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# Characterization of Metric Regularity of Subdifferentials

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We study regularity properties of the subdifferential of proper lower semicontinuous convex functions in Hilbert spaces. More precisely, we investigate the metric regularity and subregularity, the strong regularity and subregularity of such a subdifferential. We characterize each of these properties in terms of a growth condition involving the function.

*Keywords:* Set-valued mapping, subdifferential, metric regularity, quadratic growth condition

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## 1. Introduction

Let $H$ be a real Hilbert space endowed with the inner product $\langle \cdot\,,\cdot \rangle$, we denote by $\Gamma(H)$ the space of all proper lower semicontinuous convex functions from $H$ into $\mathbb{R} \cup \{\infty\}$. The subdifferential, in the sense of convex analysis, is denoted by $\partial$ and for any function $f$ in $\Gamma(H)$ and any point $x$ in the domain of $f$ it is defined by:

$$\partial f(x) := \{u \in H \mid \langle u, y - x\rangle \leq f(y) - f(x),\ \forall y \in H\}.$$

In this work, we study local properties of $\partial f$ around the points of its graph (recall that $\partial f(x)$ is the empty set whenever $x$ is not in the domain of $f$). More precisely, we investigate the link between the regularity properties of the subdifferential of $f$ around a point $(\bar{x}, \bar{v})$ in the graph of $\partial f$ and the behavior of the function $f$ around the point $\bar{x}$. We consider several regularity concepts, namely, the subregularity, the strong subregularity, the metric regularity, the strong regularity of the subdifferential operator and the Lipschitz continuity of its inverse. We show that each of these properties corresponds, and is actually equivalent, to a growth condition on the function $f$ near the point $\bar{x}$. When $\bar{v} = 0$, *i.e.,* when $\bar{x}$ is a critical point of $f$ satisfying $\partial f(\bar{x}) \ni 0$ it turns out that these regularity properties can be expressed in terms of a so-called *quadratic growth condition* involving the function $f$.

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The concepts of metric regularity and subdifferentials are closely tied and have been studied by several authors in the very last decades, we mention here the works of Ioffe [16], Jourani and Thibault [17, 18], Borwein and Zhu [8], Durea [15] and Lahrech and Benbrik [20]; see also the references therein.

A contribution to the topic we are concerned with, dating from the mid-nineteen nineties, is due to Zhang and Treiman [28] who studied functions with *upper-Lipschitz* inverse subdifferentials. Given an extended real-valued and lower semicontinuous function $f$ on $\mathbb{R}^n$, they consider the subdifferential of $f$ in the sense of Mordukhovich (see, e.g., [22]), denoted by $\partial^- f$, and they provide a necessary condition for the upper-Lipschitz continuity of the inverse of the subdifferential of $f$ at 0. We recall that, according to Robinson [24], a set-valued mapping $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is upper-Lipschitz continuous at $\bar{y} \in \mathbb{R}^m$ if there exist a neighborhood $V$ of $\bar{y}$ and a constant $c \geq 0$ such that

$$F(y) \subset F(\bar{y}) + c\|y - \bar{y}\| I\!B, \quad \forall y \in V, \tag{1}$$

where $I\!B$ denotes the closed unit ball in $\mathbb{R}^n$. Zhang and Treiman proved the following statement:

**Theorem 1.1 ([28]).** *If the multifunction* $(\partial^- f)^{-1}$ *is upper-Lipschitz at* $0$*, then for every* $C \subset (\partial^- f)^{-1}(0)$ *there exist* $c > 0$ *and* $\delta > 0$ *such that*

$$f(z) \geq \inf f + cd^2(z, (C + 2\delta I\!B) \cap (\partial^- f)^{-1}(0)), \quad \forall z \in C + \delta I\!B. \tag{2}$$

When $f$ is a convex functional they proved that the necessary condition (2) turns out to be also sufficient whenever $(\partial^- f)^{-1}(0) \neq \emptyset$ and can be rewritten as

$$f(z) \geq \inf f + cd^2(z, (\partial^- f)^{-1}(0)), \quad \forall z \in (\partial^- f)^{-1}(0) + \delta I\!B. \tag{3}$$

Our approach is somewhat different since we focus on convex functions. Nevertheless, the interest of our study lies in two main points: first, we explore several regularity properties of the subdifferential itself (and not only its inverse) and second, we work in an infinite dimensional Hilbert space.

This work is organized as follows: in Section 2 we present briefly the metric regularity properties we will consider in the paper while we provide, in Section 3, several characterizations of these properties for the subdifferential of a proper lower semicontinuous convex function on $H$. In addition, as an illustration, we apply our results to a problem of convex optimization.

## 2. Background material

Throughout, $X$ and $Y$ stand for real Banach spaces. The closed unit ball is denoted by $I\!B$ while $I\!B_r(a)$ stands for the closed ball of radius $r$ centered at $a$. We denote by $d(x, C)$ the distance from a point $x$ to a set $C$, that is, $d(x, C) = \inf_{y \in C} \|x - y\|$.

Let $F$ be a set-valued mapping from $X$ into the subsets of $Y$, indicated by $F : X \rightrightarrows Y$. Here $\operatorname{gph} F = \{(x, y) \in X \times Y \mid y \in F(x)\}$ is the graph of $F$ and the range of $F$ is the set $\operatorname{rge} F = \{y \in Y \mid \exists\, x,\ F(x) \ni y\}$. The inverse of $F$, denoted by $F^{-1}$, is defined as $x \in F^{-1}(y) \Leftrightarrow y \in F(x)$.

Our study is organized around four key notions: metric regularity, strong metric regularity, metric subregularity and strong metric subregularity. We start with the metric regularity.

**Definition 2.1.** A mapping $F : X \rightrightarrows Y$ is said to be metrically regular at $\bar{x}$ for $\bar{y}$ if $F(\bar{x}) \ni \bar{y}$ and there exist some positive constants $\kappa, a$ and $b$ such that

$$d(x, F^{-1}(y)) \leq \kappa d(y, F(x)) \quad \text{for all } x \in I\!B_a(\bar{x}),\ y \in I\!B_b(\bar{y}). \tag{4}$$

The infimum of $\kappa$ for which (4) holds is the *regularity modulus* denoted $\operatorname{reg} F(\bar{x}\,|\,\bar{y})$; the case when $F$ is not metrically regular at $\bar{x}$ for $\bar{y}$ corresponds to $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) = \infty$. The inequality (4) has direct use in providing an estimate for how far a point $x$ is from being a solution to the variational inclusion $F(x) \ni y$; the expression $d(y, F(x))$ measures the residual when $F(x) \not\ni y$. Smaller values of $\kappa$ correspond to more favorable behavior. The metric regularity of a mapping $F$ at $\bar{x}$ for $\bar{y}$ is known to be equivalent to the Aubin property of the inverse $F^{-1}$ at $\bar{y}$ for $\bar{x}$. Recall that a set-valued map $\Gamma : Y \rightrightarrows X$ has the Aubin property at $(\bar{y}, \bar{x})$ (see, *e.g.,* [1]) if there exist positive constants $\kappa, a$ and $b$ such that

$$e(\Gamma(y') \cap I\!B_a(\bar{x}), \Gamma(y)) \leq \kappa \|y' - y\| \quad \text{for all } y, y' \in I\!B_b(\bar{y}), \tag{5}$$

where $e(A, B)$ denotes the excess from a set $A$ to a set $B$ and is defined as $e(A, B) = \sup_{x \in A} d(x, B)$. For more details on metric regularity and applications to variational problems one can refer to [3, 14, 16] and the monographs [23, 26].

In order to introduce the next regularity property, we need the notion of graphical localization. A *graphical localization* of a mapping $F : X \rightrightarrows Y$ at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is a mapping $\tilde{F} : X \rightrightarrows Y$ such that $\operatorname{gph} \tilde{F} = (U \times V) \cap \operatorname{gph} F$ for some neighborhood $U \times V$ of $(\bar{x}, \bar{y})$.

**Definition 2.2.** A mapping $F : X \rightrightarrows Y$ is strongly metrically regular at $\bar{x}$ for $\bar{y}$ if the metric regularity condition in Definition 2.1 is satisfied by some $\kappa$ and neighborhoods $U$ of $x$ and $V$ of $y$ and, in addition, the graphical localization of $F^{-1}$ with respect to $U$ and $V$ is single-valued. Equivalently, the graphical localization $V \ni y \mapsto F^{-1}(y) \cap U$ is a Lipschitz continuous function whose Lipschitz constant is equal to $\kappa$.

Strong regularity implies metric regularity by definition. The simplest case is the linear and bounded mapping from $\mathbb{R}^n$ to itself represented by an $n \times n$ matrix $A$. This mapping is strongly regular precisely when the matrix $A$ is nonsingular, and metrically regular when $A$ is merely surjective. But if a square matrix is surjective, that is, of full rank, it must be nonsingular. We see here an instance of a mapping when *metric regularity automatically implies strong regularity*, that is, metric regularity and strong regularity are equivalent. It turns out, see [13], that this equivalence holds for more general mappings of the form of the sum of a smooth function and the normal cone mapping over a polyhedral convex set.

For any set-valued mapping that is *locally monotone* near the reference point metric regularity at that point implies, and hence is equivalent to, strong regularity (see [11]). This is a consequence of a deeper result by Kenderov [19] regarding single-valuedness of lower semicontinuous monotone mappings. Recall that a mapping $F : X \rightrightarrows X^*$ is said to be locally monotone at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if there is a neighborhood $W$ of $(\bar{x}, \bar{y})$ such that

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0 \quad \text{whenever } (x_1, y_1), (x_2, y_2) \in \operatorname{gph} F \cap W.$$

We will say that $F$ is *monotone* if the last inequality is valid for any pair of points in the graph. A generalization of the result obtained in [11] was shown in [21] where the equivalence between strong regularity and metric regularity together with *premonotonicity* (which is an extension of the notion of monotonicity) was established.

**Definition 2.3.** A mapping $F : X \rightrightarrows Y$ is said to be metrically subregular at $\bar{x}$ for $\bar{y}$ if $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and there exists a constant $\kappa > 0$ along with neighborhoods $U$ of $\bar{x}$ and $V$ of $\bar{y}$ such that

$$d(x, F^{-1}(\bar{y})) \leq \kappa d(\bar{y}, F(x) \cap V) \quad \text{for all } x \in U. \tag{6}$$

Obviously, metric regularity forces metric subregularity. Moreover, it is worth noting that the subregularity of a set-valued mapping $F$ at $\bar{x}$ for $\bar{y}$ is equivalent to the *calmness* of its inverse $F^{-1}$ at $\bar{y}$ for $\bar{x}$, see [14], where the constant $\kappa$ in (6) might be slightly bigger when $\operatorname{gph} F$ is not locally closed. Let us recall that the mapping $F^{-1} : Y \rightrightarrows X$ is calm at $\bar{y}$ for $\bar{x}$ if there exists a constant $\kappa \in (0, \infty)$ along with neighborhoods $U$ of $\bar{x}$ and $V$ of $\bar{y}$ such that

$$F^{-1}(y) \cap U \subset F^{-1}(\bar{y}) + \kappa \|y - \bar{y}\| I\!\!B \quad \text{for all } y \in V. \tag{7}$$

Hence, the calmness of $F^{-1}$ at $\bar{y}$ for $\bar{x}$ is a localized version of Robinson's property of upper-Lipschitz continuity that we mentioned in Section 1. Actually, relation (1) corresponds to (7) when $U = X$. In [24], Robinson showed that when $X$ and $Y$ are finite-dimensional and $\operatorname{gph} F$ is the union of finitely many convex sets that are polyhedral then $F^{-1}$ is upper-Lipschitz continuous at every $\bar{y} \in \operatorname{dom} F^{-1}$.

Now we present briefly the strong subregularity which is the last regularity property we consider here.

**Definition 2.4.** A mapping $F : X \rightrightarrows Y$ is said to be strongly subregular at $\bar{x}$ for $\bar{y}$ if $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and there exists a constant $\kappa > 0$ along with neighborhoods $U$ of $\bar{x}$ and $V$ of $\bar{y}$ such that

$$\|x - \bar{x}\| \leq \kappa d(\bar{y}, F(x) \cap V) \quad \text{for all } x \in U.$$

An equivalent description is that $F$ is metrically subregular at $\bar{x}$ for $\bar{y}$ and, in addition, $\bar{x}$ is an isolated point of $F^{-1}(\bar{y})$.

This property is equivalent to the "local Lipschitz property at a point" of the inverse mapping, a property first formally introduced in [10].

## 3. Characterization of metric regularity

Our setting, for this section, is an arbitrary Hilbert space $H$ equipped with an inner product $\langle \cdot, \cdot \rangle$. In order to establish our results we use the following version of the well known Ekeland's variational principle.

**Theorem 3.1 (Ekeland's variational principle).** *Let $(X, d)$ be a complete metric space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function bounded from below. Suppose that for some $\varepsilon > 0$ and $z \in X$, $f(z) < \inf f + \varepsilon$. Then for any $\lambda > 0$ there exists $y \in X$ such that*

*(i)* $d(z, y) \leq \lambda$,

*(ii)* $f(y) + (\varepsilon/\lambda) d(z, y) \leq f(z)$, *and*

(*iii*) $f(x) + (\varepsilon/\lambda)d(x, y) > f(y)$, *for all* $x \in X \setminus \{y\}$.

The following very simple lemma will be helpful in simplifying the proofs of our main results.

**Lemma 3.2.** *Consider a function* $f$ *in* $\Gamma(H)$ *and two points* $\bar{x}$ *and* $\bar{v}$ *in* $H$ *with* $\bar{v} \in \partial f(\bar{x})$. *Defining the function* $g(\cdot) := f(\cdot) - \langle \bar{v}, \cdot \rangle$, *we have*

(*1*) $\partial f$ *is metrically subregular at* $\bar{x}$ *for* $\bar{v}$ *if and only if* $\partial g$ *is metrically subregular at* $\bar{x}$ *for* $0$.

(*2*) $\partial f$ *is metrically regular at* $\bar{x}$ *for* $\bar{v}$ *if and only if* $\partial g$ *is metrically regular at* $\bar{x}$ *for* $0$.

**Proof.** (*1*) The mapping $\partial f$ is metrically subregular at $\bar{x}$ for $\bar{v}$ if and only if there exist positive numbers $a, b$ and $\kappa$ such that

$$d(x, (\partial f)^{-1}(\bar{v})) \le \kappa d(\bar{v}, \partial f(x) \cap \mathbb{B}_b(\bar{v})) \quad \text{for all } x \in \mathbb{B}_a(\bar{x}).$$

Since $(\partial f)^{-1}(\bar{v}) = (\partial g)^{-1}(0)$, the above relation is clearly equivalent to

$$d(x, (\partial g)^{-1}(0)) \le \kappa d(\bar{v}, (\partial g(x) + \bar{v}) \cap (\bar{v} + \mathbb{B}_b(0))) = \kappa d(0, \partial g(x) \cap \mathbb{B}_b(0)),$$

whenever $x \in \mathbb{B}_a(\bar{x})$; that is, $\partial g$ is metrically subregular at $\bar{x}$ for 0.

(*2*) The mapping $\partial f$ is metrically regular at $\bar{x}$ for $\bar{v}$ if and only if there exist positive numbers $a, b$ and $\kappa$ such that

$$d(x, (\partial f)^{-1}(v)) \le \kappa d(v, \partial f(x)) \quad \text{for all } x \in \mathbb{B}_a(\bar{x}),\ v \in \mathbb{B}_b(\bar{v}).$$

This last inequality is equivalent to

$$d(x, (\partial g)^{-1}(v - \bar{v})) \le \kappa d(v, \partial g(x) + \bar{v}) = \kappa d(v - \bar{v}, \partial g(x)),$$

for all $x \in \mathbb{B}_a(\bar{x})$ and $v \in \mathbb{B}_b(\bar{v})$, or likewise

$$d(x, (\partial g)^{-1}(w)) \le \kappa d(w, \partial g(x)) \quad \text{for all } x \in \mathbb{B}_a(\bar{x}),\ w \in \mathbb{B}_b(0),$$

which is exactly the definition of the metric regularity of $\partial g$ at $\bar{x}$ for 0. □

The following theorem characterizes the subregularity of the subdifferential of a convex lower semicontinuous function on $H$.

**Theorem 3.3.** *Consider a function* $f$ *in* $\Gamma(H)$ *and points* $\bar{x}$ *and* $\bar{v}$ *in* $H$ *such that* $\bar{v} \in \partial f(\bar{x})$. *Then* $\partial f$ *is metrically subregular at* $\bar{x}$ *for* $\bar{v}$ *if and only if there exist a neighborhood* $U$ *of* $\bar{x}$ *and a positive constant* $c$ *such that*

$$f(x) \ge f(\bar{x}) - \langle \bar{v}, \bar{x} - x \rangle + c d^2(x, (\partial f)^{-1}(\bar{v})) \quad \textit{whenever } x \in U. \tag{8}$$

**Proof.** We first prove the theorem in the case when $\bar{v} = 0$. Suppose first that $f$ has the property (8) with $\bar{v} = 0$. We can assume without loss of generality that $U$ is a closed ball centered at $\bar{x}$. Let $x \in U$, if $d(x, (\partial f)^{-1}(0)) = 0$ we are done. Hence suppose that $d(x, (\partial f)^{-1}(0)) > \alpha$, for some $\alpha > 0$. We are going to show that $d(0, \partial f(x)) \ge c\alpha$. Consider $v \in \partial f(x)$ (if $\partial f(x) = \emptyset$ there is nothing to prove). Let $x_0$ be the projection of $x$ on the closed convex set $(\partial f)^{-1}(0)$, that is, $\|x - x_0\| = d(x, (\partial f)^{-1}(0))$. Let $x_1$ be a point in the segment line from $x$ to $x_0$ such that $\|x_1 - x_0\| = \alpha$. Since $v \in \partial f(x)$, $\langle v, x - x_0\rangle \geq f(x) - f(x_0)$. Moreover, since $x_0$ and $\bar{x}$ are respectively the projections of $x$ and $\bar{x}$ on $(\partial f)^{-1}(0)$, we have $\|x_0 - \bar{x}\| \leq \|x - \bar{x}\|$ (because of the Lipschitz property with modulus 1 of the projection). Hence $x_0$ belongs to $U$ and so does $x_1$. Using these facts, the convexity of the function $f$ and the property (8) we obtain

$$\|v\| \geq \frac{\langle v, x - x_0\rangle}{\|x - x_0\|} \geq \frac{f(x) - f(x_0)}{\|x - x_0\|} \geq \frac{f(x_1) - f(x_0)}{\|x_1 - x_0\|} \geq c\alpha,$$

and thus $d(x, (\partial f)^{-1}(0)) \leq (1/c)d(0, \partial f(x))$. Then, for any neighborhood $V$ of 0, we have

$$d(x, (\partial f)^{-1}(0)) \leq \frac{1}{c} d(0, \partial f(x) \cap V),$$

that is, $\partial f$ is metrically subregular at $\bar{x}$ for 0 with growth constant $1/c$.

Now assume that $\partial f$ has the property (6) at $\bar{x}$ for 0 with growth constant $\kappa > 0$ and neighborhoods $U$ of $\bar{x}$ and $V$ of 0. If (8) is not true, then for all $n \in \mathbb{N}\backslash\{0\}$, there exists $z_n \in \bar{x} + (1/n)\mathbb{B}$ such that

$$f(z_n) < f(\bar{x}) + \frac{1}{5k} d^2(z_n, (\partial f)^{-1}(0)).$$

Hence $0 < d(z_n, (\partial f)^{-1}(0)) \leq 1/n$. If we take $\lambda_n := (1/2)d(z_n, (\partial f)^{-1}(0))$, by the Ekeland's variational principle (see Theorem 3.1), there exists $x_n$ such that $\|x_n - z_n\| \leq \lambda_n$ and for all $x \in H$,

$$\begin{aligned} f(x) &\geq f(x_n) - \frac{1}{5k\lambda_n} d^2(z_n, (\partial f)^{-1}(0))\|x - x_n\| \\ &\geq f(x_n) - \frac{2}{5k} d(z_n, (\partial f)^{-1}(0))\|x - x_n\|. \end{aligned}$$

Thus $x_n$ minimizes the convex function $f(\cdot) + 2/(5\kappa)d(z_n, (\partial f)^{-1}(0))\|\cdot - x_n\|$, and therefore

$$\begin{aligned} 0 &\in \partial\left(f(\cdot) + \frac{4}{5\kappa}\lambda_n\|\cdot - x_n\|\right)(x_n) \\ &\subset \partial f(x_n) + \frac{4}{5\kappa}\lambda_n \partial(\|\cdot - x_n\|)(x_n) \\ &\subset \partial f(x_n) + \frac{4}{5\kappa}\lambda_n \mathbb{B} \end{aligned}$$

Then there exists $v_n \in \partial f(x_n)$ such that $\|v_n\| \leq 2/(5\kappa)d(z_n, (\partial f)^{-1}(0)) \leq 2/(5\kappa n)$. Since

$$\begin{aligned} d(z_n, (\partial f)^{-1}(0)) &\leq \|z_n - x_n\| + d(x_n, (\partial f)^{-1}(0)) \\ &\leq \lambda_n + d(x_n, (\partial f)^{-1}(0)) = \frac{1}{2} d(z_n, (\partial f)^{-1}(0)) + d(x_n, (\partial f)^{-1}(0)), \end{aligned}$$

one has $d(z_n, (\partial f)^{-1}(0)) \leq 2d(x_n, (\partial f)^{-1}(0))$, and thus

$$\|v_n\| \leq 2/(5\kappa)d(z_n, (\partial f)^{-1}(0)) \leq 4/(5\kappa)d(x_n, (\partial f)^{-1}(0)).$$

Hence for all positive integer $n$ there exist $v_n \in 2/(5\kappa n)\mathbb{B}$ and $x_n \in (\partial f)^{-1}(v_n)$ such that $d(x_n, (\partial f)^{-1}(0)) \geq (5\kappa/4)\|v_n\|$. Therefore,

$$d(x_n, (\partial f)^{-1}(0)) \geq \frac{5}{4}\kappa d(0, \partial f(x_n) \cap V) \quad \text{eventually.}$$

Since $0 < d(z_n, (\partial f)^{-1}(0)) \leq 2d(x_n, (\partial f)^{-1}(0))$, we have $d(0, \partial f(x_n) \cap V) > 0$, and from

$$\|x_n - \bar{x}\| \leq \|x_n - z_n\| + \|z_n - \bar{x}\| \leq \frac{1}{2}d(z_n, (\partial f)^{-1}(0)) + \frac{1}{n} \leq \frac{3}{2n},$$

we obtain a contradiction with (6).

Now we prove Theorem 3.3 for any $\bar{v} \neq 0$. Consider a function $f \in \Gamma(H)$ and a point $\bar{x} \in H$ such that $\bar{v} \in \partial f(\bar{x})$. Set $g(\cdot) := f(\cdot) - \langle \bar{v}, \cdot \rangle$, then $g \in \Gamma(H)$, and since $\partial g(\cdot) = \partial f(\cdot) - \bar{v}$ we have $0 \in \partial g(\bar{x})$. Then we can apply Theorem 3.3, in the case when $\bar{v} = 0$, to the function $g$. We obtain that $\partial g$ is metrically subregular at $\bar{x}$ for 0 if and only if there exist a neighborhood $U$ of $\bar{x}$ and a positive number $c$ such that

$$g(x) \geq g(\bar{x}) + cd^2(x, (\partial g)^{-1}(0)) \quad \text{whenever } x \in U.$$

Keeping in mind that $\partial g$ is metrically subregular at $\bar{x}$ for 0 if and only if $\partial f$ is metrically subregular at $\bar{x}$ for $\bar{v}$ (see Lemma 3.2) and noting that $(\partial g)^{-1}(0) = (\partial f)^{-1}(\bar{v})$, we complete the proof. □

In the same way that metric subregularity 'localizes' the concept of upper-Lipschitz continuity the characterization given in Theorem 3.3 corresponds (in the case when $\bar{y} = 0$) to a local of version of the one established in [28] (see relation (3)).

Moreover, let us mention that $\partial f$ may be subregular with constant $\kappa$ but assertion (8) can be false for $c = 1/\kappa$. For example, take $\kappa = 2$ and let $f(x) = x^2$. Then $\partial f$ is metrically subregular at 0 for 0 with constant $1/2$, but relation (8) doesn't hold for $c = 2$.

**Remark 3.4.** A consequence of the proof of this theorem is that, in our case, the definition of subregularity (6) is valid for any neighborhood $V$ of $\bar{v}$ and thus can be simplified in the following way

$$d(x, (\partial f)^{-1}(\bar{v})) \leq \kappa d(\bar{v}, \partial f(x)) \quad \text{for all } x \in U. \tag{9}$$

Actually, this is the simplifying criterion given by Dontchev and Rockafellar in [14] for the metric subregularity of a mapping $F : X \rightrightarrows Y$ having

$$\liminf_{\substack{x \to \bar{x} \\ x \in \operatorname{dom} F}} F(x) \ni \bar{y}. \tag{10}$$

Nevertheless condition (10) is not valid in general in our case (for instance, take $f(x) = |x|$ and $x_n = 1/n$). In fact, it turns out that, even in the general case, the definition of subregularity can be simplified in the following way:

A mapping $F : X \rightrightarrows Y$ is metrically subregular at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with constant $\kappa$ if and only if there is some neighborhood $U$ of $\bar{x}$ such that

$$d(x, F^{-1}(\bar{y})) \leq \kappa d(\bar{y}, F(x)) \quad \text{for all } x \in U. \tag{11}$$

Indeed, if $F$ satisfies (11) for some $\kappa > 0$ and some neighborhood $U$ of $\bar{x}$, then $F$ is metrically subregular at $(\bar{x}, \bar{y})$ with constant $\kappa$ and neighborhoods $U$ and $V$, for any neighborhood $V$ of $\bar{y}$. Conversely, if $F$ is metrically subregular at $(\bar{x}, \bar{y})$ with constant $\kappa > 0$, then there are some neighborhoods $U$ of $\bar{x}$ and $V$ of $\bar{y}$ such that

$$d(x, F^{-1}(\bar{y})) \leq \kappa d(\bar{y}, F(x) \cap V) \quad \text{for all } x \in U. \tag{12}$$

Let $\delta > 0$ be such that $V' := \bar{y} + \delta I\!B \subset V$. Take $U' := U \cap (\bar{x} + \delta\kappa I\!B)$. Pick $x \in U'$. If $F(x) \cap V' \neq \emptyset$, then $d(\bar{y}, F(x) \cap V') = d(\bar{y}, F(x))$ and thus $x$ verifies the inequality (11). Otherwise, $F(x) \cap V' = \emptyset$ and hence

$$d(\bar{y}, F(x)) \geq \delta \geq \frac{1}{\kappa}\|x - \bar{x}\| \geq \frac{1}{\kappa} d(x, F^{-1}(\bar{y})),$$

and (11) holds for all $x \in U'$.

It is worth writing explicitly the following direct consequence of Theorem 3.3. It concerns the metric subregularity of $\partial f$ near a critical point $\bar{x}$ of the function $f$ (*i.e.,* a point satisfying $\partial f(\bar{x}) \ni 0$). More precisely, we have that $\partial f$ is metrically subregular at $\bar{x}$ for 0 if and only if the mapping $f$ satisfies:

$$f(x) \geq \inf f + cd^2(x, (\partial f)^{-1}(0)) \quad \text{for } x \text{ close to } \bar{x}.$$

Such a condition is called a *quadratic growth condition.* Recall that if $S$ denotes a nonempty set of minimum points of a function $f$, *i.e.*, $S \subset \operatorname{argmin} f$ then we say that $f$ satisfies the quadratic growth condition on $S$ if there exists a positive constant $c$ such that

$$f(x) \geq \inf f + cd^2(x, S) \quad \text{for all } x \text{ in a neighborhood of } S. \tag{13}$$

Particular forms of relation (13) are known to be equivalent to the second order sufficient condition in nonlinear programming problems with qualified constraints (see *e.g.,* [4]). Such a condition has also proved to be a key tool in sensitivity analysis (see [27]); for more details about the notion of quadratic growth condition one can refer to [2, 5, 9] and the references therein.

Now we examine the case when the subdifferential operator is strongly subregular.

**Theorem 3.5.** *Consider a function $f$ in $\Gamma(H)$ and points $\bar{x}$ and $\bar{v}$ in $H$ such that $\bar{v} \in \partial f(\bar{x})$. Then $\partial f$ is strongly subregular at $\bar{x}$ for $\bar{v}$ if and only if there exist a neighborhood $U$ of $\bar{x}$ and a positive constant $c$ such that*

$$f(x) \geq f(\bar{x}) - \langle \bar{v}, \bar{x} - x \rangle + c\|x - \bar{x}\|^2 \quad \textit{whenever } x \in U. \tag{14}$$

**Proof.** It is a straightforward consequence of Theorem 3.3. If $\partial f$ is strongly subregular at $\bar{x}$ for $\bar{v}$, then it is subregular at $\bar{x}$ for $\bar{v}$ and, in addition, $(\partial f)^{-1}(\bar{v}) \cap U = \{\bar{x}\}$ for some neighborhood $U$ of $\bar{x}$. But because of the convexity of $f$ we have $(\partial f)^{-1}(\bar{v}) = \{\bar{x}\}$, and thus (8) gives (14).
Conversely, the property (14) implies (8) and then $\partial f$ is metrically subregular at $\bar{x}$ for $\bar{v}$. Moreover, relation (14) yields in particular that $\bar{x}$ is an isolated point of $(\partial f)^{-1}(\bar{v})$ and therefore the mapping $\partial f$ is strongly metrically subregular at $\bar{x}$ for $\bar{v}$. □

As in Remark 3.4 the definition of strong subregularity can be simplified in our case. Actually this holds true in general:

A mapping $F : X \rightrightarrows Y$ is strongly subregular at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with constant $\kappa$ if and only if there is some neighborhood $U$ of $\bar{x}$ such that

$$\|x - \bar{x}\| \leq \kappa d(\bar{y}, F(x)) \quad \text{for all } x \in U.$$

This can be shown by repeating the same arguments we used to prove the simplified criterion (11) in Remark 3.4.

Next comes a result about the metric regularity of the subdifferential of a proper lower semicontinuous convex function on $H$.

**Theorem 3.6.** *Consider a function $f$ in $\Gamma(H)$ and points $\bar{x}$ and $\bar{v}$ in $H$ such that $\bar{v} \in \partial f(\bar{x})$. Then $\partial f$ is metrically regular at $\bar{x}$ for $\bar{v}$ if and only if there exist neighborhoods $U$ of $\bar{x}$ and $V$ of $\bar{v}$ and a positive constant $c$ such that for all $v \in V$*

$$(\partial f)^{-1}(v) \neq \emptyset, \quad \textit{and} \tag{15}$$
$$f(x) \geq f(\tilde{x}) - \langle v, \tilde{x} - x \rangle + c d^2(x, (\partial f)^{-1}(v)), \tag{16}$$

*for all $x \in U$ and $\tilde{x} \in (\partial f)^{-1}(v)$.*

**Proof.** Here again, we only present the proof for the case $\bar{v} = 0$, the general case being a straightforward consequence of the particular case $\bar{v} = 0$ and Lemma 3.2.

Assume that $\partial f$ is metrically regular at $\bar{x}$ for 0, that is, it verifies (4) for some $\kappa > 0$ and neighborhoods $U$ of $\bar{x}$ and $V$ of 0. We are going to prove that there is some $\delta > 0$ and $c > 0$ such that

$$f(\tilde{x}) - \langle v, \tilde{x} - x \rangle \leq f(x) - c d^2(x, (\partial f)^{-1}(v)), \tag{17}$$

for all $v \in (\delta/\kappa)\mathbb{B}$, $x \in (\bar{x} + \delta\mathbb{B}) \cap ((\partial f)^{-1}(v) + 2\delta\mathbb{B})$ and $\tilde{x} \in (\partial f)^{-1}(v)$. If this is not true, then for all $n \in \mathbb{N}\backslash\{0\}$, there exist $v_n \in 1/(\kappa n)\mathbb{B}$, $z_n \in (\bar{x} + (1/n)\mathbb{B}) \cap ((\partial f)^{-1}(v_n) + (2/n)\mathbb{B})$ and $\tilde{x}_n \in (\partial f)^{-1}(v_n)$ such that

$$f(z_n) < f(\tilde{x}_n) - \langle v_n, \tilde{x}_n - z_n \rangle + \frac{1}{5\kappa} d^2(z_n, (\partial f)^{-1}(v_n)). \tag{18}$$

Since $v_n \in \partial f(\tilde{x}_n)$, we have $\langle v_n, \tilde{x}_n - z_n \rangle \geq f(\tilde{x}_n) - f(z_n)$. Combining this very last inequality with relation (18) we get

$$0 < f(\tilde{x}_n) - f(z_n) - \langle v_n, \tilde{x}_n - z_n \rangle + \frac{1}{5\kappa} d^2(z_n, (\partial f)^{-1}(v_n)) \leq \frac{1}{5\kappa} d^2(z_n, (\partial f)^{-1}(v_n)).$$

Therefore $0 < d(z_n, (\partial f)^{-1}(v_n)) \leq 2/n$. Moreover, since $v_n \in \partial f(\tilde{x}_n)$, we have $\langle v_n, \tilde{x}_n - x \rangle \geq f(\tilde{x}_n) - f(x)$ for all $x \in H$, *i.e.*,

$$f(\tilde{x}_n) - \langle v_n, \tilde{x}_n \rangle \leq f(x) - \langle v_n, x \rangle \quad \text{for all } x \in H.$$

Let $g_n(\cdot) := f(\cdot) - \langle v_n, \cdot \rangle$. Then $\inf_{x \in H} g_n(x) = g_n(\tilde{x}_n)$ and from (18) we have

$$g_n(z_n) < \inf_{x \in H} g_n(x) + \frac{1}{5\kappa} d^2(z_n, (\partial f)^{-1}(v_n)).$$

Taking $\lambda_n := (1/2)d(z_n, (\partial f)^{-1}(v_n))$, the Ekeland's variational principle ensures that there is $x_n$ such that $\|x_n - z_n\| \leq \lambda_n$ and for all $x \in H$,

$$\begin{aligned} g_n(x) &\geq g_n(x_n) - \frac{1}{5\kappa\lambda_n} d^2(z_n, (\partial f)^{-1}(v_n))\|x - x_n\| \\ &\geq g_n(x_n) - \frac{2}{5\kappa} d(z_n, (\partial f)^{-1}(v_n))\|x - x_n\|. \end{aligned}$$

Thus $x_n$ minimize the convex function $g_n(\cdot) + 2/(5\kappa)d(z_n, (\partial f)^{-1}(v_n))\|\cdot - x_n\|$. Therefore

$$\begin{aligned} 0 &\in \partial\left(g_n(\cdot) + \frac{4}{5\kappa}\lambda_n\|\cdot - x_n\|\right)(x_n) \\ &\subset \partial g_n(x_n) + \frac{4}{5\kappa}\lambda_n\partial(\|\cdot - x_n\|)(x_n) \\ &\subset \partial f(x_n) - v_n + \frac{4}{5\kappa}\lambda_n I\!B. \end{aligned}$$

Then there exists $w_n \in \partial f(x_n)$ such that $\|w_n - v_n\| \leq 4\lambda_n/(5\kappa)$. Since

$$\begin{aligned} d(z_n, (\partial f)^{-1}(v_n)) &\leq \|z_n - x_n\| + d(x_n, (\partial f)^{-1}(v_n)) \\ &\leq \frac{1}{2} d(z_n, (\partial f)^{-1}(v_n)) + d(x_n, (\partial f)^{-1}(v_n)), \end{aligned}$$

we obtain $d(z_n, (\partial f)^{-1}(v_n)) \leq 2d(x_n, (\partial f)^{-1}(v_n))$, and thus

$$\|w_n - v_n\| \leq \frac{2}{5\kappa} d(z_n, (\partial f)^{-1}(v_n)) \leq \frac{4}{5\kappa} d(x_n, (\partial f)^{-1}(v_n)).$$

Therefore

$$d(x_n, (\partial f)^{-1}(v_n)) \geq \frac{5\kappa}{4}\|w_n - v_n\| \geq \frac{5\kappa}{4} d(v_n, \partial f(x_n)). \tag{19}$$

Since $2d(x_n, (\partial f)^{-1}(v_n)) \geq d(z_n, (\partial f)^{-1}(v_n)) > 0$, we have $d(v_n, \partial f(x_n)) > 0$, and thus (19) is a contradiction with (4) because $\|v_n\| \leq 1/(\kappa n)$ and

$$\|x_n - \bar{x}\| \leq \|x_n - z_n\| + \|z_n - \bar{x}\| \leq \frac{1}{2} d(z_n, (\partial f)^{-1}(v_n)) + \frac{1}{n} \leq \frac{2}{n}. \tag{20}$$

Hence (17) holds true for some $\delta > 0$ and $c > 0$. Make $\delta$ smaller if necessary so that $(\delta/\kappa)I\!B \subset V$. Then if $v \in (\delta/\kappa)I\!B$ and $x \in \bar{x} + \delta I\!B$,

$$d(x, (\partial f)^{-1}(v)) \leq \|x - \bar{x}\| + d(\bar{x}, (\partial f)^{-1}(v)) \leq \delta + \kappa d(v, \partial f(\bar{x})) \leq \delta + \kappa\|v\| \leq 2\delta. \tag{21}$$

Thus $(\bar{x} + \delta I\!B) \cap ((\partial f)^{-1}(v) + 2\delta I\!B) = \bar{x} + \delta I\!B$ for all $v \in (\delta/\kappa)I\!B$ and therefore (17) implies (16) for $U' := \bar{x} + \delta I\!B$ and $V' := (\delta/\kappa)I\!B$. Also (21) yields in particular (15).

For the converse, suppose that properties (15) and (16) are satisfied for some neighborhoods $U$ of $\bar{x}$ and $V$ of 0. Then, thanks to (15), we have that $d(x, (\partial f)^{-1}(v)) < \infty$ for any $x \in U$ and $v \in V$. Fix $x \in U$ and $v \in V$, we are going to show that

$$d(x, (\partial f)^{-1}(v)) \leq \frac{1}{c}\, d(v, \partial f(x)).$$

If $d(x,(\partial f)^{-1}(v)) = 0$ then there is nothing to prove. Otherwise, there exist a positive number $\alpha$ such that $d(x,(\partial f)^{-1}(v)) = \alpha$. Consider any $v' \in \partial f(x)$ (if $\partial f(x) = \emptyset$ we are done). Let $x_0$ be the projection of $x$ over $(\partial f)^{-1}(v)$, that is, $\|x - x_0\| = \alpha$. Applying (16) to $x, v$ and $x_0 \in (\partial f)^{-1}(v)$, we get

$$f(x) - f(x_0) \geq -\langle v, x_0 - x\rangle + cd^2(x,(\partial f)^{-1}(v)).$$

Keeping in mind that $v' \in \partial f(x)$ and using this last inequality we obtain

$$\|v' - v\| \geq \frac{\langle v' - v, x - x_0\rangle}{\|x - x_0\|} \geq \frac{f(x) - f(x_0) - \langle v, x - x_0\rangle}{\|x - x_0\|} \geq c\alpha.$$

Therefore, $d(v, \partial f(x)) \geq c\alpha = cd(x,(\partial f)^{-1}(v))$, and thus $\partial f$ is metrically regular at $\bar{x}$ for 0 with growth constant $1/c$. □

Condition (15) is necessary in order to have the metric regularity, in the sense that it is not implied by the other assumptions. For instance, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = 0$ for all $x \in \mathbb{R}$. This function satisfies all the hypotheses of Theorem 3.6 but (15), since $\partial f(x) = \{0\}$ for all $x \in \mathbb{R}$, and $\partial f$ is not metrically regular at any $x \in \mathbb{R}$ for 0.

For mappings with closed and convex graph we have the following result that extends the Banach open mapping principle and characterizes metric regularity.

**Theorem 3.7 (Robinson-Ursescu).** *For a mapping $F : X \rightrightarrows Y$ and $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, if $F$ has closed convex graph, then $F$ is metrically regular at $(\bar{x}, \bar{y})$ if and only if $\bar{y} \in \operatorname{int} \operatorname{rge} F$.*

The condition $\bar{y} \in \operatorname{int} \operatorname{rge} F$ given by this theorem is equivalent in our case to (15). Nevertheless, $\operatorname{gph}(\partial f)$ is not convex in general, and (15) is not sufficient for having metric regularity. Indeed, let $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} -x & \text{for } x < -1, \\ 1 & \text{for } x \in [-1, 1], \\ x & \text{for } x > 1. \end{cases}$$

A simple computation gives us

$$\partial f(x) = \begin{cases} -1 & \text{for } x < -1, \\ \text{[-1,0]} & \text{for } x = -1, \\ 0 & \text{for } x \in (-1, 1), \\ \text{[0,1]} & \text{for } x = 1, \\ 1 & \text{for } x > 1. \end{cases}$$

Then, condition (15) is fulfilled but $\partial f$ is not metrically regular at 0 for 0. Hence condition (16) is not implied in general by the other assumptions of Theorem 3.6.

It is well known that $\partial f$ is monotone for any $f \in \Gamma(X)$, and this fact will serve us to simplify the conditions in Theorem 3.6. Moreover, as we mentioned in Section 2, strong regularity is equivalent to metric regularity for monotone mappings.

**Proposition 3.8.** *Let $f \in \Gamma(X)$. Then $\partial f$ is metrically regular at $\bar{x}$ for $\bar{v}$ if and only if $\partial f$ is strongly regular at $\bar{x}$ for $\bar{v}$.*

Note that, for monotone mappings, the equivalence between strong subregularity and metric subregularity is not valid, as we can see for instance for the zero function (which is metrically subregular at 0 for 0 but not strongly subregular).

This last proposition leads us to the following reformulation of Theorem 3.6.

**Corollary 3.9.** *Consider a function $f \in \Gamma(H)$ and points $\bar{x}$ and $\bar{v}$ in $H$ such that $\bar{v} \in \partial f(\bar{x})$. Then $\partial f$ is (strongly) metrically regular at $\bar{x}$ for $\bar{v}$ if and only if there exist neighborhoods $U$ of $\bar{x}$ and $V$ of $\bar{v}$ and a positive constant $c$ such that, for any $v \in V$ there is $\tilde{x} \in H$ such that $(\partial f)^{-1}(v) = \{\tilde{x}\}$ and*

$$f(x) \geq f(\tilde{x}) - \langle v, \tilde{x} - x \rangle + c\|x - \tilde{x}\|^2 \quad \textit{whenever } x \in U. \tag{22}$$

**Proof.** It is clear that condition (22) implies both assertions (15) and (16). Conversely, if $\partial f$ is metrically regular at $\bar{x}$ for $\bar{v}$, by Corollary 3.8, $\partial f$ is strongly regular. Then, by Theorem 3.6, there are some positive $a$ and $b$ such that $(\partial f)^{-1}(v) \cap (\operatorname{int} I\!B_a(\bar{x}))$ is single-valued whenever $v \in I\!B_b(\bar{v})$ and $f$ verifies (15) and (16) for $U = I\!B_a(\bar{x})$ and $V = I\!B_b(\bar{v})$. But from the convexity of $f$ we have that $(\partial f)^{-1}(v)$ is a convex set for any $v \in H$. Thus $(\partial f)^{-1}(v)$ must be single-valued for any $v \in I\!B_b(\bar{v})$. This together with (15) and (16) give us (22). □

A straightforward consequence of Theorem 3.6 is a result concerning the Lipschitz continuity of the inverse of the subdifferential of a convex lower semicontinuous function around the origin. Recall that a set-valued mapping $T : Y \rightrightarrows X$ is Lipschitz continuous at $\bar{y} \in Y$ with a constant $\kappa > 0$ if there is a neighborhood $V$ of $\bar{y}$ such that

$$d(x, T(y)) \leq \kappa d(y, T^{-1}(x)) \quad \text{for all } x \in X,\ y \in V, \tag{23}$$

which can be viewed as a global version of the metric regularity of the mapping $T^{-1}$.

**Theorem 3.10.** *Consider a function $f \in \Gamma(H)$ and points $\bar{x}$ and $\bar{v}$ in $H$ such that $\bar{v} \in \partial f(\bar{x})$. Then $(\partial f)^{-1}$ is Lipschitz continuous at $\bar{v}$ if and only if there exist a neighborhood $V$ of $\bar{v}$ and a positive constant $c$ such that, for any $v \in V$ there is (only one) $\tilde{x} \in H$ with*

$$f(x) \geq f(\tilde{x}) - \langle v, \tilde{x} - x \rangle + c\|x - \tilde{x}\|^2 \quad \textit{for any } x \in H, \tag{24}$$

*which implies in particular that $(\partial f)^{-1}(v) = \{\tilde{x}\}$.*

**Proof.** The proof is twofold but since it may be performed in much the same way as the one of Theorem 3.6 we only give its main ideas. First, we prove that $(\partial f)^{-1}$ is Lipschitz continuous at $\bar{v}$ if and only if there is a neighborhood $V$ of $\bar{v}$ such that

$$(\partial f)^{-1}(v) \neq \emptyset, \quad \text{and} \tag{25}$$

$$f(\tilde{x}) - \langle v, \tilde{x} - x \rangle \leq f(x) - cd^2(x, (\partial f)^{-1}(v)), \tag{26}$$

for all $x \in H$, $v \in V$ and $\tilde{x} \in (\partial f)^{-1}(v)$. The "only if" part can be shown using the same arguments given in the proof of Theorem 3.6. The major difference here is that the point $z_n$

lies in the whole space $H$ instead of belonging to $(\bar{x}+(1/n)\mathbb{B})\cap((\partial f)^{-1}(v_n)+(2/n)\mathbb{B})$ and consequently we only have the boundedness condition $0 < d(z_n, (\partial f)^{-1}(v_n)) \leq \|z_n - \tilde{x}_n\|$. Since we do not need relation (20) we are done after establishing (19). For the converse, it suffices to replace $U$ by $H$ in the argument.

Finally, proceeding analogously to the proof of Corollary 3.9, we obtain (24). □

As an illustration, consider the following optimization problem

$$\begin{array}{c} \min f(x) \\ \text{s.t. } x \in K, \end{array} \tag{27}$$

where $f \in \Gamma(H)$ and $K$ is a nonempty closed convex subset of $H$. Under one of the following regularity conditions

$$f \text{ is finite at some point of } \operatorname{int} K, \tag{28}$$

$$f \text{ is finite and continuous at some point of } K, \tag{29}$$

a necessary and sufficient condition for $\bar{x} \in H$ to be globally optimal is

$$\partial f(\bar{x}) + N_K(\bar{x}) \ni 0, \tag{30}$$

where $N_K(\bar{x})$ is the normal cone to $K$ at $\bar{x}$ (for more details, see e.g. [7]). Let us consider the convex function $f(\cdot) + \delta_K(\cdot)$, where

$$\delta_K(x) = \begin{cases} 0 & \text{for } x \in K, \\ \infty & \text{for } x \notin K, \end{cases}$$

is the indicator function of $K$. Since $\partial \delta_K(x) = N_K(x)$ for any $x \in K$, under (28) or (29), we have

$$\partial(f + \delta_K)(x) = \partial f(x) + N_K(x) \quad \text{for any } x \in H.$$

Thus we can apply all the previous results to the function $f + \delta_K$. In particular, from Theorem 3.5 we obtain

**Corollary 3.11.** *Let $f \in \Gamma(H)$, let $K$ be a closed convex subset of $H$ such that* (28) *or* (29) *holds, and let $\bar{x} \in H$ be a solution of* (27). *Then $\partial f + N_K$ is strongly subregular at $\bar{x}$ for $0$ if and only if there exist a neighborhood $U$ of $\bar{x}$ and a positive constant $c$ such that*

$$f(x) \geq f(\bar{x}) + c\|x - \bar{x}\|^2 \quad \textit{whenever } x \in U \cap K. \tag{31}$$

For a (possibly nonconvex) function $f : H \to \mathbb{R}$ which is twice continuously differentiable everywhere, it has been shown (see [6, Theorem 3.70]) that the quadratic growth condition (31) is equivalent to the *second-order sufficient condition*

$$\langle u, \nabla^2 f(\bar{x}) u \rangle \geq \beta \|u\|^2 \quad \text{for all } x \in C(\bar{x}), \tag{32}$$

where $\beta$ is some positive constant and $C(\bar{x})$ is the *critical cone* to $K$ associated with $\bar{x}$ defined by

$$C(\bar{x}) = T_K(\bar{x}) \cap (\nabla f(\bar{x}))^{\perp},$$

and $T_K(\bar{x})$ is the tangent cone to $K$ at $\bar{x}$. Therefore, by applying Corollary 3.11, we have the following characterization.

**Corollary 3.12.** *Let $f \in \Gamma(H)$ be a twice continuously differentiable function, let $K$ be a closed convex subset of $H$, and let $\bar{x} \in H$ be a point verifying* (30)*. Then $\nabla f + N_K$ is strongly subregular at $\bar{x}$ for* 0 *if and only if the second-order sufficient condition* (32) *holds at $\bar{x}$ for some $\beta > 0$.*

In fact, when $K$ is a polyhedral subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a (possibly nonconvex) function which is twice continuously differentiable, condition (32) reduces to

$$\langle u, \nabla^2 f(\bar{x})u\rangle > 0 \quad \text{for all nonzero } x \in C(\bar{x}), \tag{33}$$

and we have the following result, where the convexity of $f$ is not a necessary condition (see [12] for details).

**Proposition 3.13.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice continuously differentiable function, let $K$ be a nonempty polyhedral subset of $\mathbb{R}^n$ and $\bar{x}$ be a (stationary) point verifying* (30)*. Then the second-order sufficient condition* (33) *holds at $\bar{x}$ if and only if $\bar{x}$ is a local minimizer of* (27) *and the mapping $\nabla f + N_K$ is strongly subregular at $\bar{x}$ for* 0*.*

The following *strong second-order sufficient condition* for local optimality has proved to be a key notion in optimization:

$$\langle u, \nabla^2 f(\bar{x})u\rangle > 0 \quad \text{for all nonzero } x \in C(\bar{x}) - C(\bar{x}), \tag{34}$$

where, because of the convexity of the critical cone, the *critical subspace* $C(\bar{x}) - C(\bar{x})$ is the smallest subspace that contains $C(\bar{x})$. It turns out, see [12], that (34) is equivalent to the strong regularity of $\nabla f + N_K$.

**Proposition 3.14.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice continuously differentiable function, let $K$ be a nonempty polyhedral subset of $\mathbb{R}^n$ and $\bar{x}$ be a (stationary) point verifying* (30)*. Then the strong second-order sufficient condition* (34) *holds at $\bar{x}$ if and only if $\bar{x}$ is a local minimizer of* (27) *and the mapping $\nabla f + N_K$ is strongly regular at $\bar{x}$ for* 0*.*

As a direct consequence of this result and Corollary 3.9 we have

**Corollary 3.15.** *Let $f \in \Gamma(\mathbb{R}^n)$ be a twice continuously differentiable function, let $K$ be a nonempty polyhedral subset of $\mathbb{R}^n$ and $\bar{x}$ be a (stationary) point verifying* (30)*. Then the strong second-order sufficient condition* (34) *holds at $\bar{x}$ if and only if there exist neighborhoods $U$ of $\bar{x}$ and $V$ of* 0 *and a positive constant $c$ such that, for any $v \in V$ there is $\tilde{x} \in K$ such that $(\nabla f + N_K)^{-1}(v) = \{\tilde{x}\}$ and*

$$f(x) \geq f(\tilde{x}) - \langle v, \tilde{x} - x\rangle + c\|x - \tilde{x}\|^2 \quad \textit{whenever } x \in U \cap K. \tag{35}$$

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