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Well-Posedness of Inverse Variational Inequalities

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Let $\Omega \subset R^P$ be a nonempty closed and convex set and $f : R^P \to R^P$ be a function. The inverse variational inequality is to find $x^* \in R^P$ such that

$$f(x^*) \in \Omega, \quad \langle f' - f(x^*), x^* \rangle \geq 0, \quad \forall f' \in \Omega.$$

The purpose of this paper is to investigate the well-posedness of the inverse variational inequality. We establish some characterizations of its well-posedness. We prove that under suitable conditions, the well-posedness of an inverse variational inequality is equivalent to the existence and uniqueness of its solution. Finally, we show that the well-posedness of an inverse variational inequality is equivalent to the well-posedness of an enlarged classical variational inequality.

Keywords: Inverse variational inequality, variational inequality, well-posedness, metric characterization

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1. Introduction and Preliminaries

Throughout this paper, unless otherwise specified, we always suppose that $\Omega \subset R^P$ is a nonempty, closed and convex set, $K \subset R^m$ is a nonempty, closed and convex set, $f : R^P \to R^P$, and $F : K \to R^m$. Denote by $\langle \cdot, \cdot \rangle$ the standard inner product. The classical variational inequality (denoted by $VI(K, F)$) is to determine a vector $u^* \in K$ such that

$$\langle F(u^*), u - u^* \rangle \geq 0, \quad \forall u \in K.$$

The classical variational inequality has many important applications in different fields and has been studied intensively (see e.g. [1, 2, 4, 11, 14, 15, 17]). Especially, the well-posedness issues of variational inequalities have been attracting attentions of researchers in the fields of mathematics and economics. Lucchetti and Patrone [17] introduced in the literature the first notion of well-posedness for a variational inequality. Lignola and Morgan [14] introduced the parametric well-posedness for a family of variational inequalities and investigated its links with the extended well-posedness [19, 20] of the corresponding minimization problems. Lignola and Morgan [15] further introduced the concept of

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α-well-posedness for variational inequalities. Recently, Fang and Hu [2] studied the well-posedness of variational inequalities by means of estimate functions for approximating solutions. For other results on the well-posedness of variational inequalities, we refer the readers to [1, 3, 16].

In this paper we consider the well-posedness of the following inverse variational inequality (denoted by $IVI(\Omega, f)$): find $x^* \in R^P$ such that

$$f(x^*) \in \Omega, \quad \langle f' - f(x^*), x^* \rangle \geq 0, \quad \forall f' \in \Omega.$$

Clearly, if f has a single-valued inverse function f^{-1}, $IVI(\Omega, f)$ can be translated into the classical variational inequality $VI(\Omega, F)$ by setting $u^* = f(x^*)$ and $F(u^*) = f^{-1}(u^*)$. This motivates the name of inverse variational inequalities. However, $f(x)$ does not allow measurements, and only $F(u)$ is available in some practical applications. In addition, an inverse variational inequality also arises in some practical problems. In fact, the primary motivation of study on inverse variational inequalities originates from the fact that the equilibrium state control problem can be interpreted as an inverse variational inequality. For details, we refer the readers to [9, 10]. Another motivation lies in the fact that an inverse variational inequality can be regarded as a special case of a general variational inequality formulated as: find $x^* \in R^P$ such that

$$f(x^*) \in \Omega, \quad \langle y - f(x^*), g(x^*) \rangle \geq 0, \quad \forall y \in \Omega.$$

where $g : R^P \to R^P$, which has been studied intensively (see e.g., [7, 18]). Compared with variational inequalities, there are only a few results on the inverse variational inequality in the literature. Recently, some numerical methods have been developed to solve the inverse variational inequality (see e.g. [5, 6, 8]). The fact that well-posedness issue is closely related to numerical methods motivates us to investigate the well-posedness of inverse variational inequalities. We generalize the concept of well-posedness to an inverse variational inequality and establish some characterizations of its well-posedness. We prove that under suitable conditions, the well-posedness of an inverse variational inequality is equivalent to the existence and uniqueness of its solution. We also prove that the well-posedness of an inverse variational inequality is equivalent to the well-posedness of an enlarged classical variational inequality.

In the sequel we always suppose that α is a nonnegative number.

Definition 1.1. A sequence $\{x_n\} \subset R^P$ is called an α-approximating sequence for $IVI(\Omega, f)$ iff there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$f(x_n) \in \Omega, \quad \langle f(x_n) - f', x_n \rangle \leq \frac{\alpha}{2} \|f(x_n) - f'\|^2 + \epsilon_n, \quad \forall f' \in \Omega, \forall n \in N.$$

When $\alpha = 0$, we say that $\{x_n\}$ is an approximating sequence for $IVI(\Omega, f)$.

Definition 1.2. We say that $IVI(\Omega, f)$ is α-well-posed iff $IVI(\Omega, f)$ has a unique solution and every α-approximating sequence converges to the unique solution. If $\alpha_1 > \alpha_2 \geq 0$, then α_1-well-posedness implies α_2-well-posedness. In the sequel 0-well-posedness is always called well-posedness.

Definition 1.3. We say that $IVI(\Omega, f)$ is α-well-posed in the generalized sense iff $IVI(\Omega, f)$ has a nonempty solution set S and every α-approximating sequence has some subsequence which converges to some point of S. In the sequel 0-well-posedness in the generalized sense is always called well-posedness in the generalized sense.

Definition 1.4. A function $f : R^P \to R^P$ is said to be monotone iff

$$\langle f(x) - f(y), x - y\rangle \geq 0, \quad \forall x, y \in R^P.$$

Definition 1.5. A function $f : R^P \to R^P$ is said to be hemicontinuous iff for any $x, y \in R^P$, the function $t \mapsto \langle f(x + t(y - x)), y - x\rangle$ from $[0, 1]$ to R is continuous at 0_+.

Denote by $P_\Omega(z)$ the metric projection of z on Ω, i.e.,

$$P_\Omega(z) = \text{argmin}\{\|z - x\| : x \in \Omega\}, \quad \forall z \in R^P.$$

It is known that $u = P_\Omega(z)$ if and only if

$$\langle u - z, v - u\rangle \geq 0, \quad \forall v \in \Omega.$$

We need the following lemma and concepts to deal with α-well-posedness of $IVI(\Omega, f)$.

Lemma 1.6. *Let $\alpha \geq 0$, $x^* \in R^P$ with $f(x^*) \in \Omega$, and let $\Omega \subset R^P$ be a nonempty convex set. Then*

$$\langle f(x^*) - f', x^*\rangle \leq 0, \quad \forall f' \in \Omega$$

if and only if

$$\langle f(x^*) - f', x^*\rangle \leq \frac{\alpha}{2}\|f(x^*) - f'\|^2, \quad \forall f' \in \Omega.$$

Proof. The necessity holds trivially. For the sufficiency, suppose that

$$\langle f(x^*) - f', x^*\rangle \leq \frac{\alpha}{2}\|f(x^*) - f'\|^2, \quad \forall f' \in \Omega.$$

For given $g \in \Omega$ and $t \in [0, 1]$, we have $f(x^*) + t(g - f(x^*)) \in \Omega$ since Ω is convex and $f(x^*) \in \Omega$. It follows that

$$\begin{aligned} t\langle f(x^*) - g, x^*\rangle &= \langle f(x^*) - (f(x^*) + t(g - f(x^*))), x^*\rangle \\ &\leq \frac{\alpha}{2}\|f(x^*) - (f(x^*) + t(g - f(x^*)))\|^2 \\ &= \frac{\alpha t^2}{2}\|f(x^*) - g\|^2. \end{aligned}$$

This yields

$$\langle f(x^*) - g, x^*\rangle \leq \frac{\alpha t}{2}\|f(x^*) - g\|^2.$$

Letting $t \to 0$ in the above inequality, we get

$$\langle f(x^*) - g, x^*\rangle \leq 0, \quad \forall g \in \Omega.$$

□

Definition 1.7 ([13]). Let A be a nonempty subset of R^P. The noncompactness measure μ of the set A is defined by

$$\mu(A) = \inf\{\epsilon > 0 : A \subset \cup_{i=1}^{n} A_i,\ \operatorname{diam} A_i < \epsilon,\ i = 1, 2, \cdots, n\},$$

where diam means the diameter of a set.

Definition 1.8. Let A, B be nonempty subsets of R^P. The Hausdorff distance $\mathcal{H}(\cdot,\cdot)$ between A and B is defined by

$$\mathcal{H}(A, B) = \max\{e(A, B), e(B, A)\},$$

where $e(A, B) = \sup_{a\in A} d(a, B)$ with $d(a, B) = \inf_{b\in B} \|a - b\|$. Let $\{A_n\}$ be a sequence of nonempty subsets of R^P. We say that A_n converges to A in the sense of Hausdorff if $\mathcal{H}(A_n, A) \to 0$. It is easy to see that $e(A_n, A) \to 0$ if and only if $d(a_n, A) \to 0$ for all selection $a_n \in A_n$. For more details on this topic, we refer the readers to [12, 13].

2. Metric Characterizations

Let α, f, Ω be defined as in the previous section. In this section we derive some metric characterizations of α-well-posedness and α-well-posedness in the generalized sense for $IVI(\Omega, f)$.

Consider the α-approximating solution set $T_\alpha(\epsilon)$ of $IVI(\Omega, f)$:

$$T_\alpha(\epsilon) = \{x \in R^P : f(x) \in \Omega, \langle f(x) - f', x\rangle \le \frac{\alpha}{2}\|f(x) - f'\|^2 + \epsilon, \forall f' \in \Omega\}, \quad \forall \epsilon \ge 0.$$

We first give a metric characterization of α-well-posedness for $IVI(\Omega, f)$.

Theorem 2.1. *Let Ω be nonempty, closed and convex, and let $f : R^P \to R^P$ be continuous. Then $IVI(\Omega, f)$ is α-well-posed if and only if*

$$T_\alpha(\epsilon) \ne \emptyset, \quad \forall \epsilon > 0 \quad \textit{and} \quad \operatorname{diam} T_\alpha(\epsilon) \to 0 \quad \textit{as } \epsilon \to 0. \tag{1}$$

Proof. Suppose that $IVI(\Omega, f)$ is α-well-posed. Then $IVI(\Omega, f)$ has a unique solution x^*. Clearly, $x^* \in T_\alpha(\epsilon)$ for all $\epsilon > 0$. If $\operatorname{diam} T_\alpha(\epsilon) \not\to 0$ as $\epsilon \to 0$, then there exist $l > 0$, $\epsilon_n > 0$ with $\epsilon_n \to 0$, and $u_n, v_n \in T_\alpha(\epsilon_n)$ such that

$$\|u_n - v_n\| > l, \quad \forall n \in N. \tag{2}$$

Since $u_n, v_n \in T_\alpha(\epsilon_n)$, both $\{u_n\}$ and $\{v_n\}$ are α-approximating sequences for $IVI(\Omega, f)$. By the α-well-posedness of $IVI(\Omega, f)$, they have to converge to the unique solution x^* of $IVI(\Omega, f)$, a contradiction to (2).

Conversely, suppose that condition (1) holds. Let $\{x_n\} \subset R^P$ be an α-approximating sequence for $IVI(\Omega, f)$. Then there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$f(x_n) \in \Omega, \quad \langle f(x_n) - f', x_n\rangle \le \frac{\alpha}{2}\|f(x_n) - f'\|^2 + \epsilon_n, \quad \forall f' \in \Omega, \forall n \in N. \tag{3}$$

This yields that $x_n \in T_\alpha(\epsilon_n)$. From (1), we know that $\{x_n\}$ is a Cauchy sequence and so it converges to a point $\bar{x} \in R^P$. Since f is continuous and Ω is closed, we have

$$f(\bar{x}) \in \Omega, \quad \langle f(\bar{x}) - f', \bar{x}\rangle \le \frac{\alpha}{2}\|f(\bar{x}) - f'\|^2, \quad \forall f' \in \Omega.$$

It follows from Lemma 1.6 that

$$f(\bar{x}) \in \Omega, \quad \langle f(\bar{x}) - f', \bar{x} \rangle \leq 0, \quad \forall f' \in \Omega.$$

Thus $\bar{x}$ is a solution of $IVI(\Omega, f)$. To complete the proof, we need only to prove that $IVI(\Omega, f)$ has a unique solution. Assume by contradiction that $IVI(\Omega, f)$ has two distinct solution x_1 and x_2. Then it is easy to see that $x_1, x_2 \in T_\alpha(\epsilon)$ for all $\epsilon > 0$ and so

$$0 < \|x_1 - x_2\| \leq \operatorname{diam} T_\alpha(\epsilon) \to 0,$$

a contradiction to (1). □

Next we establish a metric characterization of α-well-posedness in the generalized sense by considering the noncompact measure of $T_\alpha(\epsilon)$.

Theorem 2.2. *Let $\Omega \subset R^P$ be nonempty, closed and convex and let $f : R^P \to R^P$ be continuous. Then $IVI(\Omega, f)$ is α-well-posed in the generalized sense if and only if*

$$T_\alpha(\epsilon) \neq \emptyset, \quad \forall \epsilon > 0 \quad \textit{and} \quad \mu(T_\alpha(\epsilon)) \to 0 \quad \textit{as } \epsilon \to 0. \tag{4}$$

Proof. Suppose that $IVI(\Omega, f)$ is α-well-posed in the generalized sense. Let S be the solution set of $IVI(\Omega, f)$. Then S is nonempty and compact. Indeed, let $\{x_n\}$ be any sequence in S. Clearly $\{x_n\}$ is an α-approximating sequence for $IVI(\Omega, f)$. By the α-well-posedness in the generalized sense of $IVI(\Omega, f)$, $\{x_n\}$ has a subsequence which converges to some point of S. Thus S is compact. Further we have $T_\alpha(\epsilon) \supset S \neq \emptyset$ for all $\epsilon > 0$. Next we shall show that

$$\mu(T_\alpha(\epsilon)) \to 0 \quad \text{as } \epsilon \to 0.$$

Observe that for every $\epsilon > 0$,

$$\mathcal{H}(T_\alpha(\epsilon), S) = \max\{e(T_\alpha(\epsilon), S), e(S, T_\alpha(\epsilon))\} = e(T_\alpha(\epsilon), S).$$

Taking into account the compactness of S, we get

$$\mu(T_\alpha(\epsilon)) \leq 2\mathcal{H}(T_\alpha(\epsilon), S) + \mu(S) = 2e(T_\alpha(\epsilon), S).$$

To prove (4), it is sufficient to show

$$e(T_\alpha(\epsilon), S) \to 0 \text{ as } \epsilon \to 0.$$

If $e(T_\alpha(\epsilon), S) \not\to 0$ as $\epsilon \to 0$, then there exist $l > 0$, $\epsilon_n > 0$ with $\epsilon_n \to 0$, and $x_n \in T_\alpha(\epsilon_n)$ such that

$$x_n \notin S + B(0, l), \quad \forall n \in N, \tag{5}$$

where $B(0, l)$ is the closed ball centered at 0 with radius l. Being $x_n \in T_\alpha(\epsilon_n)$, $\{x_n\}$ is an α-approximating sequence for $IVI(\Omega, f)$. By the α-well-posedness in the generalized sense of $IVI(\Omega, f)$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to some point of S. This contradicts to (5) and so

$$e(T_\alpha(\epsilon), S) \to 0 \quad \text{as } \epsilon \to 0.$$

Conversely, assume that (4) holds. Since f is continuous and Ω is closed, $T_\alpha(\epsilon)$ is nonempty closed for all $\epsilon > 0$. Consider

$$S' := \cap_{\epsilon>0} T_\alpha(\epsilon) = \{x \in R^P : f(x) \in \Omega, \langle f(x) - f', x\rangle \leq \frac{\alpha}{2}\|f(x) - f'\|^2, \forall f' \in \Omega\}.$$

This together with Lemma 1.6 yields

$$S = S' = \cap_{\epsilon>0} T_\alpha(\epsilon).$$

Since $\mu(T_\alpha(\epsilon)) \to 0$, the Theorem on page 412 of [13] can be applied and one concludes that S is nonempty, compact and

$$e(T_\alpha(\epsilon), S) = \mathcal{H}(T_\alpha(\epsilon), S) \to 0 \quad \text{as } \epsilon \to 0.$$

Let $\{u_n\} \subset R^P$ be an α-approximating sequence for $IVI(\Omega, f)$. Then there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$f(u_n) \in \Omega, \quad \langle f(u_n) - f', u_n\rangle \leq \frac{\alpha}{2}\|f(u_n) - f'\|^2 + \epsilon_n, \quad \forall f' \in \Omega, \forall n \in N.$$

This means that $u_n \in T_\alpha(\epsilon_n)$. It follows that

$$d(u_n, S) \leq e(T_\alpha(\epsilon_n), S) \to 0.$$

Since S is compact, there exists $\bar{x}_n \in S$ such that

$$\|u_n - \bar{x}_n\| = d(u_n, S) \to 0.$$

Again from the compactness of S, $\{\bar{x}_n\}$ has a subsequence $\{\bar{x}_{n_k}\}$ converging to $\bar{x} \in S$. Hence the corresponding subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converges to $\bar{x}$. Thus $IVI(\Omega, f)$ is α-well-posed in the generalized sense. □

3. Conditions for Well-Posedness

In this section we shall prove that under suitable conditions, the well-posedness of an inverse variational inequality is equivalent to the existence and uniqueness of its solutions.

Theorem 3.1. *Let $\Omega \subset R^P$ be nonempty, closed and convex. Let $f : R^P \to R^P$ be hemicontinuous and monotone. Then, $IVI(\Omega, f)$ is well-posed if and only if it has a unique solution.*

Proof. The necessity holds trivially. For the sufficiency, suppose that $IVI(\Omega, f)$ has a unique solution x^*. Then

$$f(x^*) \in \Omega, \quad \langle f(x^*) - f', x^*\rangle \leq 0, \quad \forall f' \in \Omega.$$

Since f is monotone, we get

$$\langle f(x) - f', x^* - x\rangle + \langle x, f(x^*) - f'\rangle \leq \langle x^*, f(x^*) - f'\rangle \leq 0, \quad \forall x \in R^P, \forall f' \in \Omega. \tag{6}$$

Let $\{x_n\} \subset R^P$ be an approximating sequence for $IVI(\Omega, f)$. Then there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$f(x_n) \in \Omega, \quad \langle f(x_n) - f', x_n\rangle \leq \epsilon_n, \quad \forall f' \in \Omega, \forall n \in N.$$

Since f is monotone, it follows that

$$\langle f(x) - f', x_n - x\rangle + \langle x, f(x_n) - f'\rangle \leq \langle x_n, f(x_n) - f'\rangle \leq \epsilon_n, \quad \forall x \in R^P, \forall f' \in \Omega. \quad (7)$$

Take

$$u^* := (x^*, f(x^*)) \quad \text{and} \quad u_n := (x_n, f(x_n)), \quad \forall n \in N.$$

If $\{u_n\}$ is unbounded, without loss of generality, we can suppose that $\|u_n\| \to +\infty$. Set

$$\begin{aligned} t_n := \frac{1}{\|u_n - u^*\|}, \quad w_n = (z_n, g_n) &:= u^* + t_n(u_n - u^*) \\ &= (x^* + t_n(x_n - x^*), f(x^*) + t_n(f(x_n) - f(x^*))). \end{aligned}$$

Without loss of generality, we can suppose that $t_n \in (0, 1]$ and $w_n \to w = (z, g) \neq u^*$. Further we have $g \in \Omega$ since Ω is closed and convex. For any $f' \in \Omega$ and any $x \in R^P$, it follows that

$$\begin{aligned} & \langle f(x) - f', z - x\rangle + \langle x, g - f'\rangle \\ = \;& \langle f(x) - f', z - z_n\rangle + \langle f(x) - f', z_n - x^*\rangle + \langle f(x) - f', x^* - x\rangle \\ & + \langle x, g - g_n\rangle + \langle x, g_n - f(x^*)\rangle + \langle x, f(x^*) - f'\rangle \\ = \;& \{\langle f(x) - f', z - z_n\rangle + \langle x, g - g_n\rangle\} + \{\langle f(x) - f', x^* - x\rangle + \langle x, f(x^*) - f'\rangle\} \\ & + t_n\{\langle f(x) - f', x_n - x^*\rangle + \langle x, f(x_n) - f(x^*)\rangle\} \\ = \;& \{\langle f(x) - f', z - z_n\rangle + \langle x, g - g_n\rangle\} + (1 - t_n)\{\langle f(x) - f', x^* - x\rangle + \langle x, f(x^*) - f'\rangle\} \\ & + t_n\{\langle f(x) - f', x_n - x\rangle + \langle x, f(x_n) - f'\rangle\}. \end{aligned} \quad (8)$$

It follows from (6)-(8) that

$$\begin{aligned} & \langle f(x) - f', z - x\rangle + \langle x, g - f'\rangle \\ & \leq \langle f(x) - f', z - z_n\rangle + \langle x, g - g_n\rangle + t_n\epsilon_n, \forall f' \in \Omega, \quad \forall x \in R^P, \forall n \in N. \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we get

$$\langle f(x) - f', z - x\rangle + \langle x, g - f'\rangle \leq 0, \quad \forall f' \in \Omega, \forall x \in R^P. \quad (9)$$

For any $x' \in R^P$ and any $g' \in \Omega$, define $z(t) := z + t(x' - z)$ and $g(t) := g + t(g' - g)$ for all $t \in [0, 1]$. It follows from (9) that

$$\langle f(z(t)) - g(t), z - z(t)\rangle + \langle z(t), g - g(t)\rangle \leq 0,$$

which leads to

$$\langle f(z(t)) - g(t), z - x'\rangle + \langle z(t), g - g'\rangle \leq 0.$$

Since f is hemicontinuous, letting $t \to 0_+$ in the above inequality, we get

$$\langle f(z) - g, z - x'\rangle + \langle z, g - g'\rangle \leq 0, \quad \forall x' \in R^P, \forall g' \in \Omega. \quad (10)$$

From (10), since x' is arbitrary, it follows that

$$s\langle f(z) - g, r\rangle \leq \text{constant} \quad (11)$$

for every real s and every r. From (11) we get $f(z) = g$, so that

$$\langle z, f(z) - g' \rangle \leq 0, \quad \forall g' \in \Omega. \tag{12}$$

From (12), we know that z solves $IVI(\Omega, f)$ and so $z = x^*$ since x^* is the unique solution of $IVI(\Omega, f)$. This is a contradiction to $(x^*, f(x^*)) \neq (z, f(z))$.

So we can suppose that $\{u_n\}$ is bounded. Let $\{u_{n_k}\}$ be any subsequence of $\{u_n\}$ such that $u_{n_k} \to (\bar{x}, \bar{g})$ as $k \to \infty$. It follows from (7) that

$$\langle f(x) - f', x_{n_k} - x \rangle + \langle x, f(x_{n_k}) - f' \rangle \leq \epsilon_{n_k}, \quad \forall x \in R^P, \forall f' \in \Omega.$$

Letting $k \to \infty$ in the above inequality, we get

$$\langle f(x) - f', \bar{x} - x \rangle + \langle x, \bar{g} - f' \rangle \leq 0, \quad \forall x \in R^P, \forall f' \in \Omega.$$

By same arguments as in (9)-(12), we have

$$f(\bar{x}) = \bar{g} \in \Omega, \quad \langle \bar{x}, f(\bar{x}) - f' \rangle \leq 0, \quad \forall f' \in \Omega.$$

Thus $\bar{x}$ solves $IVI(\Omega, f)$. We have $\bar{x} = x^*$ since $IVI(\Omega, f)$ has a unique solution x^*. Therefore x_n converges to x^* and so $IVI(\Omega, f)$ is well-posed. □

The following simple example can illustrate the conclusion of Theorem 3.1.

Example 3.2. Let $\Omega = R^2_+$ and $f(x) = (x^2, x^4)$ for all $x \in R^2$. Clearly Ω is closed and convex, f is hemicontinuous and monotone, and $x^* = (0, 0)$ is the unique solution of $IVI(\Omega, f)$. By Theorem 3.1, $IVI(\Omega, f)$ is well-posed.

4. Links with Well-Posedness of Classical Variational Inequalities

In this section we shall prove that the well-posedness of an inverse variational inequality is equivalent to the well-posedness of an enlarged classical variational inequality.

Let $K \subset R^m$ be a nonempty, closed and convex set and $F : K \to R^m$. Consider the classical variational inequality (denoted by $VI(K, F)$): find $u^* \in K$ such that

$$\langle F(u^*), u - u^* \rangle \geq 0, \quad \forall u \in K.$$

Definition 4.1 ([14]). A sequence $\{u_n\} \subset K$ is said to be an approximating sequence for $VI(K, F)$ iff there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$\langle F(u_n), u_n - v \rangle \leq \epsilon_n, \quad \forall v \in K, \forall n \in N.$$

Definition 4.2 ([14]). We say that $VI(K, F)$ is well-posed iff $VI(K, F)$ has a unique solution and every approximating sequence converges to the unique solution. We say that $VI(K, F)$ is well-posed in the generalized sense iff $VI(K, F)$ has a nonempty solution set S and every approximating sequence has a subsequence which converges to some point of S.

Let $K = R^P \times \Omega$ and $F : K \to R^{2P}$ be defined by

$$F(u) = \begin{pmatrix} f(x) - y \\ x \end{pmatrix}, \quad \forall u = (x, y) \in K.$$

In what follows we always suppose that F and K are defined as above. The following lemma shows that every inverse variational inequality $IVI(\Omega, f)$ is equivalent to an enlarged variational inequality $VI(K, F)$.

Lemma 4.3 ([8]). *Let $x^* \in R^P$ and $u^* = (x^*, f(x^*)) \in K = R^P \times \Omega$. Then x^* is a solution of $IVI(\Omega, f)$ if and only if u^* is a solution of $VI(K, F)$.*

Theorem 4.4. *Let $\Omega \subset R^P$ be closed and let $f : R^P \to R^P$ be continuous. Then, $IVI(\Omega, f)$ is well-posed if and only if $VI(K, F)$ is well-posed.*

Proof. Let $IVI(\Omega, f)$ be well-posed. Then $IVI(\Omega, f)$ has a unique solution $x^* \in R^P$. By Lemma 4.3, $u^* = (x^*, f(x^*))$ is the unique solution of $VI(K, F)$. Let $u_n = (x_n, y_n) \in K$ be an approximating sequence for $VI(K, F)$. Then there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$\langle F(u_n), u_n - v\rangle \le \epsilon_n, \quad \forall v = (x, y) \in K, \forall n \in N.$$

This implies

$$\langle f(x_n) - y_n, x_n - x\rangle + \langle y_n - y, x_n\rangle \le \epsilon_n$$

hence

$$\langle f(x_n) - y_n, x_n - x\rangle \le \epsilon_n + \langle y - y_n, x_n\rangle, \quad \forall x \in R^P, \forall y \in \Omega, \forall n \in N.$$

Fix $y \in \Omega, z \in R^P$ and consider $x = x_n - sz$, then

$$s\langle f(x_n) - y_n, z\rangle \le \text{constant}$$

where s is arbitrary. Then $f(x_n) = y_n$, hence

$$\langle x_n, y_n - y\rangle = \langle x_n, f(x_n) - y\rangle \le \epsilon_n, \quad \forall y \in \Omega, \forall n \in N.$$

This means that $\{x_n\} \subset R^P$ is an approximating sequence for $IVI(\Omega, f)$. By the well-posedness of $IVI(\Omega, f)$, $x_n \to x^*$. Therefore, $u_n = (x_n, f(x_n)) \to (x^*, f(x^*))$ and so $VI(K, F)$ is well-posed.

Conversely, assume that $VI(K, F)$ is well-posed. Then it has a unique solution $u^* = (x^*, y^*)$ with $y^* = f(x^*)$. By Lemma 4.3, x^* is the unique solution of $IVI(\Omega, f)$. Let $\{x_n\} \subset R^P$ be an approximating sequence for $IVI(\Omega, f)$. Then there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$f(x_n) \in \Omega, \quad \langle f(x_n) - y, x_n\rangle \le \epsilon_n, \quad \forall y \in \Omega, \forall n \in N. \tag{13}$$

Take

$$y_n := f(x_n) \quad \text{and} \quad u_n := (x_n, y_n).$$

It follows from (13) that

$$u_n \in K, \quad \langle F(u_n), u_n - v\rangle \le \epsilon_n, \quad \forall v = (x, y) \in K, \forall n \in N.$$

This means that $\{u_n\}$ is an approximating sequence for $VI(K, F)$. By the well-posedness of $VI(K, F)$, $u_n = (x_n, f(x_n)) \to (x^*, f(x^*))$. Thus x_n converges to x^* and so $IVI(\Omega, f)$ is well-posed. □

For the well-posedness in the generalized sense, we have the following analogous result:

Theorem 4.5. *Let $\Omega \subset R^P$ be closed and let $f : R^P \to R^P$ be continuous. Then, $IVI(\Omega, f)$ is well-posed in the generalized sense if and only if $VI(K, F)$ is well-posed in the generalized sense.*

Proof. The conclusion follows from analogous arguments as in Theorem 4.4. □

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