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On Weakly H-Quasiconvex Functions on the Heisenberg Group

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The aim of this paper is the investigation of weakly H–quasiconvex functions in the framework of the Heisenberg group $I\!H$. The functions of this class, recently introduced by Sun and Yang, are defined via the property that their sublevels are weakly H–convex subsets of $I\!H$. Following Crouzeix's approach, we prove conditions of first and second order, easily testable, for regular weakly H–quasiconvex functions, and we provide a characterization of them.

Keywords: Heisenberg group, weak H–convexity, weak H–quasiconvexity, symmetrized horizontal Hessian

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1. Introduction

Recently, different notions of convexity for sets and functions have been introduced in the framework of the Heisenberg groups and, more generally, of Carnot groups (see, for instance, [8], [9]). The main motivations for the development of a theory of convex functions in these groups are the applications towards subelliptic fully nonlinear PDE's, given the powerful role that convexity plays in this setting. Together with convex functions, notions of convex sets were given. In particular, geodetical convexity was investigated in [13], while strong H–convexity was studied in [6]. It appears that geodetically convex sets are very rare, and few and quite peculiar are strongly H–convex sets.

There is a natural notion of convexity for functions and sets in these groups that fits the sub–Riemannian structure; indeed, the weak H–convexity can be thought of as a sort of horizontal convexity.

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In the last few years, many papers have been devoted to the study of weak H–convexity; let us recall, among them, [8] and [10]. In the first one, the authors examine appropriate notions of convexity in order to investigate the maximum principle of Alexandrov–Bakelman–Pucci in a stratified, nilpotent Lie group. In particular, it turns out that for Γ^2 functions the weak H–convexity of a function is characterized via the symmetrized horizontal Hessian at every point of the domain. In the second paper, the authors study the class of functions f defined by requiring that $[\nabla_h^2 f]^* \geq 0$ in the viscosity sense; functions of these kind are called v–convex functions; they show that v–convexity and weak H–convexity are equivalent in a $\mathcal{C}^2$ context.

In subsequent papers (see, for instance, [11] and [9]) it is proved that this equivalence is still true under the assumptions of upper semicontinuity.

Recently, Sun and Yang (see [14]) introduced a notion that generalizes the classical concept of quasiconvexity for functions on Euclidean spaces, to functions defined on subsets of $I\!H$; on the analogy of the Euclidean case, they consider functions whose sublevels are weakly H–convex. In the sequel, we will call them weakly H–quasiconvex. This class contains weakly H–convex functions, but it is larger. Sun and Yang proved, in particular, that these functions are locally bounded from above.

In this paper, inspired by previous results about weakly H–convex functions (see [8] and [5]), we prove a characterization of the first order for weakly H–quasiconvex functions in Γ^1 (Theorem 4.5). Afterwards, in Theorem 4.9, we find a second order characterization for $\mathcal{C}^2$ weakly H–quasiconvex functions, on the analogy of the result in [7], where a characterization for regular quasiconvex functions in the Euclidean setting is given in terms of the behaviour of the Hessian along particular directions.

The paper is organized as follows. In Section 2 we recall some definitions and preliminary results, and in Section 3 we introduce and state some results for weakly H–convex sets in the Heisenberg group. Section 4 is devoted to weakly H–quasiconvex functions. Under suitable regularity assumptions we provide a first order characterization, and, as a consequence of our main result (Theorem 4.6), we establish a second order characterization for weak H–quasiconvexity. In Subsection 4.1, we are able to provide a sufficient condition of weak H–quasiconvexity for the functions $f(x, y, t) = ((x^2 + y^2)^2 + z(t))^{1/4}$, that can be compared with an analogous one given in [3].

2. Preliminaries

The Heisenberg group $I\!H = I\!H^1$ is the simplest non commutative Carnot group and a privileged object of study in Analysis and Geometry; it is the Lie group given by the underlying manifold $I\!R^3$ with the non commutative group law

$$gg' = (x, y, t)(x', y', t') = (x + x', y + y', t + t' + 2(x'y - xy')) .$$

The unit element is $e = (0, 0, 0)$, and the inverse of $g = (x, y, t)$ is $g^{-1} = (-x, -y, -t)$. Left translations and anisotropic dilations are, in this setup, $L_{g_0}(g) = g_0 g$ and $\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$.

The differentiable structure on $I\!H$ is determined by the left invariant vector fields

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}, \quad \text{with } [X, Y] = -4T.$$

The vector field T commutes with the vector fields X and Y; X and Y are called *horizontal vector fields*.

The Lie algebra of $I\!H$, $\mathfrak{h}$, is the stratified algebra $\mathfrak{h} = I\!R^3 = V_1 \oplus V_2$, where $V_1 = \operatorname{span}\{X, Y\}$, $V_2 = \operatorname{span}\{T\}$; $\langle\cdot,\cdot\rangle$ will denote the inner product. Via the exponential map exp we identify the vector $\alpha X + \beta Y + \gamma T$ in $\mathfrak{h}$ with the point (α, β, γ) in $I\!H$; the inverse $\xi : I\!H \to \mathfrak{h}$ of the exponential mapping has the unique decomposition $\xi = (\xi_1, \xi_2)$ with $\xi_i : I\!H \to V_i$.

The main issue in the analysis of the Heisenberg group is that the classical first and second order differential operators are considered only in terms of horizontal fields. For a given open subset $\Omega \subseteq I\!H$, the class $\Gamma^k(\Omega)$ represents the Folland–Stein space of functions having continuous derivatives up to order k with respect to the vector fields X and Y; we denote as usual by $\mathcal{C}^k(\Omega)$ the class of functions having continuous derivatives up to order k with respect to the differential structure of $I\!R^3$.

Let us recall that the horizontal gradient of a function $u \in \Gamma^1(\Omega)$ at $g \in \Omega$ is the 2–vector

$$(\nabla_h u)(g) = ((Xu)(g),\ (Yu)(g)),$$

written with respect to the basis $\{X, Y\}$ of V_1; we denote by $\mathbf{X}u$ the element in V_1 defined as follows

$$\mathbf{X}u = (Xu)X + (Yu)Y.$$

The horizontal Hessian of $u \in \Gamma^2(\Omega)$ at $g \in \Omega$ is the 2×2 matrix

$$(\nabla_h^2 u)(g) = \begin{pmatrix} (X(Xu))(g) & (X(Yu))(g) \\ (Y(Xu))(g) & (Y(Yu))(g) \end{pmatrix},$$

while the symmetrized horizontal Hessian is the 2×2 symmetric matrix

$$\left[(\nabla_h^2 u)(g)\right]^* = \frac{1}{2}\left\{(\nabla_h^2 u)(g) + \left[(\nabla_h^2 u)(g)\right]^T\right\}.$$

Explicit calculations give us, for $u \in \mathcal{C}^1(\Omega)$ (respectively, $u \in \mathcal{C}^2(\Omega)$)

$$(\nabla_h u)(g_0) = (u_x + 2yu_t, u_y - 2xu_t)_{|(x_0,y_0,t_0)}$$

$$[(\nabla_h^2 u)(g_0)]^* = \begin{pmatrix} u_{xx} + 4yu_{xt} + 4y^2u_{tt} & u_{xy} - 2xu_{xt} + 2yu_{yt} - 4xyu_{tt} \\ u_{xy} - 2xu_{xt} + 2yu_{yt} - 4xyu_{tt} & u_{yy} - 4xu_{yt} + 4x^2u_{tt} \end{pmatrix}_{|(x_0,y_0,t_0)}.$$

A Lipschitz continuos curve $\gamma : [0, T] \to I\!H$, $T \geq 0$, $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$ is said to be *horizontal* if $\gamma'(s) \in \operatorname{span}_{I\!R}\{X(\gamma(s)), Y(\gamma(s))\}$ for almost every $s \in [0, T]$; hence, a.e.,

$$\gamma'(s) = \gamma_1'(s)X(\gamma(s)) + \gamma_2'(s)Y(\gamma(s)).$$

We define $|\gamma'(s)| = \left((\gamma_1'(s))^2 + (\gamma_2'(s))^2\right)^{1/2}$.

To this concern, the following computations of derivatives of a function restricted to a horizontal curve will be helpful in the sequel. Let u be a function in $\mathcal{C}^1(\Omega)$ and γ :

$I \subseteq I\!R \to \Omega$ a horizontal $\mathcal{C}^1$–curve. If U is the function $U(s) = u(\gamma(s))$, then easy computations show that

$$U'(s) = \langle (\nabla_h u)(\gamma(s)), (\gamma_1'(s), \gamma_2'(s)) \rangle \tag{1}$$

(here $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $I\!R^2$). Moreover, if $u \in \mathcal{C}^2(\Omega)$ and γ is a horizontal $\mathcal{C}^2$–curve, then

$$U''(s) = (\gamma_1'(s), \gamma_2'(s))[(\nabla_h^2 u)(\gamma(s))]^*(\gamma_1'(s), \gamma_2'(s))^T + \langle (\nabla_h u)(\gamma(s)), (\gamma_1''(s), \gamma_2''(s)) \rangle \tag{2}$$

Metrics in $I\!H$ which are compatible with left translations and dilations can be obtained in several ways. One of these is the non Euclidean left invariant metric, defined by $d_N(g, g') = N(g^{-1}g')$, for every $g, g' \in I\!H$, where N is the pseudo–norm, called *non–isotropic gauge*, $N(g) = N(x, y, t) = ((x^2 + y^2)^2 + t^2)^{\frac{1}{4}}$.

Another metric is the Carnot–Carathéodory distance, defined as follows:

$$d_{CC}(g, g') = \inf_{\gamma \in \Gamma_{g,g'}} \int_0^1 |\gamma'(s)| \, ds, \tag{3}$$

where $\Gamma_{g,g'}$ denotes the set of all the horizontal curves $\gamma : [0, 1] \to I\!H$ s.t. $\gamma(0) = g$ and $\gamma(1) = g'$. It is worthwhile noticing that the infimum in (3) is actually a minimum, and a curve with minimum length is a geodesic. The Heisenberg group, with the metrics above, is a locally compact metric space. It can be proved that the distances d_{CC} and d_N are equivalent (see [4]).

Recently, a notion of convexity was given via geodesic curves: $C \subseteq I\!H$ is said to be geodetically convex if and only if $\gamma \subset C$ for every $g, g' \in C$, and for all geodesic γ joining g and g'. In [13] the authors proved, in particular, that the geodetically convex envelope of three points not belonging to the same geodesic is the whole group $I\!H$.

Different notions of convexity can be defined via the twisted convex combination of two points, that plays the role in $I\!H$ of the convex combination of points in vector spaces.

Given two distinct points $g, g' \in I\!H$, the *twisted convex (tw–convex) combination* of $g = (x, y, t)$ and $g' = (x', y', t')$ based on g is the point g_λ defined by

$$g_\lambda = g\delta_\lambda(g^{-1}g'), \quad \lambda \in [0, 1]. \tag{4}$$

Easy computations show that

$$g_\lambda = ((1 - \lambda)x + \lambda x', (1 - \lambda)y + \lambda y', t + 2\lambda(x'y - xy') + \lambda^2(-t + t' - 2x'y + 2xy')).$$

From now on, $\gamma_{g,g'}$ will denote the curve in $I\!H$ defined by $\gamma_{g,g'}(\lambda) = g_\lambda$, $\lambda \in [0, 1]$.

Definition 2.1 (see, for instance, [8], 7.5). A set $C \subseteq I\!H$ is said to be strongly H–convex if and only if $\gamma_{g,g'} \in C$ for every $g, g' \in C$.

In [6] we prove that strongly H–convex sets are very scarce. In particular, a bounded strongly H–convex set should be contained in a left translated of some 2–dimensional subgroup.

Since to every notion of convexity on sets there is a naturally associated notion of convexity for functions, from the definition above we have the following:

Definition 2.2. Let Ω be a strongly H–convex subset of $\mathbb{H}$. A function $u : \Omega \to \mathbb{R}$ is called strongly H–convex if, for any $g, g' \in \Omega$, one has

$$u(g_\lambda) \leq u(g) + \lambda \left(u(g') - u(g)\right), \quad \forall \lambda \in [0,1].$$

In the sequel we will call *sublevel set* of a function $u : \Omega \to \mathbb{R}$ any set

$$\{g \in \Omega : \; u(g) \leq a\},$$

with $a \in \mathbb{R}$. It is straightforward to see that the sublevels of a strongly H–convex function are strongly H–convex. Actually, strong H–convexity is a hard request, and very few functions satisfy this property (for instance, H–affine functions are strongly H–convex, as it is shown in [8], 3.6); the gauge function N is not strongly H–convex, as shown in ([8], 6.8).

Finally, let us recall that the Pansu differential, defined for functions between Carnot groups, for a function $u : \mathbb{H} \to \mathbb{R}$ is given by

$$Du(g)(h) = \lim_{\lambda \to 0^+} \frac{u(g\delta_\lambda(h)) - u(g)}{\lambda}.$$

In [8] (see Proposition 2.3) it is proved that, if $u \in \Gamma^1(\Omega)$, then

$$Du(g)(h) = \langle \mathbf{X}u(g), \xi_1(h) \rangle \tag{5}$$

3. Weakly H–convex sets

In this section we focus on a concept of convexity of subsets of $\mathbb{H}$ that weakens the strong one and takes into account the horizontal structure of $\mathbb{H}$. As we will see, this concept has nothing to share with the idea of convexity in Euclidean spaces; in particular, the main feature of connectedness is lost.

This mentioned horizontal structure relies on the notion of horizontal plane. Given a point $g_0 \in \mathbb{H}$, the horizontal plane H_{g_0} associated to g_0 is the plane in $\mathbb{H}$ defined by $H_{g_0} = L_{g_0}(\exp(V_1)) = \{g = (x, y, t) \in \mathbb{H} : t = t_0 + 2y_0 x - 2x_0 y\}$. Notice that, given $g, g' \in \mathbb{H}$, $g \neq g'$, $\gamma_{g,g'}$ is a segment (i.e., the convex closure, in the Euclidean sense, of the set $\{g, g'\}$) if and only if $g' \in H_g$, or $(x, y) = (x', y')$; if $g' \in H_g$, $\gamma_{g,g'}$ is the horizontal geodesic whose length is $d_{CC}(g, g')$ (we call it *horizontal segment*).

Definition 3.1 (see [8], 7.1). A set $C \subseteq \mathbb{H}$ is said to be weakly H–convex if and only if $\gamma_{g,g'} \in C$ for every $g \in C$, $g' \in H_g \cap C$.

It is easy to see that the following implications hold, and none of them can be reversed:

$$\begin{array}{ccc}
C \text{ Euclidean convex} & & C \text{ geodetically convex} \\
\Uparrow & \searrow\!\!\!\searrow & \Downarrow \\
C \text{ strongly H–convex} & \Rightarrow & C \text{ weakly H–convex}
\end{array}$$

The reader can easily convince himself that the following are examples of weakly H–convex stes: $A = \gamma_{e,(1,0,1)}$, and any subset of the t–axis.

Let us consider now the closed balls $B_{CC}(e,r)$ and $B_N(e,r)$ with center $e \in I\!H$ and radius r, with respect to the Carnot–Carathéodory metric and to the gauge metric, and examine their behaviour as regards the weak H–convexity.

It is not hard to see that $B_{CC}(e,r)$ is not weakly H–convex. This is easily proved, taking into account the equation of the balls in the Carnot–Carathéodory metric (see [12]): indeed, if $g' = (x', y', t')$ is a point in $B_{CC}(e,r)$ with $t' = \max\{t : g = (x,y,t) \in B_{CC}(e,r)\}$, and $\tilde{g} = (-x', -y', t') \in B_{CC}(e,r)$, then $\gamma_{g',\tilde{g}}$ is the segment joining g' and $\tilde{g}$, and it is not contained in $B_{CC}(e,r)$.

On the contrary, the weak H–convexity of the gauge function N, the results in [6], and easy computations entail that the set $B_N(e,r)$ is connected and weakly H–convex, while is not strongly H–convex. In addition, it can be easily seen that for any $r \geq 0$, there exists $\alpha \in I\!R$ such that $B_N(e,r)$ and $B_N((0,0,\alpha),r)$ are disjoint, and $B_N(e,r) \cup B_N((0,0,\alpha),r)$ is weakly H–convex.

Given a subset of $I\!H$, the investigation of its weak H–convexity by a direct test based on the fulfilment of Definition 3.1 turns out to be quite complicated; in next section we will provide a sufficient tool to single out weakly H–convex sets, bypassing the definition.

In some special cases, where the set is described by a regular function, we can provide some more tractable conditions. Let Φ be a $\mathcal{C}^1(I\!H)$ function. Denote by C the set $C = \{g \in I\!H : \Phi(g) < 0\}$, and assume that its boundary ∂C is given by $\partial C = \{g \in I\!H : \Phi(g) = 0\}$. We say that ∂C is *regular* at $g_0 \in \partial C$ if $(\nabla\Phi)(g_0) \neq 0$; we say that ∂C is regular if it is regular at any point $g_0 \in \partial C$. Recently, domains with regularity properties were investigated in [2]. More generally, and taking into account only the horizontal structure, the point $g_0 \in \partial C$ is said to be *non characteristic* if $(\nabla_h\Phi)(g_0) \neq 0$; we say that ∂C is non characteristic if all the points $g_0 \in \partial C$ are non characteristic. In the next proposition, a sufficient condition for weak H–convexity of a subset of $I\!H$, whose boundary is non characteristic, is provided.

Proposition 3.1. *Let $\Omega \subset I\!H$ be a domain given by a $\mathcal{C}^1$–smooth defining function $u : I\!H \to I\!R$, $\Omega = \{g \in I\!H : u(g) \leq 0\}$ with boundary $\partial\Omega$ given by $\partial\Omega = \{g \in I\!H : u(g) = 0\}$.*

i) *If Ω is weakly H–convex, then, for every $g \in \partial\Omega$ and $g' \in H_g \cap \Omega$, we have*

$$\langle \mathbf{X}u(g), \xi_1(g') - \xi_1(g) \rangle \leq 0.$$

ii) *If, for every $g \in \partial\Omega$ and $g' \in H_g \cap \Omega$, we have*

$$\langle \mathbf{X}u(g), \xi_1(g') - \xi_1(g) \rangle < 0, \tag{6}$$

then Ω is weakly H–convex; in particular, $\partial\Omega$ should be non characteristic.

The set Ω is clearly a sublevel set of the function u.

Proof. Let Ω be weakly H–convex. Let $g \in \partial\Omega$, $g' \in \Omega \cap H_g$ and let us consider the function $U : [0,1] \to I\!R$ defined by $U(s) = u(\gamma_{g,g'}(s))$. From the assumption, $U(s) \leq 0$ for every $s \in [0,1]$, and $U(0) = 0$. Since U is differentiable, we conclude that $U'_+(0) \leq 0$, i.e., by (5),

$$\langle \mathbf{X}u(g), \xi_1(g') - \xi_1(g) \rangle \leq 0.$$

Now, let (6) be true and let us suppose that Ω is not weakly H–convex, i.e., there exist $g \in \Omega$, $g' \in H_g \cap \Omega$, and $\lambda_+ \in (0,1)$ such that $g_{\lambda_+} \notin \Omega$, where g_λ is defined as in (4). In other words,

$$u(g) \leq 0, \quad u(g') \leq 0, \quad u(g_{\lambda_+}) > 0.$$

Set $\overline{\lambda} = \max\{\lambda \in [0, \lambda_+] : u(g_\lambda) = 0\}$. In particular, $u(g_{\overline{\lambda}}) = 0$, therefore $g_{\overline{\lambda}} \in \partial\Omega$, and, from the assumption,

$$\langle \mathbf{X}u(g_{\overline{\lambda}}), \xi_1(g') - \xi_1(g_{\overline{\lambda}}) \rangle < 0.$$

Let $U : [0,1] \to I\!R$ denote the function $U(\lambda) = u(g_\lambda)$; we have that $U(\overline{\lambda}) = 0$, and $U(\lambda) > 0$ if $\lambda \in (\overline{\lambda}, \lambda_+]$. This implies that $U'(\overline{\lambda}) \geq 0$, i.e.,

$$\langle \mathbf{X}u(g_{\overline{\lambda}}), \xi_1(g') - \xi_1(g_{\overline{\lambda}}) \rangle \geq 0,$$

a contradiction. □

4. Weakly H–quasiconvex functions

Within this section, Ω will denote a weakly H–convex set. A notion of convexity for functions arising naturally from the weak H–convexity of sets is the following:

Definition 4.1. A function $u : \Omega \to I\!R$ is called weakly H–convex if

$$u(g_\lambda) \leq u(g) + \lambda\,(u(g') - u(g))$$

for any $g \in \Omega$, $g' \in H_g \cap \Omega$, and $\lambda \in [0,1]$.

Next theorem (see, for instance, [8]) provides useful first and second order conditions for weak H–convexity, based on the behaviour of the horizontal gradient and Hessian of u.

Theorem 4.2. *A function $u \in \Gamma^1(\Omega)$ is weakly H–convex if and only*

$$u(g') - u(g) \geq \langle \mathbf{X}u(g), \xi_1(g') - \xi_1(g) \rangle \tag{7}$$

for every $g \in \Omega$ and $g' \in H_g \cap \Omega$. Moreover, if $u \in \Gamma^2(\Omega)$, then u is weakly H–convex if and only if the symmetrized horizontal Hessian $[(\nabla_h^2 u)(g)]^*$ is positive semidefinite for every $g \in \Omega$.

From the definition of weak H–convexity, it follows that the sublevel sets of a weakly H–convex function are weakly H–convex sets. However, if all the sublevels of a function are weakly H–convex one cannot infer that the function is weakly H–convex. Let us consider, for instance, the function $u : I\!H \to I\!R$ defined by $u(x,y,t) = \ln(x^2+y^2+t^2+1)$, and notice that all the sublevel sets are Euclidean convex, in particular weakly H–convex. Since the horizontal Hessian is not positive semidefinite everywhere, we deduce that this function is not weakly H–convex.

In order to characterize all the functions whose sublevels are weakly H–convex, we give, as in [14], the following:

Definition 4.3. A function $u : \Omega \to I\!R$ is weakly H–quasiconvex if and only if

$$u(g_\lambda) \leq \max(u(g), u(g'))$$

for every $g \in \Omega$, $g' \in H_g \cap \Omega$ and $\lambda \in [0,1]$; this is equivalent to require that $\{g \in \Omega : u(g) \leq a\}$ is weakly H–convex, for every $a \in I\!R$.

While any weakly H–convex function is locally Lipschitz continuous (see [3], Th. 1.2), this is not true, in general, for weakly H–quasiconvex functions. Indeed, consider, for instance, the function $u = \chi_{A^c}$, where A is a proper, weakly H–convex subset of $I\!H$. Nevertheless, a kind of regularity is guaranteed by Sun and Yang (see [14], Th. 2.8): they prove that a weakly H–quasiconvex function is locally bounded from above.

A useful remark, whose proof is trivial, is the following:

Remark 4.4. Let u be a weakly H–quasiconvex function on $I\!H$. Then, for every $g_0 \in I\!H$, the left translated $L_{g_0}u$ is still weakly H–quasiconvex.

Similarly to (7), a first–order characterization of weak H–quasiconvexity can be provided.

Theorem 4.5. *Let Ω be an open weakly H–convex subset of $I\!H$ and let $u \in \Gamma^1(\Omega)$. Then, u is weakly H–quasiconvex on Ω if and only if for every $g \in \Omega$ and $g' \in H_g \cap \Omega$ such that $u(g) \geq u(g')$, we get*

$$\langle \mathbf{X}u(g), \xi_1(g') - \xi_1(g)\rangle \leq 0.$$

Proof. Consider $g, g' \in \Omega$, with $g' \in H_g$, and assume that u is weakly H–quasiconvex on Ω. This implies that

$$u(g_\lambda) \leq \max(u(g), u(g')), \quad \forall \lambda \in [0, 1].$$

Suppose that $u(g) \geq u(g')$; we have that $u(g_\lambda) - u(g) \leq 0$ and, by (5),

$$\lim_{\lambda \to 0^+} \frac{u(g_\lambda) - u(g)}{\lambda} = \langle \mathbf{X}u(g), \xi_1(g') - \xi_1(g)\rangle \leq 0.$$

Conversely, assume that u satisfies the first–order condition. By contradiction, suppose that u is not weakly H–quasiconvex, and denote by g and g' two points in Ω, with $g' \in H_g$, such that $u(g_\lambda) > \max(u(g), u(g'))$ for some $\lambda \in [0, 1]$. Let $u(g) \geq u(g')$. Denote by $\overline{\lambda}$ the smallest $\lambda \in [0, 1]$ such that $u(g_\lambda) = \max_{\nu \in [0,1]} u(g_\nu)$. Since the function $U(\lambda) = u(g_\lambda)$ is differentiable, then

$$U(\tilde{\lambda}) < U(\overline{\lambda}), \qquad U'(\tilde{\lambda}) > 0$$

for some $\tilde{\lambda} < \overline{\lambda}$. Consider the points $\tilde{g} = g_{\tilde{\lambda}}$ and $g' \in H_{\tilde{g}}$ and apply the first–order condition:

$$U'(\tilde{\lambda}) = \langle \mathbf{X}u(\tilde{g}), \xi_1(g') - \xi_1(\tilde{g})\rangle > 0,$$

a contradiction. □

More difficult is the proof of the second–order sufficient condition. Next theorem is devoted to this condition, that is inspired by the second order characterization given by Crouzeix ([7]) in the Euclidean setting. While the conditions in Crouzeix's result involve the Euclidean Hessian, in the sub–Riemannian framework we expect that only the horizontal Hessian plays a crucial role. As a matter of fact, this situation has already arisen in the characterization of weak H–convexity.

Theorem 4.6. *Let $u \in \mathcal{C}^2(\Omega)$, where $\Omega \subseteq I\!H$ is an open weakly H–convex set. Let us assume that the following implication holds:*

$$\forall g_0 \in \Omega,\ \forall \mathbf{v} \in I\!R^2 \setminus \{\underline{0}\} : \langle (\nabla_h u)(g_0), \mathbf{v}\rangle = 0 \quad \Longrightarrow \quad \mathbf{v}[(\nabla_h^2 u)(g_0)]^* \mathbf{v}^T > 0. \qquad (8)$$

Then u is weakly H–quasiconvex on Ω.

Proof. By contradiction, assume that u is not weakly H–quasiconvex. Denote by g_1, g_2 two points in Ω, $g_1 = (x_1, y_1, t_1)$, $g_2 = (x_2, y_2, t_2)$, with $g_2 \in H_{g_1}$, such that

$$M = \max_{\lambda \in [0,1]} u(\gamma_{g_1,g_2}(\lambda)) > \max(u(g_1), u(g_2)).$$

Let us consider the $\mathcal{C}^2-$function h defined by $h(\lambda) = u(\gamma_{g_1,g_2}(\lambda))$, and set $\overline{g} = (\overline{x}, \overline{y}, \overline{t}) = \gamma_{g_1,g_2}(\overline{\lambda})$, where $\overline{\lambda} = \min\{\lambda \in [0,1], h(\lambda) = M\}$; it follows that

$$\begin{cases} h(\lambda) < h(\overline{\lambda}) = M, & 0 \le \lambda < \overline{\lambda}, \\ h(\lambda) \le h(\overline{\lambda}) = M, & \overline{\lambda} \le \lambda \le 1. \end{cases}$$

Since $g_2 \in H_{g_1}$, from (1) and (5), we have

$$h'(\overline{\lambda}) = [(Xu)(\overline{g})](x_2 - x_1) + [(Yu)(\overline{g})](y_2 - y_1),$$

and

$$[(Xu)(\overline{g})](x_2 - x_1) + [(Yu)(\overline{g})](y_2 - y_1) = 0 \tag{9}$$

First case: Assume that

$$(\nabla_h u)(\overline{g}) = 0.$$

Then, (8) implies that $[(\nabla_h^2 u)(\overline{g})]^*$ is positive definite. Since $g_2 \in H_{g_1}$, from (2) we get that

$$h''(\overline{\lambda}) = (x_2 - x_1, y_2 - y_1)[(\nabla_h^2 u)(\overline{g})]^*(x_2 - x_1, y_2 - y_1)^T > 0.$$

Therefore, if $|\lambda - \overline{\lambda}|$ is small enough,

$$\begin{aligned} h(\lambda) &= h(\overline{\lambda}) + h'(\overline{\lambda})(\lambda - \overline{\lambda}) + \frac{h''(\overline{\lambda})}{2}(\lambda - \overline{\lambda})^2 + o\left((\lambda - \overline{\lambda})^2\right) \\ &= h(\overline{\lambda}) + \frac{h''(\overline{\lambda})}{2}(\lambda - \overline{\lambda})^2 + o\left((\lambda - \overline{\lambda})^2\right) > M, \end{aligned}$$

a contradiction.

Second case: Suppose now that $(\nabla_h u)(\overline{g}) \neq 0$. Without loss of generality, we can suppose that $(Yu)(\overline{g}) \neq 0$. Moreover, we assume that $x_1 < x_2$; indeed, if $x_1 > x_2$, we can exchange g_1 and g_2. Notice that the case $x_1 = x_2$ cannot occur if $(Yu)(\overline{g}) \neq 0$, since (9) holds.

Denote by $\tilde{g}$ a point of the segment $[g_1, \overline{g}] \subset [g_1, g_2]$, $\tilde{g} \neq \overline{g}$; standard computations show that

$$\tilde{g} = \left(\tilde{x}, -\frac{(Xu)(\overline{g})}{(Yu)(\overline{g})}(\tilde{x} - \overline{x}) + \overline{y}, \overline{t} + 2\overline{y}\tilde{x} + 2\frac{(Xu)(\overline{g})}{(Yu)(\overline{g})}(\tilde{x} - \overline{x}) - 2\overline{x}\,\overline{y}\right).$$

Actually, $\tilde{g}$ is supposed to satisfy some more conditions, that we defer to a later date for the sake of clarity.

Denote by U the $\mathcal{C}^2$–function defined by

$$U(x, y) = u(x, y, \tilde{t} + 2\tilde{y}x - 2\tilde{x}y),$$

for any $(x, y) \in I\!R^2$ such that $(x, y, \tilde{t} + 2\tilde{y}x - 2\tilde{x}y) \in H_{\tilde{g}} \cap \Omega$; in other words, U is an expression of the restriction of u to the plane $H_{\tilde{g}}$. In particular, since $\tilde{g} \in H_{\overline{g}}$, we have that $\overline{t} = \tilde{t} + 2\tilde{y}\overline{x} - 2\tilde{x}\overline{y}$, and

$$U(\overline{x}, \overline{y}) = M.$$

In order to apply the implicit function theorem to the equation above, we take into account that $(Yu)(\overline{g}) \neq 0$, that is, $u_y(\overline{g}) - 2\overline{x}u_t(\overline{g}) \neq 0$. If $u_t(\overline{g}) = 0$, then $u_y(\overline{g}) - 2\tilde{x}u_t(\overline{g}) \neq 0$, for every $\tilde{g}$; if $u_t(\overline{g}) \neq 0$, then $\tilde{g}$ can be chosen close to $\overline{g}$ enough in such a way that $u_y(\overline{g}) - 2\tilde{x}u_t(\overline{g}) \neq 0$. Denote by $\mathcal{U}_1$ a neighborhood of $\overline{g}$ such that

$$u_y(\overline{g}) - 2\tilde{x}u_t(\overline{g}) \neq 0$$

if $\tilde{g} \in \mathcal{U}_1$. This is the first demand on $\tilde{g}$. With this choice of $\tilde{g}$, there exists $\epsilon > 0$ and $\gamma_2 : (\overline{x} - \epsilon, \overline{x}] \to I\!R$ such that

$$\begin{cases} \gamma_2(\overline{x}) = \overline{y}, \\ U(x, \gamma_2(x)) = M, \\ \gamma_2'(x) = -\dfrac{U_x(x, \gamma_2(x))}{U_y(x, \gamma_2(x))}, \end{cases}$$

for every $x \in (\overline{x} - \epsilon, \overline{x}]$, where

$$U_x(x, \gamma_2(x)) = u_x(x, \gamma_2(x), \tilde{t} + 2\tilde{y}x - 2\tilde{x}\gamma_2(x)) + 2\tilde{y}u_t(x, \gamma_2(x), \tilde{t} + 2\tilde{y}x - 2\tilde{x}\gamma_2(x)),$$

and

$$U_y(x, \gamma_2(x)) = u_y(x, \gamma_2(x), \tilde{t} + 2\tilde{y}x - 2\tilde{x}\gamma_2(x)) - 2\tilde{x}u_t(x, \gamma_2(x), \tilde{t} + 2\tilde{y}x - 2\tilde{x}\gamma_2(x));$$

notice that $\gamma_2 \in \mathcal{C}^2$. Denote by $\gamma_3 : (\overline{x} - \epsilon, \overline{x}]$ the function defined by

$$\gamma_3(x) = \tilde{t} + 2\tilde{y}x - 2\tilde{x}\gamma_2(x),$$

and set $\Gamma(x) = (x, \gamma_2(x), \gamma_3(x))$. Let G denote the continuous function

$$G(g, g') = -\frac{1}{(u_y(g') - 2xu_t(g'))^3}(1, \gamma_2'(x'))A(g, g')\begin{pmatrix} 1 \\ \gamma_2'(x') \end{pmatrix},$$

where $A(g, g') = (a_{ij}(g, g'))_{i,j=1,2}$ and

$$\begin{aligned} a_{11}(g, g') &= u_{xx}(g') + 4yu_{xt}(g') + 4y^2u_{tt}(g') \\ a_{12}(g, g') &= a_{21}(g, g') = u_{xy}(g') - 2xu_{xt}(g') + 2yu_{yt}(g') - 4xyu_{tt}(g') \\ a_{22}(g, g') &= u_{yy}(g') - 4xu_{yt}(g') + 4x^2u_{tt}(g'). \end{aligned}$$

Deriving once again the function γ_2, we obtain $\gamma_2''(x) = G(\tilde{g}, \Gamma(x))$.

Notice that $A(\overline{g}, \overline{g}) = [(\nabla_h^2 u)(\overline{g})]^*$; from (9) and hypothesis (8), we have that

$$(1, \gamma_2'(\overline{x}))A(\overline{g}, \overline{g})\begin{pmatrix} 1 \\ \gamma_2'(\overline{x}) \end{pmatrix} > 0.$$

We deduce that the expression

$$-\frac{1}{(u_y(\overline{g}) - 2\overline{x}u_t(\overline{g}))^3}(1, \gamma_2'(\overline{x}))A(\overline{g}, \overline{g})\begin{pmatrix} 1 \\ \gamma_2'(\overline{x}) \end{pmatrix}$$

is non zero and has the sign opposite to the sign of $(Yu)(\overline{g})$. Assume, without loss of generality, that $(Yu)(\overline{g}) > 0$. By continuity arguments on G, we can find a neighborhood $\mathcal{U}_2$ of $\overline{g}$ such that $\gamma_2''(x) < 0$ whenever $\tilde{g} \in \mathcal{U}_1 \cap \mathcal{U}_2$; in particular, $x \mapsto \gamma_2(x)$ is strictly concave on $(\overline{x} - \epsilon_1, \overline{x}]$ for a suitable $0 < \epsilon_1 \leq \epsilon$.

Let us consider the curve Γ. This curve is not horizontal, in general, but it lies in the set

$$\{g \in I\!\!H : u(g) = M\} \cap H_{\tilde{g}}.$$

From the computations above, the curve $\{(x, \gamma_2(x)) : x \in (\overline{x} - \epsilon_1, \overline{x}]\}$ is tangent at $(\overline{x}, \overline{y})$ to the line r through $(\tilde{x}, \tilde{y})$ and $(\overline{x}, \overline{y})$, and has no intersections with it for any $x \in (\overline{x} - \epsilon_1, \overline{x})$.

Consider the line n through $(\tilde{x}, \tilde{y})$ with direction $(\nabla_h u)(\overline{g})$, i.e., $n = \{(x, y) = (\tilde{x}, \tilde{y}) + s(\nabla_h u)(\overline{g}), s \in I\!\!R\}$, that is normal to r. Denote by $(x^\#, y^\#)$ the intersection between n and the curve above, and by $g^\#$ the point in $H_{\tilde{g}}$ whose first components are $(x^\#, y^\#)$. Let W $(W(\eta) = (w_1(\eta), w_2(\eta), w_3(\eta)))$ be the horizontal geodesic in $I\!\!H$ joining $g^\#$ with $\tilde{g}$. Assume that $W(0) = g^\#$, and $W(1) = \tilde{g}$. This geodesic is actually a segment, since $g^\#$ belongs to the horizontal plane of $\tilde{g}$.

Apply the Lagrange theorem to the function $u \circ W : [0, 1] \to I\!\!R$. We get, for a suitable $\eta' \in [0, 1]$ and $K > 0$,

$$\begin{aligned} u(W(1)) &= u(W(0)) + \langle (\nabla_h u)(W(\eta')), (w_1'(\eta'), w_2'(\eta')) \rangle \\ &= u(W(0)) + K \langle (\nabla_h u)(W(\eta')), (\nabla_h u)(\overline{g}) \rangle. \end{aligned}$$

Consider a neighborhood $\mathcal{U}_3$ of $\overline{g}$ such that the angle between $(\nabla_h u)(W(\eta))$ and $(\nabla_h u)(\overline{g})$ is, for instance, less than $\pi/4$, for every $\eta \in [0, 1]$. This is the last demand on $\tilde{g}$. If we take $\tilde{g} \in \mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3$, then

$$u(W(1)) = M + K \langle (\nabla_h u)(W(\eta')), (\nabla_h u)(\overline{g}) \rangle > M,$$

but this is a contradiction. $\square$

Notice that (8) makes sense under the weaker assumption that u belongs to $\Gamma^2(\Omega)$, and this would be consistent with Theorem 4.2 by Danielli, Garofalo and Nhieu. Actually, our proof requires that u is in $\mathcal{C}^2(\Omega)$. Moreover, the second order characterization for weak H–convexity in Theorem 4.2 does not imply (8); take, for instance, any constant, or, more generally, any affine function. A similar situation occurred also in the Euclidean case investigated by Crouzeix (see [7]).

In order to provide a set of conditions that fully characterize the weak H–quasiconvexity of a function u, we follow the same line of [1], where the Euclidean case is studied. Let us first introduce the concept of semistrict local maximum.

Definition 4.7. A function $f : I \subseteq I\!\!R \to I\!\!R$, I interval, is said to have a semistrict local maximum at $\overline{t} \in I$ if there exist $t_1, t_2 \in I$, $t_1 < \overline{t} < t_2$ such that

i) $f(\overline{t}) \geq f(\lambda t_1 + (1 - \lambda) t_2)$, for every $\lambda \in [0, 1]$,
ii) $f(\overline{t}) > \max(f(t_1), f(t_2))$.

It is worthwhile noticing that any semistrict local maximum should be an interior point of I; moreover, we have:

Remark 4.8. If u is a weakly H–quasiconvex function on Ω, then for every $g, g' \in \Omega$ with $g' \in H_g$ the function $U(\lambda) = u(g_\lambda) : [0,1] \to I\!R$ has no semistrict local maxima.

We are now in the position to provide the following second–order characterization of a weakly H–quasiconvex function.

Theorem 4.9. *Let $u \in \mathcal{C}^2(\Omega)$, where Ω is an open, weakly H–convex subset of $I\!H$. Then u is weakly H–quasiconvex if and only if for every $g_0 \in \Omega$ and for every $\mathbf{v} = (v_1, v_2) \in I\!R^2$ such that $||\mathbf{v}|| = 1$ and $\langle (\nabla_h u)(g_0), \mathbf{v}\rangle = 0$, one of the following two conditions is fulfilled:*

i) $\mathbf{v}[(\nabla_h^2 u)(g_0)]^* \mathbf{v}^T > 0$,

ii) $\mathbf{v}[(\nabla_h^2 u)(g_0)]^* \mathbf{v}^T = 0$, *and the function $F(s) = u(g_0 \cdot \exp(sv_1X + sv_2Y))$ does not have a semistrict local maximum at $s = 0$.*

Proof. Assume first that u is weakly H–quasiconvex, and take $g_0 \in \Omega$ and $\mathbf{v} = (v_1, v_2) \in I\!R^2$, with $||\mathbf{v}|| = 1$, and such that $\langle (\nabla_h u)(g_0), \mathbf{v}\rangle = 0$. By contradiction, let $\mathbf{v}[(\nabla_h^2 u)(g_0)]^* \mathbf{v}^T < 0$. Since $u \in \mathcal{C}^2(\Omega)$, there exists $\epsilon > 0$ such that $\mathbf{v}(\nabla_h^2 u)(g)]^* \mathbf{v}^T < 0$ for any $g \in \Omega$ with $d_N(g, g_0) < \epsilon$. Moreover, since Ω is open, we can find two points $g_1, g_2 \in H_{g_0}$, with $d_N(g_i, g_0) < \epsilon$ and such that $g_1 = g_0 \cdot \exp(-\epsilon_1 v_1 X - \epsilon_1 v_2 Y)$, $g_2 = g_0 \cdot \exp(\epsilon_2 v_1 X + \epsilon_2 v_2 Y)$, for some positive ϵ_i. Consider $g_\lambda = g_0 \delta_\lambda (g_0^{-1} g_1)$, $\lambda \in [0,1]$. Then, for a suitable $\tilde{\lambda} \in [0,1]$ and for $K > 0$,

$$u(g_1) = u(g_0) + K(x_1 - x_0, y_1 - y_0)[(\nabla_h^2 u)(g_{\tilde{\lambda}})]^*(x_1 - x_0, y_1 - y_0)^T < u(g_0).$$

Arguing in a similar way with respect to g_2, we get that

$$u(g_0) > \max(u(g_1), u(g_2)),$$

but this is a contradiction. This implies that

$$\langle (\nabla_h u)(g_0), \mathbf{v}\rangle = 0 \quad \Longrightarrow \quad \mathbf{v}[(\nabla_h^2 u)(g_0)]^* \mathbf{v}^T \geq 0.$$

If $\mathbf{v}[(\nabla_h^2 u)(g_0)]^* \mathbf{v}^T > 0$, there is nothing more to prove. Assume that

$$\mathbf{v}[(\nabla_h^2 u)(g_0)]^* \mathbf{v}^T = 0.$$

Adapting Remark 4.8 to this case ($g_1 = g$ and $g_2 = g'$) we easily conclude that the function $F(s) = u(g_0 \cdot \exp(sv_1X + sv_2Y))$ has not a semistrict local maximum at $s = 0$.

Suppose now that condition $i)$ or $ii)$ is fulfilled whenever $\langle (\nabla_h u)(g_0), \mathbf{v}\rangle = 0$. By contradiction, suppose that there are $g_1, g_2 \in \Omega$, with $g_2 \in H_{g_1}$, such that $u(\overline{g}) > \max(u(g_1), u(g_2))$ for some $\overline{g} \in \gamma_{g_1,g_2}$. Denote by g_0 a point where u attains its maximum in γ_{g_1,g_2}; in particular, $\langle (\nabla_h u)(g_0), \mathbf{v}\rangle = 0$. From the assumption, this implies that $i)$ or $ii)$ holds. As a matter of fact, $i)$ cannot occur, since g_0 is not a minimum point for u on γ_{g_1,g_2}; even condition $ii)$ cannot be satisfied, since the point $s = 0$ is a semistrict local maximum for the function $F(s) = u(g_0 \cdot \exp(sv_1X + sv_2Y))$ defined on $[-\epsilon_1, \epsilon_2]$. This provides a contradiction. □

4.1. Example of weakly H–quasiconvex, non weakly H–convex, functions

We will now construct weakly H–quasiconvex functions which are not, in general, weakly H–convex. Following the idea in [3], we consider functions of the type

$$u(x, y, t) = ((x^2 + y^2)^2 + z(t))^{1/4},$$

where $z : I\!R \to I\!R$ is assumed to be twice continuously differentiable and positive. A simple computation shows that

$$\mathrm{Det}[(\nabla_h^2 u)(x,y,t)]^* = 3u(x,y,t)^{-10}(x^2+y^2)^2[z(t)(1+z''(t)) - 3(z'(t))^2/4]$$
$$\mathrm{tr}[(\nabla_h^2 u)(x,y,t)]^* = (x^2+y^2)[4z(t)(1+z''(t)) + (1+z''(t))(x^2+y^2)^2 - 3(z'(t))^2/4].$$

We recall (see Theorem 4.2) that u is weakly H–convex on $I\!H$ if and only if $[(\nabla_h^2 u)(x,y,t)]^*$ is positive semidefinite on $I\!H$. Since z is positive, straightforward arguments show that

$$4z(1+z'') \geq 3(z')^2, \quad \text{on } I\!R \tag{10}$$

if and only if both $\mathrm{Det}[(\nabla_h^2 u)(g)]^*$ and $\mathrm{tr}[(\nabla_h^2 u)(g)]^*$ are nonnegative on $I\!H$. Hence, (10) is equivalent to the weakly H–convexity of u.

In the following, according to our Theorem 4.9, we find conditions on z assuring that u is weakly H–quasiconvex. Since

$$(\nabla_h u)(x,y,t) = \left(\frac{(x^2+y^2)x + yz'(t)/2}{((x^2+y^2)^2+z(t))^{3/4}}, \frac{(x^2+y^2)y - xz'(t)/2}{((x^2+y^2)^2+z(t))^{3/4}} \right),$$

$(\nabla_h u)(g) = (0,0)$ if and only if $g = (0,0,t)$. In this case we get that $[(\nabla_h^2 u)(0,0,t)]^*$ is null, for every t. Hence we consider $\mathbf{v} = (v_1, v_2) \in I\!R^2$, $\|\mathbf{v}\| = 1$ and the function $F : I\!R \to I\!R$ defined by $F(s) = u(g \cdot \exp(sv_1 X + sv_2 Y)) = (s^4 + z(t))^{1/4}$. It is clear that we do not have a semistrict local maximum for F at $s = 0$.

Let us consider a point $g = (x,y,t)$ with $(x,y) \neq (0,0)$. Clearly, if we choose $\mathbf{v} = (v_1,v_2) \in I\!R^2$, with $||\mathbf{v}|| = 1$, as $(v_1,v_2) = c(-(Yu)(g), (Xu)(g))$, with $c \in I\!R \setminus \{0\}$, we have that $\langle (\nabla_h u)(g), \mathbf{v} \rangle = 0$. We obtain

$$\mathbf{v}[(\nabla_h^2 u)(g)]^* \mathbf{v}^T = \frac{c^2(x^2+y^2)^2[4(x^2+y^2)^2(z''(t)+1) + 3z'(t)^2]}{((x^2+y^2)^2+z(t))^{9/4}}.$$

Since $(x,y) \neq (0,0)$, we see that the following conditions on z are sufficient to guarantee that $u(x,y,t) = ((x^2+y^2)^2 + z(t))^{1/4}$ is weakly H–quasiconvex:

(i) $z \in \mathcal{C}^2(I\!R)$, $z > 0$ on $I\!R$;

(ii) $1 + z'' \geq 0$ on $I\!R$;

(iii) if $1 + z''(t_0) = 0$, then $z'(t_0) \neq 0$.

Observe that, for instance, the function $z : I\!R \to I\!R$ defined by $z(t) = \frac{1}{2}(te^{-t^2} + 1)$ satisfies these conditions, but the inequality (10) does not hold; therefore, $u(x,y,t) = ((x^2+y^2)^2 + z(t))^{1/4}$ is weakly H–quasiconvex, but it is not weakly H–convex.

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References

[1] M. Avriel, W. E. Diewert, I. Zang: Nine kinds of quasiconcavity and concavity, J. Econ. Theory 25 (1981) 397–420.

[2] Z. M. Balogh, R. Monti: Accessible domains in the Heisenberg group, Proc. Amer. Math. Soc. 132 (2003) 97–106.

[3] Z. M. Balogh, M. Rickly: Regularity of convex functions on Heisenberg groups, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) 2 (2003) 847–868.

[4] A. Bellaïche, J.-J. Risler (eds.): Sub–Riemannian Geometry, Progress in Mathematics 144, Birkhäuser, Basel (1996).

[5] A. Calogero, G. Carcano, R. Pini: Optimization in the Heisenberg group, Optimization 55 (2006) 387–403.

[6] A. Calogero, G. Carcano, R. Pini: Twisted convex hulls in the Heisenberg group, J. Convex Analysis 14 (2007) 607–619.

[7] J. P. Crouzeix: On second order conditions for quasiconvexity, Math. Program. 18 (1980) 349–352.

[8] D. Danielli, N. Garofalo, D. M. Nhieu: Notions of convexity in Carnot groups, Comm. Anal. Geom. 11 (2003) 263–341.

[9] P. Juutinen, G. Lu, J. J. Manfredi, B. Stroffolini: Convex functions on Carnot groups, Rev. Mat. Iberoam. 23 (2007) 191–200.

[10] G. Lu, J. J. Manfredi, B. Stroffolini: Convex functions on the Heisenberg group, Calc. Var. Partial Differ. Equ. 19 (2004) 1–22.

[11] V. Magnani: Lipschitz continuity, Aleksandrov theorem and characterizations for H–convex functions, Math. Ann. 334 (2006) 199–233.

[12] R. Monti: Some properties of Carnot–Carathéodory balls in the Heisenberg group, Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat., IX. Ser., Rend. Lincei, Mat. Appl. 11 (2000) 155–167.

[13] R. Monti, M. Rickly: Geodetically convex sets in the Heisenberg group, J. Convex Analysis 12 (2005) 187–196.

[14] M. B. Sun, X. P. Yang: Some properties of quasiconvex functions on the Heisenberg Group, Acta Math. Appl. Sin., Engl. Ser. 21(4) (2005) 571–580.