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An Evolutionary Structure of Convex Quadrilaterals

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We solve the problem of evolution of convex quadrilaterals by applying the inverse weighted Fermat-Torricelli problem, the invariance property of the weighted Fermat-Torricelli point in the plane $\mathbb{R}^2$, two-dimensional sphere S^2 and the two- dimensional hyperboloid H^2. This means that the property of plasticity is inherited by some evolutionary convex quadrilaterals. An important application is the connection of the Fermat-Torricelli point with the fundamental equation of P. de Fermat.

Keywords: Fermat-Torricelli problem, inverse Fermat-Torricelli problem, generalized convex quadrilaterals

1991 Mathematics Subject Classification: 51E12, 52A10, 52A55, 51E10

1. Introduction

Pierre de Fermat in 1643 raised the question: given three points in the Euclidean plane, find the point that the sum of distances to these three points is minimized. Contributions to the problem have been made by E. Torricelli, J. Steiner and this is the main reason that we have named the corresponding point the Fermat-Torricelli point due to the first two contributors. E. Weiszfeld found the generalized weighted Fermat-Torricelli point (A_0) for n given non-collinear points $A_i(x_i, y_i)$ with non negative weights B_i, $i = 1, 2, \ldots, n$ by using an algorithm that minimizes the function $f(x, y)$ corresponding to the coordinates of $A_0(x, y)$ (see [5] and [4]). The weighted "Fermat-Torricelli" point of a plane triangle and the inverse weighted "Fermat-Torricelli" problem have been studied in [3]. A unified approach for the weighted "Fermat-Torricelli" point of a given plane, spherical or hyperbolic triangle $\nabla A_1A_2A_3$ with non-negative weights B_i that correspond to each vertex A_i respectively is studied in [7]. Concerning the floating case and absorbed case (see below) of the weighted "Fermat-Torricelli" point in $\mathbb{R}^N$, see [4] and Chapter II of [1]. For the solution of the problem in any regular surface of $\mathbb{R}^3$ we refer to the indirect method of [6]. The invariance property of the weighted "Fermat-Torricelli" point holds for any given spherical, hyperbolic and planar (Euclidean) triangle (see [7]) and the inverse weighted "Fermat-Torricelli" problem in $\mathbb{R}^2$ was also studied in [3]. For the case of the two-dimensional sphere S^2 and two-dimensional hyperboloid H^2 you can consult [7]. In this paper we derive the evolutionary structure of some weighted convex quadrilaterals $A_1A_2A_3A_4$ in $\mathbb{R}^2$, S^2, H^2 under some conditions. The crucial point is to apply the invariance property of the weighted Fermat-Torricelli point and the inverse weighted

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Fermat-Torricelli problem of four lines that meet at the weighted Fermat-Torricelli point at A_0. The connection of the weighted "Fermat-Torricelli point" in $\mathbb{R}^2$ with the fundamental equation of P. de Fermat is a remarkable result that comes from the inverse weighted Fermat-Torricelli problem of four lines in $\mathbb{R}^2$.

2. The weighted Fermat-Torricelli point of a convex quadrilateral

Suppose that we have the convex quadrilateral $A_1A_2A_3A_4$ such that B_i is the positive weight that corresponds to the vertex A_i, $i = 1, 2, 3, 4$. We start with the notations: let a_i be the variable that represents the length of the geodesic segment A_0A_i, a_{ij} be the length of the geodesic segment A_iA_j and α_{ij} be the angle between the two geodesics A_iA_k, A_kA_j for $i, j, k = 1, 2, 3, 4$, $i \neq j \neq k$. The angles α_3, α_3', α_3'', α_3''' are defined in Figure 2.1 and their role is vital for the concluding results (see Figure 2.1).

Y. S. Kupitz and H. Martini gave an elegant proof of existence and uniqueness of the generalized Fermat-Torricelli point in $\mathbb{R}^d$ for $d = 2, \ldots, n$ and also derived conditions for the floating case and absorbed case that exist at the interior of the polygon or at a vertex. We start by stating the problem:

Problem 2.1. *Find the weighted Fermat-Torricelli point of a given weighted convex quadrilateral $A_1A_2A_3A_4$ with non-negative weights B_i that correspond to the vertex A_i, and the floating case occurs regarding the weights B_i*

$$\|\sum_{j=1}^{4} B_j \vec{u}(A_i, A_j)\| > B_i, i \neq j$$

for $i, j = 1, 2, 3, 4$.

Solution 2.2. The minimum function that corresponds to the weighted Fermat-Torricelli point A_0 is:

$$B_1a_1 + B_2a_2 + B_3a_3 + B_4a_4 = \text{minimum}. \tag{1}$$

The weighted Fermat-Torricelli point will be found by calculating the angles a_{012}, a_{023}, a_{034}, a_{041}. Let a_2, a_3, a_4 the functions that depend on the variables a_1 and α_3: we assume that the floating case of the weighted Fermat-Torricelli point A_0 occurs, in order to locate it in the interior of the convex quadrilateral $A_1A_2A_3A_4$ (see [4]) with non-negative weights B_i, $i = 1, 2, 3, 4$. The idea is to calculate the angles α_{012}, α_{023}, α_{034}, α_{041} of A_0. The intersection of at least two segments of the cycles with lengths for instance α_{012}, α_{023} that corresponds to the sides a_{12}, a_{23} is the weighted Fermat-Torricelli point A_0. Let a_2, a_3, a_4 the functions that depend on the variables a_1 and α_3:

$$a_2 = a_2(a_1, \alpha_3), \quad a_3 = a_3(a_1, \alpha_3), \quad a_4 = a_4(a_1, \alpha_3). \tag{2}$$

By differentiating the minimum function with respect to the variables α_3, a_1, two conditions are deduced:

$$B_2\frac{\partial a_2}{\partial \alpha_3} + B_3\frac{\partial a_3}{\partial \alpha_3} + B_4\frac{\partial a_4}{\partial \alpha_3} = 0$$

$$B_1 + B_2\frac{\partial a_2}{\partial a_1} + B_3\frac{\partial a_3}{\partial a_1} + B_4\frac{\partial a_4}{\partial a_1} = 0.$$

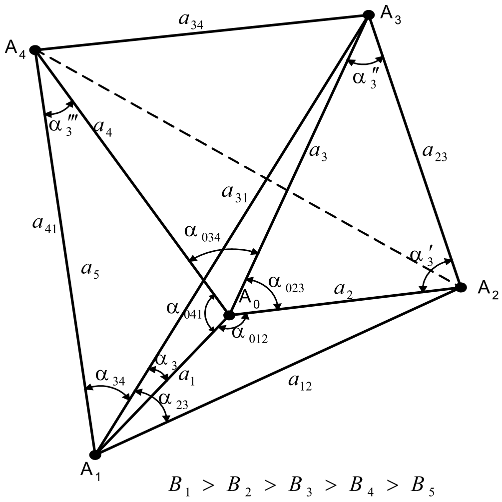

Figure 2.1

By calculating $\frac{\partial a_2}{\partial \alpha_3}$, $\frac{\partial a_3}{\partial \alpha_3}$,$\frac{\partial a_4}{\partial \alpha_3}$ and by applying the "cosine law" and "sine law" to the corresponding triangles $\nabla A_2A_1A_0$, $\nabla A_3A_1A_0$, $\nabla A_4A_1A_0$, we derive the equation:

$$-B_2\sin(\alpha_{012}) + B_3\sin(\alpha_{034}+\alpha_{041}) + B_4\sin(\alpha_{041}) = 0. \tag{3}$$

Similarly, from the differentiation with respect to the variable a_1, the second condition takes the form:

$$B_1 + B_2\cos(\alpha_{012}) + B_3\cos(\alpha_{034}+\alpha_{041}) + B_4\cos(\alpha_{041}) = 0. \tag{4}$$

From (3), (4) two equations are derived:

$$\cot(\alpha_{034}+\alpha_{041}) = \frac{B_1 + B_2\cos(\alpha_{012}) + B_4\cos(\alpha_{041}))}{B_4\sin(\alpha_{041}) - B_2\sin(\alpha_{012})} \tag{5}$$

$$B_3^2 = B_1^2 + B_2^2 + B_4^2 + 2B_2B_4\cos(\alpha_{041}+\alpha_{012}) + 2B_1B_2\cos(\alpha_{012}) + 2B_1B_4\cos(\alpha_{041}). \tag{6}$$

Two geometrical equations are used that depend on the angles α_{012}, α_{041} and they appear by applying the "sine law" to the triangles $\nabla A_1A_0A_3$, $\nabla A_1A_0A_2$, $\nabla A_1A_0A_4$:

$$\cot(\alpha_3) = \frac{\sin(\alpha_{23}) - \cos(\alpha_{23})\cot(\alpha_{012}) - \frac{a_{31}}{a_{12}}\cot(\alpha_{034}+\alpha_{041})}{-\cos(\alpha_{23}) - \sin(\alpha_{23})\cot(a_{012}) + \frac{a_{31}}{a_{12}}} \tag{7}$$

$$\cot(\alpha_3) = \frac{\sin(\alpha_{34}) - \cos(\alpha_{34})\cot(\alpha_{041}) + \frac{a_{31}}{a_{41}}\cot(\alpha_{034}+\alpha_{041})}{\cos(\alpha_{34}) + \sin(\alpha_{34})\cot(\alpha_{041}) - \frac{a_{31}}{a_{41}}}. \tag{8}$$

From (7), (8) and taking into account (5), we get the equation:

$$f_2(\alpha_{012}, \alpha_{041}) = 0. \tag{9}$$

The system of the two implicit equations (9), (6) give the angles α_{012}, α_{041}. If these values are replaced to equation (5), then α_{034} is obtained. Finally,

$$\alpha_{023} = 2\pi - \alpha_{012} - \alpha_{034} - \alpha_{41}.$$

Remark 2.3. The Newton method may be selected and the initial conditions for the values of the two angles are taken from the inequalities between B_i, $i = 1, 2, 3, 4$. The angle α_3 plays a vital role in the implementation of the computational approach of the generalized weighted "Fermat-Torricelli" point. By selecting different weights from Figure 2.1 modifications of signs will occur in equations (7), (8), by using the same method to calculate the $\cot(\alpha_3)$.

Example 2.4. For the case of tetragon the derived equations depend only on the values of the non-negative weights B_i, $i = 1, 2, 3, 4$.

$$-\cot(\alpha_{041} - \alpha_{012}) + \cot(\alpha_{034} + \alpha_{041}) = \frac{\cot(\alpha_{012}) + \cot(\alpha_{041})}{\cot(\alpha_{041}) - \cot(\alpha_{012})}$$

and

$$\cot(\alpha_{034} + \alpha_{041}) = \frac{B_1 + B_2 \cos(\alpha_{012}) + B_4 \cos(\alpha_{041})}{B_4 \sin(\alpha_{041}) - B_2 \sin(\alpha_{012})}.$$

The second equation of the variables α_{012}, α_{041} is the equation 6. The first equation depends on the orientation of the angle α_3 and it was taken as it appears in Figure 2.1. This means that by selecting different weights B_1, B_2, B_3, B_4, the signs of the first equation could change.

3. The invariance property of the weighted Fermat-Torricelli point.

Proposition 3.1. *Suppose that there is an n-convex polygon $A_1A_2 \ldots A_n$ in $\mathbb{R}^2$ and each vertex A_i has a non-negative weight B_i for $i = 1, 2, \ldots, n$. Assume that the floating case of the generalized weighted Fermat-Torricelli point A_0 point is valid:*
for each $A_i \in A_1, \ldots, A_n$

$$\|\sum_{j=1}^{n} B_j \vec{u}(A_i, A_j)\| > B_i, i \neq j.$$

If A_0 is connected with every vertex A_i for $i = 1, 2, \ldots, n$ and a point A_i' is selected randomly with non-negative weight B_i of the line that is defined by the line segment A_0A_i and the n convex polygon $A_1'A_2' \ldots A_n'$ is constructed such that:

$$\|\sum_{j=1}^{n} B_j \vec{u}(A_i', A_j')\| > B_i, i \neq j.$$

Then the generalized weighted Fermat-Torricelli point A_0' is identical with A_0 (see Figure 3.1). This is the invariance property of the weighted Fermat-Torricelli point.

Proof. The existence and uniqueness of the generalized weighted Fermat-Torricelli point given n non-collinear points $A_1, \ldots, A_n \in \mathbb{R}^d$ was studied in [4]. Furthermore, if for each point $A_i \in A_1, \ldots, A_n$

$$\|\sum_{j=1}^{n} B_j \vec{u}(A_i, A_j)\| > B_i, i \neq j$$

holds, then

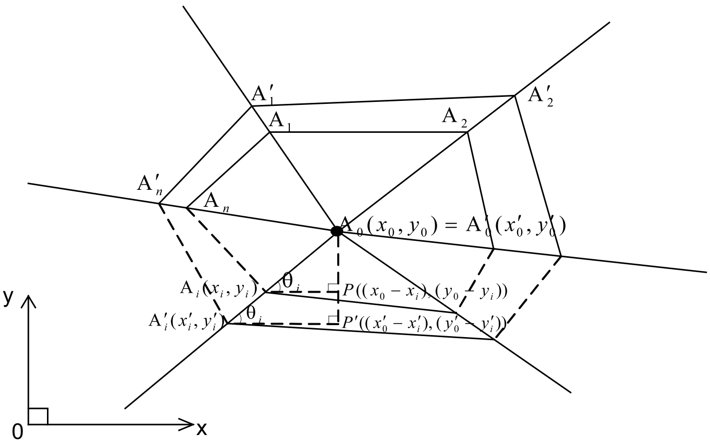

Figure 3.1

(a) the weighted minimum point A_0 does not belong to $A_i \in A_1, \ldots, A_n$

(b)
$$\sum_{i=1}^{n} B_i \vec{u}(A_0, A_i) = 0, i \neq j$$

(weighted floating case).

We consider the particular case for d=2, regarding the n-convex polygon $A_1(x_1, y_1), \ldots,$ $A_n(x_n, y_n)$. Let $A_0(x_0, y_0)$ be the coordinates of the weighted Fermat Torricelli point (critical).

The minimum conditions are:

$$\frac{\partial f}{\partial x} = \sum_{i=1}^{n} B_i \frac{(x - x_i)}{\sqrt{(x - x_i)^2 + (y - y_i)^2}} = 0$$

$$\frac{\partial f}{\partial y} = \sum_{i=1}^{n} B_i \frac{(y - y_i)}{\sqrt{(x - x_i)^2 + (y - y_i)^2}} = 0.$$

If θ_i is the angle between the line segment $(x_i - x_0)$ with the distance $A_0 A_i$, then the conditions that minimizes f(x,y) are:

$$\frac{\partial f}{\partial x} = \sum_{i=1}^{n} B_i \cos(\theta_i) = 0$$

$$\frac{\partial f}{\partial y} = \sum_{i=1}^{n} B_i \sin(\theta_i) = 0.$$

□

The invariance property of the generalized weighted Fermat-Torricelli point in $\mathbb{R}^2$ was well known by V. Viviani ([2]).

4. The inverse weighted Fermat-Torricelli problem

Definition 4.1. Given the generalized weighted Fermat-Torricelli point A_0 of the weighted n-convex polygon $A_1A_2\ldots A_n$ find the ratios between the non negative weights $\frac{B_i}{B_j}$, $i, j = 1, \ldots, n$ such that:

$$\sum_{i=1}^{n} B_i = \text{constant}.$$

This is the inverse generalized weighted Fermat-Torricelli problem.

We proceed with the case for $n = 4$ of the convex quadrilateral $A_1A_2A_3A_4$ with non-negative weights B_i, $i = 1, 2, 3, 4$. Let a_1, a_3, a_4 be functions that depend on a_2 and α_3'.

$$a_1 = a_1(a_2, \alpha_3'), \quad a_3 = a_3(a_2, \alpha_3'), \quad a_4 = a_4(a_2, \alpha_3'). \tag{10}$$

By differentiating (1) with respect to the variables α_3' and a_2, two equations are derived:

$$B_1 \frac{\partial a_1}{\partial \alpha_3'} + B_3 \frac{\partial a_3}{\partial \alpha_3'} + B_4 \frac{\partial a_4}{\partial \alpha_3'} = 0$$

$$B_1 \frac{\partial a_1}{\partial a_2} + B_2 + B_3 \frac{\partial a_3}{\partial a_2} + B_4 \frac{\partial a_4}{\partial a_2} = 0.$$

By calculating $\frac{\partial a_2}{\partial \alpha_3'}$, $\frac{\partial a_3}{\partial \alpha_3'}$,$\frac{\partial a_4}{\partial \alpha_3'}$ and by applying the "cosine law" and the "sine law" to the corresponding triangles $\nabla A_1A_2A_0$, $\nabla A_3A_2A_0$, $\nabla A_4A_2A_0$, we have the equation:

$$-B_1 \sin(\alpha_{012}) + B_3 \sin(\alpha_{023}) + B_4 \sin(\alpha_{034} + \alpha_{023}) = 0. \tag{11}$$

Additionally, by differentiating with respect to the variable a_2, the equation takes the form:

$$B_1 \cos(\alpha_{012}) + B_2 + B_3 \cos(\alpha_{023}) + B_4 \cos(\alpha_{034} + \alpha_{023}) = 0. \tag{12}$$

Similarly, we differentiate the minimum function (1) with respect to the variables (α_3''), (α_3) (see Figure 2.1), in order to get correspondingly the equations:

$$-B_1 \sin(\alpha_{034} + \alpha_{041}) + B_2 \sin(\alpha_{023}) - B_4 \sin(\alpha_{034}) = 0 \tag{13}$$

$$-B_2 \sin(\alpha_{012}) + B_3 \sin(\alpha_{041} + \alpha_{034}) + B_4 \sin(\alpha_{041}) = 0 \tag{14}$$

and

$$\alpha_{012} + \alpha_{023} + \alpha_{034} + \alpha_{041} = 2\pi. \tag{15}$$

Corollary 4.2. *Concerning the inverse 4-weighted Fermat-Torricelli problem if* $\alpha_{012} = \alpha_{034}$, $\alpha_{023} = \alpha_{041}$ *then* $B_1 = B_3, B_2 = B_4$.

It follows from the fact that $\sin(\alpha_{041} + \alpha_{034}) = 0$, $\sin(\alpha_{034} + \alpha_{041}) = 0$ and $\sin(\alpha_{012}) = \sin(\alpha_{034}) = \sin(\alpha_{023}) = \sin(\alpha_{041})$. Equations (11), (14) imply that:

$$B_1 = B_3, \quad B_2 = B_4.$$

Corollary 4.3. *Referring to the inverse weighted Fermat-Torricelli problem for the triangle* $n=3$ $(B_4=0)$*. From the equations* (14), (11) *and given the angles* α_{012}, α_{023}, α_{031} *and the weight* $B_4=0$ *the ratio of the three weights is:*

$$B_1 : B_2 : B_3 = \sin(\alpha_{023}) : \sin(\alpha_{031}) : \sin(\alpha_{012}).$$

This result was also proved in [3] *and* [7].

We conclude with a fundamental result that deals with the *plasticity* of the inverse 4-weighted Fermat-Torricelli problem.

Proposition 4.4. *Consider the inverse 4-weighted Fermat-Torricelli problem such that the angles* α_{0ij} *are given* $i,j=1,..,4$, $i\neq j$*. The following equations point out the* plasticity *of the system:*

$$(\frac{B_2}{B_1})_{1234} = (\frac{B_2}{B_1})_{123}[1-(\frac{B_4}{B_1})_{1234}(\frac{B_1}{B_4})_{134}] \tag{16}$$

$$(\frac{B_3}{B_1})_{1234} = (\frac{B_3}{B_1})_{123}[1-(\frac{B_4}{B_1})_{1234}(\frac{B_1}{B_4})_{124}]. \tag{17}$$

The weight $(B_i)_{1234}$ *corresponds to the vertex* A_i *that lie in the line* A_0A_i, $i=1,2,3,4$ *and the weight* $(B_j)_{jkl}$ *corresponds to the vertex* A_j *that lie in the* A_0A_j *regarding the triangle* $\nabla A_jA_kA_l$, $j,k,l=1,2,3,4$.

Proof. The equation (13) implies that:

$$(\frac{B_2}{B_1})_{1234} = \frac{\sin(\alpha_{034}+\alpha_{041})}{\sin(\alpha_{023})}[(\frac{B_4}{B_1})_{1234}\frac{\sin(\alpha_{034})}{\sin(\alpha_{034}+\alpha_{041})}+1].$$

Referring to Corollary 4.3, from the triangle $\nabla A_1A_2A_3$:

$$\frac{\sin(\alpha_{034}+\alpha_{041})}{\sin(\alpha_{023})} = \frac{\sin(\alpha_{031})}{\sin(\alpha_{023})} = (\frac{B_2}{B_1})_{123}.$$

Similarly, from the vague triangle $\nabla A_1A_3A_4*$ (A_4* is the projection of A_4 corresponding to the Fermat-Torricelli point A_0):

$$-(\frac{B_1}{B_4})_{134} = \frac{\sin(\alpha_{034})}{\sin(\alpha_{034}+\alpha_{041})}$$

and equation (16) is proved. To deduce the equation (17), (11) is used and the following ratio is calculated:

$$(\frac{B_3}{B_1})_{1234} = \frac{\sin(\alpha_{012})}{\sin(\alpha_{023})}(1-\frac{B_4}{B_1}\frac{\sin(\alpha_{034}+\alpha_{023})}{\sin(\alpha_{012})}).$$

From the triangles $\nabla A_1A_2A_3$, $\nabla A_1A_2A_4$ and Corollary 4.3, two ratios are used:

$$(\frac{B_3}{B_1})_{123} = \frac{\sin(\alpha_{012})}{\sin(\alpha_{023})}, \quad (\frac{B_1}{B_4})_{124} = \frac{\sin(\alpha_{034}+\alpha_{023})}{\sin(\alpha_{012})}.$$

$\square$

Corollary 4.5. *Set* $\sum_{1234} B := (B_1)_{1234}(1 + \frac{B_2}{B_1} + \frac{B_3}{B_1} + \frac{B_4}{B_1})_{1234}$. *If* $\sum_{1234} B = \sum_{123} B = \sum_{124} B = \sum_{134} B$, *then*

$$(B_i)_{1234} = x_i(B_4)_{1234} + y_i,\ \ i = 1, 2, 3:$$
$$(x_1, y_1) = (\frac{(\frac{B_1}{B_4})_{134}(\frac{B_2}{B_1})_{123} + (\frac{B_1}{B_4})_{124}(\frac{B_3}{B_1})_{123} - 1}{1 + (\frac{B_2}{B_1})_{123} + (\frac{B_3}{B_1})_{123}}, (B_1)_{123})$$
$$(x_2, y_2) = (x_1(\frac{B_2}{B_1})_{123} - (\frac{B_1}{B_4})_{134}(\frac{B_2}{B_1})_{123}, (B_2)_{123})$$
$$(x_3, y_3) = (x_1(\frac{B_3}{B_1})_{123} - (\frac{B_1}{B_4})_{124}(\frac{B_3}{B_1})_{123}, (B_3)_{123}).$$

Proposition 4.6. *Corollaries* 4.2, 4.3, 4.5 *and Proposition* 4.4 *applies also for the case of weighted convex quadrilaterals and triangles in the two-dimensional sphere and two-dimen-sional hyperboloid (Spherical and Hyperbolic Plasticity).*

Proof. We agree on the same notations that are indicated in Figure 2.1. The equations (11), (12), (13), (15) remain the same by applying the "cosine law" and "sine law" of spherical (hyperbolic) triangles to the derived equations from (1) by differentiating with respect to the spherical (hyperbolic) angles α_3, $\alpha_{3'}$, $\alpha_{3''}$ and the geodesic segment a_2. □

Example 4.7. Given the angles $\alpha_{012} = 120°$, $\alpha_{023} = 90°$, $\alpha_{034} = 50°$, $\alpha_{041} = 100°$, the weight $(B_1)_{123} = 1$ and the assumption that $\sum_{1234} B = \sum_{123} B = \sum_{124} B = \sum_{134} B$ from the Proposition 4.4 and Corollary 4.5 the following results are derived:

$$\nabla 123(A_1A_2A_3) : (B_1)_{123} = 1, (B_2)_{123} = 0.50, (B_3)_{123} = 0.866$$
$$\sum_{123} B = 2.366$$
$$\nabla 124(A_1A_2A_4) : (B_1)_{124} = 0.606, (B_2)_{124} = 0.942, (B_4)_{124} = 0.818$$
$$\sum_{124} B = 2.366$$
$$\nabla 134(A_1A_3A_4) : (B_1)_{134} = 1.432, (B_3)_{134} = 1.869, (B_4)_{134} = -0.935$$
$$\sum_{134} B = 2.366$$
$$(B_1)_{1234} - (B_1)_{123} = -0.475(B_4)_{1234} \tag{18}$$
$$(B_2)_{1234} - (B_2)_{123} = 0.53(B_4)_{1234} \tag{19}$$
$$(B_3)_{1234} - (B_3)_{123} = -1.054(B_4)_{1234} \tag{20}$$

and $\sum_{1234} B = 2.366$. The range of $(B_4)_{1234}$,$(B_1)_{1234}$, $(B_2)_{1234}$, $(B_3)_{1234}$ is:

$$0 \leqslant (B_4)_{1234} \leqslant 0.821$$
$$1 \geqslant (B_1)_{1234} \geqslant 0.61$$
$$0.5 \leqslant (B_2)_{1234} \leqslant 0.935$$
$$0.866 \geqslant (B_3)_{1234} \geqslant 0.$$

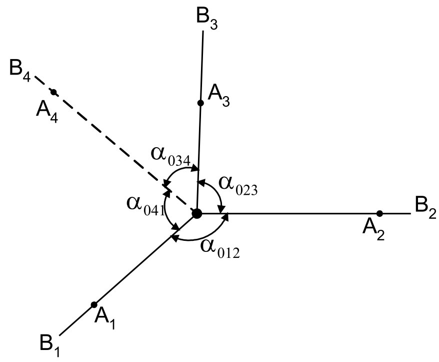

Figure 4.1

Remark 4.8. By taking into consideration in Figure 4.1 the branch with weight B_4 that lies inside the angle α_{031} and referring to the Example 4.7, the weights of the two branches with weights B_1, B_3 decrease and the weight of the opposite branch B_2 increases (see equations (18), (19), (20)).

Remark 4.9. For values of B_1, B_2, B_3 which depend on B_4 according to Corollary 4.5 and for any value of the vertex A_i which lies in the line A_0A_i such that the inequalities of the weighted floating case are satisfied (see [4]), the weighted Fermat-Torricelli point A_0 remains invariant.

Example 4.10. Let $A_1A_2A_3A_4$ be the given convex quadrilateral with $a_1 = 5$, $a_2 = 7.5$, $a_3 = 5$, $a_4 = 10$, $\alpha_{012} = 120°$, $\alpha_{023} = 90°$, $\alpha_{034} = 50°$, $\alpha_{041} = 100°$ and weights $B_1 = 0.762$, $B_2 = 0.765$, $B_3 = 0.339$, $B_4 = 0.5$ taken from Example 4.7. The point A_0 is the weighted Fermat-Torricelli point (see Figure 4.2). The convex quadrilateral $A_1A_2A_3A_4$ of Figure 4.3 has the same angles a_{0ij} and lengths a_i, $i, j = 1, 2, 3, 4$, $i \neq j$ like in Figure 4.3 with weights $B_1 = 0.81$, $B_2 = 0.712$, $B_3 = 0.444$, $B_4 = 0.4$ taken from Example 4.7. From Figure 4.2 and Figure 4.3 the weighted Fermat-Torricelli point A_0 remains invariant. Let $A_1A_2A'_3A_4$ be the convex quadrilateral in Figure 4.4 and A'_3 is the vertex that exists at the line that connects the point A_0 of $A_1A_2A'_3A_4$ with A_3 like in Figure 4.3 such that $a'_3 = 7$ with the angles a_{0ij} with the other lengths and the weights B_i, $i = 1, 2, 3, 4$, to be the same as in Figure 4.3. The weighted Fermat-Torricelli point A_0 of Figure 4.3 and Figure 4.4 remains also invariant.

Remark 4.11. Taking into consideration equations from physics, we can bound the solutions regarding the dimensions of the quadrilateral.

From statics, concerning n bars A_0A_1, $A_0A_2, \ldots, A_0A_n$, with constant density and lengths a_1, $a_2, \ldots, a_n$ respectively that connect at the Fermat-Torricelli point A_0, the sum of the projection of the moments on the x-axis and the sum of the projection on the y-axis are:

$$\sum_{i=1}^{n}(M_i)_x = 0, \quad \sum_{i=1}^{n}(M_i)_y = 0. \tag{21}$$

Definition 4.12. The moment due to the gravity of the bar A_0A_i with length a_i referring to the point A_0 is:

$$\vec{M}_i = c a_i^q \vec{u}_i, \quad q \epsilon \mathbf{Q} : q \geqslant 2.$$

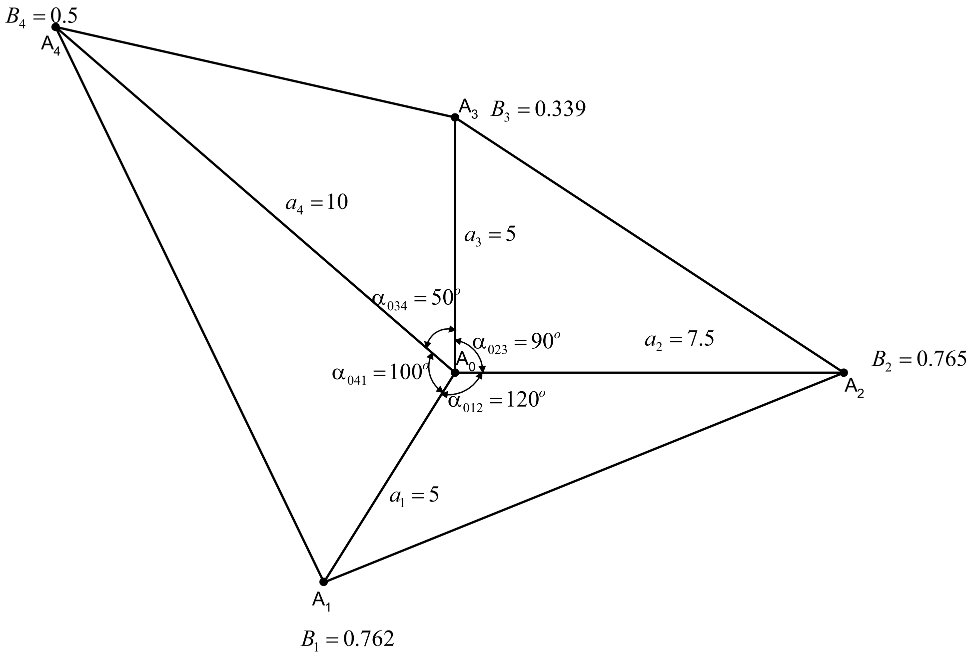

Figure 4.2

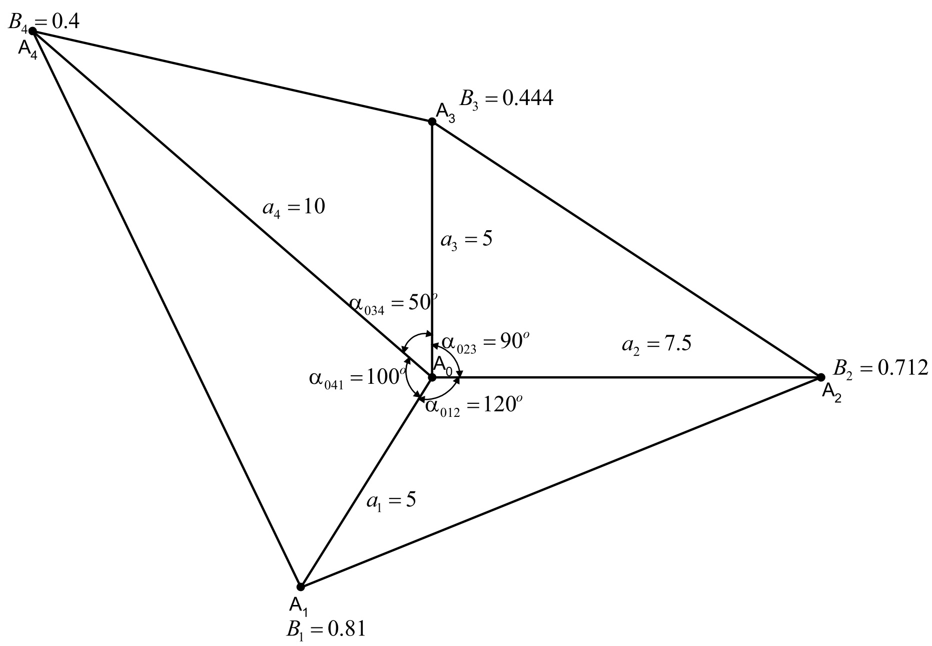

Figure 4.3

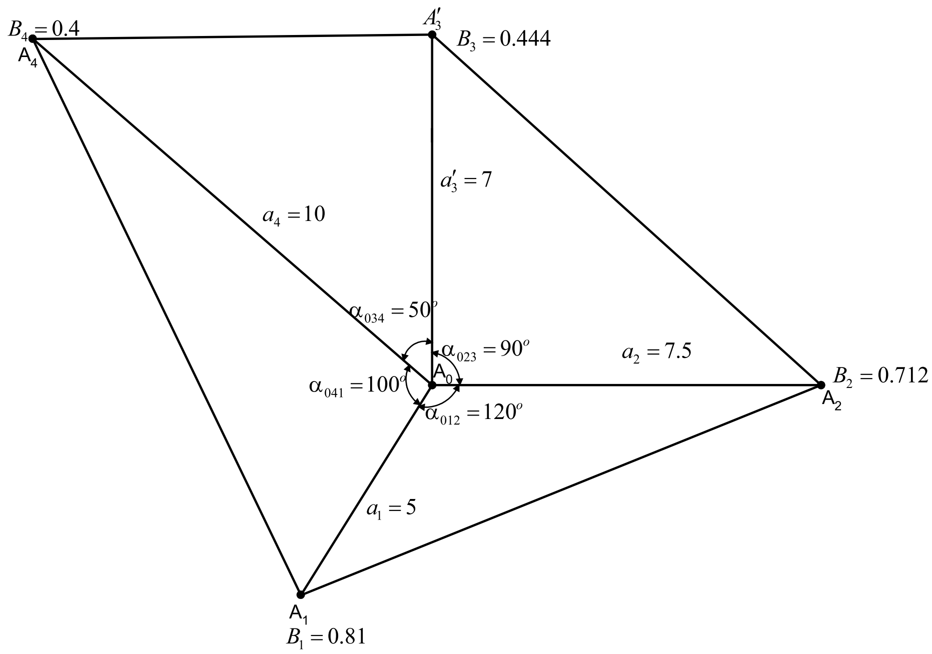

Figure 4.4

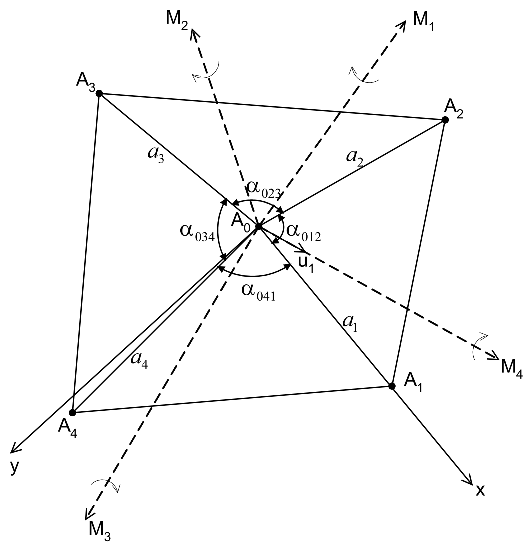

Figure 4.5

such that c is a constant and u_i is the unit vector of $\vec{M}_i$, q depends on the cross-section of the bar.

For simplicity we deal with $q = 2$ or $q = 3$. Suppose that for the case $n = 4$, we consider the projection on the x-axis (A_0A_2) of the moments and on the y-axis which is orthogonal to the x-axis (see Figure 4.5). From the conditions (21), two equations are deduced:

$$-a_1^2 \sin(\alpha_{012}) + a_3^2 \sin(\alpha_{023}) + a_4^2 \sin(\alpha_{034} + \alpha_{023}) = 0$$

$$a_1^2 \cos(\alpha_{012}) + a_2^2 + a_3^2 \cos(\alpha_{023}) + a_4^2 \cos(\alpha_{034} + \alpha_{023}) = 0.$$

Similarly, by taking as x-axis A_0A_3, A_0A_1 the following equations are derived:

$$-a_1^2 \sin(\alpha_{034} + \alpha_{041}) + a_2^2 \sin(\alpha_{023}) - a_4^2 \sin(\alpha_{034}) = 0$$

$$-a_2^2 \sin(\alpha_{012}) + a_3^2 \sin(\alpha_{041} + \alpha_{034}) + a_4^2 \sin(\alpha_{041}) = 0.$$

The form of the derived equations from the projection of moments are similar to the equations (11), (12), (13), (14) by expressing the weights $B_i \sim a_i^2$, $i = 1, 2, 3, 4$. The maximum area of the convex quadrilateral $A_1A_2A_3A_4$ can be calculated:

$$\text{Area}_1 = \frac{\alpha_1\alpha_2 \sin(\alpha_{012}) + \alpha_2\alpha_3 \sin(\alpha_{023}) + \alpha_3\alpha_4 \sin(\alpha_{034}) + \alpha_4\alpha_1 \sin(\alpha_{041})}{2}. \qquad (22)$$

The area of the quadrilateral ($A_1A_2A_3A_4$) is a composition of rational functions of the variable a_4, because the following equations are valid:

$$a_i^2 = x_i a_4^2 + y_i, \; i = 1, 2, 3 \qquad (23)$$

such that:

$$x_1 + x_2 + x_3 = 0.$$

The area of the convex quadrilateral $A_1A_2A_3A_4$ as a composition of rational functions of a_4 is a continuous function and has a maximum value for the interval $0 \leqslant a_4 \leqslant \sqrt{\frac{(B_1)_{123}}{x_1}}$ or $0 \leqslant a_4 \leqslant \sqrt{\frac{(B_3)_{123}}{x_3}}$.

Remark 4.13. Given the A_0 weighted Fermat-Torricelli point, the angles a_{012}, a_{023}, a_{034}, a_{041} of A_0 and the weights $B_i \sim a_i^q$, three different constraints are used in the following Examples 4.14, 4.15, 4.16 to calculate the distances a_1, a_2, a_3, a_4 of the A_0 weighted Fermat-Torricelli point, in order to maximize the area of the quadrilateral $A_1A_2A_3A_4$:

(i) $\sum_{i=1}^{4} a_i^q = c_1$ (Example 4.14, $q = 2$, $c_1 = 2.366$)
(ii) $a_1 = c_2$ (Example 4.15, $q = 3$, $c_2 = 1$)
(iii) The perimeter of $A_1A_2A_3A_4$ is constant:
Perimeter$=a_{12} + a_{23} + a_{34} + a_{41} = c_s$ (Example 4.16, $q = 3$, $c_s = 3$).

Example 4.14. Let $A_1A_2A_3A_4$ be the convex quadrilateral with weights $B_i \sim a_i^2$ for $i = 1, 2, 3, 4$. Given $\alpha_{012} = 120°$, $\alpha_{023} = 90°$, $\alpha_{034} = 50°$, $\alpha_{041} = 100°$, A_0 is the weighted Fermat-Torricelli point and the constraint:

$$\sum_{i=1}^{4} a_i^2 = 2.366.$$

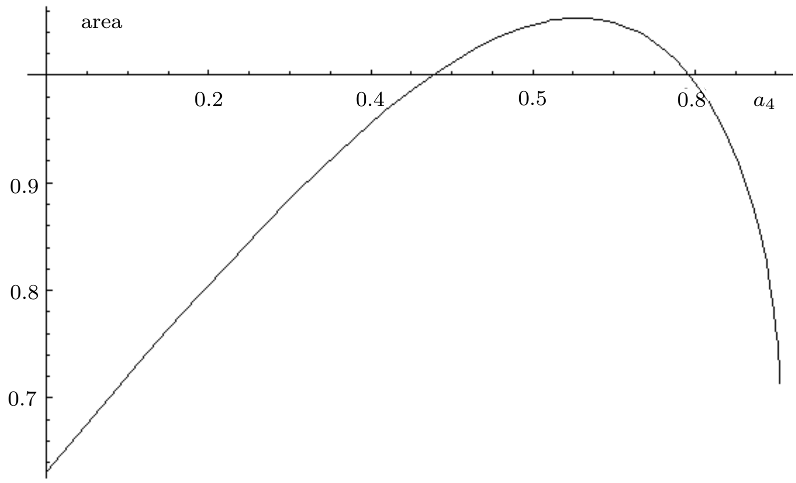

Figure 4.6

To achieve the maximum area of $A_1A_2A_3A_4$ the variables a_1, a_2, a_3 are expressed as functions of a_4: taking into account the equations (21) of the moments from the remark 4.11 and the Example 4.7, equations (23) take the form:

$$a_1^2 = 1 - 0.475a_4^2$$
$$a_2^2 = 0.50 + 0.53a_4^2$$
$$a_3^2 = 0.866 - 1.054a_4^2.$$

The maximum area of the quadrilateral is obtained when $a_4 = 0.6563$, $a_1 = 0.8918$, $a_2 = 0.8534$, $a_3 = 0.6417$ (see Figure 4.6).

Example 4.15. Consider the quadrilateral $A_1A_2A_3A_4$ with the same given angles α_{012}, α_{023}, α_{034}, α_{041} as in Example 4.14 with weights $B_i \sim a_i^3$ that correspond to each vertex A_i, $i = 1, 2, 3, 4$, A_0 is the weighted Fermat-Torricelli point and the constraint $a_1 = 1$. We use the two equations with a "sine form" (projection on the x-axis) of the moments for a_2, a_3 and $q = 3$:

$$a_2 = ((1/\sin(\alpha_{023}))(a_1^3 \sin(\alpha_{034} + \alpha_{041}) + (a_4^3)\sin(\alpha_{034})))^{1/3} \tag{24}$$

$$a_3 = ((1/\sin(\alpha_{023}))(a_1^3 \sin(\alpha_{012}) - (a_4^3)\sin(\alpha_{034} + \alpha_{023})))^{1/3}. \tag{25}$$

The maximum area of the quadrilateral is achieved as a function of a_4 by replacing the variables a_2, a_3 from (24), (25) to the equation (22). The area of the quadrilateral is maximized for the values: $a_4 = 0.982273$, $a_1 = 1$, $a_2 = 1.07029$, $a_3 = 0.635637$ (see Figure 4.7)

Example 4.16. Consider the same quadrilateral $A_1A_2A_3A_4$ as in Example 4.15, the angles α_{012}, α_{023}, α_{034}, α_{041}, the weights $B_i \sim a_i^3$, the weighted Fermat-Torricelli point A_0 and the following constraint for the perimeter of $A_1A_2A_3A_4$:

$$\text{Perimeter} = a_{12} + a_{23} + a_{34} + a_{41} = \text{constant}.$$

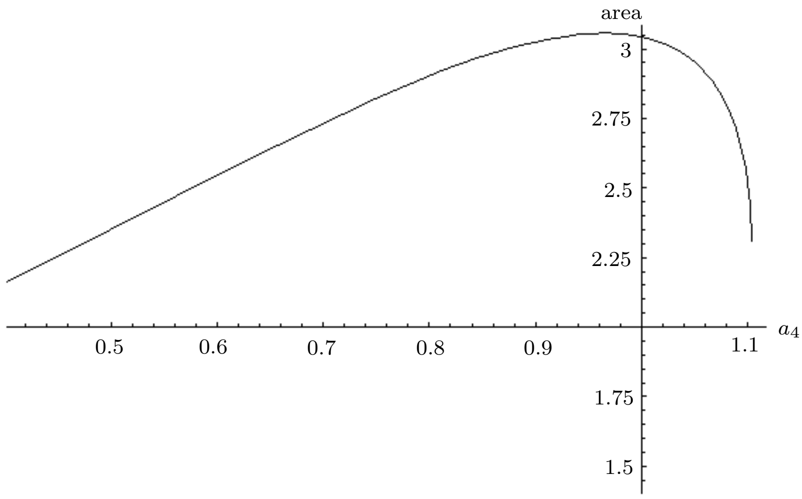

Figure 4.7

This means that the quadrilateral $A_1A_2A_3A_4$ is inscribed to a circle. We will calculate the values a_1, a_2, a_3, a_4 that maximize the area of $A_1A_2A_3A_4$. The constraint can be written as a function of a_1, a_2, a_3, a_4.

$$2S = \sum_{i=1}^{4} \sqrt{a_i^2 + a_{i+1}^2 - 2a_i a_{i+1} \cos \alpha_{0ii+1}} = C_s, \quad i = 1, 2, 3, 4. \tag{26}$$

For $i = 4$ set $i + 1 = 1$ and S is the semi perimeter of the convex quadrilateral $A_1A_2A_3A_4$. The area of the inscribed quadrilateral in a circle is given by the formula:

$$\mathrm{Area}_2 = \sqrt{(S - a_{12})(S - a_{23})(S - a_{34})(S - a_{41})}. \tag{27}$$

A system of two equations can be derived as functions of a_1 and a_4:

$$\mathrm{Area}_1 = \mathrm{Area}_2 \tag{28}$$

and the equation (26). The variables a_2 and a_3 are functions of a_1 and a_4 and they can be taken from (24), (25) respectively. The formula of Area_1 is given by (22). Let the constant of (26) be $c_s = 3$. The system of the two equations (28), (26) can be solved numerically by Newton's method. The values $a_1 = 0.595346$, $a_4 = 0.482094$, $a_2 = 0.576236$, $a_3 = 0.480186$ maximize the area of the quadrilateral $A_1A_2A_3A_4$ (see Figure 4.8 for $c_s = 3, 4, 6$).

Remark 4.17. Let $A_1A_2A_3A_4$ be the convex quadrilateral with weights $B_i \sim a_i^q$ for $i = 1, 2, 3, 4$ and angles $\alpha_{012} = 120°$, $\alpha_{023} = 120°$, $\alpha_{034} = 60°$, $\alpha_{041} = 60°$. There is a connection between the weighted Fermat-Torricelli point A_0 for the given convex quadrilateral $A_1A_2A_3A_4$ and the *fundamental equation of Fermat.* Similar equations with (11), (14) are deduced:

$$-a_1^q \sin(\alpha_{012}) + a_3^q \sin(\alpha_{023}) + a_4^q \sin(\alpha_{034} + \alpha_{023}) = 0 \tag{29}$$

$$-a_2^q \sin(\alpha_{012}) + a_3^q \sin(\alpha_{041} + \alpha_{034}) + a_4^q \sin(\alpha_{041}) = 0. \tag{30}$$

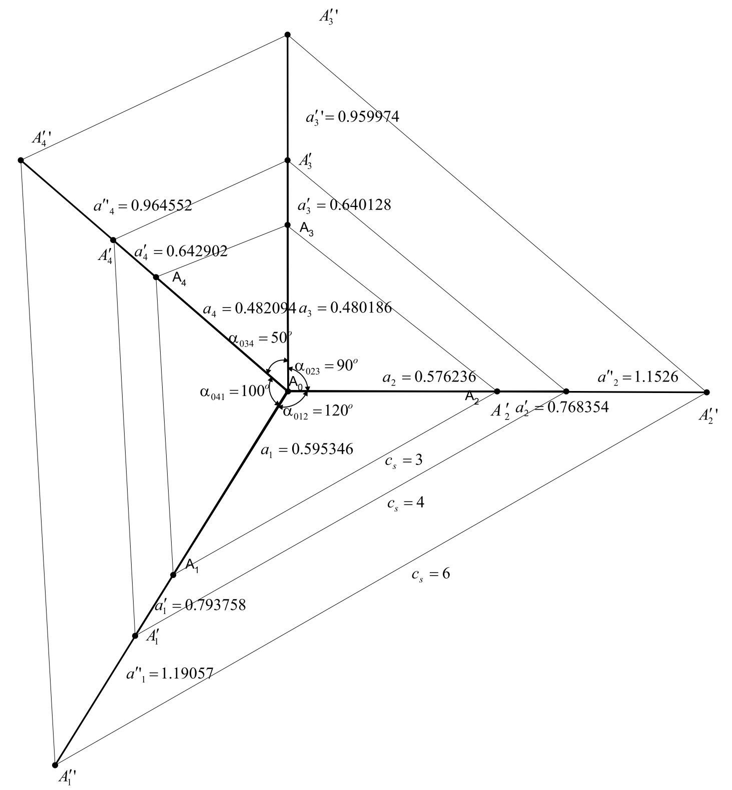

Figure 4.8

From equation (29):

$$a_1^q = a_3^q. \tag{31}$$

Equation (30) gives the equation of Fermat:

$$a_2^q = a_3^q + a_4^q \tag{32}$$

for $q \epsilon \mathbb{N}$.

Conclusion 4.18. The starting point of the evolution of weighted convex quadrilateral at time $t = 0$ is the weighted Fermat-Torricelli point A_0. This is the main reason that we have followed a computational approach which does not depend on the coordinates of the vertices $A_i(x_i, y_i)$ but on the geodesic segments a_i $i = 1, 2, 3, 4$. The result of plasticity holds in $\mathbb{R}^2$, S^2 and H^2. An open question is to derive plasticity conditions for an n-convex polygon in $\mathbb{R}^2$, S^2 and H^2 and solve the generalized inverse weighted Fermat-Torricelli problem for $n > 4$. Finally, we would like to note that taking into consideration the connection of the Fermat-Torricelli point with the fundamental equation of P. de Fermat, there cannot exist a simultaneously integer evolution concerning a_1, a_2, a_3, a_4 from A_0, because of the invariance of the weighted Fermat-Torricelli point.

A. Explanation of the moments $M = C \int_0^{a_i} r^2 x dx = r = r_0(1 - (x/a_i)^n)$

Suppose that $r_0 \sim C_1 a_i^q$ then:

$$M = C a_i^{2+q'} = C a_i^q,$$

C and C_1 are constant and $q', q \in \mathbb{Q}$. If $q' = 0$ then $q = 2$ (cylinder). If $q' = 1$ then $q = 3$ (cone).

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