\begin{aligned}\Lambda_1 &:= \{\xi \in \Lambda_n\mathbb{R}^{n+N} \mid \xi^{\bar{0}0} = 1\}\\ \Lambda_+ &:= \{\xi \in \Lambda_n\mathbb{R}^{n+N} \mid \xi^{\bar{0}0} > 0\}\\ \Lambda_0 &:= \{\xi \in \Lambda_n\mathbb{R}^{n+N} \mid \xi^{\bar{0}0} = 0\}\,.\end{aligned}$$

We also denote by Σ the class of *simple* n-vectors in $\Lambda_n\mathbb{R}^{n+N}$ and

$$\begin{aligned}\Sigma_1 &:= \{\xi \in \Sigma \mid \xi^{\bar{0}0} = 1\}\\ \Sigma_+ &:= \{\xi \in \Sigma \mid \xi^{\bar{0}0} > 0\}\,.\end{aligned}$$

If $G \in M(N,n)$, the vectors $e_i + Ge_i \in \mathbb{R}^{n+N}$, $i = 1,\dots,n$, yield a basis of the tangent n-plane to the graph of G in $\mathbb{R}^{n+N}$ that agrees with the graph of G. Letting

$$M(G) := (e_1 + Ge_1) \wedge \cdots \wedge (e_n + Ge_n) \in \Lambda_n\mathbb{R}^{n+N},$$

we find that the unit simple n-vector

$$\xi_G := \frac{M(G)}{|M(G)|}\,,$$

called the *tangent n-vector to the graph of G*, identifies the plane graph of G, and in fact orients such an n-plane. We also see that the map $M : M(N,n) \to \Lambda_n\mathbb{R}^{n+N}$ given by $G \mapsto M(G)$ is injective. Moreover, if $M_{(k)} : M(N,n) \to V_{n,k}$ is the map given by $G \mapsto M_{(k)}(G)$, it turns out that $M_{(1)}$ yields an *isometry* of linear spaces. Notice that $M_{(0)}(G) = e_1 \wedge \cdots \wedge e_n$ and

$$M_{(1)}(G) = \sum_{j=1}^{N}\sum_{i=1}^{n}(-1)^{n-i}\, G_i^j\, \widehat{e_i} \wedge \varepsilon_j\,, \qquad G = (G_i^j)_{j,i=1}^{N,n}\,.$$

Therefore, to every $\xi \in \Lambda_+$ we can associate the matrix $G_\xi \in M(N,n)$ defined by

$$G_\xi := {M_{(1)}}^{-1}\left(\frac{\xi_{(1)}}{\xi^{\bar{0}0}}\right).$$

Note that for every $\xi \in \Lambda_+$ we have

$$G_\xi = 0 \quad \text{if and only if } \xi_{(1)} = 0$$

and

$$G_{\lambda\xi} = G_\xi \quad \forall\,\lambda > 0\,.$$

Most importantly, $G_\xi = M^{-1}(\xi)$ if $\xi \in \Sigma_1$, i.e.

$$\begin{cases} G_{M(G)} = G \quad \forall\, G \in M(N,n) \\ \xi = M(G_\xi) \quad \iff \quad \xi \in \Sigma_1 \end{cases} \tag{14}$$

and finally

$$\xi \in \Lambda_+ \quad \text{is simple if and only if } \frac{\xi}{\xi^{\bar{0}0}} = M(G_\xi)\,.$$

The parametric polyconvex l.s.c. envelope. Consider the map

$$\underline{f} : \Omega \times \mathbb{R}^N \times \Sigma_1 \to \overline{\mathbb{R}}_+$$

defined, according to (14), by

$$\underline{f}(x, u, \xi) := f(x, u, G_\xi) .$$

Taking x, u as parameters, we consider the parametric polyconvex l.s.c. envelope of the integrand $G \mapsto f_{x,u}(G) := f(x, u, G)$, given by the convex l.s.c. envelope of $\xi \mapsto \overline{f}_{x,u}(\xi)$,

$$F_{x,u}(\cdot) := \Gamma C \overline{f}_{x,u}(\cdot) ,$$

where

$$\overline{f}_{x,u}(\xi) := \begin{cases} \xi^{\bar{0}0} \underline{f}(x, u, \xi/\xi^{\bar{0}0}) = \xi^{\bar{0}0} f_{x,u}(G_\xi) & \text{if } \xi \in \Sigma_+ \\ +\infty & \text{otherwise} . \end{cases}$$

Since in principle $(x, u, \xi) \mapsto F_{x,u}(\xi)$ is not l.s.c., we set

Definition 2.2. The *parametric polyconvex l.s.c. envelope of* a function $f : B^n \times \mathbb{R}^N \times M(N, n) \to \overline{\mathbb{R}}_+$ is the function $F : B^n \times \mathbb{R}^N \times \Lambda_n \mathbb{R}^{n+N} \to \overline{\mathbb{R}}_+$ defined by

$$\begin{aligned} F(x, u, \xi) := \sup\{ g(x, u, \xi) \mid\ & g : B^n \times \mathbb{R}^N \times \Lambda_n \mathbb{R}^{n+N} \to \overline{\mathbb{R}}_+ , \\ & g \text{ is l.s.c.} ,\ \ g(y, v, \cdot) \text{ is convex for any } y, v , \\ & g(y, v, \eta) \le \overline{f}_{y,v}(\eta) \text{ for any } (y, v, \eta) \} . \end{aligned}$$

We emphasize that $F(x, u, \xi)$ is l.s.c. in all variables and convex in ξ for any x, u. Notice that in general $F(x, u, \xi) \le \Gamma C \overline{f}_{x,u}(\xi)$. However, the equality

$$F(x, u, \xi) = \Gamma C \overline{f}_{x,u}(\xi) \tag{15}$$

holds if and only if $(x, u, \xi) \mapsto \Gamma C \overline{f}_{x,u}(\xi)$ is l.s.c., and we actually have that property (c') yields (15). Moreover, for every $(x, u) \in B^n \times \mathbb{R}^N$ we have

$$\begin{aligned} F(x, u, \xi) = \sup\{ \phi(\xi) \mid\ & \phi : \Lambda_n \mathbb{R}^{n+N} \to \overline{\mathbb{R}}_+ ,\ \phi \text{ linear}, \\ & \phi(M(G)) \le f(x, u, G) \quad \forall\, G \in M(N, n) \} \end{aligned} \tag{16}$$

for every $\xi \in \Lambda_n \mathbb{R}^{n+N}$. Arguing as in [8, Vol. II], we then obtain:

Proposition 2.3. *If the integrand f satisfies the properties* (a'), (b'), (c')*, its parametric polyconvex l.s.c. envelope is given for every $(x, u) \in B^n \times \mathbb{R}^N$ by*

$$\|\xi\|_{f_{x,u}} := F(x, u, \xi) = \begin{cases} \xi^{\bar{0}0} f(x, u, G_\xi) & \textit{if } \ \xi \in \Sigma_+ \\ f^\infty\big(x, u, M_{(1)}^{-1}(\xi_{(1)})\big) & \textit{if } \ \xi \in (\Lambda_+ \cup \Lambda_0) \setminus \Sigma_+ \\ +\infty & \textit{otherwise.} \end{cases} \tag{17}$$

Proof. On account of (16), we decompose every linear map $\phi : \Lambda_n\mathbb{R}^{n+N} \to \overline{\mathbb{R}}_+$ as $\phi = \sum_{k=0}^{\underline{n}} \phi_k$, where $\underline{n} := \min(n,N)$ and the ϕ_k's are linear maps on $V_{n,k}$. The condition $\phi(M(G)) \leq f(x,u,G) \leq C_2(1+|G|)$ for all $G \in M(N,n)$ yields by homogeneity that $\phi_k = 0$ for $k = 2,\dots,\underline{n}$ and

$$\phi_0(\overrightarrow{e}) + \phi_1(M_{(1)}(G)) \leq f(x,u,G) \quad \forall\, G \in M(N,n)\,,$$

where $\overrightarrow{e} := e_1 \wedge \cdots \wedge e_n$, so that for any $\xi \in \Lambda_1$ we have

$$\begin{aligned}\|\xi\|_{f_{x,u}} = \sup\big\{ a + \phi_1(\xi_{(1)}) \mid a \in \mathbb{R}\,,\ \phi_1 : V_{n,1} \to \overline{\mathbb{R}}_+ \text{ linear},\\ a + \phi_1(M_{(1)}(G)) \leq f(x,u,G) \quad \forall\, G \in M(N,n)\big\}\end{aligned}$$

and hence $\|\xi\|_{f_{x,u}} = F(x,u,M(G_\xi))$, see (16), if we take into account that $\xi_{(0)} = M_{(0)}(G_\xi) = \overrightarrow{e}$ and $\xi_{(1)} = M_{(1)}(G_\xi)$. On the other hand the maps $G \mapsto a + \phi_1(M_{(1)}(G))$ are affine and $G \mapsto f(x,u,G)$ is convex. Then, the maximum of $a + \phi_1(M_{(1)}(G))$ under the constraint $a + \phi_1(M_{(1)}(G)) \leq f(x,u,G)$ is taken for a and ϕ_1 such that

$$a + \phi_1(M_{(1)}(G)) = f^\infty(x,u,G)\,.$$

Therefore,

$$\|\xi\|_{f_{x,u}} = F(x,u,M(G_\xi)) = f^\infty(x,u,G_\xi)$$

and the claim is proved for $\xi \in \Lambda_1$. By homogeneity we get $\|\xi\|_{f_{x,u}} = f^\infty\big(x,u,M_{(1)}^{-1}(\xi_{(1)})\big)$ for every $\xi \in \Lambda_+$. The continuity of $\xi \mapsto f^\infty\big(x,u,M_{(1)}^{-1}(\xi_{(1)})\big)$ yields the result also for $\xi \in \Lambda_0$. $\square$

Remark 2.4. In the model case $f(x,u,G) := |G|$, or $f(x,u,G) := \sqrt{1+|G|^2}$, we clearly have

$$\|\xi\|_{f_{x,u}} = \begin{cases} \xi^{\bar{0}0} f(x,u,G_\xi) & \text{if } \xi \in \Sigma_+ \\ |\xi_{(1)}| & \text{if } \xi \in (\Lambda_+ \cup \Lambda_0) \setminus \Sigma_+ \\ +\infty & \text{otherwise.} \end{cases}$$

We deal with mappings that are constrained to take values into a smooth manifold $\mathcal{Y}$ isometrically embedded in $\mathbb{R}^N$. To this purpose, we replace the integrand f in Definition 2.2 with the integrand $\widehat{f} : B^n \times \mathbb{R}^N \times \Lambda_n\mathbb{R}^{n+N} \to \overline{\mathbb{R}}_+$ defined by

$$\widehat{f}(x,u,G) := \begin{cases} f(x,u,G) & \text{if } u \in \mathcal{Y} \text{ and } G \in S_u \\ +\infty & \text{otherwise}\,, \end{cases}$$

where

$$S_u := \{G \in M(N,n) \mid G \in T_u\mathcal{Y}\}\,, \quad u \in \mathcal{Y}\,,$$

$T_u\mathcal{Y}$ being the tangent space to $\mathcal{Y}$ at u. We denote by $F_f(x,u,\xi) : B^n \times \mathbb{R}^N \times \Lambda_n\mathbb{R}^{n+N} \to \overline{\mathbb{R}}_+$ the parametric polyconvex l.s.c. extension of the integrand $\widehat{f}$, i.e., for mappings from B^n into $\mathbb{R}^N$ that are constrained to take values into the given submanifold $\mathcal{Y} \subset \mathbb{R}^N$. The n-vector $M(G)$ corresponding to matrices $G \in S_u$ belongs to the subspace $\Lambda_n(\mathbb{R}^N \times T_u\mathcal{Y})$. This yields to the following property, compare [8, Vol. II, Sec. 1.2.4] or [10, Sec. 4.8].

Proposition 2.5. *We have:*

$$F_f(x,u,\xi) := \begin{cases} \|\xi\|_{f_{x,u}} & \text{if } u \in \mathcal{Y},\ \xi \in \Lambda_n(\mathbb{R}^n \times T_u\mathcal{Y}) \\ +\infty & \text{otherwise,} \end{cases} \tag{18}$$

where $\|\xi\|_{f_{x,u}}$ is given by (17) and $T_u\mathcal{Y}$ is the tangent space to $\mathcal{Y}$ at u.

Parametric variational integral. If $T \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ is such that $\|T\|_{\mathcal{E}_{1,1}} < \infty$, we denote

$$T = \|T\|_{\mathcal{E}_{1,1}} \llcorner \overrightarrow{T}$$

the Radon-Nikodym decomposition of T with respect to the $\mathcal{E}_{1,1}$-norm, see (11). Here T is identified with the $\mathbb{R}^{1+Nn}$-valued linear functional

$$T := \left(T^{\bar{0}0}, (T^{\bar{i}j})_{\mathbb{R}^{Nn}}\right), \quad i = 1, \dots n, \quad j = 1, \dots N.$$

The *parametric variational integral* associated to the integrand f is defined for every Borel set $B \subset B^n$

$$\mathcal{F}_f(T, B \times \mathcal{Y}) := \int_{B\times\mathcal{Y}} F_f\left(\pi(z), \widehat{\pi}(z), \overrightarrow{T}(z)\right) d\|T\|_{\mathcal{E}_{1,1}}(z) \tag{19}$$

where $F_f(x,u,\xi)$ is given by (18), and we let

$$\mathcal{F}_f(T) := \mathcal{F}_f(T, B^n \times \mathcal{Y}).$$

Since $\|T\|_{\mathcal{E}_{1,1}} < \infty$, by property (b') we infer that $\mathcal{F}_f(T) < \infty$. Moreover, the following lower semicontinuity property trivially holds.

Proposition 2.6. *Let $\{T_k\} \subset \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ be such that* $\sup_k \|T_k\|_{\mathcal{E}_{1,1}} < \infty$ *and $T_k \rightharpoonup T$ weakly in $\mathcal{D}_{n,1}$. Then*

$$\mathcal{F}_f(T) \le \liminf_{k\to\infty} \mathcal{F}_f(T_k).$$

An explicit formula. If $T = G \in BV-\text{graph}(B^n \times \mathcal{Y})$ for some BV-function $u = u(G) \in BV(B^n, \mathcal{Y})$ with no jump-part, $|D^J u|(B^n) = 0$, see Definition 1.3, it is readily checked that

$$\mathcal{F}_f(T, B \times \mathcal{Y}) = \int_B f(x, u(x), \nabla u(x))\, dx + \int_B f^\infty\left(x, \widetilde{u}(x), \frac{dD^C u}{d|D^C u|}(x)\right) d|D^C u|. \tag{20}$$

If $T \in \mathcal{E}_{1,1}-\text{graph}(B^n \times \mathcal{Y})$ and the singular part $S_{T,sing}$ vanishes, i.e., if (13) holds on the whole of $\mathcal{D}^{n,1}(B^n \times \mathcal{Y})$, then an explicit formula can be obtained. For example, let $\mathcal{Y} = \mathbb{S}^1 \subset \mathbb{R}^2$, the unit circle, and $G = G^a = (Id \bowtie u)_\# [\![B^n]\!]$ for some $u \in W^{1,1}(B^n, \mathcal{Y})$, i.e., $G^C = G^J = 0$. In this case

$$T = G_u + L \times [\![\mathbb{S}^1]\!]$$

for some i.m. rectifiable current $L \in \mathcal{R}_{n-1}(B^n)$, say $L = \tau(\mathcal{L}, \theta, \overrightarrow{\mathcal{L}})$. If f is *isotropic*, i.e., $f(x,u,G) = \widehat{f}(x,u,|G|)$, where $\widehat{f} : B^n \times \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}_+$, we readily obtain

$$\mathcal{F}_f(T) = \int_{B^n} \widehat{f}(x,u,|Du|)\, dx + \int_{\mathcal{L}} \theta(x) \left(\int_0^{2\pi} \widehat{f}^\infty(x, (\cos\theta, \sin\theta), 1)\, dt\right) d\mathcal{H}^{n-1}.$$

Gap phenomenon. As noticed in [9], see Sec. 5 below, in the simple case $\mathcal{Y} = \mathbb{S}^1$, the unit circle in $\mathbb{R}^2$, and in any dimension n, for any current $T \in \mathcal{E}_{1,1}$ −graph$(B^n \times \mathbb{S}^1)$ such that $\mathcal{F}_f(T) < \infty$ we can find a sequence of smooth maps $\{u_k\} \subset C^1(B^n, \mathbb{S}^1)$ such that G_{u_k} weakly converges to T in $\mathcal{D}_n(B^m \times \mathbb{S}^1)$ and the f-energies the u_k's converge to the parametric variational integral of T, i.e.,

$$\lim_{k\to\infty} \mathcal{F}_f(G_{u_k}) = \lim_{k\to\infty} \int_{B^n} f(x, u_k, Du_k)\, dx = \mathcal{F}_f(T)\,,$$

see [9] and [8, Vol. II, Sec. 6.2.2]. However, in case of general target manifolds, even in the model case $f(x, u, G) := |G|$ and in dimension $n = 1$, a *gap phenomenon* occurs, see [12]. More precisely, in general for every smooth sequence $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ we have that

$$\liminf_{k\to\infty} \int_{B^n} f(x, u, Du_k)\, dx \geq \mathcal{F}_f(T) + C$$

for some absolute constant $C > 0$. This means that in general $\mathcal{F}_f(T)$ *does not agree with the relaxed energy functional of* T*:*

$$\mathcal{F}_f(T) > \inf\Big\{ \liminf_{k\to\infty} \int_{B^n} f(x, u_k, Du_k)\, dx \mid \{u_k\} \subset C^1(B^n, \mathcal{Y})\,, \\ G_{u_k} \rightharpoonup T \text{ weakly in } \mathcal{Z}_{n,1}(B^n \times \mathcal{Y}) \Big\}\,.$$

Remark 2.7. This gap phenomenon is typical of integrands with linear growth of the gradient. Take for example $n = 1$ and $T = G_u + \delta_0 \times C$, where $u \equiv P \in \mathcal{Y}$ is a constant map and C is a integral 1-cycle in $\mathcal{Y}$. The images of smooth approximating sequences may have to "connect" the point P to the cycle C, this way paying a cost in term of the distance of P to C, see (24) and (25) below. For this reason, such a gap phenomenon does not occur if the target manifold $\mathcal{Y}$ is the unit circle $\mathbb{S}^1$.

3. Cartesian currents with finite energy

The gap phenomenon previously outlined leads us to introduce a suitable energy functional on the class of currents $\mathcal{E}_{1,1}$ −graph$(B^n \times \mathcal{Y})$. In order to recover lower semicontinuity and density properties, we shall restrict the class of integrands. More precisely, in the sequel we shall consider functions $\widetilde{f} : B^n \times \mathbb{R}^N \times \mathbb{R}^N_+ \to \mathbb{R}_+$ satisfying the following properties:

(a) $z \mapsto \widetilde{f}(x, u, z)$ is convex and lower semicontinuous in $\mathbb{R}^N_+$ for every $(x, u) \in B^n \times \mathbb{R}^N$;

(b) $C_1\,|z| \leq \widetilde{f}(x, u, z) \leq C_2\,(1+|z|)$ for every $(x, u, z) \in B^n \times \mathbb{R}^N \times \mathbb{R}^N_+$ and some absolute constants $C_i > 0$;

(c) for every $u \in \mathbb{R}^N$ there exists a continuous function $\omega_u : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\omega_u(t) \to 0$ if $t \to 0$, and depending continuously on $u \in \mathbb{R}^N$, such that

$$|\widetilde{f}(x, u, z) - \widetilde{f}(x_0, u, z)| \leq \omega_u(|x - x_0|) \cdot (1 + |z|) \quad \forall\, z \in \mathbb{R}^N_+.$$

For any matrix $G = (G_i^j)_{j,i=1}^{N,n} \in M(N,n)$, we denote by $G^j := (G_1^j, \dots, G_n^j) \in \mathbb{R}^n$ the n-vector corresponding to the j^{th}-row. The corresponding integrand $f : B^n \times \mathbb{R}^N \times M(N,n) \to \mathbb{R}_+$ is defined by

$$f(x,u,G) := \widetilde{f}(x,u,(|G^1|,\dots,|G^N|)), \qquad (x,u,G) \in B^n \times \mathbb{R}^N \times M(N,n), \tag{21}$$

the isotropic case corresponding to $\widetilde{f}(x,u,z) = \widehat{f}(x,u,|z|)$ for some $\widehat{f} : B^n \times \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}_+$. Therefore, the related variational functional $\mathcal{F}(u)$, see (1), is such that for any smooth map $u = (u^1,\dots,u^N) \in C^1(B^n,\mathcal{Y})$

$$\mathcal{F}(u) = \int_{B^n} \widetilde{f}\big(x,u(x),(|Du^1(x)|,\dots,|Du^1(x)|)\big)\,dx\,.$$

Remark 3.1. Since $f(x,u,G)$ is convex and lower semicontinuous in G for every $(x,u) \in B^n \times \mathbb{R}^N$, it satisfies the properties (a'), (b'), (c') stated at the beginning of the previous section. We notice that the continuity hypothesis (c) is used in the proof of the density theorem 5.1.

We now collect the following facts from [12], see also [10].

Jump-concentration set. Let $T \in \mathcal{E}_{1,1}$ −graph$(B^n \times \mathcal{Y})$, see Definition 1.7. If $\mathcal{L}(T)$ denotes the $(n-1)$-rectifiable set given by the union of the sets of positive multiplicity of the $\mathbb{L}_s(T)$'s, we infer that the union

$$J_c(T) := J_{u_T} \cup \mathcal{L}(T)$$

does not depend on the choice of the representative in T. The countably $\mathcal{H}^{n-1}$-rectifiable set $J_c(T)$ is said to be the set of points of *jump-concentration* of T.

Restriction over points of jump-concentration. If $n = 1$, since T has finite mass, $\eta \mapsto T(\chi_{B_r(x)} \wedge \eta)$, where $x \in B^1$ and $0 < r < 1 - |x|$, defines a current in $\mathcal{D}_1(\mathcal{Y})$. The 1-*dimensional restriction of* T *over the point* x

$$\widehat{\pi}_{\#}(T \llcorner \{x\} \times \mathcal{Y}) \in \mathcal{D}_1(\mathcal{Y})$$

is well-defined on closed 1-forms in $\mathcal{Z}^1(\mathcal{Y})$ by the limit

$$\widehat{\pi}_{\#}(T \llcorner \{x\} \times \mathcal{Y})(\eta) := \lim_{r \to 0^+} T(\chi_{B_r(x)} \wedge \eta)\,, \quad \eta \in \mathcal{Z}^1(\mathcal{Y})\,.$$

Moreover, for every $x \in J_c(T)$ there exists a 1-dimensional integral chain Γ_x on $\mathcal{Y}$ such that

$$\partial \Gamma_x = \delta_{u_T^+(x)} - \delta_{u_T^-(x)} \quad \text{and} \quad \widehat{\pi}_{\#}(T \llcorner \{x\} \times \mathcal{Y}) = \Gamma_x\,.$$

Therefore, by applying Federer's decomposition theorem, see [7, 4.2.25], we find an indecomposable 1-dimensional integral chain γ_x on $\mathcal{Y}$, satisfying $\partial\gamma_x = \delta_{u_T^+(x)} - \delta_{u_T^-(x)}$, and an integral 1-cycle C_x in $\mathcal{Y}$, satisfying $\partial C_x = 0$, such that

$$\Gamma_x = \gamma_x + C_x \quad \text{and} \quad \mathbf{M}(\Gamma_x) = \mathbf{M}(\gamma_x) + \mathbf{M}(C_x)\,. \tag{22}$$

If $n \geq 2$, we let $\nu_T : J_c(T) \to \mathbb{S}^{n-1}$ denote an extension to $J_c(T)$ of the unit normal ν_{u_T} to the Jump set J_{u_T}. For any $k = 1,\dots,n-1$, let P be an oriented k-dimensional subspace in $\mathbb{R}^n$ and $P_\lambda := P + \sum_{i=1}^{n-k} \lambda_i \nu_i$ the family of oriented k-planes parallel to P, where $\lambda := (\lambda_1, \dots, \lambda_{n-k}) \in \mathbb{R}^{n-k}$, $\mathrm{span}(\nu_1, \dots, \nu_{n-k})$ being the orthogonal space to P. Since T has finite $\mathcal{E}_{1,1}$-norm, similarly to the case of normal currents, for $\mathcal{L}^{n-k}$-a.e. λ such that $P_\lambda \cap B^n \neq \emptyset$, the *slice* $T \llcorner \pi^{-1}(P_\lambda)$ of T over $\pi^{-1}(P_\lambda)$ is a well defined k-dimensional current in $\mathcal{E}_{1,1}$ $-\mathrm{graph}((B^n \cap P_\lambda) \times \mathcal{Y})$ with finite $\mathcal{E}_{1,1}$-norm. Moreover, for any such λ we have

$$J_c(T \llcorner \pi^{-1}(P_\lambda)) = J_c(T) \cap P_\lambda \quad \text{in the } \mathcal{H}^{k-1}\text{-a.e. sense}\,,$$

whereas the BV-function associated to $T \llcorner \pi^{-1}(P_\lambda)$ is equal to the restriction $u_{T|P_\lambda}$ of u_T to P_λ. Therefore, in the particular case $k = 1$, the 1-dimensional restriction

$$\widehat{\pi}_\#\big((T \llcorner \pi^{-1}(P_\lambda)) \llcorner \{x\} \times \mathcal{Y}\big) \in \mathcal{D}_1(\mathcal{Y}) \tag{23}$$

of the 1-dimensional current $T \llcorner \pi^{-1}(P_\lambda)$ over any point $x \in J_c(T) \cap P_\lambda$ such that $\nu_T(x)$ does not belong to P is well defined. In this case, from the slicing properties of BV-functions, if $x \in (J_c(T) \setminus J_{u_T}) \cap P_\lambda$ we have $u_{T|P_\lambda}(x) = u_T(x)$. Moreover, if $x \in J_{u_T} \cap P_\lambda$, the one-sided approximate limits of u_T are equal to the one-sided limits of the restriction $u_{T|P_\lambda}$, i.e.

$$u^+_{T|P_\lambda}(x) = u^+_T(x) \quad \text{and} \quad u^-_{T|P_\lambda}(x) = u^-_T(x)\,,$$

provided that $\langle \nu, \nu_{u_T}(x) \rangle > 0$, where ν is an orienting unit vector to P, compare Theorem 1.2. We finally infer that for $\mathcal{H}^{n-1}$-a.e. point $x \in J_c(T)$ the 1-dimensional restriction (23), up to the orientation, does not depend on the choice of the oriented 1-space P and on $\lambda \in \mathbb{R}^{n-1}$, provided that $x \in P_\lambda$ and $\nu_T(x)$ does not belong to P. As a consequence we give the following

Definition 3.2. For $\mathcal{H}^{n-1}$-a.e. point $x \in J_c(T)$, the 1-dimensional restriction

$$\widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y})$$

is well-defined by (23) for any oriented 1-space P and $\lambda \in \mathbb{R}^{n-1}$ such that $x \in P_\lambda$ and $\langle \nu, \nu_T(x) \rangle > 0$, where ν is the orienting unit vector to P.

Vertical minimal connection. For every current $T \in \mathcal{E}_{1,1}$ $-\mathrm{graph}(B^n \times \mathcal{Y})$ and every point $x \in J_c(T)$ we will denote by

$$\begin{aligned}\Gamma_T(x) := \{\gamma \in \mathrm{Lip}([0,1], \mathcal{Y}) \mid \gamma(0) = u^-_T(x)\,,\ \gamma(1) = u^+_T(x)\,,\\ \gamma_\#[\![(0,1)]\!](\eta) = \widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y})(\eta) \ \ \forall \eta \in \mathcal{Z}^1(\mathcal{Y})\}\end{aligned}$$

the family of all smooth curves γ in $\mathcal{Y}$, with end points $u^\pm_T(x)$, such that their image current $\gamma_\#[\![(0,1)]\!]$ agrees with the 1-dimensional restriction $\widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y})$ on closed 1-forms in $\mathcal{Z}^1(\mathcal{Y})$. On account of (19), to every $\gamma \in \Gamma_T(x)$, say $\gamma = (\gamma^1, \dots, \gamma^N)$, we associate the parametric integrand

$$\mathcal{L}_{f,x}(\gamma) := \int_0^1 \widetilde{f}^\infty\big(x, \gamma(t), (|\dot\gamma^1(t)|, \dots, |\dot\gamma^N(t)|)\big)\, dt\,. \tag{24}$$

It turns out that $\mathcal{L}_{f,x}(\gamma)$ does not depend on the parameterization of γ. Moreover, we denote by

$$f_T(x) := \inf\{\mathcal{L}_{f,x}(\gamma) \mid \gamma \in \Gamma_T(x)\}\,, \quad x \in J_c(T)\,, \tag{25}$$

the *minimal "length"* of curves γ connecting the "vertical part" of T over x to the graph of u_T. For future use, we remark that the infimum in (25) is attained, i.e.,

$$\forall\, x \in J_c(T)\,, \quad \exists\, \gamma \in \Gamma_T(x): \quad \mathcal{L}_{f,x}(\gamma) = f_T(x)\,. \tag{26}$$

Remark 3.3. In the model case $f(x,u,G) := |G|$, or $f(x,u,G) := \sqrt{1+|G|^2}$, we have $f^\infty(x,u,G) = |G|$ and hence $\mathcal{L}_{f,x}(\gamma)$ agrees with the standard length $\mathcal{L}(\gamma)$ of the curve γ. Therefore, in this case we have

$$f_T(x) = \mathcal{L}_T(x) := \inf\{\mathcal{L}(\gamma) \mid \gamma \in \Gamma_T(x)\}\,, \quad x \in J_c(T)\,. \tag{27}$$

The energy. To any current $T \in \mathcal{E}_{1,1}\,$–graph$(B^n \times \mathcal{Y})$ we associate its f*-energy* given for every Borel set $B \subset B^n$ by

$$\begin{aligned} \mathcal{E}_f(T, B\times\mathcal{Y}) \ := \ & \int_B f(x, u_T(x), \nabla u_T(x))\, dx \\ & + \int_B f^\infty\left(x, \widetilde{u}_T(x), \frac{dD^C u_T}{d|D^C u_T|}(x)\right) d|D^C u_T| \\ & + \int_{J_c(T)\cap B} f_T(x)\, d\mathcal{H}^{n-1}(x)\,. \end{aligned} \tag{28}$$

We also let

$$\mathcal{E}_f(T) := \mathcal{E}_f(T, B^n \times \mathcal{Y})\,.$$

Moreover, if $u : B^n \to \mathcal{Y}$ is a smooth $W^{1,1}$-function, we set

$$\mathcal{E}_f(u) := \int_{B^n} f(x,u,Du)\, dx = \int_{B^n} \widetilde{f}(x,u,|Du^1|,\ldots,|Du^N|)\, dx\,.$$

Remark 3.4. The first two terms in (28), corresponding to the "diffuse" part $\nabla u_T\, dx + D^C u_T$ of Du_T, agree with the corresponding terms of the parametric variational energy $\mathcal{F}_f(T)$, see (20). Moreover, in the case $\mathcal{Y} = \mathbb{S}^1$, the unit sphere, it can be readily checked that

$$\mathcal{E}_f(T) = \mathcal{F}_f(T) \quad \forall\, T \in \mathcal{E}_{1,1}\,\text{–graph}(B^n \times \mathbb{S}^1)\,,$$

and if $\mathcal{Y} = \mathbb{S}^1$ no gap phenomenon occurs for $\mathcal{F}_f(T)$, see Remark 2.7. However, for more general target manifolds $\mathcal{Y}$, the presence of the last term in (28) yields that $\mathcal{E}_f(T)$ is not a "local" energy, i.e., it cannot be written as an integral functional depending on the components $\overrightarrow{T}$ and $\|T\|_{\mathcal{E}_{1,1}}$ of the decomposition $T = \|T\|_{\mathcal{E}_{1,1}} \llcorner \overrightarrow{T}$, as in (19).

Due to (21), the volume term in the definition of $\mathcal{E}_f(T, B\times\mathcal{Y})$ is

$$\int_B f(x,u_T,\nabla u_T)\, dx = \int_B \widetilde{f}(x,u_T,(|\nabla u_T^1|,\ldots,|\nabla u_T^N|))\, dx\,.$$

Therefore, if $T = G_u$ for some smooth function $u \in W^{1,1}(B^n,\mathcal{Y})$, we have

$$\mathcal{E}_f(G_u) = \mathcal{E}_f(u)\,.$$

As to the Cantor-type term, since $f^\infty(x,u,G) = \widetilde{f}^\infty(x,u,(|G^1|,\dots,|G^N|))$, it turns out that

$$\begin{aligned}&\int_B f^\infty\Big(x,\widetilde{u}_T,\frac{dD^C u_T}{d|D^C u_T|}\Big)\,d|D^C u_T| \\ = \quad &\int_B \widetilde{f}^\infty\Big(x,\widetilde{u}_T,\Big(\Big|\frac{dD^C u_T^1}{d|D^C u_T|}\Big|,\dots,\Big|\frac{dD^C u_T^N}{d|D^C u_T|}\Big|\Big)\Big)\,d|D^C u_T|\,.\end{aligned}$$

Moreover, in the *isotropic* case $f(x,u,G) = \widehat{f}(x,u,|G|)$, we clearly have

$$\int_B f(x,u_T,\nabla u_T)\,dx = \int_B \widehat{f}(x,u_T,|\nabla u_T|)\,dx$$

and

$$\int_B f^\infty\Big(x,\widetilde{u}_T,\frac{dD^C u_T}{d|D^C u_T|}\Big)\,d|D^C u_T| = \int_B \widehat{f}^\infty(x,\widetilde{u}_T,1)\,d|D^C u_T|\,.$$

Remark 3.5. As to the last term in (28), the so called "jump-concentration" part, it turns out that in general the "jump" part cannot be separated from the "homological" part. For instance, in the decomposition (13) of T we may define the jump part G_T^J of G_T by choosing γ_x as the 1-current integration over an oriented geodesic arc in $\mathcal{Y}$ connecting $u_T^-(x)$ and $u_T^+(x)$, see Definition 1.3. However, even in dimension $n=1$ and in the particular case $\mathcal{Y} = \mathbb{S}^1$, in general it may happen that the jump-concentration part of the energy of T cannot be recovered by the sum of the energies of its components G_T^J and S_T, see [12].

The case of simply-connected manifolds. If the first homotopy group $\pi_1(\mathcal{Y})$ is trivial, we have $H_1(\mathcal{Y}) = 0$ and hence every current $T \in \mathcal{E}_{1,1}-\text{graph}(B^n \times \mathcal{Y})$ has no homological vertical part, i.e., $S_T \equiv 0$ on $\mathcal{Z}^{n,1}(B^n \times \mathcal{Y})$, see (13). Therefore, T reduces to the vertical equivalence class of the elements $G \in BV-\text{graph}(B^n \times \mathcal{Y})$ with corresponding BV-function $u(G)$ equal to u_T, see Definition 1.3. Moreover, for every such current G, the action of the jump part G^j, on forms in $\mathcal{Z}^{n,1}(B^n \times \mathcal{Y})$, does not depend on the choice of the integral 1-chain γ_x, but only on the one-sided approximate limits $u_T^\pm(x)$.

Remark 3.6. The above facts yield that if $\pi_1(\mathcal{Y}) = 0$, for every $T \in \mathcal{E}_{1,1}-\text{graph}(B^n \times \mathcal{Y})$ the jump-concentration set $J_c(T)$ agrees with the jump set J_{u_T} of u_T and for every $x \in J_{u_T}$ the 1-dimensional restriction $\widehat{\pi}_\#(T \llcorner \{x\} \times \mathcal{Y})$, see Definition 3.2, agrees on closed 1-forms in $\mathcal{Z}^1(\mathcal{Y})$ with the current integration over *any integral 1-chain* γ_x in $\mathcal{Y}$ satisfying $\partial\gamma_x = \delta_{u_T^+(x)} - \delta_{u_T^-(x)}$, see (22).

Cartesian currents. In the model case $f(x,u,G) = |G|$, clearly $\mathcal{E}_f(T)$ agrees with

$$\mathcal{E}_{1,1}(T) := \int_{B^n} |\nabla u_T|\,dx + |D^C u_T|(B^n) + \int_{J_c(T)} \mathcal{L}_T(x)\,d\mathcal{H}^{n-1}(x)\,, \tag{29}$$

the BV*-energy* of T, see [12]. Notice that by property (b) we infer that for any $T \in \mathcal{E}_{1,1}-\text{graph}(B^n \times \mathcal{Y})$

$$\mathcal{E}_f(T) < \infty \quad \iff \quad \mathcal{E}_{1,1}(T) < \infty\,. \tag{30}$$

Definition 3.7. We denote by $\text{cart}^{1,1}(B^n \times \mathcal{Y})$ the class of Cartesian currents T in $\mathcal{E}_{1,1}$ $-\text{graph}(B^n \times \mathcal{Y})$ such that $\mathcal{E}_{1,1}(T) < \infty$.

For example, if $u \in W^{1,1}(B^n, \mathcal{Y})$ the current G_u carried by the graph of u, see (9), belongs to $\text{cart}^{1,1}(B^n \times \mathcal{Y})$ if and only if

$$\partial G_u(\omega) = 0 \quad \forall\, \omega \in \mathcal{Z}^{n-1,1}(B^n \times \mathcal{Y}) . \tag{31}$$

The previous definitions are motivated by the following lower semicontinuity property, compare [12].

Theorem 3.8. *Let $n \geq 1$ and $T \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$. For every sequence of smooth maps $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$, we have*

$$\liminf_{k\to\infty} \mathcal{E}_{1,1}(u_k) \geq \mathcal{E}_{1,1}(T) , \quad \mathcal{E}_{1,1}(u_k) := \int_{B^n} |Du_k|\, dx .$$

In addition, if we *assume that the first homotopy group $\pi_1(\mathcal{Y})$ is commutative*, the following density result holds true, see [12, 13].

Theorem 3.9. *Let $T \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$. There exists a sequence of smooth maps $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ and $\mathcal{E}_{1,1}(u_k) \to \mathcal{E}_{1,1}(T)$ as $k \to \infty$.*

Properties. As a consequence, we also obtain:

i) the functional $T \mapsto \mathcal{E}_{1,1}(T)$ is lower semicontinuous on $\text{cart}^{1,1}(B^n \times \mathcal{Y})$ with respect to the weak $\mathcal{Z}_{n,1}$-convergence;
ii) the class of Cartesian currents $\text{cart}^{1,1}(B^n \times \mathcal{Y})$ is closed under the weak $\mathcal{Z}_{n,1}$-convergence with equibounded energies;
iii) $\mathcal{E}_{1,1}$-bounded sequences in $\text{cart}^{1,1}(B^n \times \mathcal{Y})$ are relatively compact in the $\mathcal{Z}_{n,1}$-topology.

4. Lower semicontinuity of the energy

In this section we consider integrands $f(x, u, G)$ of the type (21) for some function $\widetilde{f} : B^n \times \mathbb{R}^N \times \mathbb{R}^N_+ \to \mathbb{R}_+$ satisfying the properties (a), (b), (c) of the previous section. We shall prove that the energy functional $T \mapsto \mathcal{E}_f(T)$ defined by (28) is lower semicontinuous in $\text{cart}^{1,1}(B^n \times \mathcal{Y})$ along smooth sequences.

Theorem 4.1. *Let $T \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$. For every sequence of smooth maps $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$, we have*

$$\liminf_{k\to\infty} \mathcal{E}_f(u_k) \geq \mathcal{E}_f(T) .$$

The proof follows the lines of the one given in [12] in the case $f(x, u, G) = |G|$, see Theorem 3.8. More precisely, we shall first prove Theorem 4.1 in the case of dimension $n = 1$. Secondly, applying arguments as for instance in [6], we shall deal with the case of higher dimension $n \geq 2$.

Remark 4.2. Theorem 4.1 continues to hold if we weaken the continuity assumption (c) by requiring an upper semicontinuity property similar to the one in property (c') from Sec. 2.

Proof of Theorem 4.1 (The case n = 1). We follow line by line the proof given in [12], taking into account the following facts and modifications:

i) We have $T = G_T + S_T$, where $G_T \in BV-\text{graph}(B^1 \times \mathcal{Y})$ and

$$S_T = \sum_{i=1}^{I} \delta_{x_i} \times C_i \quad \text{on } \mathcal{Z}^{1,1}(B^1 \times \mathcal{Y}),$$

$\{x_i : i = 1, \dots, I\}$ being a finite disjoint set of points in B^1, possibly intersecting the Jump set J_{u_T}, and C_i is a homologically non-trivial integral 1-cycle in $\mathcal{Y}$.

ii) If $\{x_i\}_{i>I} \subset B^1$ is the at most countable set of discontinuity points in $J_{u_T} \setminus \{x_i : i = 1, \dots, I\}$, by the properties of $\mathcal{Y}$ we have

$$\mathcal{L}_T(x_i) \le C \cdot |u_T^+(x_i) - u_T^-(x_i)| \quad \forall\, i > I,$$

where $C = C(\mathcal{Y}) > 0$ is an absolute constant, see (27). Now, property (b) yields

$$C_1 \, |\dot\gamma(t)| \le \widetilde{f}^\infty(x, \gamma(t), (|\dot\gamma^1(t)|, \dots, |\dot\gamma^N(t)|)) \le C_2 \, |\dot\gamma(t)|$$

and hence

$$C_1 \, \mathcal{L}(\gamma) \le \mathcal{L}_{f,x}(\gamma) \le C_2 \, \mathcal{L}(\gamma) \tag{32}$$

for every $\gamma \in \Gamma_T(x)$ and $x \in J_c(T)$. Therefore, for every $\varepsilon > 0$ we find again $l(\varepsilon) > I$ such that

$$\sum_{i=l(\varepsilon)+1}^{\infty} f_T(x_i) < \varepsilon. \tag{33}$$

iii) If $\{\widetilde{\gamma}_k^i\}_k$ is a sequence of Lipschitz arcs $\widetilde{\gamma}_k^i : [0,1] \to \mathcal{Y}$ uniformly converging to a Lipschitz arc $\widetilde{\gamma}^i \in \Gamma_T(x_i)$, by lower semicontinuity of the functional w.r.t. the uniform convergence, we have

$$\mathcal{L}_{f,x_i}(\widetilde{\gamma}^i) \le \liminf_{k\to\infty} \mathcal{L}_{f,x_i}(\widetilde{\gamma}_k^i).$$

By (25) we thus conclude again that

$$f_T(x_i) \le \liminf_{k\to\infty} \mathcal{L}_{f,x_i}(\gamma_k^i) \quad \forall\, i = 1, \dots, l(\varepsilon),$$

where $\gamma_k^i : [0,1] \to \mathcal{Y}$ is the Lipschitz reparametrization with constant velocity of $u_{k|[a_k^i, b_k^i]}$.

iv) By lower semicontinuity, due to the weak BV-convergence of $u_k \rightharpoonup u_T$ we have

$$\begin{aligned} &\int_{B^1} f(x, u_T, \nabla u_T)\, dx + \int_{B^1} f^\infty\left(x, \widetilde{u}_T, \frac{dD^C u_T}{d|D^C u_T|}\right) d|D^C u_T| \\ \le\; &\liminf_{k\to\infty} \int_{B^1} f(x, u_k, Du_k)\, dx. \end{aligned}$$

We finally obtain

$$\mathcal{E}_f(T) - \varepsilon \le \liminf_{k\to\infty} \int_{B^1} f(x, u_k, Du_k)\, dx$$

and hence the assertion, by letting $\varepsilon \searrow 0$. □

To prove Theorem 4.1 in the higher dimension $n \geq 2$, we shall need the following property.

One-dimensional restrictions of Cartesian currents. If T belongs to $\mathrm{cart}^{1,1}(B^n, \mathcal{Y})$, for any $\nu \in \mathbb{S}^{n-1}$ the 1-dimensional slice

$$T_y^\nu := T \llcorner (B^n)_y^\nu \times \mathcal{Y}$$

defines a Cartesian current $T_y^\nu \in \mathrm{cart}^{1,1}((B^n)_y^\nu \times \mathcal{Y})$ for $\mathcal{L}^{n-1}$-a.e. $y \in (B^n)_\nu$. By Theorem 1.2 and by the definition (28), on account of (21) we infer that the f-energy of T_y^ν is given for $\mathcal{L}^{n-1}$-a.e. $y \in (B^n)_\nu$ by

$$\begin{aligned} &\mathcal{E}_f(T_y^\nu, A_y^\nu \times \mathcal{Y}) \\ = &\int_{A_y^\nu} \widetilde{f}\big(x, u_T(y+t\nu), (|\langle \nabla u_T^1(y+t\nu), \nu\rangle|, \ldots, |\langle \nabla u_T^N(y+t\nu), \nu\rangle|)\big)\, dt \\ &\quad + \int_{A_y^\nu} \widetilde{f}^\infty\Bigg(x, \widetilde{u}_T, \bigg(\frac{dD^C(u_T^1)_y^\nu}{d|D^C(u_T)_y^\nu|}, \ldots, \frac{dD^C(u_T^N)_y^\nu}{d|D^C(u_T)_y^\nu|}\bigg)\Bigg)\, d|D^C(u_T)_y^\nu| \\ &\quad + \sum_{t \in (J_c(T) \cap A)_y^\nu} f_T(y+t\nu) \end{aligned} \tag{34}$$

for any open set $A \subset B^n$.

Proof of Theorem 4.1 (The case n $\geq$ 2). We modify the proof given in [12] in the case $f(x,u,G) = |G|$, where we followed [3, Thm. 5.4]. Since $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ is such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$, for $\mathcal{L}^{n-1}$-a.e. $y \in (B^n)_\nu$ we infer that

$$(G_{u_k})_y^\nu \rightharpoonup T_y^\nu \quad \text{weakly in } \mathcal{Z}_{1,1}((B^n)_y^\nu \times \mathcal{Y}),$$

where

$$(G_{u_k})_y^\nu = G_{(u_k)_y^\nu}, \qquad (u_k)_y^\nu(t) := u_k(y+t\nu) \in C^1((B^n)_y^\nu, \mathcal{Y}).$$

Therefore, by the case $n = 1$ we infer that

$$\mathcal{E}_f(T_y^\nu, A_y^\nu \times \mathcal{Y}) \leq \liminf_{k\to\infty} \mathcal{E}_f((u_k)_y^\nu, A_y^\nu)$$

for any open set $A \subset B^n$, where

$$\mathcal{E}_f((u_k)_y^\nu, A_y^\nu) := \mathcal{E}_f(G_{(u_k)_y^\nu}, A_y^\nu \times \mathcal{Y}) = \int_{A_y^\nu} f(x, u_k(y+t\nu), \langle \nabla u_k(y+t\nu), \nu\rangle)\, dt.$$

Denote by ν_T an extension to the countably $\mathcal{H}^{n-1}$-rectifiable set $J_c(T)$ of the outward unit normal to the Jump set J_{u_T}. We now define, for every open set $A \subset B^n$ and $\nu \in \mathbb{S}^{n-1}$,

$$\mathcal{E}_f(T, A \times \mathcal{Y}, \nu) := \mathcal{E}_f^a(T, A \times \mathcal{Y}, \nu) + \mathcal{E}_f^C(T, A \times \mathcal{Y}, \nu) + \mathcal{E}_f^{Jc}(T, A \times \mathcal{Y}, \nu),$$

where

$$\begin{aligned} \mathcal{E}_f^a(T, A \times \mathcal{Y}, \nu) &:= \int_A \widetilde{f}\big(x, u_T, (|\langle \nabla u_T^1, \nu\rangle|, \ldots, |\langle \nabla u_T^N, \nu\rangle|)\big)\, dx \\ \mathcal{E}_f^C(T, A \times \mathcal{Y}, \nu) &:= \int_A \widetilde{f}^\infty\Bigg(x, \widetilde{u}_T, \bigg(\frac{d\langle D^C u_T^1, \nu\rangle}{d|\langle D^C u_T, \nu\rangle|}, \ldots, \frac{d\langle D^C u_T^N, \nu\rangle}{d|\langle D^C u_T, \nu\rangle|}\bigg)\Bigg)\, d|\langle D^C u_T, \nu\rangle| \\ \mathcal{E}_f^{J_c}(T, A \times \mathcal{Y}, \nu) &:= \int_{J_c(T)\cap A} |\langle \nu_T(x), \nu\rangle|\, f_T(x)\, d\mathcal{H}^{n-1}(x). \end{aligned}$$

By the coarea formula we have

$$\mathcal{E}_f^{J_c}(T, A \times \mathcal{Y}, \nu) = \int_{\pi_\nu} \Bigg(\sum_{t \in (J_c(T) \cap A)_y^\nu} f_T(y + t\nu) \Bigg) \, d\mathcal{L}^{n-1}(y) \, .$$

Moreover, Theorem 1.2 gives

$$\begin{aligned} &\mathcal{E}_f^a(T, A \times \mathcal{Y}, \nu) \\ = \; &\int_{\pi_\nu} \Bigg(\int_{A_y^\nu} \widetilde{f}\big(x, u_T(y+t\nu), (|\langle \nabla u_T^1(y+t\nu), \nu \rangle|, \dots, |\langle \nabla u_T^N(y+t\nu), \nu \rangle|)\big) \, dt \Bigg) \, \mathcal{L}^{n-1}(y) \end{aligned}$$

and

$$\begin{aligned} &\mathcal{E}_f^C(T, A \times \mathcal{Y}, \nu) \\ = \; &\int_{\pi_\nu} \Bigg(\int_{A_y^\nu} \widetilde{f}^\infty \bigg(x, \widetilde{u}_T, \bigg(\frac{dD^C(u_T^1)_y^\nu}{d|D^C(u_T)_y^\nu|}, \dots, \frac{dD^C(u_T^N)_y^\nu}{d|D^C(u_T)_y^\nu|} \bigg) \bigg) \, d|D^C(u_T)_y^\nu| \Bigg) \, d\mathcal{L}^{n-1}(y) \, . \end{aligned}$$

By (34) we thus obtain the identity

$$\mathcal{E}_f(T, A \times \mathcal{Y}, \nu) = \int_{\pi_\nu} \mathcal{E}_f(T_y^\nu, A_y^\nu \times \mathcal{Y}) \, d\mathcal{L}^{n-1}(y) \, . \tag{35}$$

Similarly, setting for every k

$$\mathcal{E}_f(u_k, A, \nu) := \int_A \widetilde{f}\big(x, u_k, (|\langle \nabla u_k^1, \nu \rangle|, \dots, |\langle \nabla u_k^N, \nu \rangle|)\big) \, dx \, ,$$

we obtain

$$\mathcal{E}_f(u_k, A, \nu) = \int_{\pi_\nu} \mathcal{E}_f((u_k)_y^\nu, A_y^\nu) \, d\mathcal{L}^{n-1}(y) \, . \tag{36}$$

The rest of the proof follows exactly the one in [12] for the case $f(x, u, G) = |G|$, but taking this time $\mathcal{E}_f$ instead of $\mathcal{E}_{1,1}$. In particular, we set $\lambda := \mathcal{L}^n + f_T(\cdot) \, \mathcal{H}^{n-1} \llcorner J_c(T) + |D^C u_T|$, choose an $\mathcal{L}^n$-negligible set $E \subset B^n \setminus J_c(T)$ on which $|D^C u_T|$ is concentrated, and define

$$\varphi_i(x) := \begin{cases} \widetilde{f}\big(x, u_T, (|\langle \nabla u_T^1, \nu_i \rangle|, \dots, |\langle \nabla u_T^N, \nu_i \rangle|)\big) & \text{if } x \in B^n \setminus (E \cup J_c(T)) \\ |\langle \nu_T(x), \nu_i \rangle| \, f_T(x) & \text{if } x \in J_c(T) \\ \widetilde{f}^\infty \bigg(x, \widetilde{u}_T, \bigg(\dfrac{d\langle D^C u_T^1, \nu_i \rangle}{d|D^C u_T|}, \dots, \dfrac{d\langle D^C u_T^N, \nu_i \rangle}{d|D^C u_T|} \bigg) \bigg) & \text{if } x \in E \end{cases}$$

for a countable dense sequence $\{\nu_i\} \subset \mathbb{S}^{n-1}$. The assertion follows, as $\int_{B^n} \sup_{i \in \mathbb{N}} \varphi_i \, d\lambda = \mathcal{E}_f(T, B^n \times \mathcal{Y})$. $\square$

5. Density results for the energy

In this section we shall *assume that the first homotopy group* $\pi_1(\mathcal{Y})$ *is commutative* and prove the following density result for the energy $\mathcal{E}_f(T)$ corresponding to integrands f as in Sec. 3, see Theorem 3.9 for the model case $f(x, u, G) = |G|$, i.e., for the total variation integrand.

Theorem 5.1. *Let $T \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$. There exists a sequence of smooth maps $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ and $\mathcal{E}_f(u_k) \to \mathcal{E}_f(T)$ as $k \to \infty$.*

Remark 5.2. As noticed in [8], in the simple case of $\mathcal{Y} = \mathbb{S}^1$, the unit sphere, since the f-energy $\mathcal{E}_f(T)$ agrees with the parametric variational integral $\mathcal{F}_f(T)$, see Remark 3.4, the continuity theorem by Reshetnyak, see Proposition 5.8 and Theorem 5.9 below, yields at once Theorem 5.1 as a consequence of the corresponding theorem in [9] relative to the total variation integrand.

Relaxed functional. As a consequence of Theorems 4.1 and 5.1, setting

$$\begin{aligned} \widetilde{\mathcal{E}}_f(T) := \inf\Big\{ \liminf_{k\to\infty} \int_{B^n} f(x, u_k, Du_k)\, dx \mid \{u_k\} \subset C^1(B^n, \mathcal{Y}), \\ G_{u_k} \rightharpoonup T \text{ weakly in } \mathcal{Z}_{n,1}(B^n \times \mathcal{Y}) \Big\}, \end{aligned} \tag{37}$$

we conclude that

$$\mathcal{E}_f(T) = \widetilde{\mathcal{E}}_f(T) \quad \forall\, T \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y}).$$

Properties. By Theorems 4.1 and 5.1 we also infer the following lower semicontinuity property.

Proposition 5.3. *Let $\{T_k\} \subset \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ converge weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$, $T_k \rightharpoonup T$, to some $T \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$. Then*

$$\mathcal{E}_f(T) \le \liminf_{k\to\infty} \mathcal{E}_f(T_k).$$

On account of property (b) and of the sequential closure of $\mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ under the weak convergence with equibounded $\mathcal{E}_{1,1}$-energies, we also obtain:

Proposition 5.4. *Let $\{T_k\} \subset \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ converge weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ to some $T \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$, and $\sup_k \mathcal{E}_f(T_k) < \infty$. Then $T \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$.*

By the relative compactness of $\mathcal{E}_{1,1}$-bounded sequences in $\mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ in the $\mathcal{Z}_{n,1}$-topology, we finally infer:

Proposition 5.5. *Let $\{T_k\} \subset \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ be such that $\sup_k \mathcal{E}_f(T_k) < \infty$. Then, possibly passing to a subsequence, $T_k \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ to some $T \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$.*

Proof of Theorem 5.1 (The case $\mathbf{n = 1}$). Arguing as in the proof of the lower semicontinuity result, Theorem 4.1, we will adapt our proof from the analogous one of [12], mentioning only the necessary changes. In fact, due to the hypotheses (a), (b), (c), see Sec. 3, arguing e.g. as in Proposition 5.8 below, we apply a mollification procedure to the function u^ε_δ, defining this way a smooth map $v^\varepsilon_\delta : B^1 \to \mathbb{R}^N$ such that $\|v^\varepsilon_\delta - u^\varepsilon_\delta\|_{L^1(B^1)} \le \delta$ and

$$\int_{B^1} f(x, v^\varepsilon_\delta, Dv^\varepsilon_\delta)\, dx \le \int_{B^1} f(x, u^\varepsilon_\delta, \nabla u^\varepsilon_\delta)\, dx + \int_{B^1} f^\infty\Big(x, \widetilde{u}^\varepsilon_\delta, \frac{dD^s u^\varepsilon_\delta}{d|D^s u^\varepsilon_\delta|}\Big)\, d|D^s u^\varepsilon_\delta|,$$

where $D^s u^\varepsilon_\delta := D^C u^\varepsilon_\delta + D^J u^\varepsilon_\delta$. Since u_T is continuous outside the Jump set J_{u_T} and (33) holds true, for every $\sigma > 0$ we find again $\eta = \eta(\sigma, \delta, \varepsilon) > 0$ such that, in the a.e. sense,

$$\forall\, x, y \in B^1\,, \quad |x - y| < \eta \quad \Longrightarrow \quad |u^\varepsilon_\delta(x) - u^\varepsilon_\delta(y)| < \sigma + \varepsilon\,.$$

As a consequence, we may and do define v^ε_δ in such a way that in particular

$$\operatorname{dist}(v^\varepsilon_\delta(x), \mathcal{Y}) < \varepsilon \quad \forall\, x \in B^1\,,$$

as required. □

Proof of Theorem 5.1 (The case $\mathbf{n \geq 2}$). Following the proof of Theorem 3.9 for the model case $f(x, u, G) = |G|$, compare [12, 13], we will first prove:

Theorem 5.6. *Let $T \in \operatorname{cart}^{1,1}(B^n \times \mathcal{Y})$. We can find a sequence of currents $\{T_k\} \subset \operatorname{cart}^{1,1}(B^n \times \mathcal{Y})$ such that*

$$T_k \rightharpoonup T \quad \textit{weakly in } \mathcal{Z}_{n,1}(B^n \times \mathcal{Y})\,, \qquad \mathcal{E}_f(T_k) \to \mathcal{E}_f(T)$$

and the corresponding functions $u_k := u_{T_k}$ in $BV(B^n, \mathcal{Y})$ have no Cantor part, i.e, $|D^C u_k|(B^n) = 0$ for every k. Moreover, u_k weakly converges to u_T in the BV-sense and

$$\lim_{k \to \infty} |Du_k|(B^n) = |Du_T|(B^n)\,.$$

Secondly, we will prove:

Theorem 5.7. *Let $T \in \operatorname{cart}^{1,1}(B^n \times \mathcal{Y})$ be such that the corresponding BV-function $u_T \in BV(B^n, \mathcal{Y})$ has no Cantor part, i.e., $|D^C u_T| = 0$. There exists a sequence of smooth maps $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ and the energy $\mathcal{E}_f(u_k) \to \mathcal{E}_f(T)$ as $k \to \infty$.*

By a diagonal argument we then clearly obtain Theorem 5.1.

We shall recover Theorem 5.6 from the analogous result proved in [12, 13] for the model case $f(x, u, G) = |G|$. To this purpose, we shall make use of the following continuity property from [8, Vol. II].

A continuity property. Denote by $\mathcal{F}_{1,1}(T)$ the *parametric variational integral* associated to the total variation integral $f(x, u, G) := |G|$, see Sec. 2. According to Remark 2.4 and to (18), we have

$$\mathcal{F}_{1,1}(T) := \int_{B^n \times \mathcal{Y}} F_{TV}\big(\pi(z), \widehat{\pi}(z), \overrightarrow{T}(z)\big)\, d\|T\|_{\mathcal{E}_{1,1}}(z)\,,$$

compare (19), where for every $(x, u) \in B^n \times \mathbb{R}^N$

$$F_{TV}(x, u, \xi) := \begin{cases} |\xi_{(1)}| & \text{if } u \in \mathcal{Y},\ \xi \in \Lambda_n(\mathbb{R}^n \times T_u\mathcal{Y}) \text{ and } \xi^{\bar{0}0} \geq 0 \\ +\infty & \text{otherwise in } \Lambda_n \mathbb{R}^{n+N}. \end{cases}$$

Proposition 5.8. *Let $\{T_k\} \subset \mathcal{E}_{1,1}-\operatorname{graph}(B^n \times \mathcal{Y})$ be such that $T_k \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ to some $T \in \mathcal{E}_{1,1}-\operatorname{graph}(B^n \times \mathcal{Y})$. If $\mathcal{F}_{1,1}(T_k) \to \mathcal{F}_{1,1}(T)$, then $\mathcal{F}_f(T_k) \to \mathcal{F}_f(T)$ for every continuous integrand f satisfying the properties* (a'), (b'), (c') *of Sec. 2.*

This relies on the following continuity theorem due to Reshetnyak [15], compare Thm. 1 in Sec. 1.3.4 of [8, Vol. II]. Here, for any $\mathbb{R}^m$-valued Radon measure μ defined on an open set $U \subset \mathbb{R}^{n+N}$, we will denote by $\overrightarrow{\mu}$ its Radon Nikodym derivative with respect to the total variation $|\mu|$, and by $\mu_k \rightharpoonup \mu$ the weak convergence in the sense of the measures.

Theorem 5.9 (Reshetnyak). *Let $G(z,p)$ be a non-negative continuous function defined in $U \times \mathbb{R}^m$ satisfying the following properties:*

i) $G(z,\cdot)$ *is positively homogeneous of degree one for every z;*
ii) $G(\cdot,p)$ *is uniformly bounded as $p \in \mathbb{S}^{m-1}$;*
iii) $G(z,\cdot)$ *is* essentially convex *for every z, i.e.,*

$$G(z,p+q) \le G(z,p) + G(z,q) \quad \forall\, p,q \in \mathbb{R}^m,$$

where the equality holds if and only if $q = \lambda p$ for some $\lambda \ge 0$.

Let $F(z,p)$ be a non-negative continuous function that is homogeneous of degree one in p for every z and that satisfies

$$0 \le F(z,p) \le c_1\, G(z,p) + c_2 \quad \forall\, (z,p) \in U \times \mathbb{R}^m$$

for some absolute constants $c_i > 0$. Then we have

$$\lim_{k\to\infty} \int_U F(z, \overrightarrow{\mu}_k(z))\, d|\mu_k| = \int_U F(z, \overrightarrow{\mu}(z))\, d|\mu|$$

provided that μ_k, μ are $\mathbb{R}^m$-valued Radon measures on U satisfying

$$\mu_k \rightharpoonup \mu\,, \quad \int_U G(z, \overrightarrow{\mu}_k(z))\, d|\mu_k| \to \int_U G(z, \overrightarrow{\mu}(z))\, d|\mu| \quad \textit{as } k \to \infty\,.$$

Proof of Proposition 5.8. We set $U = B^n \times \mathbb{R}^N$, $z = (x,u)$, and $m = 1 + nN$. As before, we identify vertical homology equivalence classes of currents $T \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$, see (12), with measures $\mu_{(T)}$, and take the $\mathcal{E}_{1,1}$-norm of T instead of the total variation of $\mu_{(T)}$, so that $\overrightarrow{\mu}_{(T)} = \overrightarrow{T}$ if $\|T\|_{\mathcal{E}_{1,1}} < \infty$ and $T = \|T\|_{\mathcal{E}_{1,1}} \llcorner \overrightarrow{T}$. Set now

$$G(z,p) := F_{TV}(x,u,p)\,, \qquad F(z,p) := F_f(x,u,p)$$

if $u \in \mathcal{Y}$ and p is identified with the components $\xi_{(0)} + \xi_{(1)}$ of an n-vector satisfying $\xi \in \Lambda_n(\mathbb{R}^n \times T_u\mathcal{Y})$ and $\xi^{\bar{0}0} \ge 0$. By suitably extending G and F, it is readily checked that we can apply Theorem 5.9. Since the convergence $\mu_{(T_k)} \rightharpoonup \mu_{(T)}$ reduces to the weak convergence $T_k \rightharpoonup T$ in $\mathcal{Z}_{n,1}$, whereas

$$\int_U G(z, \overrightarrow{\mu}_{(T)}(z))\, d|\mu_{(T)}| = \mathcal{F}_{1,1}(T)\,, \qquad \int_U F(z, \overrightarrow{\mu}_{(T)}(z))\, d|\mu_{(T)}| = \mathcal{F}_f(T)\,,$$

the proof is complete. □

Proof of Theorem 5.6. We recall from [12] the main steps of the proof of Theorem 5.6 for the model case $f(x,u,G) = |G|$. In this case we have $\mathcal{L}_f(x) = \mathcal{L}_T(x)$ for $x \in J_c(T)$, see (27).

For every $m \in \mathbb{N}$ we find a closed subset $J_m \subset J_c(T)$ such that

$$J_m \subset J_{m+1} \quad \text{and} \quad \int_{J_c(T)\setminus J_m} \mathcal{L}_T(x)\, d\mathcal{H}^{n-1}(x) < \frac{1}{m} \quad \forall\, m\,. \tag{38}$$

We also find an open subset $\Omega_m \subset B^n \setminus J_m$ and a BV-function $u_m \in BV(B^n, \mathcal{Y})$ such that the following facts hold:

i) $u_m = u_T$ on $B^n \setminus \Omega_m$;
ii) Du_m has no Cantor part, $|D^C u_m|(B^n) = 0$;
iii) $u_m \rightharpoonup u_T$ weakly in the BV-sense with $|Du_m|(B^n) \to |Du_T|(B^n)$ as $m \to \infty$;
iv) $D^a u_m \rightharpoonup D^a u_T + D^C u_T$ and $|D^a u_m|(B^n) \to |D^a u_T|(B^n) + |D^C u_T|(B^n)$;
v) setting

$$T_m := G_{u_m} \llcorner \Omega_m \times \mathcal{Y} + T \llcorner (B^n \setminus \Omega_m) \times \mathcal{Y}\,,$$

then $T_m \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$ and $T_m \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$;
vi) $\mathcal{E}_{1,1}(T_m, \Omega_m \times \mathcal{Y}) \to |D^a u_T|(B^n) + |D^C u_T|(B^n)$ by iv), so that by (38) we obtain

$$\lim_{m\to\infty} \mathcal{E}_{1,1}(T_m, B^n \times \mathcal{Y}) = \mathcal{E}_{1,1}(T, B^n \times \mathcal{Y})\,.$$

From the above properties we also infer that

$$\lim_{m\to\infty} \mathcal{F}_{1,1}(T_m, B^n \times \mathcal{Y}) = \mathcal{F}_{1,1}(T, B^n \times \mathcal{Y})\,.$$

Therefore, by Proposition 5.8 we obtain

$$\lim_{m\to\infty} \mathcal{F}_f(T_m, B^n \times \mathcal{Y}) = \mathcal{F}_f(T, B^n \times \mathcal{Y})\,,$$

where $\mathcal{F}_f$ is the parametric variational integral associated to the integrand f, see (19). Now, the first two terms in $\mathcal{E}_f(T)$, corresponding to the "diffuse" part $\nabla u_T\, dx + D^C u_T$ of Du_T, agree with the corresponding terms of $\mathcal{F}_f(T)$, see Remark 3.4. Moreover, since $\Omega_m \subset B^n \setminus J_m$, property (32) yields that

$$\int_{J_c(T)\cap\Omega_m} f_T(x)\, d\mathcal{H}^{n-1}(x) \le \int_{J_c(T)\setminus J_m} f_T(x)\, d\mathcal{H}^{n-1}(x) \le C_2 \int_{J_c(T)\setminus J_m} \mathcal{L}_T(x)\, d\mathcal{H}^{n-1}(x)\,.$$

By (38) we readily conclude that $\mathcal{E}_f(T_m) \to \mathcal{E}_f(T)$, as required. $\square$

Proof of Theorem 5.7. For any $\widetilde{T} \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$ we set

$$\mu_{J_c,\widetilde{T}}(B) := \int_{J_c(\widetilde{T})\cap B} f_{\widetilde{T}}(x)\, d\mathcal{H}^{n-1}(x)\,. \tag{39}$$

We also denote by $\mathbf{F}(\widetilde{T})$ the *flat norm*

$$\mathbf{F}(\widetilde{T}) := \sup\{\widetilde{T}(\phi) \mid \phi \in \mathcal{Z}^{n-1}(B^n \times \mathcal{Y})\,,\ \mathbf{F}(\phi) \le 1\}\,,$$

where

$$\mathbf{F}(\phi) := \max\left\{ \sup_{z\in B^n\times\mathcal{Y}} \|\phi(z)\|\,,\ \sup_{z\in B^n\times\mathcal{Y}} \|d\phi(z)\| \right\},$$

and recall that the flat convergence $\mathbf{F}(T_k - T) \to 0$ yields the weak convergence $T_k \rightharpoonup T$ in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$, compare [16]. Arguing as in [12, Sec. 5], we reduce to prove:

Proposition 5.10. *Let $T \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ be such that the Cantor part $|D^C u_T|(B^n) = 0$. Let $\varepsilon \in (0, 1/2)$ and $k \in \mathbb{N}$. We can find a current $\widehat{T} \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ such that*

$$\begin{aligned} \mathcal{E}_f(\widehat{T}, B^n \times \mathcal{Y}) \leq \mathcal{E}_f(T, B^n \times \mathcal{Y}) + \varepsilon^k , \quad |D^C u_{\widehat{T}}|(B^n) = 0 , \\ \mu_{Jc,\widehat{T}}(B^n) \leq \frac{1}{2} \cdot \mu_{Jc,T}(B^n) \quad \textit{and} \quad \mathbf{F}(\widehat{T} - T) \leq \varepsilon^k , \end{aligned} \tag{40}$$

where $\mu_{Jc,T}$ is given by (39) *and* $\mathbf{F}$ *is the flat norm.*

In fact, by a diagonal argument, we find a sequence $\{T_k\} \subset \mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ that weakly converges to T with $\mathcal{E}_f(T_k) \to \mathcal{E}_f(T)$ as $k \to \infty$ and such that, if $u_k := u_{T_k}$ is the BV-function corresponding to T_k, we have $|D^C u_k|(B^n) = 0$ and $\mu_{Jc,T_k}(B^n) = 0$, so that $u_k \in W^{1,1}(B^n, \mathcal{Y})$ for every k. Therefore, T_k agrees with the current G_{u_k} given by the integration of forms in $\mathcal{Z}^{n,1}(B^n \times \mathcal{Y})$ over the rectifiable graph of u_k, see (9), so that $\mathcal{E}_f(T_k) = \mathcal{E}_f(u_k)$.

By means of Bethuel's density theorem [4], for every k we find a smooth sequence $\{u_h^{(k)}\}_h \subset C^1(B^n, \mathcal{Y})$ that strongly converges to u_k in the $W^{1,1}$-sense as $h \to \infty$. In fact, even if the first homotopy group $\pi_1(\mathcal{Y})$ is non-trivial, being commutative it is homeomorphic to the first homology group $H_1(\mathcal{Y})$. Therefore, the null-boundary condition (31) for u_k allows to remove the $(n-2)$-dimensional singularities, compare [5] and e.g. [14]. Lower dimensional singularities are removed as in [4]. Now, by the dominated convergence theorem and by property (b), we infer that the strong convergence yields $G_{u_h^{(k)}} \rightharpoonup G_{u_k}$ with $\mathcal{E}_f(u_h^{(k)}) \to \mathcal{E}_f(u_k)$. The assertion then follows by means of a diagonal argument. □

Remark 5.11. This is the exact point where the commutativity hypothesis on the first homotopy group $\pi_1(\mathcal{Y})$ is used, in addition to (31), see the counterexample in Section 6 below.

Proof of Proposition 5.10. We follow the lines of the proof of the corresponding proposition from [12] for the model case $f(x, u, G) = |G|$, but this time replacing the BV-energy $\mathcal{E}_{1,1}(T)$ with the energy $\mathcal{E}_f(T)$, taking $\mathcal{L}_f(x)$ instead of $\mathcal{L}_T(x)$, see (25), and setting $\mu_T := \mu_{d,T} + \mu_{Jc,T}$, where

$$\mu_{d,T}(B) := \int_B f(x, u_T, \nabla u_T(x))\, dx$$

and $\mu_{Jc,T}$ is given by (39), so that by (28) for every Borel set $B \subset B^n$ we have

$$\mathcal{E}_f(T, B \times \mathcal{Y}) = \mu_T(B) = \mu_{d,T}(B) + \mu_{Jc,T}(B) ,$$

as $|D^C u_T|(B^n) = 0$. The proof from [12] continues to hold, taking into account the following modifications:

Step 1: Blow-up argument. We use the estimate $C_1 |Du_T|(B) \leq \mu_T(B)$ to obtain the analogous conclusions. In addition, due to the continuity hypothesis on $\widetilde{f}$, property (c) in Sec. 3, and to the compactness of $\mathcal{Y}$ in $\mathbb{R}^N$, we may and do define the family of balls $B_j := \overline{B}(p_j, r_j)$ with radii r_j sufficiently small that for every $x \in B_j$ we have

$$|f(x, u, G) - f(p_j, u, G)| \leq \sigma\,(1 + |G|) \quad \forall\, u \in \mathcal{Y} ,\ \forall\, G \in M(N, n) . \tag{41}$$

We also remark that, since $f_T(p_j)$ is the $(n-1)$-dimensional density of $\mu_{Jc,T}$ at p_j, we have

$$|\mu_{Jc,T}(B_j) - f_T(p_j)\cdot\omega_{n-1}\,{r_j}^{n-1}| \le \sigma\cdot\omega_{n-1}\,{r_j}^{n-1}\,. \tag{42}$$

Step 2: Approximation on the balls B_j. Denote

$$B^{\pm}_{\rho} := \{x\in B^n_{\rho} : \pm x_n > 0\}\,,\qquad \partial B^{\pm}_{\rho} := \{x\in \partial B^n_{\rho} : \pm x_n > 0\}$$

and let

$$B_{\mathcal{Y}}(y,\varepsilon) := \overline{B}^N(y,\varepsilon)\cap\mathcal{Y}$$

be the intersection of $\mathcal{Y}$ with the closed N-ball of radius ε centered at a point y in $\mathcal{Y}$. Using arguments similar to Step 3 in [13], but working separately on the half balls $B^{\pm}_{\rho}$, and possibly paying a small amount of energy, we are able to deform the sliced current $\langle T^{\sigma}_j, d, r\rangle := T^{\sigma}_j \llcorner \partial B^n_r \times \mathcal{Y}$ in such a way that the support of $\langle T^{\sigma}_j, d, r\rangle \llcorner \partial B^{\pm}_r \times \mathcal{Y}$ is contained in $\partial B^n_r \times B_{\mathcal{Y}}(z^{\pm}_j, \varepsilon_\sigma)$, where $z^{\pm}_j$ are the one-sided approximate limits of u_T at p_j, and $\varepsilon_\sigma > 0$ is small with σ. In particular, we have

$$\|v^{\sigma}_j(x) - z^{\pm}_j\|_{\infty, B^{\pm}_r\cap(\Omega_\delta\setminus\widetilde{\Omega}_\delta)} \le c\cdot\varepsilon_\sigma\,.$$

Moreover, we have:

i) Since for every $v\in BV(B^n,\mathcal{Y})$

$$C_1\int_B |\nabla v|\,dx \le \int_B f(x,v,\nabla v)\,dx \le C_2\int_B (1+|\nabla v|)\,dx\,,$$

we conclude again that

$$\int_{\Omega_\delta\setminus\widetilde{\Omega}_\delta} f(x, w^{\sigma}_j, \nabla w^{\sigma}_j)\,dx$$

is small if δ and σ are small.

ii) The current $\widehat{T}^{\sigma}_j \in \mathrm{cart}^{1,1}((B^n_R\setminus\widetilde{\Omega}_\delta)\times\mathcal{Y})$ satisfies the boundary condition

$$\begin{aligned}\partial\widehat{T}^{\sigma}_j &= \partial T^{\sigma}_j\llcorner\partial B^n_R\times\mathcal{Y} - [\![\,\partial D_r\times\{0\}\,]\!]\times\Gamma_j\\ &\qquad + [\![\,\partial\widetilde{\Omega}_\delta\cap B^+_r\,]\!]\times\delta_{z^+_j} - [\![\,\partial\widetilde{\Omega}_\delta\cap B^-_r\,]\!]\times\delta_{z^-_j}\end{aligned}$$

for a suitable integral chain $\Gamma_j\in\mathcal{D}_1(\mathcal{Y})$ satisfying

$$\widehat{\pi}_{\#}(T\llcorner\{p_j\}\times\mathcal{Y}) = \Gamma_j \quad \text{on } \mathcal{Z}^{n,1}(B^n\times\mathcal{Y})\,,$$

so that $\partial\Gamma_j = \delta_{z^+_j} - \delta_{z^-_j}$. Moreover, the following energy estimate holds:

$$\mathcal{E}_f(\widehat{T}^{\sigma}_j, (B^n_R\setminus\widetilde{\Omega}_\delta)\times\mathcal{Y}) \le \int_{B^n_R} f(x, u^{\sigma}_j, \nabla u^{\sigma}_j)\,dx + c\,\sigma\,r^{n-1} + c\,\sigma\mu_{Jc,T^{\sigma}_j}(B^n_r)\,. \tag{43}$$

iii) To extend $\widehat{T}^{\sigma}_j$ to a current in $\mathrm{cart}^{1,1}(\mathrm{int}(B_j)\times\mathcal{Y})$, we take $\gamma_j\in\Gamma_T(p_j)$ satisfying (26). By the construction γ_j belongs to $\Gamma_{T^{\sigma}_j}(p_j)$ and satisfies

$$\mathcal{L}_{f,p_j}(\gamma_j) = f_{T^{\sigma}_j}(p_j) = f_T(p_j) \tag{44}$$

and $\gamma_{j\#}[\![(0,1)]\!] = \Gamma_j$. Defining again $v_j^\sigma : \widetilde{\Omega}_\delta \to \mathcal{Y}$ by

$$v_j^\sigma(x) := \gamma_j\Big(\frac{1}{2} + \frac{x_n}{\varphi_\delta(y(\widetilde{x}))}\Big), \quad \widetilde{x} \in D_r\,, \quad |x_n| \le \varphi_\delta(y(\widetilde{x}))/2\,,$$

and changing variable $t := \dfrac{1}{2} + \dfrac{x_n}{\varphi_\delta(y(\widetilde{x}))}$ for every $\widetilde{x}$, we observe that by Fubini theorem

$$\begin{aligned}
&\int_{\widetilde{\Omega}_\delta} f(p_j, v_j^\sigma(x), Dv_j^\sigma(x))\,dx \\
= \;& \int_{D_r} d\mathcal{L}^{n-1}(\widetilde{x}) \int_{-\varphi_\delta(y(\widetilde{x}))/2}^{\varphi_\delta(y(\widetilde{x}))/2} \widetilde{f}\big(p_j, \gamma_j(t), \big(\varphi_\delta(y(\widetilde{x}))^{-1}\,(|\dot\gamma^1(t)|, \dots, |\dot\gamma^N(t)|)\big)\big)\,dx_n \\
= \;& \int_{D_r} d\mathcal{L}^{n-1}(\widetilde{x}) \int_0^1 \widetilde{f}\big(p_j, \gamma_j(t), \big(\varphi_\delta(y(\widetilde{x}))^{-1}\,(|\dot\gamma^1(t)|, \dots, |\dot\gamma^N(t)|)\big)\big)\,\varphi_\delta(y(\widetilde{x}))\,dt\,,
\end{aligned}$$

the last term converging as $\delta \to 0^+$ to

$$\int_{D_r} d\mathcal{L}^{n-1}(\widetilde{x}) \int_0^1 \widetilde{f}^\infty\big(p_j, \gamma_j(t), (|\dot\gamma_j^1(t)|, \dots, |\dot\gamma_j^N(t)|)\big)\,dt = \mathcal{L}^{n-1}(D_r) \cdot \mathcal{L}_{f,p_j}(\gamma_j)\,,$$

by definition of recession function, and hence, definitely, to $\mathcal{L}^{n-1}(D_r) \cdot f_{T_j^\sigma}(p_j)$. On the other hand, by (41), (42), (32), and (44) we obtain

$$\begin{aligned}
&\left| \int_{\widetilde{\Omega}_\delta} f(x, v_j^\sigma(x), Dv_j^\sigma(x))\,dx - \int_{\widetilde{\Omega}_\delta} f(p_j, v_j^\sigma(x), Dv_j^\sigma(x))\,dx \right| \\
\le \;& \sigma \cdot \int_{\widetilde{\Omega}_\delta} (1 + |Dv_j^\sigma(x)|)\,dx \le \sigma \left(|\widetilde{\Omega}_\delta| + \mathcal{L}^{n-1}(D_r) \cdot \int_0^1 |\dot\gamma_j(t)|\,dt \right) \\
\le \;& \sigma\,\big(|\widetilde{\Omega}_\delta| + \mathcal{L}^{n-1}(D_r)\,C_2\,f_{T_j^\sigma}(p_j)\big) \le \sigma\,\big(|\widetilde{\Omega}_\delta| + c\,(\mu_{Jc,T_j^\sigma}(B_j) + r^{n-1})\big)\,,
\end{aligned}$$

where $c > 0$ is an absolute constant.

iv) As a consequence, the function v_j^σ satisfies the energy estimate

$$\int_{\widetilde{\Omega}_\delta} f(x, v_j^\sigma, Dv_j^\sigma)\,dx \le c\,\sigma\,r^{n-1} + c\,\sigma\mu_{Jc,T_j^\sigma}(B_j) + \mathcal{L}^{n-1}(D_r) \cdot f_{T_j^\sigma}(p_j)$$

if $\delta > 0$ is small. Therefore, on account of (42) and (43), the current $\widetilde{T}_j^{(\sigma)} := \widehat{T}_j^\sigma + G_{v_j^\sigma}$ belongs to $\mathrm{cart}^{1,1}(\mathrm{int}(B_j) \times \mathcal{Y})$ and satisfies

$$\mathcal{E}_f(\widetilde{T}_j^{(\sigma)}, \mathrm{int}(B_j) \times \mathcal{Y}) \le \mathcal{E}_f(T_j^\sigma, B_R^n \times \mathcal{Y}) + c\,\sigma\,r^{n-1} + c\,\sigma\mu_{Jc,T_j^\sigma}(\overline{B}_R^n)\,.$$

Step 3: Flat distance. Unchanged.

Step 4: Approximation on the whole domain. This time we obtain

$$\mathcal{E}_f(T_j^{(\sigma)}, \mathrm{int}(B_j) \times \mathcal{Y}) \le \int_{B_j} f(x, u_T, \nabla u_T)\,dx + (1 + c\,\sigma)\,\mu_{Jc,T}(B_j) + c\,\sigma\,{r_j}^{n-1}$$

and hence

$$\mathcal{E}_f(T^\sigma, B^n \times \mathcal{Y}) \leq \int_{B^n} f(x, u_T, \nabla u_T)\, dx + (1 + c\,\sigma)\, \mu_{J_c,T}(B^n) + c\,\sigma\, \mathcal{H}^{n-1}(J)\,,$$

so that if $\sigma = \sigma(\varepsilon, k, J, \mu_{Jc,T}) > 0$ is small, we conclude with

$$\mathcal{E}_f(T^\sigma, B^n \times \mathcal{Y}) \leq \mathcal{E}_f(T, B^n \times \mathcal{Y}) + \varepsilon^k\,,$$

as required. □

6. The relaxed energy of functions

In this section we analyze the lower semicontinuous envelope (2) of the variational functional (1) corresponding to integrands f defined as in Sec. 3. Of course, it may equivalently be defined for every function $u \in L^1(B^n, \mathcal{Y})$ by

$$\widetilde{\mathcal{E}}_f(u) := \inf\Big\{ \liminf_{k\to\infty} \int_{B^n} f(x, u_k, Du_k)\, dx \mid \{u_k\} \subset C^1(B^n, \mathcal{Y})\,, \\ u_k \to u \text{ strongly in } L^1(B^n, \mathbb{R}^N) \Big\}\,.$$

For any $u \in BV(B^n, \mathcal{Y})$ we denote by

$$\mathcal{T}_u^{1,1} := \{T \in \mathrm{cart}^{1,1}(B^n \times \mathcal{Y}) \mid u_T = u\} \tag{45}$$

the class of Cartesian currents T in $\mathrm{cart}^{1,1}(B^n \times \mathcal{Y})$ with underlying BV-function u_T equal to u.

In the sequel we shall *assume that the first homotopy group $\pi_1(\mathcal{Y})$ is commutative.* We first prove

Proposition 6.1. *Let $u \in L^1(B^n, \mathcal{Y})$. The following facts are equivalent:*

i) $u \in BV(B^n, \mathcal{Y})$;

ii) *the class $\mathcal{T}_u^{1,1}$ is non-empty;*

iii) $\widetilde{\mathcal{E}}_f(u) < \infty$.

Proof. The implication $i) \Longrightarrow ii)$ was proved in [12]. To prove that $ii) \Longrightarrow iii)$ we observe that, if $T \in \mathcal{T}_u^{1,1}$, by Theorem 5.1 we find a sequence $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ and $\mathcal{E}_f(u_k) \to \mathcal{E}_f(T)$ as $k \to \infty$; this yields also that $u_k \rightharpoonup u_T$ weakly in the BV-sense, where $u_T = u$, and hence $\widetilde{\mathcal{E}}_f(u) < \infty$. To show that $iii) \Longrightarrow i)$, we observe that if $u_k \to u$ with $\int_{B^n} f(x, u_k, Du_k)\, dx \to \widetilde{\mathcal{E}}_f(u) < \infty$, by property (b) we have $\limsup_{k\to\infty} \int_{B^n} |Du_k|\, dx < \infty$ whence, possibly passing to a subsequence, $u_k \rightharpoonup u$ weakly in the BV-sense and finally $u \in BV(B^n, \mathcal{Y})$. □

From the results of the previous sections we also obtain the following representation formula.

Theorem 6.2. *For every $u \in BV(B^n, \mathcal{Y})$ we have*

$$\widetilde{\mathcal{E}}_f(u) = \inf\{\mathcal{E}_f(T) \mid T \in \mathcal{T}_u^{1,1}\}\,. \tag{46}$$

Proof. Let $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ be a sequence of smooth maps with equibounded energies, $\sup_k \mathcal{E}_f(u_k) < \infty$, weakly converging to u in the BV-sense, see Proposition 6.1. Since $\sup_k \|Du_k\|_{L^1} < \infty$, see property (b), by compactness, possibly passing to a subsequence we find that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ to some $T \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$ satisfying $u_T = u$, i.e. $T \in \mathcal{T}_u^{1,1}$, see (45). Since by lower semicontinuity, Proposition 5.3,

$$\mathcal{E}_f(T) \leq \liminf_{k\to\infty} \mathcal{E}_f(u_k)\,,$$

we readily conclude that

$$\inf\{\mathcal{E}_f(T) \mid T \in \mathcal{T}_u^{1,1}\} \leq \widetilde{\mathcal{E}}_f(u)\,.$$

To prove the opposite inequality, by applying Theorem 5.1, for every $T \in \mathcal{T}_u^{1,1}$ we find a sequence $\{u_k\} \subset C^1(B^n, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{n,1}(B^n \times \mathcal{Y})$ and $\mathcal{E}_f(u_k) \to \mathcal{E}_f(T)$ as $k \to \infty$. Since the weak convergence $G_{u_k} \rightharpoonup T$ yields the convergence $u_k \to u_T$ weakly in the BV-sense, hence strongly in L^1, and $u_T = u$, we find that $\widetilde{\mathcal{E}}_f(u) \leq \mathcal{E}_f(T)$. $\square$

As a consequence, by the definition (28) of f-energy we readily obtain

Corollary 6.3. *For every $u \in BV(B^n, \mathcal{Y})$ we have*

$$\begin{aligned}\widetilde{\mathcal{E}}_f(u) \;&=\; \int_{B^n} f(x, u(x), \nabla u(x))\, dx + \int_{B^n} f^\infty\left(x, \widetilde{u}(x), \frac{dD^C u}{d|D^C u|}(x)\right) d|D^C u| \\ &\quad + \inf\left\{\int_{J_c(T)} f_T(x)\, d\mathcal{H}^{n-1}(x) \;\mid T \in \mathcal{T}_u^{1,1}\right\}.\end{aligned}$$

The case of simply-connected manifolds. If the first homotopy group $\pi_1(\mathcal{Y})$ is trivial, e.g., if $\mathcal{Y} = \mathbb{S}^p$ for some $p \geq 2$, on account of Remark 3.6 we readily infer:

Corollary 6.4. *Assume that $\pi_1(\mathcal{Y}) = 0$. Then for every $u \in BV(B^n, \mathcal{Y})$ we have*

$$\begin{aligned}\widetilde{\mathcal{E}}_f(u) \;&=\; \int_{B^n} f(x, u(x), \nabla u(x))\, dx + \int_{B^n} f^\infty\left(x, \widetilde{u}(x), \frac{dD^C u}{d|D^C u|}(x)\right) d|D^C u| \\ &\quad + \int_{J_u} \Phi_{f,u}(x)\, d\mathcal{H}^{n-1}(x)\,,\end{aligned}$$

where

$$\Phi_{f,u}(x) := \inf\{\mathcal{L}_{f,x}(\gamma) \mid \gamma \in \Gamma_u(x)\}\,,$$

$\mathcal{L}_{f,x}(\gamma)$ is given by (24), *and*

$$\Gamma_u(x) := \{\gamma \in \text{Lip}([0,1], \mathcal{Y}) \mid \gamma(0) = u^-(x)\,,\ \gamma(1) = u^+(x)\}$$

is the family of all smooth curves γ in $\mathcal{Y}$ with end points $u^\pm(x)$.

Remark 6.5. In the model cases $f(x, u, G) = |G|$, or $f(x, u, G) = \sqrt{1 + |G|^2}$, we have $\mathcal{L}_{f,x}(\gamma) = \mathcal{L}(\gamma)$, the standard length of γ. Therefore, if $\pi_1(\mathcal{Y}) = 0$, we infer that $\Phi_{f,u}(x)$ agrees with the *geodesic distance* between $u^-(x)$ and $u^+(x)$.

Properties. Let now for every function $u \in L^1(B^n, \mathcal{Y})$ and open set $A \subset B^n$, say $A \in \mathcal{A}(B^n)$,

$$\overline{\mathcal{F}}(u, A) := \inf\Big\{ \liminf_{k\to\infty} \mathcal{E}_f(u_k, A) \mid \{u_k\} \subset C^1(B^n, \mathcal{Y}) ,\\ u_k \rightharpoonup u \text{ strongly in } L^1(B^n, \mathbb{R}^N) \Big\} ,$$

where

$$\mathcal{E}_f(v, A) := \int_A f(x, v, Dv)\, dx , \quad v \in C^1(B^n, \mathcal{Y}) , \quad A \in \mathcal{A}(B^n) .$$

From the above results we readily infer the following locality property:

Corollary 6.6. *For every $u, v \in BV(B^n, \mathcal{Y})$ such that $u = v$ a.e. on B^n we have*

$$\overline{\mathcal{F}}(u, A) = \overline{\mathcal{F}}(v, A) \quad \forall\, A \in \mathcal{A}(B^n) .$$

However, a part from the case $\pi_1(\mathcal{Y}) = 0$, see Corollary 6.4, from Theorem 6.2 we infer that in general, for a given $u \in BV(B^n, \mathcal{Y})$, the set function $A \mapsto \overline{\mathcal{F}}(u, A)$ is not a measure.

A counterexample. If the first homotopy group $\pi_1(\mathcal{Y})$ is non-commutative, the density theorem 5.1 fails to hold, even in dimension $n = 2$ and with $T = G_u$ for some Sobolev map $u \in W^{1,1}(B^n, \mathcal{Y})$, see (9). For example, take as $\mathcal{Y}$ a 2-surface of genus two in $\mathbb{R}^3$, e.g.., the standard torus with two "holes", and let $\varphi : \mathbb{S}^1 \to \mathcal{Y}$ be a Lipschitz function that is *not homotopic to a constant map* in $\mathcal{Y}$ but satisfies $\varphi_\# [\![\mathbb{S}^1]\!] = 0$, hence φ is homologically trivial. The map φ can be obtained by describing a continuous loop that belongs to the nontrivial homotopy class given by the sequence of letters $ABA^{-1}B^{-1}$, where A, B are suitable generators of $\pi_1(\mathcal{Y})$. Setting $u(x) := \varphi(x/|x|)$, since $\partial G_u = -\delta_0 \times \varphi_\# [\![\mathbb{S}^1]\!]$, see [8, Vol. I, Sec. 3.2.2], the current G_u satisfies (31) and hence belongs to $\text{cart}^{1,1}(B^2 \times \mathcal{Y})$. Taking e.g. the area integrand $f(x, u, G) = \sqrt{1 + |G|^2}$, we cannot find a sequence of smooth maps $u_k : B^2 \to \mathcal{Y}$ such that $G_{u_h} \rightharpoonup G_u$ weakly in $\mathcal{Z}_{2,1}(B^2 \times \mathcal{Y})$ and $\mathcal{E}_f(u_k) \to \mathcal{E}_f(u)$. In fact, as noticed in [1], as a consequence of Theorem 5.9, one obtains that the conditions

$$u_k \to u \quad \text{in } L^1 , \quad Du_k \rightharpoonup Du \quad \text{weakly* in } L^1 ,\\ \int_{B^2} \sqrt{1 + |Du_k|^2}\, dx \to \int_{B^2} \sqrt{1 + |Du|^2}\, dx$$

yield that $u_k \to u$ strongly in $W^{1,1}$. Therefore, by B. White's results [17], for a.e. radius $0 < r < 1$ the restriction $u_{|\partial B^2_r}$ of u to the boundary of the 2-ball of radius r should have the same *homotopy type* of $u_{k|\partial B^2_r}$, a contradiction, as $u_{|\partial B^2_r} \sim \varphi$ but $u_{k|\partial B^2_r} \sim 0$. On the other hand, since $G_u \in \mathcal{T}^{1,1}_u$ we clearly have

$$\inf\{\mathcal{E}_f(T) \mid T \in \mathcal{T}^{1,1}_u\} = \mathcal{E}_f(u) := \int_{B^2} \sqrt{1 + |Du|^2}\, dx ,$$

but we have seen that $\widetilde{\mathcal{E}}_f(u) > \mathcal{E}_f(u)$, hence (46) fails to hold.

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